On Discrete Alphabets for the Two-user Gaussian Interference Channel with One Receiver Lacking Knowledge of the Interfering Codebook

Alex Dytso, Daniela Tuninetti, and Natasha Devroye
University of Illinois at Chicago, Chicago IL 60607, USA,
Email: odytso2, danielat, devroye @ uic.edu

***Abstract*—In multi-user information theory it is often assumed that every node in the network possesses all codebooks used in the network. This assumption is however impractical in distributed ad-hoc and cognitive networks. This work considers the two-user Gaussian Interference Channel with one *Oblivious Receiver* (G-IC-OR), i.e., one receiver lacks knowledge of the interfering cookbook while the other receiver knows both codebooks. We ask whether, and if so how much, the channel capacity of the G-IC-OR is reduced compared to that of the classical G-IC where both receivers know all codebooks. Intuitively, the oblivious receiver should not be able to jointly decode its intended message along with the unintended interfering message whose codebook is unavailable. We demonstrate that in strong and very strong interference, where joint decoding is capacity achieving for the classical G-IC, lack of codebook knowledge does not reduce performance in terms of generalized degrees of freedom (gDoF). Moreover, we show that the sum-capacity of the symmetric G-IC-OR is to within $O(\log(\log(\mathsf{SNR})))$ of that of the classical G-IC. The key novelty of the proposed achievable scheme is the use of a discrete input alphabet for the non-oblivious transmitter, whose cardinality is appropriately chosen as a function of SNR.**

I. Introduction

A classical assumption in multi-user information theory is that each node in the network possesses knowledge of the codebooks used by every other node. However, such assumptions might not be practical in heterogeneous, cognitive, distributed or dynamic networks For example, in very large ad-hoc networks, where nodes enter and leave at will, it might not be a practical assumption that new nodes learn the codebooks of old nodes and vice-versa. On the other hand, in cognitive radio scenarios, where new cognitive systems coexist with legacy systems, requiring the legacy system to know the codebook of the new cognitive system might not be viable. This motivates the study of networks where each node possesses only a subset of the codebooks used in the network. We will refer to such systems as networks with *partial codebook knowledge* and to nodes with only knowledge of a subset of the codebooks as *oblivious receivers*.

A. Past Work

To the best of our knowledge systems with partial codebook knowledge were first introduced in [1]. In [1] lack codebook knowledge was modeled by using *codebook indices*, which index the random encoding functions that map the messages to the codewords. If a node has codebook knowledge it knows the index (or instance) of the random encoding function used; else it does not and the codewords essentially look like the symbols were produced in an independent, identically distributed (i.i.d.) fashion from a given distribution. In [2] and [3] this concept of partial codebook knowledge was extended to model *oblivious relays*, where only multi-letter capacity expressions were obtained. As pointed out in [2, Section III.A] and [3, Remark 5], these capacity bounds are "non-computable" in the sense that it is not known how to find the optimal input distribution in general. In particular, the capacity achieving distribution for the practically relevant Gaussian noise channel remains an open problem.

In [4] we introduced the two-user Interference Channel (IC) with one Oblivious Receiver, referred to as the IC-OR. In the IC-OR, one receiver has full codebook knowledge (as in the classical IC), but the other receiver only has partial codebook knowledge (it knows the codebook of its desired message, but not that of the interfering message). The capacity region of the IC-OR was characterized to within a constant gap for the class of *injective semi-deterministic* IC in the spirit of [5]. In particular, the capacity of the real-valued Gaussian IC-OR (G-IC-OR) was characterized to within 1/2 bit per channel use per user; however, the input distribution achieving such a gap was not found. In [4, Section V.B] it was remarked that a carefully chosen i.i.d. Pulse Amplitude Modulation (PAM) can outperform i.i.d. Gaussian inputs for the given achievable rate region expression, and it was thus conjectured that discrete inputs may outperform Gaussian signaling in the strong and very strong interference regimes.

B. Contributions and Paper Outline

After formally introducing the IC-OR in Section II, we show our main contributions:

1) In Section III we introduce a new lower bound on the mutual information achievable by a discrete input on a point-to-point Gaussian noise channel, which will serve as the main tool in the derivation of our achievable rate region for the G-IC-OR.
2) To understand the utility of this new tool, in Section IV we show how to choose the cardinality of the discrete input in a point-to-point Gaussian noise channel such that the rate achieved is to within $O\left(\log\left(\log(\mathsf{SNR})\right)\right)$ of the (in this case known) capacity. This in turn shows

that a discrete input can achieve the maximum Degrees of Freedom (DoF) of the channel.

3) In Section V we evaluate the achievable rate region in [4, Lemma 3] for the G-IC-OR by using a discrete input for the non-oblivious transmitter and a Gaussian input for the other transmitter. For simplicity we only consider the *symmetric* G-IC-OR, where the direct links have the same strength and the interfering links have the same strength, but our results can be readily extended to the general asymmetric case.
4) In past work on networks with oblivious nodes no performance guarantees were provided for the Gaussian noise case. In SectionVI we study the generalized degrees of freedom (gDoF) achievable with the scheme introduced in Section V. We show that in strong and very strong interference the proposed scheme can approach the gDoF of the classical G-IC to within any degree of accuracy. This is quite surprising considering that the oblivious receiver can not perform joint decoding of the two messages, which is optimal for the classical G-IC in these regimes.
5) In Section VII we show that the sum-capacity of the G-IC-OR in strong and very strong interference is within $O\left(\log\left(\log(\mathsf{SNR})\right)\right)$ of the sum-capacity of the classical IC (which forms a natural outer bound to the oblivious channel, and where we are able to compute outer bounds). This in turn refines the gDoF result of Section V and shows that the scheme introduced in Section V is indeed gDoF optimal.

