Sorting with Fixed-Length Reversals

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1 Introduction

Due to applications in reconstructing the evolutionary history of the genome [22] and to the design of interconnection networks [14], there has been considerable recent interest in problems of sorting permutations with reversals. In this paper, we study the problem of sorting permutations and circular permutations using as few *fixed-length* reversals as possible.

Our problem is implicit in the popular TOP-SPIN^{TM} puzzle, manufactured by the Binary Arts Corporation and illustrated in Figure 1. TOP-SPIN^{TM} consists of a permutation of 20 numbered disks on an oval track, with a turnstile capable of reversing a string of 4 consecutive disks. The goal is to sort the disks to the identity permutation using reversals. We consider the more general problem, with permutations of n disks and a turnstile of size k.

Limiting the transformations to reversals of length exactly k can be very restrictive. Indeed, an easy parity argument shows that any permutation beginning $\{1, 3, 2, \ldots\}$ cannot be sorted using 3-reversals, since 1 and 2 are separated by an odd number of items and any 3-reversal changes this distance by either 0 or 2 items. Thus for a given n-permutation it is not obvious whether it can be sorted using k-reversals, or if so how many reversals may be required.

In this paper,

- We give a complete characterization, for all n and k, of the number of equivalence classes of n-permutations under k-reversal, for both permutations and circular permutations.

- We prove upper and lower bounds on the diameter of the circular permutation group induced by k-reversals. Specifically, we give an algorithm to sort all sortable circular n-permutations using $O(n^2/k + kn)$ k-reversals, while there exist permutations requiring $\Omega(n^2/k^2 + n)$ k-reversals to sort. Thus, surprisingly, $O(n^{3/2})$ reversals suffice when $k \approx \sqrt{n}$.

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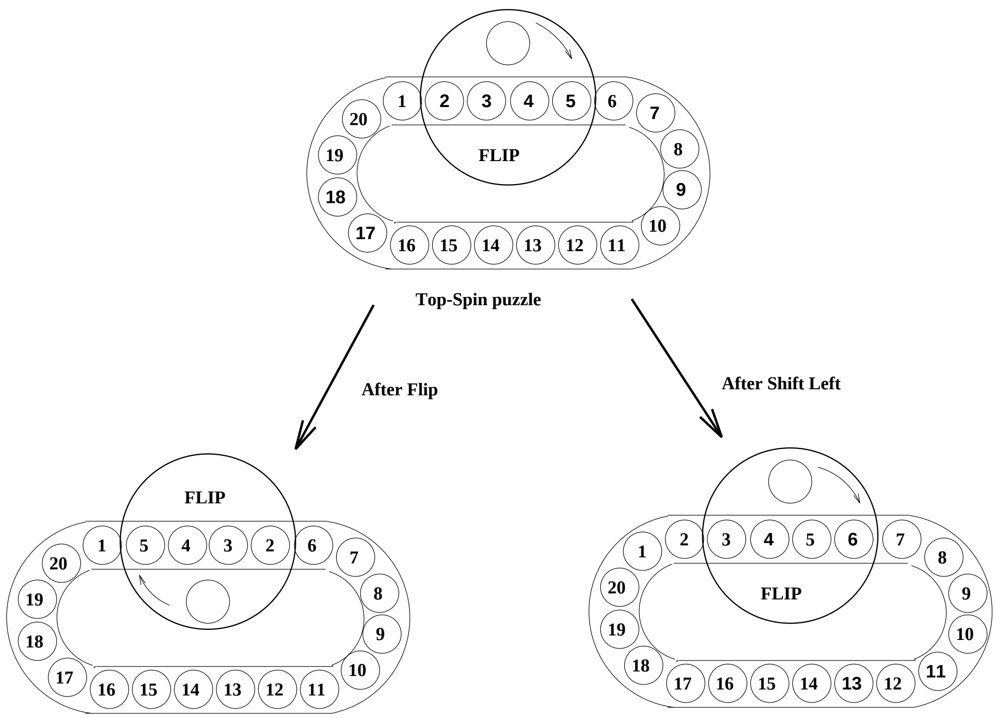

Figure 1: The Top-Spin puzzle

- We show that, when respecting parity constraints, the complexity of sorting with k-reversals is equivalent to sorting with $(n-k)$-reversals.

Previous work on sorting with fixed-length reversals has focused on the special case where $k = 2$. Thus each reversal simply transposes adjacent elements. Bubble sort [19] sorts any permutation π using exactly one transposition for each inversion in π, thus minimizing the number of reversals. Jerrum [15] presented a polynomial algorithm for the much more difficult problem of sorting circular permutations using a minimum number of transpositions.

Gates and Papadimitriou [9] (also [10]) studied the problem of sorting permutations using prefix reversals (better known as the pancake-flipping problem), where reversals may be of arbitrary length but each must start from the first element of permutation. They showed that $5n/3 + 5/3$ prefix-reversals suffice to sort any permutation, and there are permutations requiring at least $17n/16$ reversals. Heydari and Sudborough [13, 14] have tightened these bounds and proved the problem of computing the exact prefix-reversal distance between two permutations is NP-complete. Related work includes [1, 6].

Computing the exact reversal distance of a permutation is of considerable importance in reconstructing the evolutionary history of the genome. Reversal mutations occur often in chromosomes, where each reverses the order of an interval of genes. A shortest reversal sequence sorting one genome to another corresponds to the most likely evolutionary path between them. This analysis has been applied, for example, to drosophila [8, 21], plants [4, 18], viruses [12], and mammals [7, 20]. Kececloglu and Sankoff [16] gave 2-approximation algorithms on reversal distance, which Bafna and Pevzner [3] improved to a factor of 7/4 approximation. Most recently, Hannenhalli and Pevzner [11] gave a polynomial-time algorithm for signed reversal distance, although the problem for unsigned reversals (as we consider in this paper) remains open. Kececloglu and Sankoff [17] report on the success of heuristics and search in determining reversal distance for chromosomes. Bafna and Pevzner [5] present approximation algorithms for transposition distance, ie. under block moves instead of reversals.

All of these problems are special cases of determining the diameter of permutation groups. Jerrum [15] showed that the problem of computing the shortest sequence of generators for arbitrary permutation groups is PSPACE-complete, even when there are only two generators. The general problem has seen considerable attention – see the survey of Babai, et. al. [2].

The outline of this paper is as follows. In Section 2 we present our notation for describing sequences of reversal operations. In Section 3, we present our notion of equivalent transformations, which is the primary tool we need to understand sorting with fixed-length reversals. Armed with these results, we characterize the equivalence classes within the permutation groups in Section 4, and the diameter of the group in Section 5. We conclude with a list of open problems in Section 6.

