Notes on a converse to the Sunada theorem for regular graphs

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1 Introduction

A popular way to construct pairs of isospectral regular graphs is via the Sunada construction.[1] We find a finite group G with two almost-conjugate subgroups H_1 and H_2 and construct a graph Γ on which G acts by automorphisms with no fixed vertices. The quotient graphs $H_1\backslash\Gamma$ and $H_2\backslash\Gamma$ will then be isospectral, by the well-known Theorem 2 below.

Robert Brooks raised the question whether all pairs of isospectral regular graphs arise in this way, and provided a partial answer in [1].

In this note, we examine two independent proofs of Theorem 2 and consider what they have to tell us about a possible converse.

2 Notation and definitions

Let Γ be a finite, undirected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. Suppose $|V(\Gamma)| = N$. Consider the vector space $\mathbb{C}^{V(\Gamma)}$ of complex-valued functions on the vertices of Γ. The *adjacency operator* A_Γ is a linear map from $\mathbb{C}^{V(\Gamma)}$ to itself given by

$$A_\Gamma\varphi(v) \quad = \quad \sum_{w\sim v}\varphi(w) \tag{1}$$

[1] We are grateful to Bill Hoffman for pointing out that the Sunada construction, Sunada triples, and so on are also known by other names, such as Gassmann.

(for $\varphi \in \mathbb{C}^{V(\Gamma)}$) where the notation $w \sim v$ means that vertex w is joined to vertex v by an edge. The operator A_Γ is easily seen to be self-adjoint, and so it has N real eigenvalues (counted with multiplicity). The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ will be called the *adjacency spectrum* (or just the *spectrum*) of Γ.

Two graphs whose adjacency spectra are identical will be called *isospectral*.

Suppose now that we have a group H of automorphisms of Γ, acting without any fixed points. We form the *quotient graph* $H\backslash\Gamma$ in the usual way: Each vertex of $H\backslash\Gamma$ corresponds to an orbit $Hv \subset V(\Gamma)$ under the action of H on $V(\Gamma)$. Two orbits Hv and Hw are joined by an edge in $H\backslash\Gamma$ if an only if there is some $h \in H$ for which $hv \sim w$ in Γ.

3 Group representations and Sunada triples

Let G be a finite group, and let H be a subgroup of G. Let $\{g_1, g_2, \ldots, g_s\}$ be a system of right coset representatives for H in G. Let G/H denote the set of actual right cosets: $G/H = \{Hg_1, Hg_2, \ldots, Hg_s\}$.

The group G acts on the set G/H by right multiplication. This action is used to define a representation of G in $\mathbb{C}^{G/H}$ called the *quasi-regular representation of G on the cosets of H*. We will denote it as π_H^G. We remark that π_H^G can be written as $\mathrm{Ind}_H^G 1_H$, where 1_H denotes the trivial representation on H.

Two subgroups H_1 and H_2 of G will be called *representation equivalent* if $\pi_{H_1}^G$ and $\pi_{H_2}^G$ are equivalent representations of G.

The same situation may be described in terms of conjugacy classes, as follows:

Let G be a finite group, and suppose H_1 and H_2 are subgroups of G. The triple (G, H_1, H_2) is called a *Sunada triple* if each conjugacy class in G intersects H_1 and H_2 in the same number of elements.

The following standard theorem says that representation equivalence and the Sunada condition are in fact equivalent.