We conclude the paper with some final remarks and future directions in Section VIII.

C. Notation

Lower case variables are instances of upper case random variables which take values in calligraphic alphabets. We let $\delta(\cdot)$ denote the Dirac delta function, and $|A|$ denote the cardinality of a set A. The probability density function of a real-valued Gaussian random variable (r.v.) X with mean μ and variance σ^2 is denoted as

$$X \sim \mathcal{N}(x;\mu,\sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Throughout the paper $\log(\cdot)$ denotes logarithms in base 2 and $\ln(\cdot)$ in base e. We let $[x]^+ := \max(x,0)$ and $\log^+(x) := [\log(x)]^+$. The functions $\mathsf{I_d}(N,x)$ and $\mathsf{I_g}(x)$, for $N \in \mathbb{N}$ and $x \in \mathbb{R}^+$, are defined as

$$\mathsf{I_d}(N,x) := \left[\log(N) - \frac{1}{2}\log\left(\frac{\mathrm{e}}{2}\right) - \log\left(1+(N-1)\mathrm{e}^{-x}\right)\right]^+$$
$$\mathsf{I_g}(x) := \frac{1}{2}\log(1+x).$$

In the following PAM$(N, d_{\min})$ denotes the uniform distribution over a zero-mean Pulse Amplitude Modulation (PAM) constellation with N points and minimum distance $d_{\min}$ (and average energy $\mathcal{E} = d_{\min}^2\frac{N^2-1}{12}$).

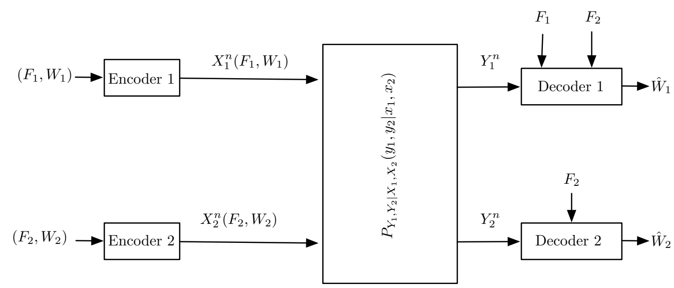

Fig. 1: The IC-OR, where F_1 and F_2 represent codebook indices known to one or both receivers.

II. Channel Model

The IC-OR consists of a two-user memoryless IC $(\mathcal{X}_1, \mathcal{X}_2, P_{Y_1Y_2|X_1X_2}, \mathcal{Y}_1, \mathcal{Y}_2)$ where receiver 2 is oblivious of transmitter 1's codebook. We model this lack of codebook knowledge as in [1], where transmitters use randomized encoding functions indexed by a message index and a *codebook index*. An oblivious receiver is unaware of the codebook index (F_1 is not given to decoder 2 in Fig. 1). The basic modeling assumption is that without the knowledge of the codebook index a codeword looks unstructured. More formally, by extending [2, Definition 2], a $(2^{nR_1}, 2^{nR_2}, n)$ code for the IC-OR with time sharing is a six-tuple $(P_{F_1|Q^n}, \sigma_1^n, \phi_1^n, P_{F_2|Q^n}, \sigma_2^n, \phi_2^n)$, where the distribution $P_{F_i|Q^n}$, $i \in [1:2]$, is over a finite alphabet $\mathcal{F}_i$ conditioned on the time-sharing sequences q^n from some finite alphabet $\mathcal{Q}$, and where the encoders σ_i^n and the decoders ϕ_i^n, $i \in [1:2]$, are mappings

$$\sigma_1^n : [1:2^{nR_1}] \times [1:|\mathcal{F}_1|] \to \mathcal{X}_1^n,$$
$$\sigma_2^n : [1:2^{nR_2}] \times [1:|\mathcal{F}_2|] \to \mathcal{X}_2^n,$$
$$\phi_1^n : [1:|\mathcal{F}_1|] \times [1:|\mathcal{F}_2|] \times \mathcal{Y}_1^n \to [1:2^{nR_1}],$$
$$\phi_2^n : [1:|\mathcal{F}_2|] \times \mathcal{Y}_2^n \to [1:2^{nR_2}].$$

Moreover, when transmitter 1's codebook index is unknown at decoder 2, the encoder σ_1^n and distribution $P_{F_1|Q^n}$ satisfy

$$\sum_{w_1=1}^{2^{nR_1}} \sum_{f_1=1}^{|\mathcal{F}_1|} P_{F_1|Q^n}(f_1|q^n)\ 2^{-nR_1}\ \delta\big(x_1^n - \sigma_1^n(w_1, f_1)\big)$$
$$=: \mathbb{P}[X_1^n = x_1^n | Q^n = q^n] = \prod_{t\in[1:n]} P_{X_1|Q}(x_{it}|q_t), \qquad (1)$$

according to some distribution $P_{X_1|Q}$. In other words, when averaged over the probability of selecting a given codebook and over a uniform distribution on the message set, the transmitted codeword conditioned on any time sharing sequence has a product distribution. Besides the restriction in (1) on the allowed class of codes, the probability of error, achievable rates and capacity region are defined in the usual way [6].

In this work we consider the practically relevant real-valued single-antenna symmetric Gaussian noise case. The restriction to symmetric channel gains is just for ease of exposition; all the results in the following can be extended straightforwardly to the general asymmetric case. For the symmetric G-IC-OR, the input-output relationship is

$$Y_1 = \sqrt{\mathsf{SNR}}\ X_1 + \sqrt{\mathsf{INR}}\ X_2 + Z_1 \quad (2a)$$

$$Y_2 = \sqrt{\mathsf{INR}}\ X_1 + \sqrt{\mathsf{SNR}}\ X_2 + Z_2 \quad (2b)$$

where the channel inputs are subject to the average power constraint $\mathbb{E}[|X_i|^2] \le 1, i \in [1:2]$, and the noise are i.i.d. $Z_i \sim \mathcal{N}(z;0,1), i \in [1:2]$. The real-valued parameters SNR and INR represent the received signal-to-noise ratio of the intended and interfering signal, respectively, at each receiver.