2 Notation

Sorting with reversals is properly described by a permutation group with a specific set of generators. We will be interested in both the *symmetric group* $Sym(n)$ comprising all permutations of size n, and the corresponding *circular permutation group* $CPG(n)$. Each permutation in $CPG(n)$ represents a set of n permutations on $Sym(n)$ equivalent under the *shift* operation:

$$shift = \begin{pmatrix} 1 & 2 & \ldots & i & \ldots & n-1 & n \\ 2 & 3 & \ldots & i+1 & \ldots & n & 1 \end{pmatrix}$$

Any permutation of n numbers can be rearranged to exactly n arrangements by *shift.* There are $n!$ permutations in $Sym(n)$ and $n!/n = (n-1)!$ in $CPG(n)$.

The k-reversal operation on a permutation in $CPG(n)$ will generate n different permutations, parameterized by the starting element of the reversal and denoted $Rev(1)$, $Rev(2)$, $Rev(3)$, ..., $Rev(n)$ where:

$$Rev(i) = \{1, ..., i-1, \underline{i+k-1, i+k-2, ..., i+1, i}, i+k, ..., n\}$$

The *diameter* of the group $G =< Rev(1), Rev(2), ..., Rev(n) >$ is the least integer d such that every permutation in the group can be expressed as product of generators with length less than or equal to d, denoted $diam(G)$. The *Cayley graph* $\Gamma(G)$ is the graph whose vertices are the elements of G, with an edge between vertices p and q iff $p \cdot g_i = q$, for some generator g_i. Throughout this paper, $\{a_1, a_2, a_3, \ldots, a_n\}$ represents a *permutation*, while $\{c_1, c_2, c_3, \ldots, c_n\}$ represents a *circular permutation.* Let $I = \{1, 2, 3, ..., n\}$ denote the identity permutation or circular permutation.

We use the symbol $\longrightarrow$ to denote the result of applying a generator to the permutation to the left of the arrow yielding the permutation to the right of the arrow. The generators we will be most interested in are described below. For clarity of exposition, we have provided a descriptor for each generator above each $\longrightarrow$ in our derivations:

shift circular shifts the permutation one element to the right:

$$\{a_1, a_2, ..., a_{n-1}, a_n\} \quad \overset{Shift}{\longrightarrow} \quad \{a_n, a_1, a_2, ..., a_{n-1}\}$$

k-reversal reverses a sequence of k elements. $Rev_k(i)$ denotes the k-reversal beginning at the ith position:

$$\{a_1, ..., a_{i-1}, \underline{a_i, a_{i+1}, ..., a_{i+k-1}}, a_{i+k}, ...\} \quad \xrightarrow{Rev_k(i)} \quad \{a_1, ..., a_{i-1}, \underline{a_{i+k-1}, ..., a_{i+1}, a_i}, a_{i+k}, ...\}$$

transposition $\tau_{s,t}$ moves a sequence of s elements right over the subsequent t elements. $\tau_{s,t}(i)$ denotes transposing s and t elements beginning with the ith position.

$$\{a_1 \ldots a_i, \underline{a_{i+1}, \ldots, a_{i+s}}, \underline{a_{i+s+1}, \ldots, a_{i+s+t}}, \ldots\} \quad \xrightarrow{\tau_{s,t}(i)} \quad \{a_1 \ldots a_i, \underline{a_{i+s+1}, \ldots, a_{i+s+t}}, \underline{a_{i+1}, \ldots, a_{i+s}}, \ldots\}$$

Most frequently used in this paper are $\tau_{2,k-1}$, $\tau_{1,k}$, $\tau_{1,2}$, $\tau_{1,1}$, $\tau_{2,2}$, $\tau_{2,4}$ and their *inverse* permutations $\tau_{k-1,2}$, $\tau_{k,1}$, $\tau_{2,1}$, $\tau_{1,1}$, $\tau_{2,2}$, $\tau_{4,2}$. Transposition $\tau_{1,1}$ is the 2-reversal or *adjacent transposition.*

3-Cycle rotates the values of three elements. $Cyc(i, j, k)$ denotes transforming the ith a_i, jth a_j, kth a_k elements into ith a_j, jth a_k, kth a_i elements:

$$\{..., a_{i-1}, \underline{a_i}, ..., a_{j-1}, \underline{a_j}, ..., a_{k-1}, \underline{a_k}, ...\} \quad \xrightarrow{Cyc(i,j,k)} \quad \{..., a_{i-1}, \underline{a_j}, ..., a_{j-1}, \underline{a_k}, ..., a_{k-1}, \underline{a_i}, ...\}$$

Transposition $\tau_{1,2}$ is the same as the *adjacent 3-Cycle* $Cyc(i, i+1, i+2)$.

3 Equivalent Transformations

Let P and Q be two sets of permutations (or generators) in a group. We use $P \Longrightarrow Q$ to denote that the permutations of Q can be *implemented* using multiple applications of P. For example

$$\tau_{1,1} \Longrightarrow \tau_{1,2}, \tau_{2,1}$$

since

$$\{a_1, ..., a_{i-1}, \underline{a_i}, \underline{a_{i+1}}, a_{i+2}, a_{i+3}, ..., a_n\} \quad \xrightarrow{\tau_{1,1}(i)} \quad \{a_1, ..., a_{i-1}, \underline{a_{i+1}}, \underline{a_i}, a_{i+2}, a_{i+3}, ..., a_n\}$$

$$\{a_1, ..., a_{i-1}, a_{i+1}, \underline{a_i}, \underline{a_{i+2}}, a_{i+3}, ..., a_n\} \quad \xrightarrow{\tau_{1,1}(i+1)} \quad \{a_1, ..., a_{i-1}, a_{i+1}, \underline{a_{i+2}}, \underline{a_i}, a_{i+3}, ..., a_n\}$$

Thus $\Longrightarrow$ is by definition a transitive relation. We say two permutations (or generators) P and Q are *equivalent transformations* $P \Longleftrightarrow Q$ iff $P \Longrightarrow Q$ and $Q \Longrightarrow P$. We introduce the relation $\Longrightarrow$ to derive useful transformations from k-reversal. If $P \Longleftrightarrow Q$, then any permutation that can be sorted by P can be sorted by Q too.

The simplest transformation capable of sorting is the 2-reversal or adjacent transposition. *Bubble sort* (or *insertion sort*) [19] demonstrates that each permutation can be sorted in $O(n^2)$ steps of 2-reversals. We also can use it to establish whether a set of generators is sufficient for sorting.

Lemma 1 *A set of generators P is sufficient for sorting iff $P \Longrightarrow 2-reversal$*

Proof: 2-reversal is sufficient to sort both permutations and circular permutations. If 2-reversal can be implemented by composition of P, we can repeat these operations to generate a sequence of 2-reversals to sort any permutations.

