On Decidability and Expressiveness of Propositional Interval Neighborhood Logics

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Abstract. Interval-based temporal logics are an important research area in computer science and artificial intelligence. In this paper we investigate decidability and expressiveness issues for Propositional Neighborhood Logics (PNLs). We begin by comparing the expressiveness of the different PNLs. Then, we focus on the most expressive one, namely, $\mathrm{PNL}^{\pi+}$, and we show that it is decidable over various classes of linear orders by reducing its satisfiability problem to that of the two-variable fragment of first-order logic with binary relations over linearly ordered domains, due to Otto. Next, we prove that $\mathrm{PNL}^{\pi+}$ is expressively complete with respect to such a fragment. We conclude the paper by comparing $\mathrm{PNL}^{\pi+}$ expressiveness with that of other interval-based temporal logics.
Keywords: neighbourhood interval logics, decidability, expressiveness.

1 Introduction

Interval-based temporal logics over ordered domains are an important research area in various fields of computer science and artificial intelligence. Unfortunately, even when restricted to the case of *propositional languages* and *linear time*, they usually exhibit a bad computational behavior where undecidability rules. The main species of studied propositional interval temporal logics include Moszkowski's Propositional Interval Logic (PITL) [17], Halpern and Shoham's modal logic of time intervals (HS) [12], Venema's CDT logic [22] (extended to branching-time frames with linear intervals by Goranko, Montanari and Sciavicco [9]), Lodaya's Begins/Ends fragment of HS (BE) [15], and Montanari, Goranko and Sciavicco's Propositional Neighborhood Logics [7]. Many expressiveness and (un)decidability results for these logics substantially depend on the assumptions about the class of frames over which they are interpreted. Typical classes of frames are the class of all (resp., dense, discrete, Dedekind complete) linear frames, and of specific linear orders such as $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$. Basic results

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about these logics are the undecidability of HS and CDT over most of classes of frames, of PITL over dense and discrete frames, and of BE over dense frames. A comprehensive survey of the main developments, results, and open problems in the area of propositional interval temporal logics can be found in [8].

In this paper we focus our attention on expressiveness and decidability issues for Propositional Neighborhood Logics (PNLs) over various classes of linear orders. PNLs are fragments of HS which feature two modalities, corresponding to Allen's relations *meets* and *met by*, and (possibly) the interval operator π. Sound and complete axiomatic systems for PNLs and a tableau-based semi-decision procedure for them have been developed in [7]. A tableau-based decision procedure for the future fragments of PNLs, interpreted over $\mathbb{N}$, together with a proof of NEXPTIME-completeness, have been given in [2, 4] and later extended to full PNLs over $\mathbb{Z}$ [3]. To the best of our knowledge, these are the first non-trivial decidability results for propositional interval logics interpreted over *fully-instantiated* temporal structures, that is, temporal structures containing all intervals that can be built up from a given linearly ordered set of points, which do not resort to any *projection principle*, such as locality or homogeneity [12].

Here is a summary of the paper. First, we compare the expressive power of three PNLs, namely, $\text{PNL}^{\pi+}$, PNL^{+}, and PNL^{-}, and we show that $\text{PNL}^{\pi+}$ is strictly more expressive than PNL^{+} and PNL^{-}. Then, we prove that the satisfiability problem for $\text{PNL}^{\pi+}$ over the classes of all linear orders, of all well-orders, and of all finite linear orders, can be decided in NEXPTIME by reducing it to the satisfiability problem for the two-variable fragment of first-order logic over the same classes of structures [18]. Next, we focus on expressive completeness, in the spirit of Kamp's theorem [14]. Kamp proved the functional completeness of the *Since* (S) and *Until* (U) temporal logic with respect to first-order definable connectives over Dedekind-complete linear orders. This result has been later re-proved and generalized in several ways (see [13, 6]). In particular, Stavi extended Kamp's result to the class of all linear orders by adding the binary operators S' and U' (see [6] for details), while Etessami et al. proved the functional completeness of the *future* (F) and *past* (P) temporal logic (TL[F,P] for short) with respect to the monadic two-variable fragment of first-order logic $\text{MFO}^2[<]$ over $\mathbb{N}$ [5]. As for interval-based logics, Venema showed the functional completeness of CDT with respect to the three-variable (with at most two of them free) fragment of first-order logic $\text{FO}^3_{x,y}[<]$ over all linear orders. In this paper we prove the expressive completeness of $\text{PNL}^{\pi+}$ with respect to the full two-variable fragment of first-order logic over various classes of linear orders. We conclude the paper with a comparison of $\text{PNL}^{\pi+}$ expressive power with that of other HS fragments.

2 Basics

Propositional Neighborhood Logics. The syntax and semantics of propositional neighborhood logics (PNLs for short), interpreted over linear orders, are defined as follows. Let $\mathbb{D} = \langle D, < \rangle$ be a linearly ordered set. An *interval* over $\mathbb{D}$ is an ordered pair $[a, b]$, where $a, b \in D$ and $a \leq b$. An interval $[a, b]$ is a *strict interval* if $a < b$, while it is a *point interval* if $a = b$. We denote the set of all (resp., strict) intervals over $\mathbb{D}$ by $\mathbb{I}(\mathbb{D})^+$ (resp., $\mathbb{I}(\mathbb{D})^-$). The language of *Full Propositional Neighborhood Logic* ($\mathrm{PNL}^{\pi+}$) consists of a set $\mathcal{AP}$ of propositional letters, the propositional connectives $\neg, \vee$, the modal constant π, and the modal operators $\Diamond_r$ and $\Diamond_l$. The other propositional connectives, as well as the logical constants $\top$ (*true*) and $\bot$ (*false*) and the dual modal operators $\Box_r$ and $\Box_l$, are defined as usual. *Formulas* of $\mathrm{PNL}^{\pi+}$, denoted by $\varphi, \psi, \ldots$, are recursively defined by the following grammar: $\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \pi \mid \Diamond_r\varphi \mid \Diamond_l\varphi$. The language of *Non-strict Propositional Neighborhood Logic* (PNL^+) is the fragment of $\mathrm{PNL}^{\pi+}$ devoid of the modal constant π, while the language of *Strict Propositional Neighborhood Logic* (PNL^-) is obtained from that of PNL^+ by replacing the modalities $\Diamond_r$ and $\Diamond_l$ with the modalities $\langle A\rangle$ and $\langle\overline{A}\rangle$ (with dual modalities $[A]$ and $[\overline{A}]$), respectively. We adopt different notations for the modalities of $\mathrm{PNL}^{\pi+}/\mathrm{PNL}^+$ and PNL^- to reflect their historical links and to make it easier to distinguish between their non-strict/strict semantics from the syntax. We will write PNLs when referring to either $\mathrm{PNL}^{\pi+}$, PNL^+, or PNL^-. The semantics of $\mathrm{PNL}^{\pi+}/\mathrm{PNL}^+$ is given in terms of *non-strict interval models* $\mathbf{M}^+ = \langle\mathbb{I}(\mathbb{D})^+, V^+\rangle$, while that of PNL^- is given in terms of *strict interval models* $\mathbf{M}^- = \langle\mathbb{I}(\mathbb{D})^-, V^-\rangle$. The *valuation function* $V^+ : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})^+}$ (resp., $V^- : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})^-}$) assigns to every propositional variable p the set of (all, resp. strict) intervals $V(p)$ over which p holds. Instead of V^+ and V^-, we will write just V whenever there is no risk of confusion; likewise we will write $\mathbb{I}(\mathbb{D})$ for either $\mathbb{I}(\mathbb{D})^+$ or $\mathbb{I}(\mathbb{D})^-$. Note that for every p, $V(p)$ can be viewed as a binary relation on D, and we will use that later on. When referring to either the strict or the non-strict interval model, we will use $\mathbf{M}$. The *truth relation* of a formula at a given interval in a model $\mathbf{M}$ is defined by structural induction on formulas:

- $\mathbf{M}, [a,b] \Vdash p$ iff $[a,b] \in V(p)$, for all $p \in \mathcal{AP}$;
- $\mathbf{M}, [a,b] \Vdash \neg\psi$ iff it is not the case that $\mathbf{M}, [a,b] \Vdash \psi$;
- $\mathbf{M}, [a,b] \Vdash \varphi \vee \psi$ iff $\mathbf{M}, [a,b] \Vdash \varphi$ or $\mathbf{M}, [a,b] \Vdash \psi$;
- $\mathbf{M}, [a,b] \Vdash \Diamond_r\psi$ (resp., $\langle A\rangle\psi$) iff there exists c such that $c \geq b$ (resp., $c > b$) and $\mathbf{M}, [b,c] \Vdash \psi$;
- $\mathbf{M}, [a,b] \Vdash \Diamond_l\psi$ (resp., $\langle\overline{A}\rangle\psi$) iff there exists c such that $c \leq a$ (resp., $c < a$) and $\mathbf{M}, [c,a] \Vdash \psi$;
- $\mathbf{M}^+, [a,b] \Vdash \pi$ iff $a = b$.

A formula is *satisfiable* if it is true over some interval in some interval model (for the respective language) and it is *valid* if it is true over every interval in every interval model. As shown in [7], PNLs are powerful enough to express interesting temporal properties, e.g., they allow one to constrain the structure of the underlying linear ordering. In particular, $\mathrm{PNL}^{\pi+}$ and PNL^- allow one to express the *difference* operator and thus to simulate *nominals*.

The two-variable fragment of first-order logic. In this section we give some basic definitions about fragments of first-order logic. Let us denote by FO^2 (resp., $\mathrm{FO}^2[=]$) the fragment of first-order logic (resp., first-order logic with equality) whose language uses only two distinct (possibly reused) variables. We denote its formulas by $\alpha, \beta, \ldots$. For example, the formula $\forall x(P(x) \rightarrow \forall y \exists x Q(x, y))$ belongs to FO^2, while the formula $\forall x(P(x) \rightarrow \forall y \exists z(Q(z, y) \wedge Q(z, x)))$ does not. We focus our attention on the logic $\mathrm{FO}^2[<]$ over a purely relational vocabulary $\{=, <, P, Q, \ldots\}$ including equality and a distinguished binary relation $<$ interpreted as a linear ordering. Since atoms in the two-variable fragment can involve at most two distinct variables, we may further assume without loss of generality that the arity of every relation is exactly 2.

Let x and y be the two variables of the language. The formulas of $\mathrm{FO}^2[<]$ can be defined recursively as follows:

$$\begin{aligned} \alpha &::= A_0 \mid A_1 \mid \neg\alpha \mid \alpha \vee \beta \mid \exists x \alpha \mid \exists y \alpha \\ A_0 &::= x = x \mid x = y \mid y = x \mid y = y \mid x < y \mid y < x \\ A_1 &::= P(x, x) \mid P(x, y) \mid P(y, x) \mid P(y, y), \end{aligned}$$

where A_1 deals with (uninterpreted) binary predicates. For technical convenience, we assume that both variables x and y occur as (possibly vacuous) free variables in every formula $\alpha \in \mathrm{FO}^2[<]$, that is, $\alpha = \alpha(x, y)$.

Formulas of $\mathrm{FO}^2[<]$ are interpreted over *relational models* of the form $\mathcal{A} = \langle \mathbb{D}, V_{\mathcal{A}} \rangle$, where $\mathbb{D}$ is a linear ordering and $V_{\mathcal{A}}$ is a *valuation function* that assigns to every *binary relation* P a subset of $D \times D$. When we evaluate a formula $\alpha(x, y)$ on a pair of elements a, b, we write $\alpha(a, b)$ for $\alpha[x := a, y := b]$.

The satisfiability problem for FO^2 without equality was proved decidable by Scott [19] by a satisfiability preserving reduction of any FO^2-formula to a formula of the form $\forall x \forall y \psi_0 \wedge \bigwedge_{i=1}^{m} \forall x \exists y \psi_i$, which belongs to the Gödel's prefix-defined decidable class of first-order formulas [1]. Later, Mortimer extended this result by including equality in the language [16]. More recently, Grädel, Kolaitis, and Vardi improved Mortimer's result by lowering the complexity bound [11]. Finally, by building on techniques from [11] and taking advantage of an in-depth analysis of the basic 1-types and 2-types in $\mathrm{FO}^2[<]$-models, Otto proved the decidability of $\mathrm{FO}^2[<]$ over the class of all linear orderings as well as on some natural subclasses of it [18].

Theorem 1 ([18]). *The satisfiability problem for formulas in* $\mathrm{FO}^2[<]$ *is decidable in NEXPTIME on each of the classes of structures where* $<$ *is interpreted as (i) any linear ordering, (ii) any well-ordering, (iii) any finite linear ordering, and (iv) the linear ordering on* $\mathbb{N}$.

Comparing the expressive power of interval logics. In the following we will compare the expressive power of $\mathrm{PNL}^{\pi+}$ with that of PNL^{+} and PNL^{-} as well as with that of other classical/temporal logics. There are several ways to compare the expressive power of different modal languages/logics, e.g., they can be compared with respect to frame validity, that is, with respect to the properties of frames that they can express (such a comparison for PNLs has been done in [7]). Here we compare the considered logics with respect to truth at a given element of a model. We distinguish three different cases: the case in which we compare two interval logics over the same class of models, e.g., $\mathrm{PNL}^{\pi+}$ and PNL^{+}, the case in which we compare strict and non-strict interval logics, e.g., PNL^{-} and $\mathrm{PNL}^{\pi+}$, and the case in which we compare an interval logic with a first-order logic, e.g., $\mathrm{PNL}^{\pi+}$ and $\mathrm{FO}^2[<]$.

Given two interval logics L and L' interpreted over the same class of models $\mathcal{C}$, we say that L' is *at least as expressive as* L (with respect to $\mathcal{C}$), denoted by $\mathrm{L} \preceq_{\mathcal{C}} \mathrm{L}'$ ($\mathcal{C}$ is omitted if clear from the context), if there exists an effective translation τ from L to L' (inductively defined on the structure of formulas) such that for every model $\mathbf{M}$ in $\mathcal{C}$, any interval $[a,b]$ in $\mathbf{M}$, and any formula φ of L, $\mathbf{M},[a,b] \Vdash \varphi$ iff $\mathbf{M},[a,b] \Vdash \tau(\varphi)$. Furthermore, we say that L is *as expressive as* L', denoted by $\mathrm{L} \equiv_{\mathcal{C}} \mathrm{L}'$, if both $\mathrm{L} \preceq_{\mathcal{C}} \mathrm{L}'$ and $\mathrm{L}' \preceq_{\mathcal{C}} \mathrm{L}$, while we say that L is *strictly more expressive than* L', denoted by $\mathrm{L}' \prec_{\mathcal{C}} \mathrm{L}$, if $\mathrm{L}' \preceq_{\mathcal{C}} \mathrm{L}$ and $\mathrm{L} \not\preceq_{\mathcal{C}} \mathrm{L}'$.