Theorem 1 *Let G be a finite group. Let H_1 and H_2 be subgroups of G. Then*

(G, H_1, H_2) *is a Sunada triple if and only if* $\pi^G_{H_1}$ *and* $\pi^G_{H_2}$ *are equivalent.*

Proof We'll adopt the following convention in the proof. Let P be a proposition. Then $\delta(P)$ will be given by

$$\delta(P) \quad = \quad \begin{cases} 1 & \text{if} \quad P \text{ is true} \\ 0 & \text{if} \quad P \text{ is false} \end{cases} \tag{2}$$

Let G be a finite group, and H a subgroup of G. Let $G_R = \{g_1, g_2, \ldots, g_s\}$ be a system of right coset representatives for H in G. Then for each x in G, we have

$$\mathrm{Tr}\,(\pi^G_H(x)) \quad = \quad \sum_{g_i \in G_R} \delta(Hg_i x = Hg_i) \tag{3}$$

Now write $H = \{h_1, h_2, \ldots, h_s\}$ and observe that $Hh_j = H$ for every j, so that

$$\sum_{h_j \in H} \sum_{g_i \in G_R} \delta(Hh_j g_i x = Hh_j g_i) \quad = \quad \sum_{h_j \in H} \sum_{g_i \in G_R} \delta(Hg_i x = Hg_i) \tag{4}$$

The sum on the left in (4) is the same as

$$\sum_{g \in G} \delta(Hgx = Hg)$$

and the sum on the right is simply $|H|$ times the expression on the right side of (3). Thus we have

$$|H|\mathrm{Tr}\,(\pi^G_H(x)) \quad = \quad \sum_{g \in G} \delta(Hgx = Hg) \tag{5}$$

$$= \quad \sum_{g \in G} \delta(gxg^{-1} \in H). \tag{6}$$

Now as g runs through G, the expression gxg^{-1} hits every conjugate of x exactly $|C(x)|$ times, where $C(x)$ denotes the centralizer of x in G. Applying this to the right side of (6), we get

$$|H|\mathrm{Tr}\,(\pi^G_H(x)) \quad = \quad |C(x)| \sum_{y \in [x]} \delta(y \in H) \tag{7}$$

$$= \quad |C(x)| \cdot \sharp([x] \cap H). \tag{8}$$

Suppose now that (G, H_1, H_2) is a Sunada triple. Then for each $x \in G$, we have

$$\sharp([x] \cap H_1) \quad = \quad \sharp([x] \cap H_2) \tag{9}$$

and summing (9) over all conjugacy classes, we find that $|H_1| = |H_2|$. Then for each $x \in G$, we get

$$\begin{aligned} \mathrm{Tr}\,(\pi^G_{H_1}(x)) &= \frac{|C(x)|}{|H_1|} \cdot \sharp([x] \cap H_1) & (10) \\ &= \frac{|C(x)|}{|H_2|} \cdot \sharp([x] \cap H_2) & (11) \\ &= \mathrm{Tr}\,(\pi^G_{H_2}(x)). & (12) \end{aligned}$$

Two representations whose characters are equal are equivalent, so we can conclude that $\pi^G_{H_1}$ and $\pi^G_{H_2}$ are equivalent.

Conversely, if $\pi^G_{H_1}$ and $\pi^G_{H_2}$ are equivalent, then they must have the same dimension, so $|H_1| = |H_2|$, and again we can use (8) to get

$$\begin{aligned} \sharp([x] \cap H_1) &= \frac{|H_1|}{|C(x)|} \mathrm{Tr}\,(\pi^G_{H_1}) & (13) \\ &= \frac{|H_2|}{|C(x)|} \mathrm{Tr}\,(\pi^G_{H_2}) & (14) \\ &= \sharp([x] \cap H_2) & (15) \end{aligned}$$

for each $x \in G$. □

4 Statement of the Theorem

Here is the theorem whose converse we propose to explore:

Theorem 2 *Let Γ be a finite, regular, undirected graph. Let G be a group of automorphisms of Γ, acting with no fixed vertices, and let H_1 and H_2 be subgroups of G.*

If (G, H_1, H_2) is a Sunada triple, or, equivalently, if $\pi^G_{H_1}$ and $\pi^G_{H_2}$ are equivalent, then the graphs $H_1 \backslash \Gamma$ and $H_2 \backslash \Gamma$ are isospectral.