To show that 2-reversal is necessary, observe that any transformation T in a permutation group must have an inverse transformation T'. Let P_1 and P_2 be permutations which differ by a 2-reversal. If P is sufficient to sort, then there is a sequence of transformations from P_1 and P_2 to the identity, and vice-versa, giving an implementation of 2-reversal. ∎

3.1 Equivalent transformations for $Sym(n)$

In this section, we consider transformations equivalent to k-reversal for $Sym(n)$, and hence which can be used in algorithms to sort either permutations or circular permutations. We will assume that $n > k + 2$, to allow sufficient freedom to ignore certain boundary conditions. Equivalent transformations for $CPG(n)$ will be discussed in Section 3.2.

Theorem 1 *The following equivalent transformations exist for permutation groups:*

$$(2+4l) - reversal \Longleftrightarrow 2 - reversal \Longleftrightarrow \tau_{1,1}$$

$$4l - reversal \Longleftrightarrow 4 - reversal \Longleftrightarrow \tau_{1,2}, \tau_{2,1}$$

$$(3+4l) - reversal \Longleftrightarrow 3 - reversal$$

$$(5+8l) - reversal \Longleftrightarrow 5 - reversal \Longleftrightarrow \tau_{2,2}$$

$$(9+8l) - reversal \Longleftrightarrow 9 - reversal \Longleftrightarrow \tau_{2,4}, \tau_{4,2}$$

The rest of this section gives the proofs of these equivalent transformations.

Lemma 2 *Two steps of k-reversal can generate transpositions $\tau_{2,k-1}$ and $\tau_{k-1,2}$, ie. $k-reversal \Longrightarrow \tau_{2,k-1}, \tau_{k-1,2}$.*

Proof: We give the proof for transposition $\tau_{2,k-1}(1)$. Other starting positions follow analogously:

$$\{a_1, \underline{a_2, a_3, ..., a_k, a_{k+1}}, a_{k+2}, ...\} \xrightarrow{Rev_k(2)} \{a_1, \underline{a_{k+1}, a_k, ..., a_3, a_2}, a_{k+2}, ...\}$$

$$\{\underline{a_1, a_{k+1}, a_k, ..., a_3}, a_2, a_{k+2}, ...\} \xrightarrow{Rev_k(1)} \{\underline{a_3, ..., a_k, a_{k+1}, a_1}, a_2, a_{k+2}, ...\}$$

To implement the inverse transposition $\tau_{k-1,2}$, simply reverse these two operations. ∎

Observe that applying transpositions $\tau_{2,k-1}$ and $\tau_{k-1,2}$ will maintain the order of all the elements in a permutation except the two elements moved.

Lemma 3 *k steps of $2l$-reversal ($k = 2l$) can generate transpositions $\tau_{1,k}$ and $\tau_{k,1}$. $8k$ steps of $2l$-reversal can generate adjacent 3-cycle transpositions $\tau_{1,2}$ and $\tau_{2,1}$, ie.:*

$$2l - reversal \Longrightarrow \tau_{1,2l}, \tau_{2l,1} \Longrightarrow \tau_{1,2}, \tau_{2,1}$$

Proof: Lemma 2 provides transpositions $\tau_{2,2l-1}$ and $\tau_{2l-1,2}$ from $2l$-reversal. We use them to construct $\tau_{1,2l}$.

$$\{\underline{a_1, a_2, a_3, \ldots, a_{2l-1}}, \underline{a_{2l}, a_{2l+1}}, \ldots\} \quad \overset{\tau_{2l-1,2}(1)}{\longrightarrow} \quad \{\underline{a_{2l}, a_{2l+1}}, \underline{a_1, a_2, a_3, \ldots, a_{2l-1}}, \ldots\}$$

$$\{\underline{a_{2l}, a_{2l+1}, a_1, \ldots, a_{2l-3}}, \underline{a_{2l-2}, a_{2l-1}}, \ldots\} \quad \overset{\tau_{2l-1,2}(1)}{\longrightarrow} \quad \{\underline{a_{2l-2}, a_{2l-1}}, \underline{a_{2l}, a_{2l+1}, a_1, \ldots, a_{2l-3}}, \ldots\}$$

Repeat a total of $k/2$ times

$$\{\underline{a_4, a_5, \ldots, a_{2l}, a_{2l+1}, a_1}, \underline{a_2, a_3}, \ldots\} \quad \overset{\tau_{2l-1,2}(1)}{\longrightarrow} \quad \{\underline{a_2, a_3}, \underline{a_4, a_5, \ldots, a_{2l}, a_{2l+1}, a_1}, \ldots\}$$

Reaching the target permutation $\tau_{1,2}$ uses $k/2$ steps of $\tau_{2l-1,2}$, each $\tau_{2l-1,2}$ using 2 steps of $2l$-reversal, for a total of k reversals. We emphasize that this transformation holds only for even k. The inverse permutation, as always, follows by reversing the construction.

Using $\tau_{1,2l}$ and $\tau_{2l,1}$ we can generate transposition $\tau_{1,2}$, and its inverse $\tau_{2,1}$:

$$\{a_1, \underline{a_2}, \underline{a_3, a_4, \ldots, a_{2l+2}}, \ldots\} \quad \overset{\tau_{1,2l}(2)}{\longrightarrow} \quad \{a_1, \underline{a_3, a_4, \ldots, a_{2l+2}}, \underline{a_2}, \ldots\}$$

$$\{a_1, \underline{a_3}, \underline{a_4, \ldots, a_{2l+2}, a_2}, \ldots\} \quad \overset{\tau_{1,2l}(2)}{\longrightarrow} \quad \{a_1, \underline{a_4, \ldots, a_{2l+2}, a_2}, \underline{a_3}, \ldots\}$$

$$\{\underline{a_1, a_4, \ldots, a_{2l+2}}, \underline{a_2}, a_3, \ldots\} \quad \overset{\tau_{2l,1}(1)}{\longrightarrow} \quad \{\underline{a_2}, \underline{a_1, a_4, \ldots, a_{2l+2}}, a_3, \ldots\}$$

$$\{a_2, \underline{a_1, a_4, \ldots, a_{2l+2}}, \underline{a_3}, \ldots\} \quad \overset{\tau_{2l,1}(2)}{\longrightarrow} \quad \{a_2, \underline{a_3}, \underline{a_1, a_4, \ldots, a_{2l+2}}, \ldots\}$$

Four steps of $\tau_{1,2l}$ suffice to reach the target $\tau_{1,2}$, costing a total of $4 \times 2l = 4k$ k-reversals. These transformations only require $n \geq k + 2$ or $n \geq 2l + 2$.

Special care must be taken when $\tau_{1,2l}$ or $\tau_{21,1}$ is restricted by the left or right boundaries of the permutation. Consider transposing a_i, a_{i+1} and a_{i+2} in a segment of length $2l + 2$:

$$\{a_1, a_2, \ldots, a_{i-1}, \underline{a_i, a_{i+1}, a_{i+2}}, a_{i+3}, \ldots, a_{2l+1}, a_{2l+2}\}$$

If $(i-1)$ is even, we apply $\tau_{2,2l-1}(1)$ to this segment $(i-1)/2$ times, moving the three elements to the left boundary so we can use $\tau_{2l,1}$ and $\tau_{1,2l}$ to generate $\tau_{1,2}$ and $\tau_{2,1}$. Afterwards, we apply $\tau_{2l-1,2}(1)$ $(i-1)/2$ times to move a_1, ..., a_{i-1} back to their original positions.