When comparing an interval logic L^- interpreted over *strict* interval models with an interval logic L^+ interpreted over *non-strict* ones, we need to slightly revise the above definitions. Given a strict interval model $\mathbf{M}^- = \langle \mathbb{I}(\mathbb{D})^-, V^- \rangle$, we say that a non-strict interval model $\mathbf{M}^+ = \langle \mathbb{I}(\mathbb{D})^+, V^+ \rangle$ is *a non-strict extension* of $\mathbf{M}^-$ (and that $\mathbf{M}^-$ is *the strict restriction* of $\mathbf{M}^+$) if V^- and V^+ agree on the valuation of strict intervals, that is, if for every strict interval $[a,b] \in \mathbb{I}(\mathbb{D})^-$ and propositional letter $p \in \mathcal{AP}$, $[a,b] \in V^-(p)$ if and only if $[a,b] \in V^+(p)$. We say that L^+ is *at least as expressive as* L^-, and we denote it by $\mathrm{L}^- \preceq_I \mathrm{L}^+$, if there exists an effective translation τ from L^- to L^+ such that for any strict interval model $\mathbf{M}^-$, any interval $[a,b]$ in $\mathbf{M}^-$, and any formula φ of L^-, $\mathbf{M}^-,[a,b] \Vdash \varphi$ iff $\mathbf{M}^+,[a,b] \Vdash \tau(\varphi)$ for every non-strict extension $\mathbf{M}^+$ of $\mathbf{M}^-$. Conversely, we say that L^- is *at least as expressive as* L^+, and we denote it by $\mathrm{L}^+ \preceq_I \mathrm{L}^-$, if there exists an effective translation τ' from L^+ to L^- such that for any non-strict interval model $\mathbf{M}^+$, any strict interval $[a,b]$ in $\mathbf{M}^+$, and any formula φ of L^+, $\mathbf{M}^+,[a,b] \Vdash \varphi$ iff $\mathbf{M}^-,[a,b] \Vdash \tau'(\varphi)$, where $\mathbf{M}^-$ is the strict restriction of $\mathbf{M}^+$. $\mathrm{L}^- \equiv_I \mathrm{L}^+$, $\mathrm{L}^- \prec_I \mathrm{L}^+$, and $\mathrm{L}^+ \prec_I \mathrm{L}^-$ are defined in the usual way.

Finally, we compare interval logics with first-order logics interpreted over relational models. In this case, the above criteria are no longer adequate, since we need to compare logics which are interpreted over different types of models (interval models and relational models). We deal with this complication by following the approach outlined by Venema in [22]. First, we define suitable model transformations (from interval models to relational models and vice versa); then, we compare the expressiveness of interval and first-order logics modulo these transformations. To define the mapping from interval models to relational models, we associate a binary relation P with every propositional variable $p \in \mathcal{AP}$ of the considered interval logic [22].

Definition 1. *Given an interval model* $\mathbf{M} = \langle \mathbb{I}(\mathbb{D}), V_{\mathbf{M}} \rangle$*, the corresponding relational model* $\eta(\mathbf{M})$ *is a pair* $\langle \mathbb{D}, V_{\eta(\mathbf{M})} \rangle$*, where for all* $p \in \mathcal{AP}$*,* $V_{\eta(\mathbf{M})}(P) = \{(a,b) \in D \times D : [a,b] \in V_{\mathbf{M}}(p)\}$.

As a matter of fact, the above relational models can be viewed as 'point' models for logics over $\mathbb{D}^2$ and the above transformation as a mapping of propositional letters of the interval logic, interpreted over $\mathbb{I}(\mathbb{D})$, into propositional letters of the target logic, interpreted over $\mathbb{D}^2$ [21, 20].

To define the mapping from relational models to interval ones, we have to solve a technical problem: the truth of formulas in interval models is evaluated only on ordered pairs $[a,b]$, with $a \leq b$, while in relational models there is not such a constraint. To deal with this problem, we associate two propositional letters $p^{\leq}$ and $p^{\geq}$ of the interval logic with every binary relation P.

Definition 2. *Given a relational model $\mathcal{A} = \langle \mathbb{D}, V_{\mathcal{A}} \rangle$, the corresponding non-strict interval model $\zeta(\mathcal{A})$ is a pair $\langle \mathbb{I}(\mathbb{D})^+, V_{\zeta(\mathcal{A})} \rangle$ such that for any binary relation P and any interval $[a,b]$, $[a,b] \in V_{\zeta(\mathcal{A})}(p^{\leq})$ iff $(a,b) \in V_{\mathcal{A}}(P)$ and $[a,b] \in V_{\zeta(\mathcal{A})}(p^{\geq})$ iff $(b,a) \in V_{\mathcal{A}}(P)$.*

Given an interval logic L_I and a first-order logic L_{FO}, we say that L_{FO} is *at least as expressive as* L_I, denoted by $\mathrm{L}_I \preceq_R \mathrm{L}_{FO}$, if there exists an effective translation τ from L_I to L_{FO} such that for any interval model $\mathbf{M}$, any interval $[a,b]$, and any formula φ of L_I, $\mathbf{M}, [a,b] \Vdash \varphi$ iff $\eta(\mathbf{M}) \models \tau(\varphi)(a,b)$. Conversely, we say that L_I is *at least as expressive as* L_{FO}, denote by $\mathrm{L}_{FO} \preceq_R \mathrm{L}_I$, if there exists an effective translation τ' from L_{FO} to L_I such that for any relational model $\mathcal{A}$, any pair (a,b) of elements, and any formula φ of L_{FO}, $\mathcal{A} \models \varphi(a,b)$ iff $\zeta(\mathcal{A}), [a,b] \Vdash \tau'(\varphi)$ if $a \leq b$ or $\zeta(\mathcal{A}), [b,a] \Vdash \tau'(\varphi)$ otherwise. We say that L_I is *as expressive as* L_{FO}, denoted by $\mathrm{L}_I \equiv_R \mathrm{L}_{FO}$, if $\mathrm{L}_I \preceq_R \mathrm{L}_{FO}$ and $\mathrm{L}_{FO} \preceq_R \mathrm{L}_I$. $\mathrm{L}_I \prec_R \mathrm{L}_{FO}$ and $\mathrm{L}_{FO} \prec_R \mathrm{L}_I$ are defined in the usual way.

3 PNL$^{\pi+}$, PNL$^+$, and PNL$^-$ expressiveness

In this section we compare the relative expressive power of PNL$^{\pi+}$, PNL$^+$, and PNL$^-$. The comparison of the expressive power of PNL$^{\pi+}$ and PNL$^+$ is based on an application of the bisimulation game for modal logics [10]. More precisely, we exploit a game-theoretic argument to show that there exist two models that can be distinguished by a PNL$^{\pi+}$ formula, but not by a PNL$^+$ formula. To this end, we define the notion of *k-round* PNL$^+$- *bisimulation game* to be played on a pair of PNL$^+$ models $(\mathbf{M_0}^+, \mathbf{M_1}^+)$, with $\mathbf{M_0}^+ = \langle \mathbb{I}(\mathbb{D}_0)^+, V_0 \rangle$ and $\mathbf{M_1}^+ = \langle \mathbb{I}(\mathbb{D}_1)^+, V_1 \rangle$, which starts from a given *initial configuration*, where a configuration is a pair of intervals $([a_0,b_0],[a_1,b_1])$, with $[a_0,b_0] \in \mathbb{I}(\mathbb{D}_0)^+$ and $[a_1,b_1] \in \mathbb{I}(\mathbb{D}_1)^+$. The game is played by two players, Player I and Player II. If after any given round the current position is not a local isomorphism between the submodels of $\mathbf{M_0}^+$ and $\mathbf{M_1}^+$ induced by the corresponding configuration, Player I wins the game; otherwise, Player II wins. At every round, given a current configuration $([a_0,b_0],[a_1,b_1])$, Player I plays one of the following two moves:

$\Diamond_r$**-move:** Player I chooses $\mathbf{M_i}^+$, with $i \in \{0,1\}$, and an interval $[b_i, c_i]$;
$\Diamond_l$**-move:** Player I chooses $\mathbf{M_i}^+$, with $i \in \{0,1\}$, and an interval $[c_i, a_i]$.