III. Main Tool

In this section we present a new lower bound on the mutual information achievable by a discrete input on a point-to-point Gaussian noise channel that will serve as the main tool in evaluating our inner bound for the G-IC-OR. We are not the first to consider discrete inputs for Gaussian noise channels; however, to best of our knowledge, prior to this, no firm lower bounds existed. [7, Theorems 6 and 7] asymptotically characterize the optimal input distribution over N masses at high and low SNR, respectively, for a point-to-point power-constrained Gaussian noise channel; [7, Theorem 8] gives a mutual information lower bound that holds for the *Gauss quadrature* distribution for all SNRs; [8] considers arbitrary input constellations with distribution *independent of* SNR and finds exact asymptotic expressions for the rate in the high-SNR limit. Here we can not use these results as we need *firm* lower bounds that hold for all distributions of N distinct masses and *for all* SNR. Our bound is as follows.

Theorem 1. *Let X_D be a discrete random variable with support $\{s_i \in \mathbb{R}, i \in [1:N]\}$, minimum distance $d_{\min}$ and average energy $\mathcal{E}_D := \sum_{i\in[1:N]} s_i^2 \mathbb{P}[X_D = s_i]$. Let $Z_G \sim \mathcal{N}(z;0,1)$ and SNR be a non-negative constant. Then*

$$\mathsf{I_d}\left(N, \mathsf{SNR}\ \frac{d_{\min}^2}{4}\right) \le I(X_D; \sqrt{\mathsf{SNR}}\ X_D + Z_G) \quad (3)$$

$$\le \min\Big(\log(N), \mathsf{I_g}\left(\mathsf{SNR}\ \mathcal{E}_D\right)\Big). \quad (4)$$

Proof: Let $p_i := \mathbb{P}[X_D = s_i], i \in [1:N]$. The output $Y = \sqrt{\mathsf{SNR}}\ X_D + Z_G$ has density

$$Y \sim P_Y(y) := \sum_{i\in[1:N]} p_i \mathcal{N}(y; \sqrt{\mathsf{SNR}}\ s_i, 1). \quad (5)$$

The upper bound in (4) follows from the well known facts that 'Gaussian maximizes the differential entropy for a given second moment constraint' and that 'a uniform input maximizes the entropy of a discrete random variable' [6]. To prove the lower bound in (3) we first find a lower bound on the differential entropy $h(Y) := -\int P_Y(y)\log(P_Y(y))dy$, where the output density is the Gaussian mixture in (5). We have

$$\begin{aligned}
&-h(Y) = \int P_Y(y)\log(P_Y(y))dy \\
&\overset{(a)}{\le} \log \int P_Y(y)P_Y(y)dy \\
&= \log \int \left(\sum_{i\in[1:N]} p_i \mathcal{N}(y;\sqrt{\mathsf{SNR}}s_i,1)\right)^2 dy \\
&= \log\Big(\sum_{(i,j)\in[1:N]^2} p_i p_j \int \mathcal{N}(y;\sqrt{\mathsf{SNR}}s_i,1) \\
&\quad \cdot \mathcal{N}(y;\sqrt{\mathsf{SNR}}s_j,1)dy\Big) \\
&= \log\Big(\sum_{(i,j)\in[1:N]^2} p_i p_j \frac{1}{\sqrt{4\pi}} \mathrm{e}^{\frac{-\mathsf{SNR}(s_i-s_j)^2}{4}} \\
&\quad \cdot \int \mathcal{N}(y;\sqrt{\mathsf{SNR}}\frac{s_i+s_j}{2},\frac{1}{2})dy\Big) \\
&\overset{(b)}{=} \log\left(\sum_{(i,j)\in[1:N]^2} p_i p_j \frac{1}{\sqrt{4\pi}} \mathrm{e}^{-\frac{\mathsf{SNR}(s_i-s_j)^2}{4}}\right) \\
&\overset{(c)}{\le} \log\Big(\sum_{i\in[1:N]} p_i^2 \frac{1}{\sqrt{4\pi}} + \sum_{i\in[1:N]} p_i(1-p_i)\frac{1}{\sqrt{4\pi}}\mathrm{e}^{-\frac{\mathsf{SNR}d_{\min}^2}{4}}\Big) \\
&\overset{(d)}{\le} -\log(N\sqrt{4\pi}) + \log\left(1+(N-1)\mathrm{e}^{-\frac{\mathsf{SNR}d_{\min}^2}{4}}\right), \\
&\iff I(X_D;\sqrt{\mathsf{SNR}}\ X_D + Z_G) = h(Y) - h(Z_G) \ge \\
&\log(N) - \frac{1}{2}\log\left(\frac{\mathrm{e}}{2}\right) - \log\left(1+(N-1)\mathrm{e}^{-\frac{\mathsf{SNR}d_{\min}^2}{4}}\right),
\end{aligned}$$

where the (in)equalities follow from: (a) Jensen's inequality, (b) $\int \mathcal{N}(y;\mu,\sigma^2)dy = 1$, (c) upper bounding by maximizing the exponential with $d_{\min} := \min_{i\neq j}|s_i - s_j|$, (d) by maximizing over the $\{p_i, i \in [1:N]\}$. Combining this bound with the fact that mutual information is non-negative proves the lower bound in (3). ■

