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Spatial Reasoning with Propositional Logics

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Abstract

I present a method for reasoning about spatial relationships on the basis of entailments in propositional logic. Formalisms for representing topological and other spatial information (e.g. [2] [10] [11]) have generally employed the 1st-order predicate calculus. Whilst this language is much more expressive than 0-order (propositional) calculi it is correspondingly harder to reason with. Hence, by encoding spatial relationships in a propositional representation automated reasoning becomes more effective. I specify representations in both classical and intuitionistic propositional logic, which — together with well-defined meta-level reasoning algorithms — provide for efficient reasoning about a large class of spatial relations.

1 INTRODUCTION

This work has developed out of research done by Randell, Cui and Cohn (henceforth RCC) on formalising spatial and temporal concepts used in describing physical situations [11]. A set of classical 1st-order logic axioms has been formulated in which a large number of spatial and temporal relations can be defined [10]. One problem with this formalism is that computing inferences in the theory is far from easy — see e.g. [12]. Of course one can use any general purpose 1st-order theorem prover, but the complexity of the theory means that for many significant problems this approach is impractical.

In this paper I present an alternative approach to the logical representation of spatial relationships. Whilst the system of relations that are represented is essentially the same as that identified by the RCC work, the way in which they are represented is substantially different. Rather than using 1st-order logic, spatial relations are encoded into purely propositional formulae together with certain meta-level constraints concerning entailments between these formulae. I show first how a limited set of relations can be defined by means of classical propositional logic and then show how by using intuitionistic logic a more expressive representation is obtained.

The main motivation for using this alternative approach is that automated reasoning becomes far more efficient. In fact, given a finite set of spatial relationships characterisable in the propositional representation, there is an effective procedure for deciding whether this set describes a possible situation.

This paper can be regarded as a response to the challenge laid down in [12] (*Computing Transitivity Tables: a challenge for automated theorem provers*). However the approach taken is quite different from that envisaged in [12] in that, rather than enhancing or adapting a 1st-order theorem prover to suit the domain of spatial reasoning, a substantially different logical system is used to reason about this domain.

Since the taxonomy of spatial relations which I represent is identical to a family of relations dealt with in the RCC work, I use the same names to refer to these relations. Figure 1 gives 2-dimensional examples of the set of 8 jointly exhaustive and pairwise-disjoint relations which forms the basis of a lattice of topological relations definable in the RCC formalism (see [10] for more details).

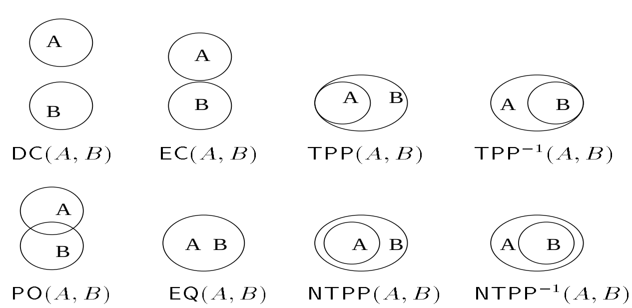

Figure 1: Basic Relations in the RCC Theory

1.1 PRELIMINARY DEFINITIONS

We shall need some precise terminology for referring to topological relationships and expressions describing those relationships:

- A *space* is a non-empty set. (In the intended interpretation the space will be the set of points constituting Cartesian 3 dimensional space.)[1]
- A *situation* is a triple $\langle \mathcal{U}, \Sigma, f \rangle$, where $\mathcal{U}$ is a space, Σ is a set of constant symbols and f is an assignment function which maps each constant in Σ to a subset of $\mathcal{U}$.
- A *situation-description* is a triple $\langle \mathcal{L}, \Sigma, \Theta \rangle$, where $\mathcal{L}$ is a logical language whose vocabulary includes the constant symbols Σ and whose semantics interprets these symbols as denoting sets; Θ is a theory expressed in $\mathcal{L}$.
- A situation $\langle \mathcal{U}, \Sigma, f \rangle$ *exemplifies* a situation-description $\langle \mathcal{L}, \Sigma, \Theta \rangle$ iff the assignment f of subsets of $\mathcal{U}$ to the constants Σ (together with some auxiliary assignment to any non-logical symbols of $\mathcal{L}$ occurring in Θ but not in Σ) satisfies Θ according to the semantics of $\mathcal{L}$. [2]
- A situation-description $\langle \mathcal{L}, \Sigma, \Theta \rangle$ is *impossible* iff it is not exemplified by any situation $\langle \mathcal{U}, \Sigma, f \rangle$.

2 TOPOLOGICAL INTERPRETATION OF PROPOSITIONAL LOGIC

There is a close connection between classical propositional calculus, which I shall refer to as $\mathcal{C}_0$, and set theory [8, p14]. The simplest semantics for $\mathcal{C}_0$ is to take propositions as denoting truth values and to correlate the connectives with truth functions. However, if we interpret propositional letters as denoting arbitrary subsets of some universal set $\mathcal{U}$ and the connectives $\neg$, $\wedge$ and $\vee$ respectively as the set operations *complement*, *intersection* and *union* then the classical tautologies will be those formulae whose value is $\mathcal{U}$ whatever the assignment of set values to propositional letters. To give content to this interpretation one can regard $\mathcal{U}$ as a set of all *possible worlds*. Then propositional letters denote the set of worlds in which they are true.

[1] We shall later adopt a richer notion of space: what mathematicians call a *topological space*. This is a pair $\langle \mathcal{U}, i \rangle$, where i is a function which maps subsets of $\mathcal{U}$ to their *interiors*.

[2] This *exemplification* relation is very similar to but slightly more general than the usual *satisfaction* relation between models and theories. It allows one to speak of models as satisfying (*exemplifying*) descriptions given in a number of different formal languages.

This semantics can be formally characterised as follows: a model for the logic $\mathcal{C}_0$ is a structure, $\langle \mathcal{U}, \mathcal{P}, d \rangle$, where $\mathcal{U}$ is a non-empty set, $\mathcal{P}$ is a denumerably infinite set of propositional constants, and d is a denotation function which assigns to each constant in $\mathcal{P}$ a subset of $\mathcal{U}$. The domain of d is extended to all $\mathcal{C}_0$ formulae formed from the propositional constants by stipulating that:

1. $d(\neg P) = \overline{d(P)}$
2. $d(P \wedge Q) = d(P) \cap d(Q)$
3. $d(P \vee Q) = d(P) \cup d(Q)$

where for any set S, $\overline{S}$ is the set of all elements of $\mathcal{U}$ which are not elements of S.

Intuitively, tautologous formulae ought to be true in any possible world; and indeed it can be shown that F is a theorem of $\mathcal{C}_0$ if and only if $d(F) = \mathcal{U}$ in all models.

This interpretation induces a simple correspondence between propositional formulae and *set-terms* — i.e. terms comprised of set-constants combined with the operations: union, intersection and complement. I use the notation ${}_{C_0}\!\!\Longrightarrow^{ST}$ to refer to the mapping between propositional formulae and set-terms; thus we can write e.g. $(P \vee \neg Q) \;{}_{C_0}\!\!\Longrightarrow^{ST}\; (P \cup \overline{Q})$.

I now introduce some further notation in order to state the theorem which provides the foundation for my reasoning system.

- A *universal set-equation* is an expression of the form $\phi = \mathcal{U}$ which asserts that the set-term ϕ denotes the set of all elements in the universe..
- $P_1, \ldots, P_n \models_{C_o} P_0$ means that in the calculus, $\mathcal{C}_0$, the formula P_0 is entailed by the set of formulae, $\{P_1, \ldots, P_n\}$. (Thus $\models_{C_o} P$ means that P is a theorem of $\mathcal{C}_0$.)
- $E_1, \ldots, E_n \models_S E_0$, where $E_0, \ldots, E_n$ are set-equation, means that in any model for which the equations $E_1, \ldots, E_n$ hold, the equation E_0 also holds. ($\models_S E$ means that E holds in every model.)

It can then be shown that:

Theorem 1 $P_1, \ldots, P_n \models_{C_o} P_0$ *if and only if* $\pi_1 = \mathcal{U}, \ldots, \pi_n = \mathcal{U} \models_S \pi_0 = \mathcal{U}$, *where* $P_i \;{}_{C_0}\!\!\Longrightarrow^{ST}\; \pi_i$ *for each* i.

I first establish that:

Lemma 1 *If* $\models_{C_o} P$ *then* $\models_S t = \mathcal{U}$, *where* $P \;{}_{C_0}\!\!\Longrightarrow^{ST}\; t$.

Proof: If P is a tautology then if it is converted to *conjunctive normal form* (CNF) each conjunct will

Table 1: Definitions of Four Topological Relations in $\mathcal{C}_0$

Relation	*Description*	*Set Equation*	*Model Constraint*
$\mathsf{DR}(X,Y)$	X and Y are discrete	$\overline{X \cap Y} = \mathcal{U}$	$\neg(X \wedge Y)$
$\mathsf{P}(X,Y)$	X is part of Y	$\overline{X} \cup Y = \mathcal{U}$	$X \rightarrow Y$
$\mathsf{P}^{-1}(X,Y)$	Y is part of X	$X \cup \overline{Y} = \mathcal{U}$	$Y \rightarrow X$
$\mathsf{EQ}(X,Y)$	X and Y are equal	$(\overline{X} \cup Y) \cap (X \cup \overline{Y}) = \mathcal{U}$	$X \leftrightarrow Y$

contain a pair of complementary literals (L and $\neg L$). Set-terms can also be converted to an analogous normal form, *intersection normal form* (INF): by means of simple re-write rules any set-term can be expressed as an intersection of unions of set-constants and their complements.