In the first case, Player II replies by choosing an interval $[b_{1-i}, c_{1-i}]$, which leads to the new configuration $([b_0,c_0],[b_1,c_1])$; in the other case, Player II chooses an interval $[c_{1-i}, a_{1-i}]$, which leads to the new configuration $([c_0,a_0],[c_1,a_1])$. Roughly speaking, Player II has a *winning strategy* in the k-round PNL$^+$-bisimulation game on the models $\mathbf{M_0}^+$ and $\mathbf{M_1}^+$ with a given initial configuration if she can win regardless of the moves played by Player I; otherwise, Player I has a winning strategy. A formal definition of winning strategy can be found in [10]. The following key property of the k-round PNL$^+$-bisimulation game directly follows from standard results for bisimulation games in modal logics [10].

Proposition 1. *Let $\mathcal{P}$ be a finite set of propositional letters. For all $k \geq 0$, Player II has a winning strategy in the k-round* PNL$^+$*-bisimulation game on $\mathbf{M_0}^+$ and $\mathbf{M_1}^+$, with initial configuration $([a_0, b_0], [a_1, b_1])$, iff $[a_0, b_0]$ and $[a_1, b_1]$ satisfy the same* PNL$^+$*-formulas over $\mathcal{P}$ with operator depth at most k.*

We exploit Proposition 1 to prove that the π operator of PNL$^{\pi+}$ cannot be expressed in PNL$^+$. We choose two models $\mathbf{M_0}^+$ and $\mathbf{M_1}^+$ that can be distinguished with a PNL$^{\pi+}$ formula which makes an essential use of π, but not by a PNL$^+$ formula. The claim is proved by showing that for all k, Player II has a winning strategy in the k-round PNL$^+$-bisimulation game on $\mathbf{M_0}^+$ and $\mathbf{M_1}^+$.

Theorem 2. *The interval operator π cannot be defined in* PNL$^+$.

Proof. Let $\mathbf{M}^+ = \langle \mathbb{I}(\mathbb{Z})^+, V \rangle$, where V is such that p holds everywhere, be a non-strict model. Consider the k-round PNL$^+$-bisimulation game on $(\mathbf{M}^+, \mathbf{M}^+)$ with initial configuration $([0, 1], [1, 1])$. The intervals $[0, 1]$ and $[1, 1]$ can be easily distinguished in PNL$^{\pi+}$, since π holds in $[1, 1]$ but not in $[0, 1]$. We show that this pair of intervals cannot be distinguished in PNL$^+$ by providing a simple winning strategy for Player II in the k-round PNL$^+$-bisimulation game on $(\mathbf{M}^+, \mathbf{M}^+)$ with initial configuration $([0, 1], [1, 1])$, as follows: if Player I plays a $\Diamond_r$-move on a given structure, then Player II arbitrarily chooses a right-neighbor of the current interval on the other structure. Likewise, if Player I plays a $\Diamond_l$-move on a given structure, then Player II arbitrarily chooses a left-neighbor of the current interval on the other structure. Since the valuation V is such that p holds everywhere, in any case the new configuration is a local isomorphism. $\square$

The next theorem shows that PNL$^-$ is strictly less expressive than PNL$^{\pi+}$.

Theorem 3. PNL$^-$ $\prec_I$ PNL$^{\pi+}$.

Proof. We prove the claim by showing that PNL$^-$ $\preceq_I$ PNL$^{\pi+}$ and PNL$^{\pi+}$ $\not\preceq_I$ PNL$^-$. To prove the former, we provide a translation τ from PNL$^-$ to PNL$^{\pi+}$. Consider the mapping τ_0 defined as follows:

$$\begin{aligned} \tau_0(p) &= p & \tau_0(\langle A \rangle \varphi) &= \Diamond_r(\neg\pi \wedge \tau_0(\varphi)) \\ \tau_0(\neg\varphi) &= \neg\tau_0(\varphi) & \tau_0(\langle \overline{A} \rangle \varphi) &= \Diamond_l(\neg\pi \wedge \tau_0(\varphi)) \\ \tau_0(\varphi_1 \vee \varphi_2) &= \tau_0(\varphi_1) \vee \tau_0(\varphi_2) \end{aligned}$$

For every PNL$^-$-formula φ, let $\tau(\varphi) = \neg\pi \wedge \tau_0(\varphi)$. Given a strict model $\mathbf{M}^- = \langle \mathbb{I}(\mathbb{D})^-, V^- \rangle$, let $\mathbf{M}^+ = \langle \mathbb{I}(\mathbb{D})^+, V^+ \rangle$ be a non-strict extension of $\mathbf{M}^-$. It is immediate to show that for any interval $[a, b]$ in $\mathbf{M}^-$ and any PNL$^-$-formula φ, $\mathbf{M}^-, [a, b] \Vdash \varphi$ if and only if $\mathbf{M}^+, [a, b] \Vdash \tau(\varphi)$. The proof is an easy induction on the structure of φ. This proves that PNL$^-$ $\preceq_I$ PNL$^{\pi+}$.

To prove that $\mathrm{PNL}^{\pi+} \not\preceq_I \mathrm{PNL}^-$, suppose by contradiction that there exists a translation τ' from $\mathrm{PNL}^{\pi+}$ to PNL^- such that, for any non-strict model $\mathbf{M}^+$, any strict interval $[a,b]$, and any formula φ of $\mathrm{PNL}^{\pi+}$, $\mathbf{M}^+, [a,b] \Vdash \varphi$ iff $\mathbf{M}^-, [a,b] \Vdash \tau'(\varphi)$, where $\mathbf{M}^-$ is the strict restriction of $\mathbf{M}^+$. Consider the non-strict models $\mathbf{M}_0^+ = \langle \mathbb{I}(\mathbb{Z})^+, V_0 \rangle$ and $\mathbf{M}_1^+ = \langle \mathbb{I}(\mathbb{Z})^+, V_1 \rangle$, where $V_0(p) = \{[a,b] \in \mathbb{I}(\mathbb{Z})^+ : a \leq b\}$ and $V_1(p) = \{[a,b] \in \mathbb{I}(\mathbb{Z})^+ : a < b\}$. It is immediate to see that $\mathbf{M}_0^+, [0,1] \Vdash \Box_r p$, while $\mathbf{M}_1^+, [0,1] \not\Vdash \Box_r p$. Let $\mathbf{M}^- = \langle \mathbb{I}(\mathbb{Z})^-, V^- \rangle$ be a strict interval model such that p holds everywhere in $\mathbb{I}(\mathbb{Z})^-$. We have that $\mathbf{M}^-$ is the strict restriction of both $\mathbf{M}_0^+$ and $\mathbf{M}_1^+$. Hence, we may conclude that $\mathbf{M}^-, [0,1] \Vdash \tau'(\Box_r p)$ and $\mathbf{M}^-, [0,1] \not\Vdash \tau'(\Box_r p)$, which is a contradiction. □

Finally, we show that neither $\mathrm{PNL}^+ \preceq_I \mathrm{PNL}^-$ nor $\mathrm{PNL}^- \preceq_I \mathrm{PNL}^+$.