We will recount two proofs of this theorem – one using the representation-theory condition and one using the conjugacy-class condition – and consider the extent to which each proof is convertible.

5 Pesce's proof and a converse

The first proof is slightly adapted from a paper ([2]) by Hubert Pesce, and uses the language of representations.

Proof For notational convenience, denote $H_1\backslash\Gamma$ by Γ_1 and $H_2\backslash\Gamma$ by Γ_2.

We first observe that if λ is an eigenvalue of the adjacency operator on any quotient $H\backslash\Gamma$ of Γ by an automorphism subgroup H, then λ is also an eigenvalue of the adjacency operator on Γ. This is because any λ-eigenfunction on $H\backslash\Gamma$ lifts to a λ-eigenfunction on Γ.

It follows that the spectra of Γ_1 and Γ_2 are both subsets of the spectrum of Γ. To show that Γ_1 and Γ_2 are isospectral, we need only show that for each eigenvalue λ of Γ, the dimensions of the λ-eigenspaces on Γ_1 and Γ_2 are equal.

We have a natural representation ρ_G of G on $\mathbb{C}^{V(\Gamma)}$ by

$$[\rho_G(g)\varphi](v) \quad = \quad \varphi(g^{-1}v) \tag{16}$$

for $\varphi \in \mathbb{C}^{V(\Gamma)}$. Since each $g \in G$ preserves adjacencies in Γ, each λ-eigenspace is invariant under the representation ρ_G. Thus for each eigenvalue λ, we have a subrepresentation ρ_G^λ of G on the λ-eigenspace on Γ. The degree of this representation is just the dimension of the λ-eigenspace on Γ.

Now consider a λ-eigenspace on Γ_1. Let $\varphi_1, \varphi_2, \ldots, \varphi_t$ be a basis for this space and consider the lifts $\tilde{\varphi}_i$ of these basis functions to Γ. The functions $\tilde{\varphi}_i$ remain linearly independent, they remain λ-eigenfunctions, and each one is invariant under $\operatorname{Res}_{H_1}^{G}\rho_G^\lambda$. Conversely, any λ-eigenfunction on Γ that's invariant under $\operatorname{Res}_{H_1}^{G}\rho_G^\lambda$ projects to a λ-eigenfunction on $H_1\backslash\Gamma$.

From this, we conclude that the dimension of the λ-eigenspace on Γ_1 is in fact equal to the dimension of that part of the λ-eigenspace on Γ left fixed by $\operatorname{Res}_{H_1}^{G}\rho_G^\lambda$. This latter number is given by $[1_{H_1} : \operatorname{Res}_{H_1}^{G}\rho_G^\lambda]$, where 1_{H_1} is the trivial representation on H_1. Similarly, the dimension of the λ-eigenspace on Γ_2 is equal to $[1_{H_2} : \operatorname{Res}^G_{H_2} \rho^\lambda_G]$.

By Frobenius reciprocity ([4]), we have

$$[1_{H_i} : \operatorname{Res}^G_{H_i} \rho^\lambda_G] = [\operatorname{Ind}^G_{H_i} 1_{H_i} : \rho^\lambda_G] \tag{17}$$

for $i = 1, 2$. By our earlier remark, $\operatorname{Ind}^G_{H_i} 1_{H_i}$ is just our quasi-regular representation $\pi^G_{H_i}$, so we conclude that

$$\begin{pmatrix} \text{dimension of} \\ \lambda\text{-eigenspace on } \Gamma_i \end{pmatrix} = [\pi^G_{H_i} : \rho^\lambda_G] \tag{18}$$

for $i = 1, 2$.