If a set-term corresponds to a tautological proposition then when expressed in INF each union in the expression must contain some pair, τ and $\overline{\tau}$, of a set constant and its complement. So whatever the assignment to the set constants each union and hence the intersection of these unions will have the value $\mathcal{U}$. This ensures that lemma 1 must hold. □

I now return to the proof of theorem 1.

Proof: If $P_1, \ldots, P_n \models_{C_0} P_0$ then the formula $(P_1 \wedge \ldots \wedge P_n) \rightarrow P_0$ must be a tautology; hence, by lemma 1, the equation $\overline{t_1 \cap \ldots \cap t_n} \cup t_0 = \mathcal{U}$ must hold. But in any model satisfying $t_1 = \mathcal{U}, \ldots, t_n = \mathcal{U}$ one must have $\overline{t_1 \cap \ldots \cap t_n} = \emptyset$. Therefore $t_0 = \mathcal{U}$.

On the other hand suppose $P_1, \ldots, P_n \not\models_{C_0} P_0$; this means that there is some truth-functional assignment, f, under which $P_1, \ldots, P_n$ are all true whilst P_0 is false. Given such an assignment we construct a set assignment, s, such that $s(P) = \mathcal{U}$ if $f(P) = true$ and $s(P) = \emptyset$ if $f(P) = false$. Clearly, the values of complex set-terms under s will correspond directly with the truth values of the associated propositions under f. Hence s is an assignment such that $t_1 = \mathcal{U}, \ldots, t_n = \mathcal{U}$ and $t_0 = \emptyset$. So $t_1 = \mathcal{U}, \ldots, t_n = \mathcal{U} \not\models_S t_0 = \mathcal{U}$. □

This correspondence theorem allows us to use classical propositional formulae to reason about universal set-equations.

2.1 FROM POSSIBLE WORLDS TO SPATIAL CONSTRAINTS

The basis of the topological representation system presented below is to exploit this semantics of propositional logic in terms of sets; but rather than taking $\mathcal{U}$ to be a set of possible worlds, $\mathcal{U}$ will be interpreted as a *space* of points and propositional letters will thus be interpreted as referring to *regions* within that space.

A universal set-equation can be regarded as a constraint on possible models — i.e. possible assignments of subsets of $\mathcal{U}$ to set-constants. If the set-constants denote regions, this allows one to specify relationships between these regions. For example the constraint $\overline{A} \cup B = \mathcal{U}$ will be satisfied by all and only those models in which set A is a subset of set B — i.e. region A is part of region B. In terms of $\mathcal{C}_0$ this constraint could be represented by the formula $\neg A \vee B$ (or equivalently $A \rightarrow B$). Thus, if $\mathcal{L}_{use}$ refers to the language whose expressions are all universal set-equations, a set of these equations can form the Θ component of a *situation-description*, $\langle \mathcal{L}_{use}, \Sigma, \Theta \rangle$.

3 DEFINING TOPOLOGICAL RELATIONS

The basic method is as follows: certain constraints associated with topological relationships are represented by propositional formulae; further constraints are then added at the meta-level in terms of restrictions on entailments of these formulae. A topological relation is thus defined by a set of formula called *model constraints* together with a further set of formulae called *entailment constraints*. A situation involving a number of topological relations is possible if and only if the set of model-constraints associated with all of the relations does *not* entail any of the entailment constraint formulae.

3.1 MODEL CONSTRAINTS

Suppose we have a situation in which a region A is part of another region B. Then the union of B with the complement of A must fill the entire space, $\mathcal{U}$. This can be represented by the set equation $\overline{A} \cup B = \mathcal{U}$. Hence because of the correspondence with $\mathcal{C}_0$ we can represent this as $\neg A \vee B$ (or equivalently $A \rightarrow B$). This formula is the model constraint associated with the situation A is part of B since in any model $A \rightarrow B$ denotes $\mathcal{U}$ if and only if A is part of B.

By means of such formulae four topological relations can be defined as shown in table 1. The relations defined here strictly correspond to the RCC relations of the same names only if we constrain all propositional letters to denote non-null regions. This rules out pathological cases such as where X is part of Y and X and Y are also discrete, which is only possible if X is null. More will be said about null regions below.

3.2 ENTAILMENT CONSTRAINTS

As it stands, our representation is very limited: many simple spatial relations cannot be defined solely by means of universal set-equations specifying model constraints. For example the relation $\mathsf{PP}(X, Y)$, X is a *proper* part of Y cannot be so expressed. Nevertheless, informally this relation can be defined quite straightforwardly as that relation which holds whenever $\mathsf{P}(X, Y)$ is true but not $\mathsf{EQ}(X, Y)$. So it would seem that we can characterise the proper part relation if we can find a way to represent the absence of a relation which we can already define.

We must now ask how the negations of $\mathcal{C}_0$ model constraints should be represented. Take for example $\neg\mathsf{P}(X, Y)$ (X is not part of Y). Suppose we simply negate the model-constraint corresponding to $\mathsf{P}(X, Y)$; we would then get $\neg(X \rightarrow Y)$. But this formula corresponds to the set equation $\overline{\overline{X} \cup Y} = \mathcal{U}$ or equivalently $X \cap \overline{Y} = \mathcal{U}$; and this will only hold when $X = \mathcal{U}$ and $Y = \emptyset$. So we see that the negation of a model-constraint formula does not correspond to the absence of the relation enforced by that constraint.

In terms of sets, what we really wanted to represent was $\overline{X} \cup Y \neq \mathcal{U}$ which is the direct negation of the set equation for $\mathsf{P}(X, Y)$. But negating the formula in the propositional representation does not give us this because such a negation is interpreted as a complement operation on the set-term rather than a negation of the whole equation. This means that the absence of the relations defined so far cannot be represented directly as model-constraints.

We need to increase the expressive capabilities of our representation language so we can represent situations in which we specify not only that a number of universal set-equations hold but also that certain such equations do not hold. Thus, we employ the language $\mathcal{L}_{usei}$ of universal set-equations and inequalities. A situation-description in this language is a structure $\langle \mathcal{L}_{usei}, \Sigma, \Theta \rangle$ where Θ is a set of universal set-equations and inequalities which are negations of universal set-equations. Such a situation description can be represented by a pair $\langle \mathcal{M}, \mathcal{E} \rangle$ where $\mathcal{M}$ and $\mathcal{E}$ are sets of $\mathcal{C}_0$ formulae obtained respectively from the set-terms involved in the set-equations and inequalities in Θ according to $_{C_0}\!\!\Longrightarrow^{ST}$. The language consisting of pairs of sets of $\mathcal{C}_0$ formulae will be called $\mathcal{C}_0^+$.

3.3 CONSISTENCY OF $\mathcal{C}_0^+$ SITUATION DESCRIPTIONS

What we now need is a method of determining from a pair of formula sets, $\langle \mathcal{M}, \mathcal{E} \rangle$, whether the corresponding situation-description, $\langle \mathcal{L}_{usei}, \Sigma, \Theta \rangle$, is possible.

Suppose Θ is a set $\{m_1 = \mathcal{U}, \ldots, m_j = \mathcal{U},\ e_1 \neq \mathcal{U}, \ldots, e_k \neq \mathcal{U}\}$ then Θ describes an impossible situation if and only if the following entailment holds:

$$m_1 = \mathcal{U}, \ldots, m_j = \mathcal{U} \models_S e_1 = \mathcal{U} \ \vee\ \ldots\ \vee\ e_k = \mathcal{U}$$

The r.h.s. is a disjunction of set-equations and as such cannot be translated into a union at the level of set-terms (just as negating a set equation is not equivalent to applying the complement operation to its set term).

However, it can be established that in the domain of sets, entailments of this kind are *convex* [3] in the sense of [9]. A class of entailments is convex iff whenever $\Gamma \models \phi_1 \vee \ldots \vee \phi_n$ then $\Gamma \models \phi_i$, for some i in $1 \ldots n$.

Consider the entailment associated with the impossibility of Θ. Suppose none of the disjuncts on the r.h.s. are entailed by the equations on the l.h.s.. This means that for each disjunct $e_i = \mathcal{U}$ there is a model, $\langle \mathcal{U}_i, \mathcal{P}, d_i \rangle$ in which it is false whilst all the l.h.s. equations are true. We can assume that the universes for each of these models are disjoint. We now construct a new model, $\langle \mathcal{U}_*, \mathcal{P}, d_* \rangle$, such that $\mathcal{U}_* = \bigcup_i \mathcal{U}_i$ and $d_*(X) = \bigcup_i d_i(X)$. The $\mathcal{U}_i$'s thus form discrete subspaces of $\mathcal{U}_*$. A consideration of this new model will reveal that it provides a counter-example to the entailment, since it must satisfy all the l.h.s. equations whilst making each of the disjuncts on the r.h.s. false.

Thus the r.h.s. will be entailed if and only if at least one of the disjuncts is individually entailed. So for each e_i we need to check whether

$$m_1 = \mathcal{U}, \ldots, m_j = \mathcal{U} \models_S e_i = \mathcal{U}$$

Thus, because of the equivalence between $\models_S$ and entailment between corresponding $\mathcal{C}_0$ formulae given by Theorem 1, we have the following:

Theorem 2 *A $\mathcal{C}_0^+$ representation $\langle \mathcal{M}, \mathcal{E} \rangle$ corresponds to a possible situation description (specified in $\mathcal{L}_{usei}$) if and only if there is no formula $F \in \mathcal{E}$ such that $\mathcal{M} \models_{C_0} F$.*

This theorem should make clear why the formulae in the set $\mathcal{E}$ are called entailment constraints.