Theorem 4. *The expressive powers of* PNL^+ *and* PNL^- *are incomparable, namely,* $\mathrm{PNL}^- \not\preceq_I \mathrm{PNL}^+$ *and* $\mathrm{PNL}^+ \not\preceq_I \mathrm{PNL}^-$.

Proof. We first prove that $\mathrm{PNL}^- \not\preceq_I \mathrm{PNL}^+$. Let $\mathbf{M_0}^+ = \langle \mathbb{I}(\mathbb{Z})^+, V_0 \rangle$ and $\mathbf{M_1}^+ = \langle \mathbb{I}(\mathbb{Z} \setminus \{2\})^+, V_1 \rangle$, where V_0 is such that $V_0(p) = \{[1,1],[1,2],[2,2]\}$ and V_1 is such that $V_1(p) = \{[1,1]\}$, be two PNL^+-models. For any $k \geq 0$, consider the k-round PNL^+-bisimulation game between $\mathbf{M_0}^+$ and $\mathbf{M_1}^+$, with initial configuration $([0,1],[0,1])$. Player II has the following winning strategy: at any round, if Player I chooses an interval $[a,b] \in \mathbb{I}(\mathbb{Z} \setminus \{2\})^+$ in one of the models, then Player II chooses the same interval on the other model, while if Player I chooses an interval $[a,2]$ (resp., $[2,b]$) in $\mathbf{M_0}^+$, then Player II chooses the interval $[a,1]$ (resp., $[1,b]$) in $\mathbf{M_1}^+$. On the contrary, the strict restrictions $\mathbf{M_0}^-$ of $\mathbf{M_0}^+$ and $\mathbf{M_1}^-$ of $\mathbf{M_1}^+$ can be easily distinguished by PNL^-: we have that $\mathbf{M_0}^-, [0,1] \Vdash \langle A \rangle p$, while $\mathbf{M_1}^-, [0,1] \not\Vdash \langle A \rangle p$. Since $\mathbf{M_0}^+$ and $\mathbf{M_1}^+$ satisfy the same formulas over the interval $[0,1]$, there cannot exist a translation τ' from PNL^- to PNL^+ such that $\mathbf{M_0}^+, [0,1] \Vdash \tau'(\langle A \rangle p)$ and $\mathbf{M_1}^+, [0,1] \not\Vdash \tau'(\langle A \rangle p)$.

As for $\mathrm{PNL}^+ \not\preceq_I \mathrm{PNL}^-$, we can exploit the very same proof we gave to show that $\mathrm{PNL}^{\pi+} \not\preceq_I \mathrm{PNL}^-$ (it suffices to notice that $\Box_r p$ is a PNL^+ formula). □

4 Decidability of PNLs

In this section we prove the decidability of $\mathrm{PNL}^{\pi+}$, and thus that of its proper fragments PNL^+ and PNL^-, by embedding it into the two-variable fragment of first-order logic interpreted over linearly ordered domains. $\mathrm{PNL}^{\pi+}$ can be translated into $\mathrm{FO}^2[<]$ as follows. Let $\mathcal{AP}$ be the set of propositional letters in $\mathrm{PNL}^{\pi+}$. The signature for $\mathrm{FO}^2[<]$ includes a binary relational symbol P for every $p \in \mathcal{AP}$. The translation function $ST_{x,y}$ is defined as follows:

$$ST_{x,y}(\varphi) = x \leq y \wedge ST'_{x,y}(\varphi),$$

where x, y are two first-order variables and

$$\begin{array}{rlrl} ST'_{x,y}(p) &= P(x,y) & ST'_{x,y}(\varphi \vee \psi) &= ST'_{x,y}(\varphi) \vee ST'_{x,y}(\psi) \\ ST'_{x,y}(\pi) &= (x = y) & ST'_{x,y}(\Diamond_r \varphi) &= \exists x (y \leq x \wedge ST'_{y,x}(\varphi)) \\ ST'_{x,y}(\neg\varphi) &= \neg ST'_{x,y}(\varphi) & ST'_{x,y}(\Diamond_l \varphi) &= \exists y (y \leq x \wedge ST'_{y,x}(\varphi)) \end{array}$$

Two variables are thus sufficient to translate $\mathrm{PNL}^{\pi+}$ into $\mathrm{FO}^2[<]$. As we will show later, this is not the case with other interval temporal logics, such as, for instance, HS and CDT. The next theorem proves that that $\mathrm{FO}^2[<]$ is at least as expressive as $\mathrm{PNL}^{\pi+}$ (η is the model transformation defined in Section 2).

Theorem 5. *For any* $\mathrm{PNL}^{\pi+}$*-formula* φ*, any non-strict interval model* $\mathbf{M}^+ = \langle \mathbb{I}(\mathbb{D})^+, V\rangle$*, and any interval* $[a,b]$ *in* $\mathbf{M}^+$*:*

$$\mathbf{M}^+, [a,b] \Vdash \varphi \textit{ iff } \eta(\mathbf{M}^+) \models ST_{x,y}(\varphi)[x := a, y := b].$$

Proof. The proof is by structural induction on φ. The base case, as well as the cases of Boolean connectives, are straightforward, and thus omitted. Let $\varphi = \Diamond_r \psi$. From $\mathbf{M}^+, [a,b] \Vdash \varphi$, it follows that there exists an element c such that $c \geq b$ and $\mathbf{M}^+, [b,c] \Vdash \psi$. By inductive hypothesis, we have that $\eta(\mathbf{M}^+) \models ST_{y,x}(\psi)[y := b, x := c]$. By definition of $ST_{y,x}(\psi)$, this is equivalent to $\eta(\mathbf{M}^+) \models y \leq x \wedge ST'_{y,x}(\psi)[y := b, x := c]$. This implies that $\eta(\mathbf{M}^+) \models \exists x(y \leq x \wedge ST'_{y,x}(\psi))[y := b]$. Since $a \leq b$ ($[a,b]$ in $\mathbf{M}^+$), we can conclude that $\eta(\mathbf{M}^+) \models ST_{x,y}(\Diamond_r \psi)[x := a, y := b]$. The converse direction can be proved in a similar way. The case $\varphi = \Diamond_l \psi$ is completely analogous and thus omitted. □

Corollary 1. *A* $\mathrm{PNL}^{\pi+}$*-formula* φ *is satisfiable in a class of non-strict interval structures built over a class of linear orderings* $\mathcal{C}$ *iff* $ST_{x,y}(\varphi)$ *is satisfiable in the class of all* $\mathrm{FO}^2[<]$*-models expanding linear orderings from* $\mathcal{C}$*.*

Since the above translation is polynomial in the size of the input formula, decidability of $\mathrm{PNL}^{\pi+}$ follows from Theorem 1.

Corollary 2. *The satisfiability problem for* $\mathrm{PNL}^{\pi+}$ *is decidable in NEXPTIME for each of the classes of non-strict interval structures built over (i) the class of all linear orderings, (ii) the class of all well-orderings, (iii) the class of all finite linear orderings, and (iv) the linear ordering on* $\mathbb{N}$*.*

This result can be extended to decide the satisfiability problem for $\mathrm{PNL}^{\pi+}$ over any class of linear orderings, definable in $\mathrm{FO}^2[<]$ within any of the above, e.g., the class of all (un)bounded (above, below) linear orderings or all (un)bounded above well-orderings, etc. On the contrary, the decidability of the satisfiability problem for $\mathrm{PNL}^{\pi+}$ on any of the classes of all discrete, dense, or Dedekind complete linear orderings is still open.