IV. Discrete inputs for the power-constrained point-to-point Gaussian noise channel

In this section we give a flavor of how we intend to use discrete inputs on the G-IC-OR by considering the familiar point-to-point Gaussian noise channel. Specifically, we will show that, for a unit-variance additive white Gaussian noise channel, the unit-energy discrete input $X_D \sim \mathrm{PAM}\left(N, \sqrt{\frac{12}{N^2-1}}\right)$ with a properly chosen number of points N as a function of $\mathsf{SNR} := |h|^2$ achieves

$$I(X_D; hX_D + Z_G) \approx \log(N), \quad (6)$$

$$I(X_G; hX_G + X_D + Z_G) \approx I(X_G; hX_G + Z_G), \quad (7)$$

What this implies is that the discrete input X_D is a "good" input and a "good" interference. To put it more clearly, when we use a discrete constellation with uniform distribution as input, as in (6), the mutual information is roughly equal to the entropy of the constellation, which is highly desirable. On the other hand, when the same constellation is used as interference/noise, as in (7), the mutual information is roughly as if there was no interference, which is again highly desirable. In contrast, a Gaussian r.v. is considered to be the “best” input but the “worst” interference/noise when subject to a second moment constraint [9].

Consider the point-to-point Gaussian channel

$$Y = \sqrt{\mathsf{SNR}}\ X + Z, \tag{8a}$$

$$\mathbb{E}[X^2] \le 1,\ Z \sim \mathcal{N}(z; 0, 1), \tag{8b}$$

whose capacity $C = \mathsf{I}_\mathsf{g}(\mathsf{SNR})$ is achieved by $X \sim \mathcal{N}(x; 0, 1)$ at all SNRs. For this channel the gDoF is

$$d := \lim_{\mathsf{SNR}\to\infty} \frac{C}{\frac{1}{2}\log(1+\mathsf{SNR})} = 1. \tag{9}$$

Consider now the performance of the input

$$X \sim \mathrm{PAM}\left(N, \sqrt{\frac{12}{N^2-1}}\right). \tag{10}$$

It was shown in [10, Th. 10] that for any fixed N *independent* of SNR the gDoF is zero. Similar conclusions were found in [7] by considering high-SNR approximations of the *finite constellation* capacity of the point-to-point Gaussian channel, defined as the maximum rate achieved by a discrete input constrained to have a finite support. A question left open in [7] is what happens if N is allowed to be a function of SNR. In the following we address this question.

Fig. 2 shows that, through clever picking of N as a function of SNR, we seem to be able to follow to within an additive gap the capacity $C = \mathsf{I}_\mathsf{g}(\mathsf{SNR})$. This is formally shown in the next theorem.

Theorem 2. *For the channel in* (8)*, the input in* (10) *with*

$$N = \lfloor \sqrt{1+\mathsf{SNR}^{1-\epsilon}} \rfloor \tag{11}$$

achieves $d = 1-\epsilon$*, for any* $\epsilon \in (0,1)$.

Proof: By Theorem 1, the proposed input achieves

$$R \ge \mathsf{I}_\mathsf{d}\left(\lfloor \sqrt{1+\mathsf{SNR}^{1-\epsilon}} \rfloor,\ 3\mathsf{SNR}^{\epsilon}\right) \tag{12}$$

Next, by using the definition of gDoF and the fact that

$$\frac{\log\left(1+(\lfloor\sqrt{1+\mathsf{SNR}^{1-\epsilon}}\rfloor-1)\mathrm{e}^{-\frac{3\mathsf{SNR}}{\lfloor\sqrt{1+\mathsf{SNR}^{1-\epsilon}}\rfloor^2-1}}\right)}{\frac{1}{2}\log(1+\mathsf{SNR})} \to 0,$$

for any $\epsilon > 0$, we see that

$$d = \lim_{\mathsf{SNR}\to\infty} \frac{\log(N)}{\frac{1}{2}\log(1+\mathsf{SNR})} = 1-\epsilon$$

as claimed. This concludes the proof. ■

Theorem 2 shows that the input in (10) with the number of points chosen as in (11) approaches 1 gDoF to within any degree of accuracy. We next show that, with a clever choice of ϵ in (11) as a function of SNR, the rate in (12) is to within an additive gap of $O(\log(\log(\mathsf{SNR})))$ of the capacity $C = \mathsf{I}_\mathsf{g}(\mathsf{SNR})$, thus showing that indeed a discrete input can *exactly achieve* 1 gDoF. As a matter of fact this gap (given precisely

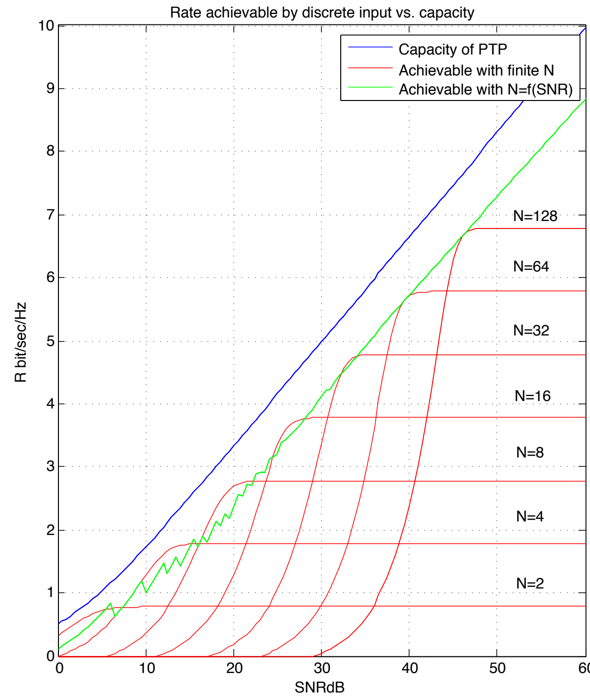

Fig. 2: Comparing achievable strategies: 1) Gaussian input (blue line), 2) Discrete input with fixed N (red lines), 3) Discrete input with N as in Theorem 3 (green line).

in (15)) grows very slowly with SNR. For example, the gap reaches value of 3 bits at $\mathsf{SNR} \approx 50$dB. Hence, for all practical purposes, this gap can be considered a small constant. We have

Theorem 3. *For the channel in* (8)*, the input in* (10) *with the number of points chosen as in* (11) *and with*

$$\epsilon = \left[\frac{\log(\frac{1}{6}\ln(\mathsf{SNR}))}{\log(\mathsf{SNR})}\right]^+ \tag{13}$$

the achievable rate in (12) *achieves the capacity* $C = \mathsf{I}_\mathsf{g}(\mathsf{SNR})$ *to within an additive gap of* $O(\log(\log(\mathsf{SNR})))$.