3.4 THE RCC RELATIONS DEFINED

We can now give $\mathcal{C}_0^+$ representations for a significant sub-class of the RCC relations. Let us first look at how the situation type "A is a proper part of B" is represented. We can say that $\mathsf{PP}(A, B)$ holds when A is part of B but the two regions are not equal. This gives us the equality $\overline{A} \cup B = \mathcal{U}$ and the inequality $(\overline{A} \cup B) \cap (A \cup \overline{B}) \neq \mathcal{U}$. Also as noted above, to rule out cases where either A or B is the null set, we also need $\overline{A} \neq \mathcal{U}$ and $\overline{B} \neq \mathcal{U}$. Equalities are encoded as

[3] Note that later in the paper I use the term *convex* with its ordinary sense, as a property of the surface of a region. Hopefully this will not cause too much confusion.

Table 2: A Five Relation Basis Defined in $\mathcal{C}_0^+$

Relation	*Model Constraint*	*Entailment Constraints*
$\mathsf{DR}(X, Y)$	$\neg(X \wedge Y)$	$\neg X$, $\neg Y$
$\mathsf{PO}(X, Y)$	—	$\neg X \vee \neg Y$, $X \to Y$, $Y \to X$, $\neg X$, $\neg Y$
$\mathsf{PP}(X, Y)$	$X \to Y$	$Y \to X$, $\neg X$, $\neg Y$
$\mathsf{PP}^{-1}(X, Y)$	$Y \to X$	$X \to Y$, $\neg X$, $\neg Y$
$\mathsf{EQ}(X, Y)$	$X \leftrightarrow Y$	$\neg X$, $\neg Y$

model-constraints and inequalities as entailment constraints so our propositional representation for the relation $\mathsf{PP}(A, B)$ is the pair

$$\langle\{A \to B\}, \{A \leftrightarrow B, \neg A, \neg B\}\rangle.$$

The $\mathcal{C}_0^+$ representation allows us to define five jointly exhaustive and pairwise disjoint topological relations from the RCC lattice of spatial relations. The definitions are shown in table 2.

The model constraint associated with a relation is the strongest formula which holds in all models in which the relation holds. The entailment constraints serve to exclude cases which, although consistent with the model constraint are incompatible with the relation. Thus the entailment constraints correspond to model constraints of other relations (plus the non-null constraints). The relation PO has no model constraint and is defined by excluding all of the other relations.

Certain entailment constraints which one might expect to be required can be eliminated or weakened because they are indirectly captured by other constraints. For example, in table 2 the entailment constraint $A \leftrightarrow B$, which occurred in the representation of PP worked out above, is replaced by the weaker formula $B \to A$, since in the presence of the model constraint $A \to B$, $B \to A$ would immediately entail $A \leftrightarrow B$.

4 REASONING WITH $\mathcal{C}_0^+$

By making use of the results obtained so far one can use a $\mathcal{C}_0$ theorem prover as the basis of an effective automated spatial reasoning system. For clarity I concisely summarise the consistency checking algorithm for $\mathcal{C}_0^+$. Given a situation description consisting of a set of relations of the form $R(\alpha, \beta)$, where R is one of the relations characterisable in $\mathcal{C}_0^+$, and α and β are constants denoting regions, the following simple algorithm will decide whether the description describes a possible situation:

- For each relation $R_i(\alpha_i, \beta_i)$ in the situation description find the corresponding propositional representation $\langle\mathcal{M}_i, \mathcal{E}_i\rangle$.
- Construct the overall $\mathcal{C}_0^+$ representation $\langle\bigcup_i \mathcal{M}_i,\ \bigcup_i \mathcal{E}_i\rangle$.
- For each formula $F \in \bigcup_i \mathcal{E}_i$ use a classical propositional theorem prover to determine whether the entailment $\bigcup_i \mathcal{M}_i \models_{C_O} F$ holds.
- If any of the entailments determined in the last step does hold then the situation is impossible.

For example we may want to know whether the following situation is possible: A, is a proper part of B; B is disjoint with C; and, A is a proper part of C. The $\mathcal{C}_0^+$ representations of the three spatial relations are respectively:

$\langle\{A \to B\}, \{B \to A, \neg A, \neg B\}\rangle$,
$\langle\{\neg(B \wedge C)\}, \{\neg B, \neg C\}\rangle$,
$\langle\{A \to C\}, \{C \to A, \neg A, \neg C\}\rangle$.

So the overall $\mathcal{C}_0^+$ representation is

$$\langle\{A \to B,\ \neg(B \wedge C), A \to C\},\\ \{B \to A, C \to A, \neg A, \neg B, \neg C\}\rangle.$$

We determine that this situation is impossible since

$$A \to B,\ \neg(B \wedge C),\ A \to C\ \models_{C_O} \neg A.$$

4.1 DETERMINING ENTAILMENTS

Computing inconsistency of situations is a special case of determining entailments between situation descriptions characterisable in $\mathcal{C}_0^+$. To refer to such an entailment, I shall use the notation $\langle\mathcal{M}, \mathcal{E}\rangle \models_{C_o^+} \langle\mathcal{M}', \mathcal{E}'\rangle$. We can express the meaning of this as an entailment between set-equations corresponding to the formulae in the $\mathcal{C}_0^+$ representation:

$$m_1 = \mathcal{U}\ \wedge\ \ldots\ \wedge m_h = \mathcal{U}\ \wedge e_1 \neq \mathcal{U}\ \wedge\ \ldots\ \wedge e_i \neq \mathcal{U}$$
$$\models_S$$
$$m'_1 = \mathcal{U}\ \wedge\ \ldots\ \wedge m'_j = \mathcal{U}\ \wedge e'_1 \neq \mathcal{U}\ \wedge\ \ldots\ \wedge e'_k \neq \mathcal{U}$$

If we then bring the r.h.s. over to the left and move the resulting negation inwards we get:

$$m_1 = \mathcal{U}\ \wedge\ \ldots\ \wedge m_h = \mathcal{U}\ \wedge e_1 \neq \mathcal{U}\ \wedge\ \ldots\ \wedge e_i \neq \mathcal{U}\ \wedge\\ (m'_1 \neq \mathcal{U}\ \vee \ldots \vee\ m'_j \neq \mathcal{U}\ \vee e'_1 = \mathcal{U}\ \vee \ldots \vee e'_k = \mathcal{U}\,) \models_S$$

To show the validity of this we must show that whichever of the equations in the disjunction is chosen the resulting equation set is inconsistent. This is equivalent to showing that:

for all $p \in \mathcal{M}'$ we have $\langle\mathcal{M}, \mathcal{E} \cup \{p\}\rangle \models_{C_o^+}$
and for all $q \in \mathcal{E}'$ we have $\langle\mathcal{M} \cup \{q\}, \mathcal{E}\rangle \models_{C_o^+}$

Table 3: Composition table for the 5 relation basis

$R(a,b)$ \ $R(b,c)$	DR	PO	EQ	PP	PP^{-1}
DR	$*$	DR $\vee$ PO $\vee$ PP	DR	DR $\vee$ PO $\vee$ PP	DR
PO	DR $\vee$ PO $\vee$ PP^{-1}	$*$	PO	PO $\vee$ PP	DR $\vee$ PO $\vee$ PP^{-1}
EQ	DR	PO	EQ	PP	PP^{-1}
PP	DR	DR $\vee$ PO $\vee$ PP	PP	PP	$*$
PP^{-1}	DR $\vee$ PO $\vee$ PP^{-1}	PO $\vee$ PP^{-1}	PP^{-1}	O	PP^{-1}

Another equivalent way of expressing these which is more convenient from the point of view of actually calculating the entailments is the following:

Theorem 3 $\langle \mathcal{M}, \mathcal{E} \rangle \models_{C_o^+} \langle \mathcal{M}', \mathcal{E}' \rangle$ *iff*
either $\langle \mathcal{M}, \mathcal{E} \rangle \models_{C_o^+}$
or (*for all* $p \in \mathcal{M}'$: $\langle \mathcal{M}, \{p\} \rangle \models_{C_o^+}$
and for all $q \in \mathcal{E}'$: $\langle \mathcal{M} \cup \{q\}, \mathcal{E} \rangle \models_{C_o^+}$)

Determining the validity of a $\mathcal{C}_0^+$ entailment has thus been reduced to determining the impossibility of certain situation descriptions derived from the constraints involved; and we already know that a description is impossible iff one of its entailment constraints is entailed by its model constraints.

4.2 COMPUTING LOCI OF COMPOSITION

Given a particular theory Θ supporting a set $\mathcal{B}$ of mutually exhaustive and pairwise disjoint dyadic relations (a *basis* set), for each pair of relations R_1 and R_2 taken from $\mathcal{B}$, the *locus of composition* [4] of R_1 and R_2, $Comp(R_1, R_2)$, is the disjunction of all relations R_3 in $\mathcal{B}$, such that, for arbitrary individual constants a, b, c, the formula $R_1(a,b) \wedge R_2(b,c) \wedge R_3(a,c)$ is consistent with Θ. In other words $Comp(R_1, R_2)$ is the disjunction of all possible base relations which could hold between a and c. Computing loci of composition for spatial relations is the "challenge for automated theorem provers" proposed in [12].