Now by hypothesis, $\pi^G_{H_1}$ and $\pi^G_{H_2}$ are equivalent, so their inner products with ρ^λ_G must be equal. Thus we have what we want: for each λ in the spectrum of Γ, the dimensions of the λ-eigenspaces on Γ_1 and Γ_2 are equal. The two quotient graphs are isospectral. □

Now suppose we have a regular, undirected graph Γ, a freely-acting group of automorphisms G of Γ, and two subgroups H_1 and H_2 of G such that the quotient graphs $H_1\backslash\Gamma$ and $H_2\backslash\Gamma$ are isospectral. What can we say about H_1 and H_2? In particular, are $\pi^G_{H_1}$ and $\pi^G_{H_2}$ necessarily equivalent?

According to proof above, if the graphs $H_1\backslash\Gamma$ and $H_2\backslash\Gamma$ are isospectral, then we get

$$[\pi^G_{H_1} : \rho^{\lambda_i}_G] = [\pi^G_{H_2} : \rho^{\lambda_i}_G] \tag{19}$$

for each $i \in \{1, \cdots, N\}$. Let $\hat{G} = \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ be a complete set of irreducible representations of G. Then for each i, we get the decomposition

$$[\pi^G_{H_1} : \rho^{\lambda_i}_G] = \sum_{\sigma_j \in \hat{G}} [\sigma_j : \pi^G_{H_1}][\sigma_j : \rho^{\lambda_i}_G] \tag{20}$$

and similarly for $\pi^G_{H_2}$.

We combine (19) and (20) to get a system of linear equations relating the numbers $[\sigma_j : \pi^G_{H_1}]$ and $[\sigma_j : \pi^G_{H_2}]$. Specifically, we find that the isospectrality of $H_1\backslash\Gamma$ and $H_2\backslash\Gamma$ implies that

$$\begin{pmatrix} [\sigma_1 : \rho_G^{\lambda_1}] & [\sigma_2 : \rho_G^{\lambda_1}] & \cdots & [\sigma_r : \rho_G^{\lambda_1}] \\ [\sigma_1 : \rho_G^{\lambda_2}] & [\sigma_2 : \rho_G^{\lambda_2}] & \cdots & [\sigma_r : \rho_G^{\lambda_2}] \\ \vdots & \vdots & \ddots & \vdots \\ [\sigma_1 : \rho_G^{\lambda_N}] & [\sigma_2 : \rho_G^{\lambda_N}] & \cdots & [\sigma_r : \rho_G^{\lambda_N}] \end{pmatrix} \begin{pmatrix} [\sigma_1 : \pi_{H_1}^G] - [\sigma_1 : \pi_{H_2}^G] \\ [\sigma_2 : \pi_{H_1}^G] - [\sigma_2 : \pi_{H_2}^G] \\ \vdots \\ [\sigma_r : \pi_{H_1}^G] - [\sigma_r : \pi_{H_2}^G] \end{pmatrix} \quad (21)$$

equals the zero matrix.

Since $\pi_{H_1}^G$ and $\pi_{H_2}^G$ are equivalent exactly when the column vector in (21) is 0, we have identified a condition under which the isospectrality of $H_1\backslash\Gamma$ and $H_2\backslash\Gamma$ will guarantee that $\pi_{H_1}^G$ and $\pi_{H_2}^G$ are equivalent. We have

Theorem 3 *Let Γ be a finite, undirected, regular graph, G a group of freely-acting automorphisms of Γ, and H_1 and H_2 subgroups of G such that $H_1\backslash\Gamma$ and $H_2\backslash\Gamma$ are isospectral. Let $\hat{G} = \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ be the set of irreducible representations of G. If the matrix*

$$\begin{pmatrix} [\sigma_1 : \rho_G^{\lambda_1}] & [\sigma_2 : \rho_G^{\lambda_1}] & \cdots & [\sigma_r : \rho_G^{\lambda_1}] \\ [\sigma_1 : \rho_G^{\lambda_2}] & [\sigma_2 : \rho_G^{\lambda_2}] & \cdots & [\sigma_r : \rho_G^{\lambda_2}] \\ \vdots & \vdots & \ddots & \vdots \\ [\sigma_1 : \rho_G^{\lambda_N}] & [\sigma_2 : \rho_G^{\lambda_N}] & \cdots & [\sigma_r : \rho_G^{\lambda_N}] \end{pmatrix}$$

has full rank, then $\pi_{H_1}^G$ and $\pi_{H_2}^G$ are equivalent.