If $(i-1)$ is odd, we perform an initial step of $\tau_{1,2l}(1)$:

$$\{\underline{a_1}, \underline{a_2, a_3, ..., a_i, ..., a_{2l+1}}, a_{2l+2}\} \xrightarrow{\tau_{1,2l}(1)} \{\underline{a_2, a_3, ..., a_i, ..., a_{2l+1}}, \underline{a_1}, a_{2l+2}\}$$

and proceed with the sequence when $(i-1)$ is even, concluding with an extra $\tau_{2l,1}(1)$ as the last step. This adds $(i-1)$ steps of $\tau_{2,2l-1}$ and 2 steps of $\tau_{1,2l}$, a total of $k \times 2 + 2 \times k + 4k = 8k$ k-reversals. ∎

Now we give the first two transformations of Theorem 1:

Lemma 4 *The following equivalent transformations exist for permutation groups:*

$$2 - reversal \Longleftrightarrow (2+4l) - reversal$$

$$\tau_{1,2}, \tau_{2,1} \Longleftrightarrow 4 - reversal \Longleftrightarrow 4l - reversal$$

Proof: We can reverse the first $2l+4$ elements of a permutation using $2l$-reversals, and Lemmas 2 and 3 as follows:

$$\{a_1, a_2, a_3, a_4, \underline{a_5, ..., a_{2l+3}, a_{2l+4}}, a_{2l+5}, ...\} \xrightarrow{Rev_{2l}(5)} \{a_1, a_2, a_3, a_4, \underline{a_{2l+4}, a_{2l+3}, ..., a_5}, a_{2l+5}, ...\}$$

$$\{a_1, a_2, \underline{a_3, a_4}, \underline{a_{2l+4}, a_{2l+3}, ..., a_6}, a_5, a_{2l+5}, ...\} \xrightarrow{\tau_{2,2l-1}(3)} \{a_1, a_2, \underline{a_{2l+4}, a_{2l+3}, ..., a_6}, \underline{a_3, a_4}, a_5, a_{2l+5}, ...\}$$

$$\{\underline{a_1, a_2}, \underline{a_{2l+4}, a_{2l+3}, ..., a_6}, a_3, a_4, a_5, a_{2l+5}, ...\} \xrightarrow{\tau_{2,2l-1}(1)} \{\underline{a_{2l+4}, a_{2l+3}, ..., a_6}, \underline{a_1, a_2}, a_3, a_4, a_5, a_{2l+5}, ...\}$$

$$\{a_{2l+4}, a_{2l+3}, ..., a_6, a_1, a_2, \underline{a_3, a_4}, \underline{a_5}, a_{2l+5}, ...\} \xrightarrow{\tau_{2,1}(2l+2)} \{a_{2l+4}, a_{2l+3}, ..., a_6, a_1, a_2, \underline{a_5}, \underline{a_3, a_4}, a_{2l+5}, ...\}$$

$$\{a_{2l+4}, a_{2l+3}, ..., a_6, \underline{a_1, a_2}, \underline{a_5}, a_3, a_4, a_{2l+5}, ...\} \xrightarrow{\tau_{2,1}(2l)} \{a_{2l+4}, a_{2l+3}, ..., a_6, \underline{a_5}, \underline{a_1, a_2}, a_3, a_4, a_{2l+5}, ...\}$$

$$\{a_{2l+4}, a_{2l+3}, ..., a_6, a_5, \underline{a_1}, \underline{a_2, a_3}, a_4, a_{2l+5}, ...\} \xrightarrow{\tau_{1,2}(2l+1)} \{a_{2l+4}, a_{2l+3}, ..., a_6, a_5, \underline{a_2, a_3}, \underline{a_1}, a_4, a_{2l+5}, ...\}$$

$$\{a_{2l+4}, a_{2l+3}, ..., a_6, a_5, a_2, \underline{a_3, a_1}, \underline{a_4}, a_{2l+5}, ...\} \xrightarrow{\tau_{2,1}(2l+2)} \{a_{2l+4}, a_{2l+3}, ..., a_6, a_5, a_2, \underline{a_4}, \underline{a_3, a_1}, a_{2l+5}, ...\}$$

$$\{a_{2l+4}, a_{2l+3}, ..., a_6, a_5, \underline{a_2}, \underline{a_4, a_3}, a_1, a_{2l+5}, ...\} \xrightarrow{\tau_{1,2}(2l+1)} \{a_{2l+4}, a_{2l+3}, ..., a_6, a_5, \underline{a_4, a_3}, \underline{a_2}, a_1, a_{2l+5}, ...\}$$

To show equivalence the other way, we reverse the first $2l$ elements of a permutation using $(2l+4)$-reversals, as follows:

$$\{\underline{a_1, ..., a_{2l}, a_{2l+1}, a_{2l+2}, a_{2l+3}, a_{2l+4}}, a_{2l+5}, ...\} \xrightarrow{Rev_{2l+4}(1)} \{\underline{a_{2l+4}, a_{2l+3}, a_{2l+2}, a_{2l+1}, a_{2l}, ..., a_1}, a_{2l+5}, ...\}$$

$$\{\underline{a_{2l+4}, a_{2l+3}}, \underline{a_{2l+2}, a_{2l+1}, a_{2l}, ..., a_1, a_{2l+5}}, ...\} \xrightarrow{\tau_{2,2l+3}(1)} \{\underline{a_{2l+2}, a_{2l+1}, a_{2l}, ..., a_1, a_{2l+5}}, \underline{a_{2l+4}, a_{2l+3}}, ...\}$$

$$\{\underline{a_{2l+2}, a_{2l+1}}, \underline{a_{2l}, ..., a_1, a_{2l+5}, a_{2l+4}, a_{2l+3}}, ...\} \xrightarrow{\tau_{2,2l+3}(1)} \{\underline{a_{2l}, ..., a_1, a_{2l+5}, a_{2l+4}, a_{2l+3}}, \underline{a_{2l+2}, a_{2l+1}}, ...\}$$

$$\{a_{2l}, ..., a_1, \underline{a_{2l+5}}, \underline{a_{2l+4}, a_{2l+3}}, a_{2l+2}, a_{2l+1}, ...\} \xrightarrow{\tau_{1,2}(2l+1)} \{a_{2l}, ..., a_1, \underline{a_{2l+4}, a_{2l+3}}, \underline{a_{2l+5}}, a_{2l+2}, a_{2l+1}, ...\}$$