By using the consistency algorithm described above, the $\mathcal{C}_0^+$ representation enables loci of composition for spatial relations to be computed very efficiently. Given R_1 and R_2, which are members of some basis set $\mathcal{B}$, one simply checks for all values of R_3 taken from $\mathcal{B}$, whether the situation described by $R_1(a,b)$, $R_2(b,c)$, $R_3(a,c)$ is possible. Table 3 gives the loci of composition for the 5 relation basis $\{\mathsf{DR}, \mathsf{PO}, \mathsf{PP}, \mathsf{PP}^{-1}, \mathsf{EQ}\}$. The symbol '$*$' stands for the disjunction of all 5 relations. This table was computed in under 6.7 seconds on a SPARC1 workstation.

[4] What is here called the *locus of composition* is the same as what in [12] was referred to as the 'transitive closure' of two base relations. This terminology derives from Allen's 'transitivity table' for temporal intervals [1]. However,'transitive closure' already has a meaning different from what is intended here, so a new term is required to avoid potential confusion. In describing the more general problem of determining possible values of unknown relations in the context of a partial situation description I have adopted the phrase 'locus of an unspecified relation'. The 'locus of composition' is a special case of such a locus.

5 MORE EXPRESSIVENESS WITH INTUITIONISTIC LOGIC

In his paper "Sentential Calculus and Topology" [13] Tarski has shown that the intuitionist propositional calculus (henceforth $\mathcal{I}_0$) can be given an interpretation in which propositional letters correspond to *open* sets within a *topological space.*

The spatial interpretation of intuitionistic logic requires a richer notion of a *space* than the classical. Specifically, whereas before a space was simply a set of elements, a space is now a set of elements for which the notions of *interior* and *closure* are defined for each subset of spatial elements.

A topological space can be described by a structure $\langle \mathcal{U}, i \rangle$, where $\mathcal{U}$ is an arbitrary set of elements whose *topology* is defined by a function i which maps each subset of $\mathcal{U}$ to another subset of $\mathcal{U}$, its *interior.* i must satisfy certain well known axioms (see e.g. [6, p.129]). The closure of a set $c(X)$ is defined as equivalent to $\overline{i(\overline{X})}$.

5.1 INTERPRETATION OF $\mathcal{I}_0$

The topological interpretation of $\mathcal{I}_0$ is very similar to the interpretation of $\mathcal{C}_0$ given above. Again propositional formulae will denote subsets of a space, although admissible subsets will be limited to those which are *open* under the topology of the space. A set X is open if and only if $i(X) = X$.

A model for $\mathcal{I}_0$ is a structure $\langle \mathcal{U}, i, \mathcal{P}, d \rangle$, where $\mathcal{U}$ is a non-empty set, i is a function satisfying the appropriate axioms, which maps subsets of $\mathcal{U}$ to their interiors, $\mathcal{P}$ is a denumerably infinite set of propositional constants, and d is a denotation function which assigns to each constant in $\mathcal{P}$ an *open* subset of $\mathcal{U}$. The domain of d is extended to all $\mathcal{I}_0$ formulae formed from these variables by stipulating that:

1. $d(\sim P) = i(\ \overline{d(P)}\ )$ [5]
2. $d(P \wedge Q) = d(P) \cap d(Q)$
3. $d(P \vee Q) = d(P) \cup d(Q)$
4. $d(P \Rightarrow Q) = i(\overline{d(P)} \cup d(Q))$

where for any set S, $\overline{S}$ is the set of all elements of $\mathcal{U}$ which are not elements of S.

Just as for the classical logic we can consider the topological interpretation of $\mathcal{I}_0$ as associating each intuitionistic formula with a set-term; but set-terms may now contain the interior operator. I refer to the mapping between $\mathcal{I}_0$ formulae and set-terms induced by this interpretation with the notation ${}_{I_0}\!\!\Longrightarrow^{ST}$.

Tarski's "Second Principal Theorem" [13, p.448] establishes that a propositional formula is a theorem of $\mathcal{I}_0$ if and only if the corresponding set-term has the value $\mathcal{U}$ in any topological space under any assignment of open sets to the set constants occurring in the term. The proof of this is fairly involved and is not reconstructed. I use the notation '$\vdash_{I_0}$' to denote entailment in $\mathcal{I}_0$ and '$\models_T$' to denote topological entailment — i.e. entailment between set-equations which may contain the interior operator, i. Tarski's theorem can then be written formally as:

Theorem 4 $\vdash_{I_0}\ P$ *if and only if* $\models_T\ \pi = \mathcal{U}$, *where* $P\ {}_{I_0}\!\!\Longrightarrow^{ST}\ \pi$. [6]

In using $\mathcal{I}_0$ to represent spatial relations we shall exploit very similar correspondence relations to those holding between the $\mathcal{C}_0$ and the Boolean algebra of sets. In order to secure the correspondence between entailment in $\mathcal{I}_0$ and entailment between set equations in the topological algebra of sets, we need to generalise Tarski's result to a correspondence between entailments:

Theorem 5 $P_1, \ldots, P_n\ \vdash_{I_0}\ P_0$
if and only if $\pi_1 = \mathcal{U}, \ldots, \pi_n = \mathcal{U}\ \models_T\ \pi_0 = \mathcal{U}$

Proof: The positive half is simple:

An $\mathcal{I}_0$ entailment $P_1, \ldots, P_n\ \vdash_{I_0}\ P_0$ holds iff $\vdash_{I_0}$ $(P_1 \wedge \ldots \wedge P_n) \Rightarrow P_0$, so by Theorem 4 we have $\models_T\ i(\overline{\pi_1 \cap \ldots \cap \pi_n} \cup \pi_0) = \mathcal{U}$. But if a set has $\mathcal{U}$ as its interior then it must be equal to $\mathcal{U}$, so the equation $(\overline{\pi_1 \cap \ldots \cap \pi_n} \cup \pi_0) = \mathcal{U}$ must hold in every model. Thus whenever $\pi_i = \mathcal{U}$ for $i = 1 \ldots n$ we must also have $\pi_0 = \mathcal{U}$ — in other words $\pi_1 = \mathcal{U}, \ldots, \pi_n = \mathcal{U}\ \models_T\ \pi_0 = \mathcal{U}$.

Suppose on the other hand $P_1, \ldots, P_n \not\vdash_{I_0}\ P_0$. Theorem 4 gives us $\not\models_T\ i(\overline{\pi_1 \cap \ldots \cap \pi_n} \cup \pi_0) = \mathcal{U}$, which means that there is some model, $\mathcal{M} = \langle \mathcal{U}, i, \mathcal{P}, d \rangle$, in which there is at least one element of $\pi_1 \cap \ldots \cap \pi_n$ which is not an element of π_0. On the basis of this model we now construct a model $\mathcal{M}' = \langle \mathcal{U}', i', \mathcal{P}, d' \rangle$ whose universe, $\mathcal{U}'$ is the set denoted by $\pi_1 \cap \ldots \cap \pi_n$ in $\mathcal{M}$. We set $i'(X) = i(X)$ for all $X \subseteq \mathcal{U}'$ and for all propositional constants P_i we set $d'(P_i) = d(P_i) \cap \mathcal{U}'$. The interpretations of the logical operators given above will ensure that for all formulae F, $d'(F) = d(F) \cap \mathcal{U}'$.

Thus in particular for each $i = 1 \ldots n$, $d'(P_i) = d(P_i) \cap \mathcal{U}' = \pi_i \cap \mathcal{U}' = \mathcal{U}'$; i.e. in the new model all antecedent formulae denote the universe. We also have $d'(P_0) = d(P_0) \cap \mathcal{U}' = \pi_0 \cap \mathcal{U}'$. Furthermore, we know that there is at least one element of $\mathcal{U}'$ which is not an element of π_0. This means that $d'(P_0) \neq \mathcal{U}'$; so $\mathcal{M}'$ provides a counter-example to the entailment. This concludes the proof of theorem 5. □

5.2 $\mathcal{I}_0$ REPRESENTATION OF RCC RELATIONS

We can now translate the topological relations defined by 1st-order logic in the RCC system into a 0-order representation in which intuitionistic formulae represent constraints on possible situations.

The basis of the interpretation is as follows:

- A *region* is identified with an open set of points. (So regions are denoted by propositional letters in the $\mathcal{I}_0$ representation.)
- Regions *overlap* if they share at least one point.
- Regions are *connected* if their *closures* share at least one point.

This interpretation is in accord with that suggested for the RCC theory in [10].

Because the topological interpretation of $\mathcal{I}_0$ involves set-terms containing the interior operator, i, it allows us to make some distinctions which are not possible with the classical calculus. In particular we can now distinguish the case where two non-overlapping regions are connected (i.e. touch at one or more points) from that in which they are totally disconnected. And, in a similar manner, we can specify whether a region which is a proper part of another is a tangential or non-tangential proper part.

[5] Under this interpretation one can see why the law of excluded middle fails in intuitionistic logic. $A \vee \sim A$ is interpreted as $A \cup i(\ \overline{A}\ )$. But the union of A with its exterior, $i(\ \overline{A}\ )$, does not exhaust the space, since the points in $c(A) - A$, the *boundary* of A, are neither included in A nor its exterior.

[6] This theorem holds for any topology whatsoever. Adding conditions to the topology would mean the corresponding logic would be stronger. The limiting case is the *discrete* topology corresponding to classical logic.