Proof: Picking ϵ as in (13) ensures that

$$\log\left(1+(\lfloor\sqrt{1+\mathsf{SNR}^{1-\epsilon}}\rfloor-1)\mathrm{e}^{-\frac{3\mathsf{SNR}}{\lfloor\sqrt{1+\mathsf{SNR}^{1-\epsilon}}\rfloor^2-1}}\right) \le 1,$$

and hence the achievable rate satisfies

$$R \ge \left[\log(N) - \frac{1}{2}\log\left(\frac{\mathrm{e}}{2}\right) - 1\right]^+. \tag{14}$$

Next, the difference between the capacity and the achievable rate in (14) for $\mathsf{SNR} \ge 1$ (if $\mathsf{SNR} < 1$ a trivial gap of 1 bit/sec/Hz can be shown) can be upper bounded as

$$\frac{1}{2}\log(1+\mathsf{SNR}) - \left[\log(\lfloor\sqrt{1+\mathsf{SNR}^{1-\epsilon}}\rfloor) - \frac{1}{2}\log\left(\frac{\mathrm{e}}{2}\right) - 1\right]^+$$
$$\le \frac{1}{2}\log(1+\mathsf{SNR}) - \log(\lfloor\sqrt{1+\mathsf{SNR}^{1-\epsilon}}\rfloor) + \frac{1}{2}\log\left(\frac{\mathrm{e}}{2}\right) + 1$$
$$\le \frac{1}{2}\log(1+\mathsf{SNR}) - \frac{1}{2}\log(1+\mathsf{SNR}^{1-\epsilon}) + \frac{1}{2}\log\left(\frac{\mathrm{e}}{2}\right) + 2$$

where used $\lfloor x \rfloor \geq \frac{1}{2}x$ for $x \geq 1$; next, since $\frac{1+x}{1+x^{1-\epsilon}} \leq x^{\epsilon}$ for $x \geq 1$ (for $\mathsf{SNR} \leq 1$ capacity can be trivially achievable to within 1 bit) we have

$$\begin{aligned} \mathsf{gap}(\mathsf{SNR}) &\leq \frac{\epsilon}{2}\log(\mathsf{SNR}) + \frac{1}{2}\log(8e) \\ &= \left[\frac{1}{2}\log(\frac{1}{6}\ln(\mathsf{SNR}))\right]^{+} + \frac{1}{2}\log(8e), \end{aligned} \quad (15)$$

as claimed. This concludes the proof. ■

Theorem 3 showed that a discrete input is a "good" input in the sense alluded to by (6). We now show that that a discrete interference is a "good" interference in the sense alluded to by (7). We study an extension of the channel in (8) by considering a *state T available neither at the encoder nor at the decoder*. The input-output relationship is

$$Y = \sqrt{\mathsf{SNR}}\ X + hT + Z: \quad (16a)$$

$$\mathbb{E}[X^2] \leq 1,\ Z \sim \mathcal{N}(z;0,1), \quad (16b)$$

$$T \sim \mathrm{PAM}\left(N, \sqrt{\frac{12}{N^2-1}}\right). \quad (16c)$$

It is well known [6] that capacity of channel with random state is $C = \max_{P_X} I(X;Y) \leq \max_{P_X} I(X;Y|T) = \mathsf{I_g}(\mathsf{SNR})$. We can show

Theorem 4. *For the channel with unknown states in* (16) *the input $X \sim \mathcal{N}(x;0,1)$ achieves*

$$\begin{aligned} R \geq & \mathsf{I_g}(\mathsf{SNR}) + \mathsf{I_d}\left(N, \frac{3|h|^2}{(1+\mathsf{SNR})(N^2-1)}\right) \\ & - \min\left(\log(N), \mathsf{I_g}(|h|^2)\right). \end{aligned} \quad (17)$$

Proof: By using Theorem 1 we have

$$\begin{aligned} I(X;Y) &= h(\sqrt{\mathsf{SNR}}X + hT + Z) - h(hT+Z) \\ &= h\left(\frac{h}{\sqrt{1+\mathsf{SNR}}}T + Z\right) - h(Z) + \frac{1}{2}\log(1+\mathsf{SNR}) \\ &- \Big(h\left(hT+Z\right) - h(Z)\Big) \\ &\geq \mathsf{I_d}\left(N, \frac{3|h|^2}{(1+\mathsf{SNR})(N^2-1)}\right) + \mathsf{I_g}(\mathsf{SNR}) \\ &- \min\Big(\log(N), \mathsf{I_g}(|h|^2)\Big), \end{aligned}$$

as claimed. ■

Note that the result of Theorem 4 can be readily used to lower bound the achievable rate in a G-IC where one user has a Gaussian input and the other a discrete input and where the discrete input is 'treated as noise', as we shall do in the next Section for the G-IC-OR. Before concluding we show that

Corollary 5. *For the channel with unknown states in* (16)*, the achievable rate from Theorem 4 attains 1 gDoF.*