Since $\mathrm{PNL}^+ \prec \mathrm{PNL}^{\pi+}$ and $\mathrm{PNL}^- \prec_I \mathrm{PNL}^{\pi+}$, both PNL^+ and PNL^- are decidable in NEXPTIME (at least) over the same classes of orderings as $\mathrm{PNL}^{\pi+}$. Moreover, a translation from PNL^+ to $\mathrm{FO}^2[<]$ can be obtained from that for $\mathrm{PNL}^{\pi+}$ by simply removing the rule for π, while a translation from PNL^- to $\mathrm{FO}^2[<]$ can be obtained from that for $\mathrm{PNL}^{\pi+}$ by removing the rule for π, by substituting $<$ for $\leq$, and by replacing $\Diamond_r$ (resp., $\Diamond_l$) with $\langle A\rangle$ (resp., $\langle \overline{A}\rangle$).

The NEXPTIME-hardness of the satisfiability problem for $\mathrm{PNL}^{\pi+}$, PNL^+, and PNL^- can be proved by exploiting the very same reduction from the exponential tiling problem given by Bresolin et al. for PNLs future fragments [4].

Theorem 6. *The satisfiability problem for* PNL^{-}*,* PNL^{+}*, and* $\mathrm{PNL}^{\pi+}$ *interpreted in the class of all linear orderings, the class of all well-orderings, the class of all finite linear orderings, and the linear ordering on* $\mathbb{N}$ *is NEXPTIME-complete.*

5 Expressive Completeness

In this section, we show that $\mathrm{PNL}^{\pi+}$ is at least as expressive as $\mathrm{FO}^2[<]$, that is, we show that every formula of $\mathrm{FO}^2[<]$ can be translated into an equivalent formula of $\mathrm{PNL}^{\pi+}$ (see Section 2). This allows us to conclude that $\mathrm{PNL}^{\pi+}$ is as expressive as $\mathrm{FO}^2[<]$. A similar result for CDT was given by Venema in [22], where the expressive completeness of CDT with respect to $\mathrm{FO}^3_{x,y}[<]$ (the fragment of first-order logic interpreted over linear orderings whose language features only three, possibly reused variables and at most two of them, x and y, can be free) was proved. Both results can be viewed as interval-based counterparts of Kamp's theorem for propositional point-based linear time temporal logic [14].

The translation τ from $\mathrm{FO}^2[<]$ to $\mathrm{PNL}^{\pi+}$ is given in the following table:

Basic formulas	Non-basic formulas
$\tau[x,y](x=x)=\tau[x,y](y=y)=\top$	$\tau[x,y](\neg\alpha)=\neg\tau[x,y](\alpha)$
$\tau[x,y](x=y)=\tau[x,y](y=x)=\pi$	$\tau[x,y](\alpha\vee\beta)=\tau[x,y](\alpha)\vee\tau[x,y](\beta)$
$\tau[x,y](y<x)=\bot$	$\tau[x,y](\exists x\beta)=$
$\tau[x,y](x<y)=\neg\pi$	$\Diamond_r(\tau[y,x](\beta))\vee\Box_r\Diamond_l(\tau[x,y](\beta))$
$\tau[x,y](P(x,x))=\Diamond_l(\pi\wedge p^{\leq}\wedge p^{\geq})$	$\tau[x,y](\exists y\beta)=$
$\tau[x,y](P(y,y))=\Diamond_r(\pi\wedge p^{\leq}\wedge p^{\geq})$	$\Diamond_l(\tau[y,x](\beta))\vee\Box_l\Diamond_r(\tau[x,y](\beta))$
$\tau[x,y](P(x,y))=p^{\leq}$	
$\tau[x,y](P(y,x))=p^{\geq}$	

As formally stated by Theorem 7 below, every $\mathrm{FO}^2[<]$-formula $\alpha(x,y)$ is mapped into two distinct $\mathrm{PNL}^{\pi+}$-formulas $\tau[x,y](\alpha)$ and $\tau[y,x](\alpha)$. The first one captures all and only the models of $\alpha(x,y)$ where $x\leq y$ (if any), while the second one captures all and only the models of $\alpha(x,y)$ where $y\leq x$ (if any).

Example 1. Consider the formula $\alpha=\exists x\neg\exists y(x<y)$, which constrains the model to be right-bounded. Let $\beta=\exists y(x<y)$. We have that

$$\begin{aligned}\tau[x,y](\beta)&=\Diamond_l(\tau[y,x](x<y))\vee\Box_l\Diamond_r(\tau[x,y](x<y)) =\\ &=\Diamond_l\bot\vee\Box_l\Diamond_r\neg\pi\ (\equiv\ \Box_l\Diamond_r\neg\pi)\end{aligned}$$

and that

$$\begin{aligned}\tau[y,x](\beta)&=\Diamond_r(\tau[x,y](x<y))\vee\Box_r\Diamond_l(\tau[y,x](x<y)) =\\ &=\Diamond_r\neg\pi\vee\Box_r\Diamond_l\bot\ (\equiv\ \Diamond_r\neg\pi)\end{aligned}$$

The resulting translation of α is:

$$\begin{aligned}\tau[x,y](\alpha) &= \Diamond_r(\tau[y,x](\neg\beta)) \vee \Box_r\Diamond_l(\tau[x,y](\neg\beta)) = \\ &= \Diamond_r(\neg\tau[y,x](\beta)) \vee \Box_r\Diamond_l(\neg\tau[x,y](\beta)) = \\ &= \Diamond_r\neg\Diamond_r\neg\pi \vee \Box_r\Diamond_l\neg\Box_l\Diamond_r\neg\pi = \\ &= \Diamond_r\Box_r\pi \vee \Box_r\Diamond_l\Diamond_l\Box_r\pi \ (\equiv\ \Diamond_r\Box_r\pi \vee \Box_r\pi)\end{aligned}$$

which is a $\mathrm{PNL}^{\pi+}$-formula which constrains the model to be right-bounded.

Let $\mathcal{A} = \langle \mathbb{D}, V_{\mathcal{A}}\rangle$ be a $\mathrm{FO}^2[<]$-model and let $\zeta(\mathcal{A}) = \langle \mathbb{I}(\mathbb{D})^+, V_{\zeta(\mathcal{A})}\rangle$ be the corresponding $\mathrm{PNL}^{\pi+}$-model (see Section 2).

Theorem 7. *For every* $\mathrm{FO}^2[<]$*-formula* $\alpha(x,y)$*, every* $\mathrm{FO}^2[<]$*-model* $\mathcal{A} = \langle \mathbb{D}, V_{\mathcal{A}}\rangle$*, and every pair* $a, b \in D$*, with* $a \leq b$*, (i)* $\mathcal{A} \models \alpha(a,b)$ *if and only if* $\zeta(\mathcal{A}), [a,b] \Vdash \tau[x,y](\alpha)$ *and (ii)* $\mathcal{A} \models \alpha(b,a)$ *if and only if* $\zeta(\mathcal{A}), [a,b] \Vdash \tau[y,x](\alpha)$.

Proof. The proof is by simultaneous induction on the complexity of α.