Table 4: Some RCC Relations Defined in $\mathcal{I}_0^+$ (including the 8 relation basis)

Relation	*Model Constraint*	*Entailment Constraints*
$\mathsf{DC}(X,Y)$	$\sim X \vee \sim Y$	$\sim X,\ \sim Y$
$\mathsf{EC}(X,Y)$	$\sim(X \wedge Y)$	$\sim X \vee \sim Y, \sim X,\ \sim Y$
$\mathsf{PO}(X,Y)$	—	$\sim(X \wedge Y),\ X \Rightarrow Y,\ Y \Rightarrow X, \sim X, \sim Y$
$\mathsf{TPP}(X,Y)$	$X \Rightarrow Y$	$\sim X \vee Y,\ Y \Rightarrow X, \sim X, \sim Y$
$\mathsf{TPP}^{-1}(X,Y)$	$Y \Rightarrow X$	$\sim Y \vee X,\ X \Rightarrow Y, \sim X, \sim Y$
$\mathsf{NTPP}(X,Y)$	$\sim X \vee Y$	$Y \Rightarrow X, \sim X, \sim Y$
$\mathsf{NTPP}^{-1}(X,Y)$	$\sim Y \vee X$	$X \Rightarrow Y, \sim X, \sim Y$
$\mathsf{EQ}(X,Y)$	$X \Leftrightarrow Y$	$\sim X, \sim Y$
$\mathsf{C}(X,Y)$	—	$\sim X \vee \sim Y, \sim X, \sim Y$
$\mathsf{EQ}(X, \mathsf{sum}(Y,Z))$	$X \Leftrightarrow (Y \vee Z)$	$\sim X, \sim Y, \sim Z$

If two regions share no points they cannot overlap (although they may be connected). In such a case the equation $i(\ \overline{X \cap Y}\ ) = \mathcal{U}$ must hold; this can be represented by the $\mathcal{I}_0$ formula $\sim(X \wedge Y)$. In $\mathcal{I}_0$ (unlike $\mathcal{C}_0$) this formula is not equivalent to $\sim X \vee \sim Y$. The latter corresponds to the set-equation $i(\ \overline{X}\ ) \cup i(\ \overline{Y}\ ) = \mathcal{U}$, which says that the union of the exteriors of two regions exhaust the space. If the regions touch at one or more points, then these points of contact will not be in the exterior of either region so this equation will not hold. Hence the second (stronger) formula can be employed as a model constraint to describe situations where two regions are completely disconnected.

5.3 THE $\mathcal{I}_0^+$ REPRESENTATION LANGUAGE

To represent relations using $\mathcal{I}_0$ we can use essentially the same type of encoding as we employed for $\mathcal{C}_0$. As before, restrictions on possible models corresponding to the presence of topological relationships between regions are enforced by means of model constraints and entailment constraints. The role of these two types of constraint in reasoning about situations is exactly as in the classical case. In fact the arguments given in sections 3.2, 3.3 and 4.1 regarding the representation of negative constraints and the correct procedures for reasoning in $\mathcal{C}_0^+$ apply equally when $\mathcal{I}_0$ is employed as a language for representing set equations. Most of the arguments rely only upon the correspondence expressed in theorem 1, so parallel arguments for $\mathcal{I}_0$ can be given on the basis of theorem 5. The convexity property shown in section 3.3 can also be similarly demonstrated for the topological interpretation of $\mathcal{I}_0$. Hence we already have the apparatus for reasoning with the language $\mathcal{I}_0^+$, whose expressions are pairs of sets of $\mathcal{I}_0$ formulae specifying model-constraints and entailment-constraints. Counterparts of theorems 2 and 3 apply to the language $\mathcal{I}_0^+$ as well as to $\mathcal{C}_0^+$.

Table 4 gives the $\mathcal{I}_0^+$ representation of each of the 8 basic relations shown in figure 1. The definition of C plus another example using the RCC function sum are also given. That the model constraints given in this table must hold if the corresponding RCC relation holds is easily verified by considering the interpretation of the formulae given in section 5.1. As with $\mathcal{C}_0^+$, the set of entailment constraints represent negative conditions needed to exclude unwanted situations which are compatible with the model-constraint.

6 IMPLEMENTATION OF A $\mathcal{I}_0^+$ REASONING SYSTEM

A spatial reasoner using this technique has been implemented in Prolog using a Horn clause representation of a restricted Gentzen calculus for $\mathcal{I}_0$ and a look-up table to translate topological relations into the appropriate model and entailment constraints. Running on a SPARC1 workstation the program generated the full *composition table* for the 8 base relations shown in Figure 1 in under 244 seconds.

This is a substantial improvement over the method described in [12]. In generating the table given there, the theorem prover OTTER [7] was used, working with the 1st-order axiomatisation of the RCC theory. OTTER took a total of 2460 seconds to prove the required theorems but some proofs required human assistance (addition of hand chosen lemmas and restriction of the set of axioms used). Furthermore this method involves not only theorem proving but also *model building* in order to ensure the minimality of table entries (see [12]) and this is also computationally intensive. This method cannot really compete with reasoning using the $\mathcal{I}_0^+$ representation, since unlike $\mathcal{I}_0^+$ no decision procedure is known for the 1st-order RCC theory.

6.1 THEOREM PROVING IN $\mathcal{I}_0$

Clearly, to use $\mathcal{I}_0^+$ as a representation language for effective spatial reasoning we need to be able to reason efficiently in $\mathcal{I}_0$. Theorem proving in $\mathcal{I}_0$ is decidable but potentially very hard (see [5]). A proof-theory for the language can be specified in terms of a simple cut-free Gentzen sequent calculus which is only a slight modification of the corresponding classical system. The formalisation I use is essentially the same as that given in [4].

Theorem proving in the $\mathcal{I}_0$ sequent calculus is more complex than that of $\mathcal{C}_0$: in $\mathcal{C}_0$ all connectives can be eliminated deterministically because the rules produce Boolean combinations of sequents which are logically equivalent to the original sequent. However with certain rules in the $\mathcal{I}_0$ calculus the resulting combination of proofs is not necessarily provable even if the original sequent is valid. In other words the rule gives a sufficient but not necessary condition for validity. Consequently theorem proving in $\mathcal{I}_0$ is non-deterministic and involves a much larger search space.

However, given that the representation of many spatial constraints involves only a very limited class of $\mathcal{I}_0$ formulae, much of the potential complexity of theorem proving can be avoided. This is achieved by employing a proof system which, although not complete for the full language of $\mathcal{I}_0$, is complete for sequents containing only formulae used to represent the RCC spatial relations. Specifically, we need handle formulae of the following forms: $\sim X$, $\sim X \vee \sim Y$, $\sim(X \wedge Y)$, $X \Rightarrow Y$, $\sim X \vee Y$.

Given this restriction, the non-deterministic and extremely computationally expensive rule for eliminating implications from the left hand side of a sequent can be replaced by other rules which can be applied deterministically (space does not permit a fuller explanation). Use of this restricted proof system dramatically increases the effectiveness of reasoning in $\mathcal{I}_0^+$.

7 EXTENDING THE REPRESENTATION

In the rest of the paper I indicate how the $\mathcal{I}_0^+$ representation can be extended to incorporate extra concepts which are not directly reducible to $\mathcal{I}_0^+$ but for which we do have a set of axioms specified in the (more expressive) 1st-order classical logic, $\mathcal{C}_1$. To illustrate the method I show how the notions of 'inside' and 'outside' can be represented.

7.1 'INSIDE' AND 'OUTSIDE'

Following the approach taken in [10] I define the relations 'inside' and 'outside' in terms of a *convex-hull* operator which is introduced as a new primitive. The convex-hull, $\mathsf{conv}(X)$, of a region X can be informally defined as that region which would be enclosed by a taught rubber membrane stretched around X. [7] In terms of the relations $\mathsf{P}(x,y)$ (x is a part of y) and $\mathsf{TP}(x,y)$ (x is a tangential part of y) and $\mathsf{C}(x,y)$ (x is connected to y) an axiomatisation of $\mathsf{conv}(x)$, the convex-hull operator can be given in $\mathcal{C}_1$ as follows:

1. $\forall x \mathsf{TP}(x, \mathsf{conv}(x))$
2. $\forall x[\mathsf{conv}(\mathsf{conv}(x)) = \mathsf{conv}(x)]$
3. $\forall x \forall y[\mathsf{P}(x,y) \;\rightarrow\; \mathsf{P}(\mathsf{conv}(x), \mathsf{conv}(y))]$
4. $\forall x \forall y[\mathsf{conv}(x) = \mathsf{conv}(y) \;\rightarrow\; \mathsf{C}(x,y)]$ [8]

Whether these axioms are indeed faithful in characterising the convex-hull is not completely certain. The first three are very simple and undoubtably true. 4) is more difficult to see. It states that, if two (finite) regions have the same convex-hull they must be connected.

To show this I introduce the notion of the *convex-hull defining points* of a (finite) region. These are points in the closure of a region which do not lie between any two other points in its closure. Such points will always lie on the surface of a region (i.e. $c(X) - X$) and will always be points where the surface is convex.

The convex-hull defining points of a region uniquely determine its convex-hull. Also every convex-hull has a unique set of defining points. Consequently, two regions have the same convex-hull if and only if they have the same defining points. We may also note that an n dimensional region must have at least $n+1$ defining points. From these observations it is clear that if two regions have the same convex-hull then their closures must share certain points; they must have at least the convex-hull defining points in common. This being so, regions with the same convex-hull must be connected.

So there are compelling arguments for the truth of all the axioms given above. What is less certain is whether this axiom set is complete: it is possible that there are properties (expressible in terms of C and conv) of the convex-hull in Euclidean space that are not captured. If this were the case then there would be situation descriptions consistent with the axioms but impossible under the intended interpretation of the conv operator[9].

7.2 RELATIONS DEFINABLE WITH conv

A large number of new binary relations can be defined in terms of the conv together with other RCC relations.