$$\{a_{2l}, ..., a_1, a_{2l+4}, a_{2l+3}, \underline{a_{2l+5}}, \underline{a_{2l+2}, a_{2l+1}}, ...\} \xrightarrow{\tau_{1,2}(2l+3)} \{a_{2l}, ..., a_1, a_{2l+4}, a_{2l+3}, \underline{a_{2l+2}, a_{2l+1}}, \underline{a_{2l+5}}, ...\}$$

$$\{a_{2l}, ..., a_1, \underline{a_{2l+4}}, \underline{a_{2l+3}, a_{2l+2}}, a_{2l+1}, a_{2l+5}, ...\} \xrightarrow{\tau_{1,2}(2l+1)} \{a_{2l}, ..., a_1, \underline{a_{2l+3}, a_{2l+2}}, \underline{a_{2l+4}}, a_{2l+1}, a_{2l+5}, ...\}$$

$$\{a_{2l}, ..., a_1, a_{2l+3}, \underline{a_{2l+2}, a_{2l+4}}, \underline{a_{2l+1}}, a_{2l+5}, ...\} \xrightarrow{\tau_{2,1}(2l+2)} \{a_{2l}, ..., a_1, a_{2l+3}, \underline{a_{2l+1}}, \underline{a_{2l+2}, a_{2l+4}}, a_{2l+5}, ...\}$$

$$\{a_{2l}, ..., a_1, \underline{a_{2l+3}}, \underline{a_{2l+1}, a_{2l+2}}, a_{2l+4}, a_{2l+5}, ...\} \xrightarrow{\tau_{1,2}(2l+1)} \{a_{2l}, ..., a_1, \underline{a_{2l+1}, a_{2l+2}}, \underline{a_{2l+3}}, a_{2l+4}, a_{2l+5}, ...\}$$

By cascading these transformations, we can prove the equivalence of $2l$-reversals and $(2l+4i)$-reversals for all i, which implies $2\text{-}reversal \iff (2+4l)\text{-}reversal$ and $4\text{-}reversal \iff 4l\text{-}reversal$.

For the second part of the lemma, we note that $\tau_{1,2}, \tau_{2,1} \Longleftarrow 4\text{-}reversal$ follows from Lemma 3. To implement the 4-reversal using $\tau_{1,2}$:

$$\{\underline{a_1}, \underline{a_2, a_3}, a_4, ...\} \xrightarrow{\tau_{1,2}(1)} \{\underline{a_2, a_3}, \underline{a_1}, a_4, ...\}$$

$$\{a_2, \underline{a_3, a_1}, \underline{a_4}, ...\} \xrightarrow{\tau_{2,1}(2)} \{a_2, \underline{a_4}, \underline{a_3, a_1}, ...\}$$

$$\{\underline{a_2}, \underline{a_4, a_3}, a_1, ...\} \xrightarrow{\tau_{1,2}(1)} \{\underline{a_4, a_3}, \underline{a_2}, a_1, ...\}$$

With the equivalence of 4-reversals and $4l$-reversals, the general result follows. ∎

This concludes all transformations of even reversals. The remainder of this section deals with odd reversals.

Lemma 5 *The following transformations exist for $Sym(n)$:*

$$k - reversal \Longrightarrow \tau_{2,4}, \tau_{4,2} \Longrightarrow \tau_{4p,2q}, \tau_{2q,4p}$$

Proof: By Lemma 2, we know $\tau_{2,k-1}$,$\tau_{k-1,2}$ can be derived using k-reversals. We show k-$reversal \Longrightarrow \tau_{2,4}, \tau_{4,2}$:

$$\{a_1, a_2, \underline{a_3, a_4}, \underline{a_5, a_6, a_7, ..., a_{k+3}}, ...\} \xrightarrow{\tau_{2,k-1}(3)} \{a_1, a_2, \underline{a_5, a_6, a_7, ..., a_{k+3}}, \underline{a_3, a_4}, ...\}$$

$$\{\underline{a_1, a_2}, \underline{a_5, a_6, a_7, ..., a_{k+3}}, a_3, a_4, ...\} \xrightarrow{\tau_{2,k-1}(1)} \{\underline{a_5, a_6, a_7, ..., a_{k+3}}, \underline{a_1, a_2}, a_3, a_4, ...\}$$

$$\{a_5, a_6, \underline{a_7, ..., a_{k+3}, a_1, a_2}, \underline{a_3, a_4}, ...\} \xrightarrow{\tau_{k-1,2}(3)} \{a_5, a_6, \underline{a_3, a_4}, \underline{a_7, ..., a_{k+3}, a_1, a_2}, ...\}$$

$$\{a_5, a_6, \underline{a_3, a_4, a_7, ..., a_{k+3}}, \underline{a_1, a_2}, ...\} \xrightarrow{\tau_{k-1,2}(3)} \{a_5, a_6, \underline{a_1, a_2}, \underline{a_3, a_4, a_7, ..., a_{k+3}}, ...\}$$

Although we have implemented $\tau_{4,2}(1)$, the boundary restrictions can be eliminated. When k is even, it follows from Lemma 3, where $2l$-$reversal \Longrightarrow \tau_{1,2}, \tau_{2,1}$, since $\tau_{1,2} \Longrightarrow \tau_{4,2}$. When k is odd, we can apply $\tau_{2,k-1}(1)$ repeatly to move the ith element to the left end, as in the proof of Lemma 4. When there are an odd number of elements to the left, one initial k-reversal $Rev_k(1)$ leaves an even number and the previous analysis suffices.

The inverse follows since $\tau_{2,4}(i)$=$\tau_{4,2}(i)\tau_{4,2}(i)$. By repeating $\tau_{4,2}$, it is easily shown that $\tau_{4,2} \Longrightarrow \tau_{4p,2}$, and by repeating $\tau_{4p,2}$, it is shown that $\tau_{4p,2} \Longrightarrow \tau_{4p,2q}$. ∎

Lemma 6 *The following transformations exist for $Sym(n)$:*

$$5 - reversal \Longleftrightarrow \tau_{2,2}$$

$$9 - reversal \Longleftrightarrow \tau_{2,4}, \tau_{4,2}$$

$$3 - reversal \Longleftrightarrow 7 - reversal$$

Proof: To show 5-*reversal* $\Longrightarrow \tau_{2,2}$, we use $\tau_{2,4}$ and $\tau_{4,2}$ as generated in Lemma 5:

$$\{a_1, \underline{a_2, a_3}, \underline{a_4, a_5, a_6, a_7}, ...\} \xrightarrow{\tau_{2,4}(2)} \{a_1, \underline{a_4, a_5, a_6, a_7}, \underline{a_2, a_3}, ...\}$$

$$\{\underline{a_1, a_4}, \underline{a_5, a_6, a_7, a_2}, a_3, ...\} \xrightarrow{\tau_{2,4}(1)} \{\underline{a_5, a_6, a_7, a_2}, \underline{a_1, a_4}, a_3, ...\}$$

$$\{a_5, \underline{a_6, a_7}, \underline{a_2, a_1, a_4, a_3}, ...\} \xrightarrow{\tau_{2,4}(2)} \{a_5, \underline{a_2, a_1, a_4, a_3}, \underline{a_6, a_7}, ...\}$$

$$\{\underline{a_5, a_2, a_1, a_4, a_3}, a_6, a_7, ...\} \xrightarrow{Rev_5(1)} \{\underline{a_3, a_4, a_1, a_2, a_5}, a_6, a_7, ...\}$$