- $\alpha = (x = x)$ or $\alpha = (y = y)$. Both α and $\tau[x,y](\alpha) = \top$ are true.
- $\alpha = (x < y)$. As for claim *(i)*, $\mathcal{A} \models \alpha(a,b)$ iff $a < b$ iff $\zeta(\mathcal{A}), [a,b] \Vdash \neg\pi$. As for claim *(ii)* $\mathcal{A} \not\models \alpha(b,a)$, since $a \leq b$, and $\zeta(\mathcal{A}), [a,b] \nVdash \tau[y,x](x < y) (= \bot)$. Likewise, for $\alpha = (y < x)$.
- $\alpha = P(x,y)$ or $\alpha = P(y,x)$. Both claims follow from the valuation of $p^{\leq}$ and $p^{\geq}$ (given in Section 2).
- $\alpha = P(x,x)$. As for claim *(i)*, $\mathcal{A} \models \alpha(a,b)$ iff $\mathcal{A} \models P(a,a)$ iff $\zeta(\mathcal{A}), [a,a] \Vdash \pi \wedge p^{\leq} \wedge p^{\geq}$ iff $\zeta(\mathcal{A}), [a,b] \Vdash \Diamond_l(\pi \wedge p^{\leq} \wedge p^{\geq})$. A similar argument can be used to prove claim *(ii)*. Likewise for $\alpha = P(y,y)$.
- The Boolean cases are straightforward.
- $\alpha = \exists x\beta$. As for claim *(i)*, suppose that $\mathcal{A} \models \alpha(a,b)$. Then, there is $c \in \mathcal{A}$ such that $\mathcal{A} \models \beta(c,b)$. There are two (non-exclusive) cases: $b \leq c$ and $c \leq b$. If $b \leq c$, by the inductive hypothesis, we have that $\zeta(\mathcal{A}), [b,c] \Vdash \tau[y,x](\beta)$ and thus $\zeta(\mathcal{A}), [a,b] \Vdash \Diamond_r(\tau[y,x](\beta))$. Likewise, if $c \leq b$, by the inductive hypothesis, we have that $\zeta(\mathcal{A}), [c,b] \Vdash \tau[x,y](\beta)$ and thus for every d such that $b \leq d$, $\zeta(\mathcal{A}), [b,d] \Vdash \Diamond_l(\tau[x,y](\beta))$, that is, $\zeta(\mathcal{A}), [a,b] \Vdash \Box_r\Diamond_l(\tau[x,y](\beta))$. Hence $\zeta(\mathcal{A}), [a,b] \Vdash \Diamond_r(\tau[y,x](\beta)) \vee \Box_r\Diamond_l(\tau[x,y](\beta))$, that is, $\zeta(\mathcal{A}), [a,b] \Vdash \tau[x,y](\alpha)$. For the converse direction, it suffices to note that the interval $[a,b]$ has at least one right neighbor, viz. $[b,b]$, and thus the above argument can be reversed. Claim *(ii)* can be proved in a similar way.
- $\alpha = \exists y\beta$. Analogous to the previous case. □

Corollary 3. *For every formula* $\alpha(x,y)$ *and every* $\mathrm{FO}^2[<]$*-model* $\mathcal{A} = \langle \mathbb{D}, V_{\mathcal{A}}\rangle$, $\mathcal{A} \models \forall x\forall y\alpha(x,y)$ *if and only if* $\zeta(\mathcal{A}) \Vdash \tau[x,y](\alpha) \wedge \tau[y,x](\alpha)$.

Definition 3. *We say that a* $\mathrm{PNL}^{\pi+}$*-model* $\mathbf{M}$ *of the considered language is* synchronized on a pair of variables $(p^{\leq}, p^{\geq})$ *if these variables are equally true at any point interval* $[a,a]$ *in* $\mathbf{M}$*;* $\mathbf{M}$ *is* synchronized for a $\mathrm{FO}^2[<]$-formula α *if it is synchronized on every pair of variables* $(p^{\leq}, p^{\geq})$ *corresponding to a predicate* p *occurring in* α*;* $\mathbf{M}$ *is* synchronized *if it is synchronized on every pair* $(p^{\leq}, p^{\geq})$.

It is immediate to see that every model $\zeta(\mathcal{A})$, where $\mathcal{A}$ is a $\mathrm{FO}^2[<]$-model, is synchronized. Conversely, every synchronized $\mathrm{PNL}^{\pi+}$-model $\mathbf{M}$ can be represented as $\zeta(\mathcal{A})$ for some model $\mathcal{A}$ for $\mathrm{FO}^2[<]$: the linear ordering of $\mathcal{A}$ is inherited from $\mathbf{M}$ and the interpretation of every binary predicate P is defined in accordance with Theorem 7, that is, for any $a, b \in \mathcal{A}$ we set $P(a, b)$ to be true precisely when $a \leq b$ and $\mathbf{M}, [a, b] \Vdash p^{\leq}$ or $b \leq a$ and $\mathbf{M}, [b, a] \Vdash p^{\geq}$. Due to the synchronization, these two conditions agree when $a = b$. Furthermore, the condition that a $\mathrm{PNL}^{\pi+}$-model $\mathbf{M}$ is synchronized on a pair of variables $p^{\leq}$ and $p^{\geq}$ can be expressed by the validity in $\mathbf{M}$ of the formula $[U](\pi \rightarrow (p^{\leq} \leftrightarrow p^{\geq}))$, where $[U]$ is the *universal modality*, which is definable in $\mathrm{PNL}^{\pi+}$ as follows [7]:

$$[U]\varphi ::= \Box_r\Box_r\Box_l\varphi \wedge \Box_r\Box_l\Box_l\varphi \wedge \Box_l\Box_l\Box_r\varphi \wedge \Box_l\Box_r\Box_r\varphi.$$

Building on this observation, we associate with every $\mathrm{FO}^2[<]$-formula α the formulas

$$\sigma_v(\alpha) = \left(\bigwedge_{p^{\leq}, p^{\geq}} [U](\pi \rightarrow (p^{\leq} \leftrightarrow p^{\geq})) \right) \rightarrow (\tau[x, y](\alpha) \wedge \tau[y, x](\alpha))$$

and

$$\sigma_s(\alpha) = \left(\bigwedge_{p^{\leq}, p^{\geq}} [U](\pi \rightarrow (p^{\leq} \leftrightarrow p^{\geq})) \right) \wedge (\tau[x, y](\alpha) \vee \tau[y, x](\alpha)),$$

where the conjunctions range over all pairs $p^{\leq}, p^{\geq}$ corresponding to predicates occurring in α.

Corollary 4. *For any* $\mathrm{FO}^2[<]$*-formula* α*, (i)* α *is valid in all* $\mathrm{FO}^2[<]$*-models iff* $\sigma_v(\alpha)$ *is a valid* $\mathrm{PNL}^{\pi+}$*-formula, and (ii)* α *is satisfiable in some* $\mathrm{FO}^2[<]$*-model iff* $\sigma_s(\alpha)$ *is a satisfiable* $\mathrm{PNL}^{\pi+}$*-formula.*

Notice that the proposed translation from $\mathrm{FO}^2[<]$ to $\mathrm{PNL}^{\pi+}$ is exponential, due to the clause for the existential quantifier. We do not know whether there exists a polynomial translation or not.

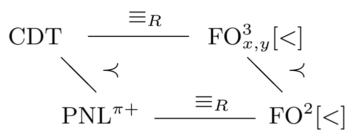

Fig. 1. Expressive completeness results for interval logics.

In Figure 1 we put together the expressive completeness results for CDT and $\mathrm{PNL}^{\pi+}$, using the notation introduced in Section 2. Since $\mathrm{FO}^2[<]$ is a proper fragment of $\mathrm{FO}^3_{x,y}[<]$, from the equivalences between CDT and $\mathrm{FO}^3_{x,y}[<]$ and between $\mathrm{PNL}^{\pi+}$ and $\mathrm{FO}^2[<]$ it immediately follows that CDT is strictly more expressive than $\mathrm{PNL}^{\pi+}$.