[7] More formally, in terms of point sets, $\mathsf{conv}(X)$ is the closure of X with respect to the relation of *betweenness*, that is $\mathsf{conv}(X) = \{x : \exists y \exists z[y \in X \;\wedge\; z \in X \;\wedge\; B(y,x,z)]\}$, where $B(x,y,z)$ means that point y lies on the straight line between x and z.

[8] Actually this is not necessarily true for infinite regions.

[9] One way to demonstrate adequacy of the axioms would be to show that they are faithful to the interpretation in terms of the betweenness relation, which has a straightforward algebraic definition in a model which is a Cartesian space over the real numbers (see [14]).

For example [10] gives the following definitions:

- $\mathsf{INSIDE}(X, Y) \equiv_{def} \mathsf{DR}(X, Y) \wedge \mathsf{P}(X, \mathsf{conv}(Y))$
- $\mathsf{P\text{-}INSIDE}(X, Y) \equiv_{def}$
  $\mathsf{DR}(X, Y) \wedge \mathsf{PO}(X, \mathsf{conv}(Y))$
- $\mathsf{OUTSIDE}(X, Y) \equiv_{def} \mathsf{DR}(X, \mathsf{conv}(Y))$ [10]

More generally by combining the 8 basic RCC relation with the conv operator we can specify a total of 8^4 relations of the form $R_1(X, Y) \wedge R_2(X, \mathsf{conv}(Y)) \wedge R_3(\mathsf{conv}(X), Y) \wedge R_4(\mathsf{conv}(X), \mathsf{conv}(Y))$.

To keep the number of relations dealt with manageable, I identify a set of 18 mutually exclusive relations which are refinements of the DR. Following [10] I represent these by expressions of the form $[\sigma_1, \sigma_2, \tau](X, Y)$, where σ_1 is either 'I', 'P' or 'O' according as either $\mathsf{INSIDE}(X, Y)$, $\mathsf{P\text{-}INSIDE}(X, Y)$ or $\mathsf{OUTSIDE}(X, Y)$; σ_2 refers to the corresponding inverse relation (i.e. one of these 3 relations but with the arguments reversed); and τ is either 'D' or 'E' according to whether the regions are completely disconnected or externally connected. Thus, for example, $[\text{P,I,E}](X, Y)$ means that $\mathsf{P\text{-}INSIDE}(X, Y)$, $\mathsf{INSIDE}(Y, X)$ and $\mathsf{EC}(X, Y)$.

Actually the relation $[\text{I,I,D}](X, Y)$ is impossible, since if two regions are both inside each other they must share the same convex-hull and therefore, according to axiom 4., must be connected. Thus we can identify a basis of 23 pairwise disjoint and mutually exhaustive relations consisting of the 17 possible refinements of DR, plus the six remaining relations of the RCC 8 relation basis.

7.3 ENCODING conv IN $\mathcal{I}_0^+$

Suppose we treat the expression $\mathsf{conv}(Y)$ simply as referring to an arbitrary region. Then the relation INSIDE(X,Y) as defined above could be represented by two model constraints: $\sim(X \wedge Y)$ and $X \Rightarrow \mathsf{conv}(Y)$, corresponding to $\mathsf{DR}(X, Y)$, and $\mathsf{P}(X, \mathsf{conv}(Y))$, respectively. So we can assimilate references to convex-hulls into the $\mathcal{I}_0^+$ representation simply by introducing propositional expressions of the form $\mathsf{conv}(X)$ into $\mathcal{I}_0$ formulae. But, as regards correct reasoning, this is inadequate, since the meaning of $\mathsf{conv}(X)$ relative to X is not fixed — they are just two regions.

This can be remedied by adding extra constraints to $\mathcal{I}_0^+$ situation representations which enforce the axioms given above. This extra information means that situations which are inconsistent in virtue of these axioms can be detected by means of a $\mathcal{I}_0$ theorem prover. In so far as the axioms adequately characterise the intended interpretation of conv this will serve to fix the meaning of the operator.

[10] Note that these relations are not purely topological, since they are not preserved by *rubber* deformations of the regions involved.

Axiom 1. can be enforced as follows: for each region X mentioned in the initial situation description, augment the description with extra model and entailment constraints corresponding to the situation $\mathsf{TP}(X, \mathsf{conv}(X))$. Any model which satisfies this extended model will clearly satisfy axiom 1.

Axiom 2. is taken into account implicitly. In enforcing axiom 1. we introduce extra regions into the situation description corresponding to the convex-hulls of each region in the initial description. Axiom 2. tells us that these are the only additional regions we need consider, since iterating the conv functions does not produce any more new regions.

Treatment of axioms 3. and 4. is encompassed by a general procedure which enables enforcement of all axioms of the form:

$$\forall x_1, \ldots, x_n[\Phi(x_1, \ldots, x_n) \rightarrow \Psi(x_1, \ldots, x_n)],$$

where $\Phi(x_1, \ldots, x_n)$ and $\Psi(x_1, \ldots, x_n)$ specify situations which can be described by means of $\mathcal{I}_0^+$.

To test whether a given $\mathcal{I}_0^+$ situation description satisfies such an axiom an iterative fixed-point method can be used:

- Test the $\mathcal{I}_0^+$ description for consistency
- Check whether some instance of the antecedent is entailed by an the initial description. This involves translating $\Phi(\ldots)$ into $\mathcal{I}_0^+$ and substituting all combinations of constants occurring in the description for the free variables.
- If any such $\Phi(\ldots)$ is entailed add the corresponding $\mathcal{I}_0^+$ representation of $\Psi(\ldots)$, under the same substitution, to the situation description.
- Repeat until either the situation description becomes inconsistent or no new information is added by the previous step.

This process must terminate; and if the final situation description is still consistent then clearly the axiom is satisfiable, since for all substitutions either the antecedent is not entailed by the description or the consequent has been explicitly added.

Clearly the convex-hull axioms 3. and 4. are of the form which can be captured in this manner. In fact, since their antecedents are quite simple, they can be enforced quite efficiently.

7.4 AN AUTOMATICALLY GENERATED 23 RELATION COMPOSITION TABLE

Table 5 gives the full composition table (i.e. table of loci of composition) for the basis of 23 relations described in section 7.2. If $R_1(A, B)$ and $R_2(B, C)$, where R_1 is the relation specified in the left hand column and R_2 is specified along the top, the corresponding table entry encodes the possible values of the relation $R_3(A, C)$. Each of the 23 relations is represented