To show $\tau_{2,2} \Longrightarrow$ 5-*reversal*:

$$\{a_1, \underline{a_2, a_3}, \underline{a_4, a_5}, a_6, ...\} \xrightarrow{\tau_{2,2}(2)} \{a_1, \underline{a_4, a_5}, \underline{a_2, a_3}, a_6, ...\}$$

$$\{\underline{a_1, a_4}, \underline{a_5, a_2}, a_3, a_6, ...\} \xrightarrow{\tau_{2,2}(1)} \{\underline{a_5, a_2}, \underline{a_1, a_4}, a_3, a_6, ...\}$$

$$\{a_5, \underline{a_2, a_1}, \underline{a_4, a_3}, a_6, ...\} \xrightarrow{\tau_{2,2}(2)} \{a_5, \underline{a_4, a_3}, \underline{a_2, a_1}, a_6, ...\}$$

Lemma 5 demonstrates that $\tau_{2,4}$, $\tau_{4,2}$ can be generated by 9-reversal. To show that $\tau_{2,4} \Longrightarrow$ 9-*reversal*:

$$\{a_1, \underline{a_2, a_3}, \underline{a_4, a_5, a_6, a_7}, a_8, a_9, ...\} \xrightarrow{\tau_{2,4}(2)} \{a_1, \underline{a_4, a_5, a_6, a_7}, \underline{a_2, a_3}, a_8, a_9, ...\}$$

$$\{\underline{a_1, a_4}, \underline{a_5, a_6, a_7, a_2}, a_3, a_8, a_9, ...\} \xrightarrow{\tau_{2,4}(2)} \{\underline{a_5, a_6, a_7, a_2}, \underline{a_1, a_4}, a_3, a_8, a_9, ...\}$$

$$\{a_5, a_6, a_7, \underline{a_2, a_1, a_4, a_3}, \underline{a_8, a_9}, ...\} \xrightarrow{\tau_{4,2}(4)} \{a_5, a_6, a_7, \underline{a_8, a_9}, \underline{a_2, a_1, a_4, a_3}, ...\}$$

$$\{a_5, \underline{a_6, a_7}, \underline{a_8, a_9, a_2, a_1}, a_4, a_3, ...\} \xrightarrow{\tau_{2,4}(2)} \{a_5, \underline{a_8, a_9, a_2, a_1}, \underline{a_6, a_7}, a_4, a_3, ...\}$$

$$\{\underline{a_5, a_8}, \underline{a_9, a_2, a_1, a_6}, a_7, a_4, a_3, ...\} \xrightarrow{\tau_{2,4}(1)} \{\underline{a_9, a_2, a_1, a_6}, \underline{a_5, a_8}, a_7, a_4, a_3, ...\}$$

$$\{a_9, \underline{a_2, a_1, a_6, a_5}, \underline{a_8, a_7}, a_4, a_3, ...\} \xrightarrow{\tau_{4,2}(2)} \{a_9, \underline{a_8, a_7}, \underline{a_2, a_1, a_6, a_5}, a_4, a_3, ...\}$$

$$\{a_9, a_8, a_7, \underline{a_2, a_1}, \underline{a_6, a_5, a_4, a_3}, ...\} \xrightarrow{\tau_{2,4}(4)} \{a_9, a_8, a_7, \underline{a_2, a_1}, \underline{a_6, a_5, a_4, a_3}, ...\}$$

Note that Lemma 2 implies $3 - reversal \Longrightarrow \tau_{2,2}$. Since 5-$reversal \Longleftrightarrow \tau_{2,2}$, we can derive 3-$reversal \Longrightarrow$ 5-$reversal$. To show that 3-$reversal \Longrightarrow$ 7-$reversal$:

$$\{\underline{a_1, a_2, a_3, a_4, a_5}, a_6, a_7, ...\} \xrightarrow{Rev_5(1)} \{\underline{a_5, a_4, a_3, a_2, a_1}, a_6, a_7, ...\}$$

$$\{a_5, \underline{a_4, a_3, a_2, a_1}, \underline{a_6, a_7}, ...\} \xrightarrow{\tau_{4,2}(2)} \{a_5, \underline{a_6, a_7}, \underline{a_4, a_3, a_2, a_1}, ...\}$$

$$\{\underline{a_5, a_6, a_7}, a_4, a_3, a_2, a_1, ...\} \xrightarrow{Rev_3(1)} \{\underline{a_7, a_6, a_5}, a_4, a_3, a_2, a_1, ...\}$$

By Lemma 5, and Lemma 6, 7-$reversal \Longrightarrow$ 9-$reversal$. By Lemma 2, 7-$reversal \Longrightarrow \tau_{6,2}$. To complete the argument that 7-$reversal \Longrightarrow$ 3-$reversal$,

$$\{\underline{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9}, ...\} \xrightarrow{Rev_9(1)} \{\underline{a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1}, ...\}$$

$$\{\underline{a_9, a_8, a_7, a_6, a_5, a_4, a_3}, a_2, a_1, ...\} \xrightarrow{Rev_7(1)} \{\underline{a_3, a_4, a_5, a_6, a_7, a_8, a_9}, a_2, a_1, ...\}$$

$$\{a_3, \underline{a_4, a_5, a_6, a_7, a_8, a_9}, \underline{a_2, a_1}, ...\} \xrightarrow{\tau_{6,2}(2)} \{a_3, \underline{a_2, a_1}, \underline{a_4, a_5, a_6, a_7, a_8, a_9}, ...\}$$

∎

Lemma 7 *In Sym(n),* $(2l+1)$*-reversal and* $(2l+9)$*-reversal have the following equivalent transformations, for all* $l \geq 0$*:*

$$3 - reversal \Longleftrightarrow 7 - reversal \Longleftrightarrow (3+4l) - reversal$$

$$\tau_{2,2} \Longleftrightarrow 5 - reversal \Longleftrightarrow (5+8l) - reversal$$

$$\tau_{2,4}, \tau_{4,2} \Longleftrightarrow 9 - reversal \Longleftrightarrow (9+8l) - reversal$$