We remark that the full-rank requirement is a condition on G and Γ only, so that if a group G and a graph Γ satisfy this condition, then every pair of isospectral quotients $H_1\backslash\Gamma$ and $H_2\backslash\Gamma$ (with H_1 and H_2 subgroups of G) must arise from a Sunada triple.

6 Brooks's proof and a converse

The second proof of Theorem 2, due to Robert Brooks, is more geometric in flavor, and stays well clear of any representation theory.

Before getting started, we'll need a few definitions.

An n-*walk* in a graph Γ is a sequence $v_0e_1v_1e_2v_2e_3\cdots e_nv_n$ of vertices v_i and edges e_j in Γ such that e_i is incident on v_{i-1} and v_i for each $i \in \{1,\ldots,n\}$. The n-walk is called *closed* if $v_0 = v_n$. It is called *non-backtracking* (abbreviated here as NBT) if $e_i \neq e_{i+1}$ for any $i \in \{1,\ldots,n-1\}$.

It is standard that the number of closed n-walks in a graph is equal to the n^{th} moment of the adjacency spectrum. Thus the adjacency spectrum of a graph and the numbers of closed n-walks (for n going from 0 to the number of vertices) determine one another.

A similar result holds for closed NBT walks.

Theorem 4 *(See [3].) Let Γ be a finite, undirected, regular graph. For each non-negative integer n, let D_n denote the total number of closed* NBT *n-walks in Γ. Then the adjacency spectrum of Γ determines the sequence $D_0, D_1, D_2, \ldots$, and the numbers $D_1, D_2, \ldots, D_N$ determine the adjacency spectrum of Γ.*

Brooks ([1]) proved Theorem 2 by using the conjugacy-class version of the Sunada condition to count closed NBT n-walks. Here is an outline of the proof.

Proof Let G be a freely-acting group of automorphisms on a finite graph Γ, and let H be a subgroup of G.

Fix n and let B_n denote the set of all NBT n-walks in Γ whose endpoints differ by some element of G. It is not hard to see that

$$\sharp\begin{pmatrix}\text{closed NBT}\\ n\text{-walks in } H\backslash\Gamma\end{pmatrix} = \frac{1}{|H|}\cdot\sharp\begin{pmatrix}\text{walks in } B_n \text{ that project}\\ \text{to closed walks}\\ \text{in } H\backslash\Gamma\end{pmatrix}. \quad (22)$$

Thus the goal is to count the number of walks $\alpha \in B_n$ whose projections to $H\backslash\Gamma$ close up.

We partition B_n into G-orbits: $B_n = B_{n,1} \cup B_{n,2} \cup \cdots \cup B_{n,m}$, and select a representative walk α_i from each $B_{n,i}$. Each α_i determines a unique element g_{α_i} of G such that g_{α_i} takes the first vertex of α_i to the last vertex of α_i.

For an arbitrary $g \in G$, the walk $g(\alpha_i)$ projects to a closed walk in $H\backslash\Gamma$ if and only if $gg_{\alpha_i}g^{-1} \in H$. Thus the number of elements of $B_{n,i}$ that project to closed walks in $H\backslash\Gamma$ is $|C(g_{\alpha_i})| \cdot \sharp([g_{\alpha_i}] \cap H)$.

From this it follows that the total number of closed NBT n-walks in $H\backslash\Gamma$ is given by

$$\frac{1}{|H|} \sum_i |C(g_{\alpha_i})| \cdot \sharp([g_{\alpha_i}] \cap H). \tag{23}$$

Since this number depends on H only through the sizes of the intersections of H with the conjugacy classes of G, we have shown that two graphs $H_1\backslash\Gamma$ and $H_2\backslash\Gamma$ are isospectral if the numbers $\sharp(H_1 \cap [g])$ and $\sharp(H_2 \cap [g])$ in the Sunada condition are equal. □

What happens if we read this second proof backward?