Proof: By taking achievable rate in Theorem 4 we have $d = 1$ since both $\mathsf{I_d}\left(N, \frac{3|h|^2}{(1+\mathsf{SNR})(N^2-1)}\right)$ and $\min\Big(\log(N), \mathsf{I_g}(|h|^2)\Big)$ tend to zero as $\mathsf{SNR} \to \infty$ (here N and $|h|$ do not depend on SNR). ■

Theorem 5 shows that even with lack of state knowledge at both the receiver and transmitter, if the state is discrete and *its support is not a function of* SNR, then its effect can be 'removed' at high-SNR. From the proof of Theorem 5 it is immediate that the channel with unknown states in (16) has 1 DoF also when N and $|h|$ vary with SNR as long as $\mathsf{I_d}\left(N, \frac{3|h|^2}{(1+\mathsf{SNR})(N^2-1)}\right) - \min\Big(\log(N), \mathsf{I_g}(|h|^2)\Big)$ tends to zero as $\mathsf{SNR} \to \infty$ in the rate expression in (17).

In the next Section we shall use a discrete input to characterize the sum-capacity of the G-IC-OR.

V. An achievable region for the G-IC-OR

With the tools and insights developed form the previous Sections, we are ready to analyze the G-IC-OR.

Theorem 6. *For the G-IC-OR the following rate region is achievable*

$$R_1 \leq \mathsf{I_d}\left(N, \frac{3\ \mathsf{SNR}}{N^2-1}\right) \quad (18a)$$

$$\begin{aligned} R_2 \leq & \mathsf{I_d}\left(N, \frac{3\ \mathsf{INR}}{(1+\mathsf{SNR})(N^2-1)}\right) + \mathsf{I_g}(\mathsf{SNR}) \\ & - \min\left(\log(N), \mathsf{I_g}(\mathsf{INR})\right) \end{aligned} \quad (18b)$$

$$R_1 + R_2 \leq \mathsf{I_d}\left(N, \frac{3\ \mathsf{SNR}}{(1+\mathsf{INR})(N^2-1)}\right) + \mathsf{I_g}(\mathsf{INR}) \quad (18c)$$

Proof: From [4, Lemma 3 with $U_2 = X_2$] the following region is achievable

$$R_1 \leq I(X_1;Y_1|X_2,Q) \quad (19a)$$

$$R_2 \leq I(X_2;Y_2|Q) \quad (19b)$$

$$R_1 + R_2 \leq I(X_1,X_2;Y_1|Q), \quad (19c)$$

for all $P_Q P_{X_1|Q} P_{X_2|Q}$. We now evaluate the region in (19) without time sharing, i.e., $Q = \emptyset$, and with inputs

$$X_1 \sim \mathrm{PAM}\left(N, \sqrt{\frac{12}{N^2-1}}\right), \quad (20a)$$

$$X_2 \sim \mathcal{N}(x;0,1). \quad (20b)$$

The bound in (18a) is a direct application of Theorem 1 with $d_{\min}^2 = \frac{12}{N^2-1}$. The bound in (18b) follows from Theorem 4 with $|h|^2 = \mathsf{INR}$. The bound in (18c) follows since $I(X_1,X_2;Y_1) = I(X_1;Y_1) + I(X_2;Y_1|X_1)$, where $I(X_1;Y_1)$ is evaluated with Theorem 1 (here the Gaussian input X_2 is treated as noise hence the SNR is $\frac{\mathsf{SNR}}{1+\mathsf{INR}}$; the minimum distance is $d_{\min}^2 = \frac{12}{N^2-1}$) and $I(X_2;Y_1|X_1) = \mathsf{I_g}(\mathsf{INR})$. ■

VI. High SNR performance

In this Section we analyze the performance of the scheme in Theorem 6 at high-SNR by using the gDoF region as metric. For each rate R_i we define a gDoF d_i as in (9), for $i \in [1:2]$, where we parameterize $\mathsf{INR} = \mathsf{SNR}^{\alpha}$ for some $\alpha \geq 0$ [11]. In a spirit of Theorem 2, we take $N = \lfloor\sqrt{1+\mathsf{SNR}^{\beta}}\rfloor$. With this, we have that the following achievable gDoF region

Theorem 7. *From Theorem 6, the following* (d_1, d_2) *pairs are achievable*

$$d_1 \le \left\{ \begin{array}{ll} \beta & \textit{if } 1-\beta > 0 \\ 0 & \textit{if } 1-\beta \le 0 \end{array} \right\}, \tag{21a}$$

$$d_2 \le \left\{ \begin{array}{ll} \beta & \textit{if } [\alpha-1]^+ - \beta > 0 \\ 0 & [\textit{if } \alpha-1]^+ - \beta \le 0 \end{array} \right\} + 1 - \min(\beta, \alpha), \tag{21b}$$

$$d_1 + d_2 \le \left\{ \begin{array}{ll} \beta & \textit{if } [1-\alpha]^+ - \beta > 0 \\ 0 & \textit{if } [1-\alpha]^+ - \beta \le 0 \end{array} \right\} + \alpha. \tag{21c}$$

union over all $\beta \ge 0$.