Proof: First, we show that $(2l+1)$-$reversal \Longleftrightarrow (2l+9)$-$reversal$. By Lemma 6, $\tau_{2,4}$,$\tau_{4,2}$ are equivalent to 9-reversal, and by Lemma 5, $\tau_{2,4}$, $\tau_{4,2}$ can be derived from all reversals. Thus 9-reversal can be generated by all odd length reversals. To show that $(2l+1)$-$reversal \Longrightarrow (2l+9)$-$reversal$:

$$\{a_1, a_2, ..., a_8, \underline{a_9, a_{10}, ..., a_{2l+8}, a_{2l+9}}, ...\} \xrightarrow{Rev_{2l+1}(9)} \{a_1, a_2, ..., a_8, \underline{a_{2l+9}, a_{2l+8}, ..., a_{10}, a_9}, ...\}$$

$$\{\underline{a_1, a_2, ..., a_8}, \underline{a_{2l+9}, a_{2l+8}, ..., a_{10}}, a_9, ...\} \xrightarrow{\tau_{8,2l}(9)} \{\underline{a_{2l+9}, a_{2l+8}, ..., a_{10}}, \underline{a_1, a_2, ..., a_8}, a_9, ...\}$$

$$\{a_{2l+9}, a_{2l+8}, ..., a_{10}, \underline{a_1, a_2, ..., a_8, a_9}, ...\} \xrightarrow{Rev_9(2l+1)} \{a_{2l+9}, a_{2l+8}, ..., a_{10}, \underline{a_9, a_8, ..., a_2, a_1}, ...\}$$

To show $(2l+9)$-$reversal \Longleftrightarrow (2l+1)$-$reversal$:

$$\{\underline{a_1, a_2, ..., a_{2l}, a_{2l+1}, a_{2l+2}, ..., a_{2l+8}, a_{2l+9}}, ...\} \xrightarrow{Rev_{2l+9}(1)} \{\underline{a_{2l+9}, a_{2l+8}, ..., a_{2l+2}, a_{2l+1}, a_{2l}, ..., a_2, a_1}, ...\}$$

$$\{\underline{a_{2l+9}, a_{2l+8}, ..., a_{2l+2}, a_{2l+1}}, a_{2l}, ..., a_2, a_1, ...\} \xrightarrow{Rev_9(1)} \{\underline{a_{2l+1}, a_{2l+2}, ..., a_{2l+8}, a_{2l+9}}, a_{2l}, ..., a_2, a_1, ...\}$$

$$\{a_{2l+1}, \underline{a_{2l+2}, ..., a_{2l+8}, a_{2l+9}}, \underline{a_{2l}, ..., a_2, a_1}, ...\} \xrightarrow{\tau_{8,2l}(2)} \{a_{2l+1}, \underline{a_{2l}, ..., a_2, a_1}, \underline{a_{2l+2}, ..., a_{2l+8}, a_{2l+9}}, ...\}$$

This transformation gives:

$$3 - reversal \Longleftrightarrow (3+8l) - reversal$$

$$5 - reversal \Longleftrightarrow (5+8l) - reversal$$

$$7 - reversal \Longleftrightarrow (7+8l) - reversal$$

$$9 - reversal \Longleftrightarrow (9+8l) - reversal$$

Combining these equivalent transformations with Lemma 6, 3-$reversal \Longleftrightarrow$ 7-$reversal$, 5-$reversal \Longleftrightarrow \tau_{2,2}$, and 9-$reversal \Longleftrightarrow \tau_{2,4}$, completes the proof. ∎

This concludes all transformations of odd reversals and the proof of Theorem 1.

3.2 Equivalent transformations for $CPG(n)$

All the relations in Section 3.1 hold for both permutations and circular permutations. In this section, we develop additional transformations which hold for $CPG(k,n)$. These transformations are a function of n as well as the reversal-length k.

Lemma 8 *If $S \Longleftrightarrow T$ in $PG(k,n)$ then $S \Longleftrightarrow T$ in $CPG(k,n)$ when $n > k+2$.*

Proof: Clearly any permutation sortable under S or T in $PG(k,n)$ remains sortable in $CPG(k,n)$. Any transformation of T in $CPG(k,n)$ is equivalent to an initial set of circular shifts (so the altered region of T does not wrap around the ends of the permutation) followed by the implementation of T using S in $PG(k,n)$. This is true since circular permutations are unchanged under circular shifts. ∎

Lemma 9 *For $CPG(k,n)$, $\tau_{2,4}$ and $\tau_{4,2}$ can be generated by 8 steps of k-reversal.*

Proof: By Lemma 5, *k-reversal* $\Longrightarrow \tau_{2,4}, \tau_{4,2}$ in $Sym(n)$. If $n > k+2$, there are no boundary constraints for $CPG(k,n)$. Thus 4 steps of $\tau_{2,k-1}$, or 8 steps of k-reversal are sufficient to derive $\tau_{4,2}$ or $\tau_{4,2}$. ∎

Lemma 10 *For $CPG(k,2m)$, the following equivalence transformation exists:*

$$4l - reversal \Longrightarrow \tau_{1,1}$$

Proof: From Theorem 1, we know *$4l$-reversal* $\Longleftrightarrow \tau_{1,2}, \tau_{2,1}$. We may repeatedly apply transposition $\tau_{1,2}$ to c_{i+1}, skipping over two elements at time until it arrives in the proper position, since $n = 2m$:

$$\{c_1, ..., c_i, \underline{c_{i+1}}, \underline{c_{i+2}, c_{i+3}}, ..., c_{2m}\} \quad \overset{\tau_{1,2}(i+1)}{\longrightarrow} \quad \{c_1, ..., c_i, \underline{c_{i+2}, c_{i+3}}, \underline{c_{i+1}}, ..., c_{2m}\}$$

Repeat $\tau_{1,2}((i+1+2j) \bmod 2m)$ for $1 < j < m$

$$\{c_1, ..., c_{i-3}, \underline{c_{i+1}}, \underline{c_{i-2}, c_{i-1}}, c_i, c_{i+2}, ..., c_{2m}\} \quad \overset{\tau_{1,2}(i-2)}{\longrightarrow} \quad \{c_1, ..., c_{i-3}, \underline{c_{i-2}, c_{i-1}}, \underline{c_{i+1}}, c_i, c_{i+2}, ..., c_{2m}\}$$

∎

We note that for $n = 2m+1$, the transposition $\tau_{1,1}$ cannot be generated for all k.