Suppose we have an isospectral pair $H_1\backslash\Gamma$ and $H_2\backslash\Gamma$ in the usual setting.

It is trivial that isospectrality implies $|H_1| = |H_2|$, so the Brooks proof tells us that

$$\sum_i |C(g_{\alpha_i})| \sharp([g_{\alpha_i}] \cap H_1) \quad = \quad \sum_i |C(g_{\alpha_i})| \sharp([g_{\alpha_i}] \cap H_2) \tag{24}$$

for each n, where the sum is taken over representatives α_i of the equivalence classes of NBT n-walks in Γ.

We rewrite the left-hand sum in (24) as a sum over conjugacy classes. We get

$$\sum_{[g]\subset G} \left(\sum_{g_{\alpha_i}\in[g]} |C(g_{\alpha_i})| \right) \sharp([g] \cap H_1) \tag{25}$$

where the outer sum runs through all the conjugacy classes in G. Now for each $g_{\alpha_i} \in [g]$, $|C(g_{\alpha_i})| = |C(g)|$, so the inner sum in (25) is just $|C(g)|$ times the number of g_{α_i} in $[g]$. We denote this number by $c(n, g)$ and rewrite equation 24 as

$$\sum_{[g]\subset G} c(n, g)|C(g)| \cdot \sharp([g] \cap H_1) \quad = \quad \sum_{[g]\subset G} c(n, g)|C(g)| \cdot \sharp([g] \cap H_2). \tag{26}$$

Once again, we have a system of linear equations. In this case, the numbers of interest are $\sharp([g] \cap H_1)$ and $\sharp([g] \cap H_2)$; if $\sharp([g] \cap H_1) = \sharp([g] \cap H_2)$ for each conjugacy class $[g] \subset G$, then (G, H_1, H_2) is a Sunada triple. Putting the system in matrix form, we have

Theorem 5 *Let Γ be a finite, undirected, regular graph, G a group of freely-acting automorphisms of Γ, and H_1 and H_2 subgroups of G such that $H_1 \backslash \Gamma$ and $H_2 \backslash \Gamma$ are isospectral.*

For each non-negative integer n and each $g \in G$, let $c(n, g)$ denote the number of equivalence classes of NBT *n-walks in Γ whose associated group elements are conjugate to g. Let $[g_1], [g_2], \ldots, [g_r]$ denote the conjugacy classes in G. If there is some matrix*

$$\begin{pmatrix} c(n_1, g_1)|C(g_1)| & c(n_1, g_2)|C(g_2)| & \ldots & c(n_1, g_r)|C(g_r)| \\ c(n_2, g_1)|C(g_1)| & c(n_2, g_2)|C(g_2)| & \ldots & c(n_2, g_r)|C(g_r)| \\ \vdots & \vdots & \ddots & \vdots \\ c(n_m, g_1)|C(g_1)| & c(n_m, g_2)|C(g_2)| & \ldots & c(n_m, g_r)|C(g_r)| \end{pmatrix} \tag{27}$$

with full rank, then (G, H_1, H_2) is a Sunada triple.

We remark that we can factor a diagonal matrix (with entries $|C(g_i)|$) out of matrix (27), so we are really looking for a full-rank matrix of the form

$$\begin{pmatrix} c(n_1, g_1) & c(n_1, g_2) & \ldots & c(n_1, g_r) \\ c(n_2, g_1) & c(n_2, g_2) & \ldots & c(n_2, g_r) \\ \vdots & \vdots & \ddots & \vdots \\ c(n_m, g_1) & c(n_m, g_2) & \ldots & c(n_m, g_r) \end{pmatrix} \tag{28}$$

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