Table 5: Loci of Composition for a Basis of 23 Spatial Relations

Rel.	*Rep.*	1 ★·· ··· ··· ···	2 ○·· ··· ··· ···	3 ·○· ··· ··· ···	4 ·★· ··· ··· ···	5 ··★ ··· ··· ···	6 ··○ ··· ··· ···	7 ··· ★·· ··· ···	8 ··· ·★· ··· ···	9 ··· ·○· ··· ···	10 ··· ··★ ··· ···	11 ··· ··○ ··· ···	12 ··· ··· ★·· ···	13 ··· ··· ○·· ···	14 ··· ··· ·★· ···	15 ··· ··· ·○· ···	16 ··· ··· ··★ ···	17 ··· ··· ··○ ···	18 ··· ··· ··· ★··	19 ··· ··· ··· ○··	20 ··· ··· ··· ·★·	21 ··· ··· ··· ·○·	22 ··· ··· ··· ··★	23 ··· ··· ··· ··○
1 NTPP	★·· ··· ··· ···	★·· ··· ··· ···	★·· ··· ··· ···	●○· ★●● ●●● ●●●	★·· ··· ··· ···	●●● ★●● ●●● ●●●	●○· ★●● ●●● ●●●	··· ·○○ ··· ···	··· ·○○ ··· ···	··· ·○○ ··· ···	··· ··○ ··· ···	··· ··○ ··· ···	··· ·○○ ○○○ ○○○	··· ·○○ ○○○ ○○○	··· ·○○ ·○○ ·○○	··· ·○○ ·○○ ·○○	··· ··○ ··○ ··○	··· ··○ ··○ ··○	··· ··· ··· ○○○	··· ··· ··· ○○○	··· ··· ··· ·○○	··· ··· ··· ·○○	··· ··· ··· ··○	··· ··· ··· ··○
2 TPP	○·· ··· ··· ···	★·· ··· ··· ···	●·· ··· ··· ···	●○· ★●● ●●● ●●●	○·· ··· ··· ···	·○● ★●● ●●● ●●●	○●○ ★●● ●●● ●●●	··· ★●● ··· ···	··· ·●● ··· ···	··· ·○○ ··· ···	··· ··● ··· ···	··· ··○ ··· ···	··· ★●● ●●● ●●●	··· ·○○ ○○○ ○○○	··· ·●● ·●● ·●●	··· ·○○ ·○○ ·○○	··· ··● ··● ··●	··· ··○ ··○ ··○	··· ··· ··· ●●●	··· ··· ··· ○○○	··· ··· ··· ·●●	··· ··· ··· ·○○	··· ··· ··· ··●	··· ··· ··· ··○
3 PO	·○· ··· ··· ···	●○· ··· ··· ···	●○· ··· ··· ···	●●● ★●● ●●● ●●●	·○· ··· ··· ···	·○● ★●● ●●● ●●●	·○● ★●● ●●● ●●●	·○● ★●● ●●● ···	·○● ★●● ●●● ···	·○● ★●● ●●● ···	·○● ★●● ●●● ···	·○● ★●● ●●● ···	·○● ★●● ●●● ●●●	·○● ★●● ●●● ●●●	·○● ★●● ●●● ●●●	·○● ★●● ●●● ●●●	·○● ★●● ●●● ●●●	·○● ★●● ●●● ●●●	·○● ··· ●●● ●●●	·○● ··· ●●● ●●●	·○● ··· ●●● ●●●	·○● ··· ●●● ●●●	·○● ··· ●●● ●●●	·○● ··· ●●● ●●●
4 EQ	·★· ··· ··· ···	★·· ··· ··· ···	○·· ··· ··· ···	·○· ··· ··· ···	·★· ··· ··· ···	··★ ··· ··· ···	··○ ··· ··· ···	··· ★·· ··· ···	··· ·★· ··· ···	··· ·○· ··· ···	··· ··★ ··· ···	··· ··○ ··· ···	··· ··· ★·· ···	··· ··· ○·· ···	··· ··· ·★· ···	··· ··· ·○· ···	··· ··· ··★ ···	··· ··· ··○ ···	··· ··· ··· ★··	··· ··· ··· ○··	··· ··· ··· ·★·	··· ··· ··· ·○·	··· ··· ··· ··★	··· ··· ··· ··○
5 NTPP^{-1}	··★ ··· ··· ···	●●● ··· ··· ···	·○● ··· ··· ···	·○● ··· ··· ···	··★ ··· ··· ···	··★ ··· ··· ···	··★ ··· ··· ···	·○● ··· ··· ···	·○● ··· ··· ···	·○● ★●· ●●· ···	·○● ··· ··· ···	·○● ★●● ●●● ···	·○● ··· ··· ···	·○● ··· ●·· ···	·○● ··· ··· ···	·○● ··· ●●· ···	·○● ··· ··· ···	·○● ··· ●●● ···	·○● ··· ··· ···	·○● ··· ●·· ●··	·○● ··· ··· ···	·○● ··· ●●· ●●·	·○● ··· ··· ···	·○● ··· ●●● ●●●
6 TPP^{-1}	··○ ··· ··· ···	●○· ··· ··· ···	○●○ ··· ··· ···	·○● ··· ··· ···	··○ ··· ··· ···	··★ ··· ··· ···	··● ··· ··· ···	·○● ★·· ★·· ···	·○● ★★· ★★· ···	·○● ★●· ●●· ···	·○● ★★★ ★★★ ···	·○● ★●● ●●● ···	·○● ··· ★·· ···	·○● ··· ●·· ···	·○● ··· ★★· ···	·○● ··· ●●· ···	·○● ··· ★★★ ···	·○● ··· ●●● ···	·○● ··· ★·· ★··	·○● ··· ●·· ●··	·○● ··· ★★· ★★·	·○● ··· ●●· ●●·	·○● ··· ★★★ ★★★	·○● ··· ●●● ●●●
7 [IIE]	··· ★·· ··· ···	●○· ··· ··· ···	●○· ★★· ··· ···	●○· ★●· ●●· ●●·	··· ★·· ··· ···	··· ··· ○·· ○··	··· ★·· ●·· ●··	○●○ ★·· ··· ···	○○· ·●· ··· ···	·○· ·●· ··· ···	··· ··● ··· ···	··· ··● ··· ···	·○○ ··· ●·· ●··	·○● ··· ●·· ●··	·○· ··· ·●· ·●·	·○· ··· ·●· ·●·	··· ··· ··● ··●	··· ··· ··● ··●	·○○ ··· ●·· ●··	·○● ··· ●·· ●··	·○· ··· ·●· ·●·	·○· ··· ·●· ·●·	··· ··· ··● ··●	··· ··· ··● ··●
8 [IPE]	··· ·★· ··· ···	●○· ··· ··· ···	●○· ·★· ··· ···	●○· ★●● ●●● ●●●	··· ·★· ··· ···	··· ·○○ ○○○ ○○○	··· ★●● ●●● ●●●	○○· ·●● ··· ···	○○· ·●● ··· ···	·○· ·●● ··· ···	··· ··● ··· ···	··· ··● ··· ···	○●○ ★●● ●●● ●●●	·○● ★●● ●●● ●●●	○○· ·●● ·●● ·●●	·○· ·●● ·●● ·●●	··· ··● ··● ··●	··· ··● ··● ··●	·○○ ··· ●●● ●●●	·○● ··· ●●● ●●●	·○· ··· ·●● ·●●	·○· ··· ·●● ·●●	··· ··· ··● ··●	··· ··· ··● ··●
9 [IPD]	··· ·○· ··· ···	●○· ·●· ··· ···	●○· ·●· ··· ···	●○· ★●● ●●● ●●●	··· ·○· ··· ···	··· ·○○ ○○○ ○○○	··· ·○○ ○○○ ○○○	●○· ·●● ··· ···	●○· ·●● ··· ···	●○· ·●● ··· ···	··· ··● ··· ···	··· ··● ··· ···	●○· ★●● ●●● ●●●	●●● ★●● ●●● ●●●	●○· ·●● ·●● ·●●	●○· ·●● ·●● ·●●	··· ··● ··● ··●	··· ··● ··● ··●	·○· ··· ●●● ●●●	·○● ··· ●●● ●●●	·○· ··· ·●● ·●●	·○· ··· ·●● ·●●	··· ··· ··● ··●	··· ··· ··● ··●
10 [IOE]	··· ··★ ··· ···	●○· ··· ··· ···	●○· ·★★ ··· ···	●○· ·●● ·●● ·●●	··· ··★ ··· ···	··· ··○ ··○ ··○	··· ··● ··● ··●	○○· ·●● ··· ···	○○· ·●● ··· ···	·○· ·●● ··· ···	··· ··● ··· ···	··· ··● ··· ···	○○· ·●● ·●● ·●●	·○· ·●● ·●● ·●●	○○· ·●● ·●● ·●●	·○· ·●● ·●● ·●●	··· ··● ··● ··●	··· ··● ··● ··●	○●○ ★●● ●●● ●●●	·○● ★●● ●●● ●●●	○○· ·●● ·●● ·●●	·○· ·●● ·●● ·●●	··· ··● ··● ··●	··· ··● ··● ··●
11 [IOD]	··· ··○ ··· ···	●○· ·●● ··· ···	●○· ·●● ··· ···	●○· ·●● ·●● ·●●	··· ··○ ··· ···	··· ··○ ··○ ··○	··· ··○ ··○ ··○	●○· ·●● ··· ···	●○· ·●● ··· ···	●○· ·●● ··· ···	··· ··● ··· ···	··· ··● ··· ···	●○· ·●● ·●● ·●●	●○· ·●● ·●● ·●●	●○· ·●● ·●● ·●●	●○· ·●● ·●● ·●●	··· ··● ··● ··●	··· ··● ··● ··●	●○· ★●● ●●● ●●●	●●● ★●● ●●● ●●●	●○· ·●● ·●● ·●●	●○· ·●● ·●● ·●●	··· ··● ··● ··●	··· ··● ··● ··●
12 [PIE]	··· ··· ★·· ···	●○· ··· ··· ···	●○· ★★· ★★· ···	●○· ★●· ●●· ●●·	··· ··· ★·· ···	··· ··· ○·· ○··	··· ··· ●·· ●··	·○○ ··· ●·· ···	○●○ ★●· ●●· ···	·○● ★●· ●●· ···	·○○ ★●● ●●● ···	·○● ★●● ●●● ···	·○○ ··· ●·· ●··	·○● ··· ●·· ●··	·○○ ··· ●●· ●●·	·○● ··· ●●· ●●·	·○○ ··· ●●● ●●●	·○● ··· ●●● ●●●	·○○ ··· ●·· ●··	·○● ··· ●·· ●··	·○○ ··· ●●· ●●·	·○● ··· ●●· ●●·	·○○ ··· ●●● ●●●	·○● ··· ●●● ●●●
13 [PID]	··· ··· ○·· ···	●○· ★●· ●●· ···	●○· ★●· ●●· ···	●○· ★●· ●●· ●●·	··· ··· ○·· ···	··· ··· ○·· ○··	··· ··· ○·· ○··	·○· ··· ●·· ···	●○· ★●· ●●· ···	●●● ★●· ●●· ···	·○· ★●● ●●● ···	·○● ★●● ●●● ···	·○· ··· ●·· ●··	·○● ··· ●·· ●··	·○· ··· ●●· ●●·	·○● ··· ●●· ●●·	·○· ··· ●●● ●●●	·○● ··· ●●● ●●●	·○· ··· ●·· ●··	·○● ··· ●·· ●··	·○· ··· ●●· ●●·	·○● ··· ●●· ●●·	·○· ··· ●●● ●●●	·○● ··· ●●● ●●●
14 [PPE]	··· ··· ·★· ···	●○· ··· ··· ···	●○· ·★· ·★· ···	●○· ★●● ●●● ●●●	··· ··· ·★· ···	··· ··· ○○○ ○○○	··· ··· ●●● ●●●	·○· ··· ·●● ···	○○· ·●● ·●● ···	·○· ·●● ·●● ···	·○· ·●● ·●● ···	·○· ·●● ·●● ···	·○○ ··· ●●● ●●●	·○● ··· ●●● ●●●	○●○ ★●● ●●● ●●●	·○● ★●● ●●● ●●●	·○○ ★●● ●●● ●●●	·○● ★●● ●●● ●●●	·○○ ··· ●●● ●●●	·○● ··· ●●● ●●●	·○○ ··· ●●● ●●●	·○● ··· ●●● ●●●	·○○ ··· ●●● ●●●	·○● ··· ●●● ●●●
15 [PPD]	··· ··· ·○· ···	●○· ·●· ·●· ···	●○· ·●· ·●· ···	●○· ★●● ●●● ●●●	··· ··· ·○· ···	··· ··· ○○○ ○○○	··· ··· ○○○ ○○○	·○· ··· ·●● ···	●○· ·●● ·●● ···	●○· ·●● ·●● ···	·○· ·●● ·●● ···	·○· ·●● ·●● ···	·○· ··· ●●● ●●●	·○● ··· ●●● ●●●	●○· ★●● ●●● ●●●	●●● ★●● ●●● ●●●	·○· ★●● ●●● ●●●	·○● ★●● ●●● ●●●	·○· ··· ●●● ●●●	·○● ··· ●●● ●●●	·○· ··· ●●● ●●●	·○● ··· ●●● ●●●	·○· ··· ●●● ●●●	·○● ··· ●●● ●●●
16 [POE]	··· ··· ··★ ···	●○· ··· ··· ···	●○· ·★★ ·★★ ···	●○· ·●● ·●● ·●●	··· ··· ··★ ···	··· ··· ··○ ··○	··· ··· ··● ··●	·○· ··· ·●● ···	○○· ·●● ·●● ···	·○· ·●● ·●● ···	·○· ·●● ·●● ···	·○· ·●● ·●● ···	·○· ··· ·●● ·●●	·○· ··· ·●● ·●●	○○· ·●● ·●● ·●●	·○· ·●● ·●● ·●●	·○· ·●● ·●● ·●●	·○· ·●● ·●● ·●●	·○○ ··· ●●● ●●●	·○● ··· ●●● ●●●	○●○ ★●● ●●● ●●●	·○● ★●● ●●● ●●●	·○○ ★●● ●●● ●●●	·○● ★●● ●●● ●●●
17 [POD]	··· ··· ··○ ···	●○· ·●● ·●● ···	●○· ·●● ·●● ···	●○· ·●● ·●● ·●●	··· ··· ··○ ···	··· ··· ··○ ··○	··· ··· ··○ ··○	·○· ··· ·●● ···	●○· ·●● ·●● ···	●○· ·●● ·●● ···	·○· ·●● ·●● ···	·○· ·●● ·●● ···	·○· ··· ·●● ·●●	·○· ··· ·●● ·●●	●○· ·●● ·●● ·●●	●○· ·●● ·●● ·●●	·○· ·●● ·●● ·●●	·○· ·●● ·●● ·●●	·○· ··· ●●● ●●●	·○● ··· ●●● ●●●	●○· ★●● ●●● ●●●	●●● ★●● ●●● ●●●	·○· ★●● ●●● ●●●	·○● ★●● ●●● ●●●
18 [OIE]	··· ··· ··· ★··	●○· ··· ··· ···	●○· ★★· ★★· ★★·	●○· ★●· ●●· ●●·	··· ··· ··· ★··	··· ··· ··· ○··	··· ··· ··· ●··	··· ··· ··· ●··	○○· ★●· ●●· ●●·	·○· ★●· ●●· ●●·	○●○ ★●● ●●● ●●●	·○● ★●● ●●● ●●●	··· ··· ··· ●··	··· ··· ··· ●··	·○· ··· ●●· ●●·	·○· ··· ●●· ●●·	·○○ ··· ●●● ●●●	·○● ··· ●●● ●●●	··· ··· ··· ●··	··· ··· ··· ●··	·○· ··· ●●· ●●·	·○· ··· ●●· ●●·	·○○ ··· ●●● ●●●	·○● ··· ●●● ●●●
19 [OID]	··· ··· ··· ○··	●○· ★●· ●●· ●●·	●○· ★●· ●●· ●●·	●○· ★●· ●●· ●●·	··· ··· ··· ○··	··· ··· ··· ○··	··· ··· ··· ○··	··· ··· ··· ●··	●○· ★●· ●●· ●●·	●○· ★●· ●●· ●●·	●○· ★●● ●●● ●●●	●●● ★●● ●●● ●●●	··· ··· ··· ●··	··· ··· ··· ●··	·○· ··· ●●· ●●·	·○· ··· ●●· ●●·	·○· ··· ●●● ●●●	·○● ··· ●●● ●●●	··· ··· ··· ●··	··· ··· ··· ●··	·○· ··· ●●· ●●·	·○· ··· ●●· ●●·	·○· ··· ●●● ●●●	·○● ··· ●●● ●●●
20 [OPE]	··· ··· ··· ·★·	●○· ··· ··· ···	●○· ·★· ·★· ·★·	●○· ★●● ●●● ●●●	··· ··· ··· ·★·	··· ··· ··· ○○○	··· ··· ··· ●●●	··· ··· ··· ·●●	○○· ·●● ·●● ·●●	·○· ·●● ·●● ·●●	○○· ·●● ·●● ·●●	·○· ·●● ·●● ·●●	··· ··· ··· ●●●	··· ··· ··· ●●●	○○· ★●● ●●● ●●●	·○· ★●● ●●● ●●●	○●○ ★●● ●●● ●●●	·○● ★●● ●●● ●●●	··· ··· ··· ●●●	··· ··· ··· ●●●	·○· ··· ●●● ●●●	·○· ··· ●●● ●●●	·○○ ··· ●●● ●●●	·○● ··· ●●● ●●●
21 [OPD]	··· ··· ··· ·○·	●○· ·●· ·●· ·●·	●○· ·●· ·●· ·●·	●○· ★●● ●●● ●●●	··· ··· ··· ·○·	··· ··· ··· ○○○	··· ··· ··· ○○○	··· ··· ··· ·●●	●○· ·●● ·●● ·●●	●○· ·●● ·●● ·●●	●○· ·●● ·●● ·●●	●○· ·●● ·●● ·●●	··· ··· ··· ●●●	··· ··· ··· ●●●	●○· ★●● ●●● ●●●	●○· ★●● ●●● ●●●	●○· ★●● ●●● ●●●	●●● ★●● ●●● ●●●	··· ··· ··· ●●●	··· ··· ··· ●●●	·○· ··· ●●● ●●●	·○· ··· ●●● ●●●	·○· ··· ●●● ●●●	·○● ··· ●●● ●●●
22 [OOE]	··· ··· ··· ··★	●○· ··· ··· ···	●○· ·★★ ·★★ ·★★	●○· ·●● ·●● ·●●	··· ··· ··· ··★	··· ··· ··· ··○	··· ··· ··· ··●	··· ··· ··· ·●●	○○· ·●● ·●● ·●●	·○· ·●● ·●● ·●●	○○· ·●● ·●● ·●●	·○· ·●● ·●● ·●●	··· ··· ··· ·●●	··· ··· ··· ·●●	○○· ·●● ·●● ·●●	·○· ·●● ·●● ·●●	○○· ·●● ·●● ·●●	·○· ·●● ·●● ·●●	··· ··· ··· ●●●	··· ··· ··· ●●●	○○· ★●● ●●● ●●●	·○· ★●● ●●● ●●●	○●○ ★●● ●●● ●●●	·○● ★●● ●●● ●●●
23 [OOD]	··· ··· ··· ··○	●○· ·●● ·●● ·●●	●○· ·●● ·●● ·●●	●○· ·●● ·●● ·●●	··· ··· ··· ··○	··· ··· ··· ··○	··· ··· ··· ··○	··· ··· ··· ·●●	●○· ·●● ·●● ·●●	●○· ·●● ·●● ·●●	●○· ·●● ·●● ·●●	●○· ·●● ·●● ·●●	··· ··· ··· ·●●	··· ··· ··· ·●●	●○· ·●● ·●● ·●●	●○· ·●● ·●● ·●●	●○· ·●● ·●● ·●●	●○· ·●● ·●● ·●●	··· ··· ··· ●●●	··· ··· ··· ●●●	●○· ★●● ●●● ●●●	●○· ★●● ●●● ●●●	●○· ★●● ●●● ●●●	●●● ★●● ●●● ●●●