Proof: Due to the space limitations we show the proof for d_2 in (21b) only; proofs for the other constraints follow similarly. By using $\mathsf{INR} = \mathsf{SNR}^\alpha$ and $N = \lfloor \sqrt{1+\mathsf{SNR}^\beta} \rfloor$ and by noting that

$$\lim_{\mathsf{SNR}\to\infty} \frac{\log(N^2)}{\log(1+\mathsf{SNR})} = \beta, \quad \lim_{\mathsf{SNR}\to\infty} \frac{\log(1+\mathsf{INR})}{\log(1+\mathsf{SNR})} = \alpha,$$

we compute d_2 in the following way

$$\begin{aligned} d_2 &= \lim_{\mathsf{SNR}\to\infty} \frac{\text{left hand side of eq.(18b)}}{\frac{1}{2}\log(1+\mathsf{SNR})} \\ &= \beta - \lim_{\mathsf{SNR}\to\infty} \frac{\log\left(1+(N-1)\exp\left(\frac{-3\mathsf{INR}}{(1+\mathsf{SNR})(N^2-1)}\right)\right)}{\frac{1}{2}\log(\mathsf{SNR})} \\ &\quad + 1 - \min(\beta, \alpha) \\ &= \beta - \left\{ \begin{array}{ll} 0 & \alpha - 1 - \beta > 0 \\ \beta & \alpha - 1 - \beta \le 0 \end{array} \right\} - \min(\beta, \alpha) + 1. \end{aligned}$$

This concludes the proof. ■

Determining analytically which β's attain the closure of the gDoF region in (21) is a bit involved, but it can be done very easily numerally. Fig 3 shows (for example) the gDoF region of Theorem 7 for $\alpha = \frac{4}{3}$; we see that while the sum-gDoF is the same as that of the classical G-IC, that the region is not, which makes intuitive sense as d_2 corresponds to the achieved gDoF of the oblivious receiver, which is much more constrained in our model and our achievability scheme.

It is interesting to compare the achievable sum-gDoF for the G-IC-OR based on Thereom 7 with the outer bound given by sum-gDoF of the classical G-IC [11]

$$\frac{\max(d_1+d_2)}{2} \le \min\left(1, \max\left(\frac{\alpha}{2}, 1-\frac{\alpha}{2}\right), \max(\alpha, 1-\alpha)\right).$$

We next demonstrates the sum-gDoF of the G-IC-OR.

Lemma 8. *The following gDoF is achievable by G-IC-OR*

$$\max_\beta(d_1+d_2) = \left\{ \begin{array}{ll} 1-\epsilon & 0 \le \alpha < 1, \ 0 < \epsilon \le 1-\alpha \\ \alpha - \epsilon & 1 \le \alpha < 2, \ 0 < \epsilon \le \alpha - 1 \\ 2-\epsilon & \alpha \ge 2, \ 0 < \epsilon \le 1 \end{array} \right. . \tag{22}$$

Proof: By setting $\beta = \min(1, |\alpha-1|) - \epsilon \ge 0$ in Theorem 7 one can verify that

$$\begin{aligned} d_1 &\le \beta, \\ d_2 &\le \min(1, \max(\alpha, 1-\alpha)), \\ d_1 + d_2 &\le \max(1, \alpha), \end{aligned}$$

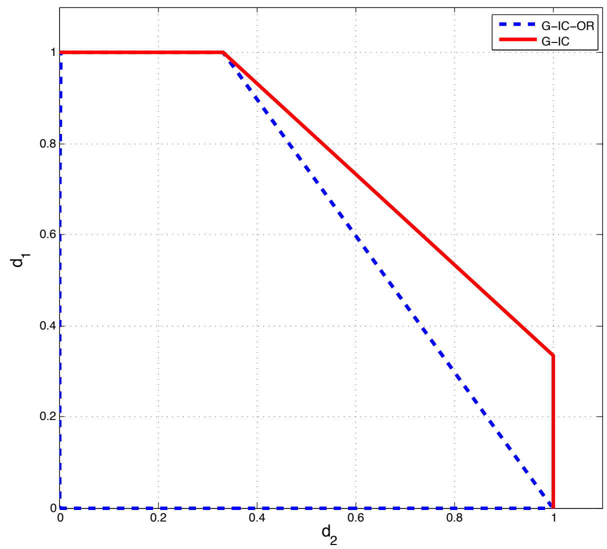

Fig. 3: gDof region achievable by Theorem 7 at $\alpha = \frac{4}{3}$. Note that one of the corner points is not achieved, as might be expected since d_2 corresponds to the oblivious user.

from which the claim follows. ■

To compare performance of the classical G-IC and of the G-IC-OR we plot the corresponding gDoFs in Fig. 4. We observe that in strong and very strong interference ($\alpha \ge 1$) the gDoFs are the same (up to an arbitrary small ϵ); hence, in this regime lack of codebook knowledge does not impact performance in the gDoF sense. We also note that in weak interference ($\alpha < 1$) our proposed scheme only has 1 gDoF, which means that the same performance can be achieved by silencing one of the users. For reference we also plot the achievable sum-gDoF when both users use a Gaussian input and treat interference as noise, i.e., $d_1 = d_2 = [1-\alpha]^+$, which are known to be optimal for the classical G-IC in very weak interference ($\alpha \le 1/2$) [11]; we see that the proposed scheme does not achieve this sum-gDoF in this regime; indeed, in weak interference it is not optimal to set $U_2 = X_2$ in [4, Lemma 3] as we did in Theorem 6; different choices of discrete inputs for the general region in [4, Lemma 3] are reported in [12].

VII. Finite SNR performance

In the previous section we showed that in strong interference ($\alpha \ge 1$) the sum-gDoF of the classical G-IC can be approached with any precision even when one receiver lacks knowledge of the interfering codebook. Thus, it is interesting to ask whether one can exactly achieve the sum-gDoF of the classical G-IC in strong interference by showing an additive gap to the capacity of the classical G-IC which is $o(\log(\mathsf{SNR}))$. In light of the results for the point-to-point channel, we ask whether we the sum-capacity of the G-IC-OR is to within $O(\log(\log(\mathsf{SNR})))$ of that of the classical G-IC. We next answer this in the positive, thus showing that in strong interference there is no penalty in term of sum-gDoF when one receiver is oblivious.