6 PNL$^{\pi+}$ and other HS fragments

In this section we explore the relationships between PNL$^{\pi+}$ and other fragments of HS. More precisely, we describe the fragments of HS which are fragments of PNL$^{\pi+}$ as well. To this end, we consider all other interval modalities of HS, namely, $\langle B\rangle$, $\langle E\rangle$, $\langle O\rangle$, $\langle D\rangle$, $\langle L\rangle$, and their transposes, which correspond to Allen's relations *begins, ends, overlaps, during*, and *after*, and their inverse relations. The semantics of such modalities can be given by their *standard translations* into first-order logic:

$$ST_{x,y}(\langle B\rangle\varphi) = x \leq y \wedge \exists z(z < y \wedge ST_{x,z}(\varphi))$$
$$ST_{x,y}(\langle E\rangle\varphi) = x \leq y \wedge \exists z(x < z \wedge ST_{z,y}(\varphi))$$
$$ST_{x,y}(\langle O\rangle\varphi) = x \leq y \wedge \exists z(x < z < y \wedge \exists y(y < x \wedge ST_{y,z}(\varphi)))$$
$$ST_{x,y}(\langle D\rangle\varphi) = x \leq y \wedge \exists z(x < z < y \wedge \exists y(x < y \wedge ST_{y,z}(\varphi)))$$
$$ST_{x,y}(\langle L\rangle\varphi) = x \leq y \wedge \exists x(y < x \wedge \exists y ST_{x,y}(\varphi))$$

The standard translation of $\langle L\rangle$ is a two-variable formula, while the standard translations of the other modalities are three-variable formulas. By taking advantage of the translation from FO$^2[<]$ to PNL$^{\pi+}$, $\langle L\rangle$ can be defined in PNL$^{\pi+}$ as follows: $\langle L\rangle\varphi = \Diamond_r(\neg\pi \wedge \Diamond_r\varphi)$. We show that the other interval modalities cannot be defined in PNL$^{\pi+}$ by a game-theoretic argument similar to the one of Theorem 2. To this end, we define the *k-round* PNL$^{\pi+}$*-bisimulation game* played on a pair of PNL$^{\pi+}$ models $(\mathbf{M_0}^+, \mathbf{M_1}^+)$ starting from a given initial configuration as follows: the rules of the game are the same of the k-round PNL$^+$-bisimulation game described in Section 3; the only difference is that a configuration $([a_0, b_0], [a_1, b_1])$ constitutes a local isomorphism between $\mathbf{M_0}^+$ and $\mathbf{M_1}^+$ if and only if *(i)* $[a_0, b_0]$ and $[a_1, b_1]$ share the same valuation of propositional variables, and *(ii)* $a_0 = b_0$ iff $a_1 = b_1$, that is, $\mathbf{M_0}^+, [a_0, b_0] \Vdash \pi$ iff $\mathbf{M_1}^+, [a_1, b_1] \Vdash \pi$. The following proposition is analogous to Proposition 1.

Proposition 2. *Let $\mathcal{P}$ be a finite set of propositional letters. For all $k \geq 0$, Player II has a winning strategy in the k-round* PNL$^{\pi+}$*-bisimulation game on* $\mathbf{M_0}^+$ *and* $\mathbf{M_1}^+$ *with initial configuration* $([a_0, b_0], [a_1, b_1])$ *iff* $[a_0, b_0]$ *and* $[a_1, b_1]$ *satisfy the same formulas of* PNL$^{\pi+}$ *over $\mathcal{P}$ with operator depth at most k.*

We exploit Proposition 2 to prove that none of the interval modalities $\langle B\rangle$, $\langle E\rangle$, $\langle O\rangle$, and $\langle D\rangle$ is expressible in PNL$^{\pi+}$. The proof structure is always the same: for every operator $\langle X\rangle$, we choose two models $\mathbf{M_0}^+$ and $\mathbf{M_1}^+$ that can be distinguished with a formula containing $\langle X\rangle$ and we prove that Player II has a winning strategy in the k-rounds PNL$^{\pi+}$-bisimulation game.

Theorem 8. *Neither of $\langle B\rangle$, $\langle E\rangle$, $\langle O\rangle$, and $\langle D\rangle$ can be defined in* PNL$^{\pi+}$.

Proof. We prove the claim for $\langle B\rangle$ and $\langle D\rangle$; the other cases are analogous. Consider the PNL$^{\pi+}$-models $\mathbf{M_0}^+ = \langle \mathbb{I}(\mathbb{Z} \setminus \{1, 2\})^+, V_0\rangle$ and $\mathbf{M_1}^+ = \langle \mathbb{I}(\mathbb{Z})^+, V_1\rangle$, where V_1 is such that p holds for all intervals $[a, b]$ such that $a < b$ and V_0 is the restriction of V_1 to $\mathbb{I}(\mathbb{Z} \setminus \{1, 2\})^+$. Note that $\mathbf{M_1}^+, [0, 3] \Vdash \langle B\rangle p$, while

$\mathbf{M_0}^+, [0,3] \not\Vdash \langle B\rangle p$; likewise for $\langle D\rangle p$. Thus, to prove the claims it suffices to show that Player II has a winning strategy for the k-round $\mathrm{PNL}^{\pi+}$-bisimulation game between $\mathbf{M_0}^+$ and $\mathbf{M_1}^+$, with initial configuration $([0,3],[0,3])$. In fact, Player II has a *uniform* strategy to play forever that game: at any position, assuming that Player I has not won yet, if he chooses a $\diamond_r$-move then Player II arbitrarily chooses a right-neighbor of the current interval on the other structure, with the only constraint to take a point-interval if and only if Player I has taken a point-interval as well. If Player I chooses a $\diamond_l$-move, Player II acts likewise. $\square$

7 Conclusions

In this paper we explored expressiveness and decidability issues for PNLs. First, we compared $\mathrm{PNL}^{\pi+}$ with PNL^+ and PNL^-, and we showed that the former is strictly more expressive than the other two. Then, we proved that $\mathrm{PNL}^{\pi+}$ is decidable by embedding it into $\mathrm{FO}^2[<]$. Next, we proved that $\mathrm{PNL}^{\pi+}$ is as expressive as $\mathrm{FO}^2[<]$. Finally, we compared $\mathrm{PNL}^{\pi+}$ with other interval logics.

A number of open questions remain. To mention just two: Is the satisfiability problem for $\mathrm{PNL}^{\pi+}$ over the classes of all discrete, dense, or Dedekind complete linear orders decidable? Can we extend $\mathrm{PNL}^{\pi+}$ with any modality in the set $\{\langle B\rangle, \langle E\rangle, \langle O\rangle, \langle D\rangle\}$ to preserve decidability? We can foresee various natural further developments stemming from the present work. In particular, the tableau systems that have been developed in [2–4] for PNLs over specific structures, such as $\mathbb{N}$ and $\mathbb{Z}$, can be considered for adaptation to deal with $\mathrm{FO}^2[<]$ over these and related classes of linear orders. As for expressiveness, here we have only partially explored the relationships between PNLs and other fragments of HS. A comparison with instant-based temporal logics can be of interest as well. For example, there is an obvious embedding of the standard instant-based temporal logic TL[F,P] into $\mathrm{PNL}^{\pi+}$. The (non-)existence of the opposite embedding is more interesting, but also more difficult to state in a precise way.

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