by one of the two symbols '$\star$' and '$\circ$' at a certain position in a 3×4 matrix. These representations are shown in the second column. Table entries are constructed by superimposing the representations for each of the possible relations. Where '$\star$' and '$\circ$' should both be present in the same position, the symbol '$\bullet$' is used.

The table was generated using meta-level enforcement of the conv axioms in a Prolog implemented $\mathcal{I}_0^+$ reasoning program. It was produced in 3h 31mins on a SPARC10 workstation. Such a table has hitherto never been computed by a proof oriented method. [3] contains a similar table constructed using a model building approach but it has subsequently been found that the table given there is too strict in that it rules out certain configurations, which are in fact possible for 3D spatial regions. My table has not been found to contain any false entries.

8 CONCLUSIONS

I have shown how a significant family of spatial relations can be represented in a logical representation which is decidable. The computational effectiveness of this representation has been demonstrated by generating tables of loci of composition for a number of sets of spatial relations.

The divergence between expressiveness and tractability of logical languages is perhaps the greatest obstacle to the development of AI systems. I believe that this problem can be mitigated to some extent by ensuring that the expressive power of a representation does not exceed what is really needed. In particular, much of the power of 1st-order logic is unnecessary for reasoning in many domains. Hence, it is likely that encoding information in a (non-classical) propositional logic rather than 1st-order calculus may provide a mechanism for effective reasoning in other areas of knowledge representation.

There are many ways in which the system presented here could be enhanced. The efficiency of the system could be improved by optimising its theorem proving performance. Also expressivity could be increased by developing a more general framework for meta-level enforcement of 1st-order axioms.

I am currently exploring the possibility of using the modal logic $S4$ for spatial reasoning. This may well prove to be better suited to the task than $\mathcal{I}_0$.

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