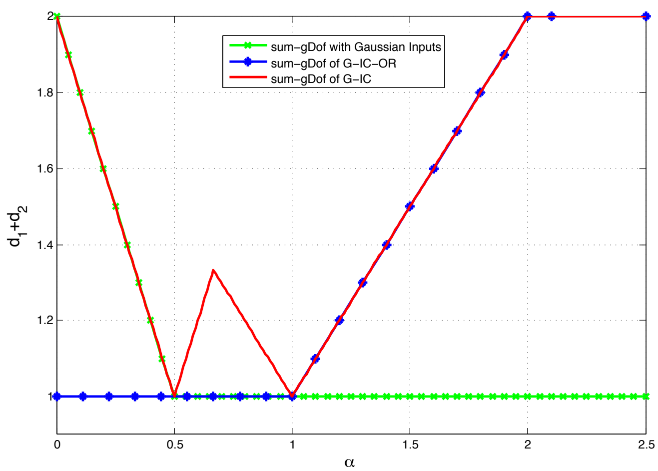

Fig. 4: The sum-gDoF of the G-IC-OR compared to the sum-gDoF of the G-IC.

Theorem 9. *The sum-capacity of the G-IC-OR in the strong and very strong interference regimes is to within $O(\log(\log(\mathsf{SNR})))$ of that of the classical G-IC.*

Proof: In very strong interference, $\mathsf{SNR}(1+\mathsf{SNR}) \le \mathsf{INR}$, we choose

$$N = \lfloor\sqrt{1+\mathsf{SNR}^{1-\epsilon}}\rfloor, \ \epsilon = \left[\frac{\log\left(\frac{1}{6}\ln(\mathsf{SNR})\right)}{\log(\mathsf{SNR})}\right]^+,$$

for $N \ge 3$ in Theorem 7 so that the following rates are achievable for the G-IC-OR

$$R_1 \le \log(\lfloor\sqrt{1+\mathsf{SNR}^{1-\epsilon}}\rfloor) - \frac{1}{2}\log\left(\frac{\mathrm{e}}{2}\right) - 1,$$
$$R_2 \le \frac{1}{2}\log(1+\mathsf{SNR}) - \frac{1}{2}\log\left(\frac{\mathrm{e}}{2}\right) - 1.$$

In this regime, the classical G-IC has capacity [13]

$$R_1 \le \frac{1}{2}\log(1+\mathsf{SNR}),$$
$$R_2 \le \frac{1}{2}\log(1+\mathsf{SNR}).$$

Clearly, the gap for R_2 is a constant (with respect to $(\mathsf{SNR}, \mathsf{INR})$) given by $\frac{1}{2}\log\left(\frac{\mathrm{e}}{2}\right) + 1 = 1.2213$; the gap for R_1 is as in (15). Although the theorem statement is for the sum-capacity, the proof holds for the whole capacity region.

In strong interference, $\mathsf{SNR} \le \mathsf{INR} < \mathsf{SNR}(1+\mathsf{SNR})$, we choose

$$N = \left\lfloor\sqrt{1+\left(\frac{\mathsf{INR}}{1+\mathsf{SNR}}\right)^{1-\epsilon}}\right\rfloor, \ \epsilon = \left[\frac{\log\left(\frac{1}{6}\ln\left(\frac{\mathsf{INR}}{1+\mathsf{SNR}}\right)\right)}{\log\left(\frac{\mathsf{INR}}{1+\mathsf{SNR}}\right)}\right]^+,$$

for $N \ge 3$ in Theorem 7 so that the following sum-rate is achievable for the G-IC-OR

$$R_1 + R_2 \le \log\left(\left\lfloor\sqrt{1+\left(\frac{\mathsf{INR}}{1+\mathsf{SNR}}\right)^{1-\epsilon}}\right\rfloor\right) + \frac{1}{2}\log(1+\mathsf{SNR}) - \log\left(\frac{\mathrm{e}}{2}\right) - 2.$$

In this regime, the classical G-IC has sum-capacity [14]

$$R_1 + R_2 \le \frac{1}{2}\log(1+\mathsf{SNR}+\mathsf{INR}).$$

The gap is hence

$$\frac{1}{2}\log(1+\mathsf{SNR}+\mathsf{INR}) - \log\left(\left\lfloor\sqrt{1+\left(\frac{\mathsf{INR}}{1+\mathsf{SNR}}\right)^{1-\epsilon}}\right\rfloor\right) - \frac{1}{2}\log(1+\mathsf{SNR}) + 2 + \log\left(\frac{\mathrm{e}}{2}\right)$$
$$\le \left[\frac{1}{2}\log\left(\frac{1}{6}\ln\left(\frac{\mathsf{INR}}{1+\mathsf{SNR}}\right)\right)\right]^+ + 3 + \log\left(\frac{\mathrm{e}}{2}\right)$$
$$\le \left[\frac{1}{2}\log\left(\frac{1}{6}\ln(\mathsf{SNR})\right)\right]^+ + \log(4\mathrm{e}),$$

since the steps are the same as those leading to (15) if one substitutes SNR in (15) with $\frac{\mathsf{INR}}{1+\mathsf{SNR}}$. Since $\frac{\mathsf{INR}}{1+\mathsf{SNR}} \le \mathsf{SNR}$ in strong interference, we obtain an $O(\log(\log(\mathsf{SNR}))$ gap in this regimeas well. This concludes the proof. ■

VIII. Conclusion

In the paper we focused on deriving capacity results for the Gaussian interference channel where one of the receivers is lacking knowledge of the interfering codebook, in contrast to a classical model where both receivers possess full codebook knowledge. To that end we derived a novel inequality on the achievable rate in a point-to-point Gaussian noise channel with discrete inputs, that we believe might be of an interest on its own. We surprisingly demonstrated that lack of codebook knowledge is not as detrimental as one might believe in strong and very strong interference.

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