Lemma 11 *The following transformation exists for $CPG(2l+1, 2m+1)$:*

$$(2l+1) - reversal \Longrightarrow \tau_{1,2}, \tau_{2,1}$$

Proof: We have proven in Lemma 5 that

$$(2l+1) - reversal \Longrightarrow \tau_{2q,4p}, \tau_{4p,2q}$$

We show that we can transpose a_1 with a_2, a_3. If m is odd, we can do $\tau_{2,2m-2}$ since $2m-2 \equiv 0 \bmod 4$, giving:

$$\{a_1, \underline{a_2, a_3}, \underline{a_4, ..., a_{2m-1}, a_{2m}, a_{2m+1}}\} \quad \overset{\tau_{2,2m-2}(2)}{\longrightarrow} \quad \{a_1, \underline{a_4, ..., a_{2m-1}, a_{2m}, a_{2m+1}}, \underline{a_2, a_3}\}$$

$$\{a_1, a_4, ..., a_{2m-1}, a_{2m}, a_{2m+1}, \underline{a_2, a_3}\} \xrightarrow{Shift} \{\underline{a_2, a_3}, a_1, a_4, ..., a_{2m-1}, a_{2m}, a_{2m+1}\}$$

If m is even, we can do $\tau_{2m-4,2}$, since $2m-4 \equiv 0 \bmod 4$, so:

$$\{a_1, \underline{a_2, a_3}, a_4, a_5, a_6, a_7, ..., a_{2m}, a_{2m+1}\} \xrightarrow{Shift} \{a_4, a_5, a_6, a_7, ..., a_{2m}, a_{2m+1}, a_1, \underline{a_2, a_3}\}$$

$$\{a_4, a_5, a_6, \underline{a_7, ..., a_{2m}, a_{2m+1}, a_1}, \underline{a_2, a_3}\} \xrightarrow{\tau_{2m-4,2}(4)} \{a_4, a_5, a_6, \underline{a_2, a_3}, \underline{a_7, ..., a_{2m}, a_{2m+1}, a_1}\}$$

$$\{a_4, a_5, a_6, \underline{a_2, a_3}, a_7, ..., a_{2m}, a_{2m+1}, a_1\} \xrightarrow{Shift} \{a_1, a_4, a_5, a_6, \underline{a_2, a_3}, a_7, ..., a_{2m}, a_{2m+1}\}$$

$$\{\underline{a_1, a_4, a_5, a_6}, \underline{a_2, a_3}, a_7, ..., a_{2m}, a_{2m+1}\} \xrightarrow{\tau_{4,2}(1)} \{\underline{a_2, a_3}, \underline{a_1, a_4, a_5, a_6}, a_7, ..., a_{2m}, a_{2m+1}\}$$

Thus $\tau_{1,2}$ can be generated in $G_{CPG}(2l+1, 2m+1)$ and by reversing the derivations, so can $\tau_{2,1}$. ∎

Lemma 12 *The following transformation exists for $CPG(4l+1, 2m+1)$:*

$$(4l+1) - reversal \Longleftrightarrow \tau_{1,2}, \tau_{2,1}$$

Proof: In Lemma 11, we have proved that $(4l+1)$-$reversal \Longrightarrow \tau_{1,2}, \tau_{2,1}$ in $CPG(4l+1, 2m+1)$. By Theorem 1,

$$(8l+9) - reversal \Longleftrightarrow \tau_{2,4}, \tau_{4,2}$$

$$(8l+5) - reversal \Longleftrightarrow \tau_{2,2}$$

Combining these two transformations, we get

$$(4l+1) - reversal \Longleftarrow \tau_{2,2}$$

To complete the argument, we show $\tau_{1,2} \Longrightarrow \tau_{2,2}$:

$$\{a_1, \underline{a_2}, \underline{a_3, a_4}, a_5, ...\} \xrightarrow{\tau_{1,2}(2)} \{a_1, \underline{a_3, a_4}, \underline{a_2}, a_5, ...\}$$

$$\{\underline{a_1}, \underline{a_3, a_4}, a_2, a_5, ...\} \xrightarrow{\tau_{1,2}(1)} \{\underline{a_3, a_4}, \underline{a_1}, a_2, a_5, ...\}$$

∎

We conclude with two transformations which are independent of k-reversals, but which will prove useful in establishing bounds for circular permutations.

Lemma 13 *The following equivalent transformation exists:* $\tau_{2,2} \Longrightarrow Cyc(i, i+2, i+4)$.

Proof: To show $\tau_{2,2} \Longrightarrow Cyc(i, i+2, i+4)$:

$$\{..., c_{i-3}, \underline{c_{i-2}, c_{i-1}}, \underline{c_i, c_{i+1}}, c_{i+2}, c_{i+3}, ...\} \xrightarrow{\tau_{2,2}(i-2)} \{..., c_{i-3}, \underline{c_i, c_{i+1}}, \underline{c_{i-2}, c_{i-1}}, c_{i+2}, c_{i+3}, ...\}$$

$$\{..., c_{i-3}, c_i, \underline{c_{i+1}, c_{i-2}}, \underline{c_{i-1}, c_{i+2}}, c_{i+3}, ...\} \xrightarrow{\tau_{2,2}(i-1)} \{..., c_{i-3}, c_i, \underline{c_{i-1}, c_{i+2}}, \underline{c_{i+1}, c_{i-2}}, c_{i+3}, ...\}$$

$\blacksquare$

Lemma 14 *The following equivalent transformation exists:* $\tau_{2,4}, \tau_{4,2} \Longleftrightarrow Cyc(i, i+2, i+4)$.

Proof: To show $Cyc(i, i+2, i+4) \Longrightarrow \tau_{2,4}$:

$$\{\underline{a_1}, a_2, \underline{a_3}, a_4, \underline{a_5}, a_6, a_7, ...\} \xrightarrow{Cyc(1,3,5)} \{\underline{a_3}, a_2, \underline{a_5}, a_4, \underline{a_1}, a_6, a_7, ...\}$$

$$\{a_3, \underline{a_2}, a_5, \underline{a_4}, a_1, \underline{a_6}, a_7, ...\} \xrightarrow{Cyc(2,4,6)} \{a_3, \underline{a_4}, a_5, \underline{a_6}, a_1, \underline{a_2}, a_7, ...\}$$

To show $\tau_{2,4} \Longrightarrow Cyc(i, i+2, i+4)$:

$$\{a_1, \underline{a_2, a_3}, \underline{a_4, a_5, a_6, a_7}, ...\} \xrightarrow{\tau_{2,4}(2)} \{a_1, \underline{a_4, a_5, a_6, a_7}, \underline{a_2, a_3}, ...\}$$

$$\{\underline{a_1, a_4}, \underline{a_5, a_6, a_7, a_2}, a_3, ...\} \xrightarrow{\tau_{2,4}(1)} \{\underline{a_5, a_6, a_7, a_2}, \underline{a_1, a_4}, a_3, ...\}$$

$$\{a_5, \underline{a_6, a_7}, \underline{a_2, a_1, a_4, a_3}, ...\} \xrightarrow{\tau_{2,4}(2)} \{a_5, \underline{a_2, a_1, a_4, a_3}, \underline{a_6, a_7}, ...\}$$

$$\{\underline{a_5, a_2}, \underline{a_1, a_4, a_3, a_6}, a_7, ...\} \xrightarrow{\tau_{2,4}(1)} \{\underline{a_1, a_4, a_3, a_6}, \underline{a_5, a_2}, a_7, ...\}$$

$\blacksquare$

4 Connected Components under Fixed-Length Reversals

In this section, we consider the number of connected components on the Cayley graphs of fixed-length reversals of permutations and circular permutations. We note that each of these connected components represents a subgroup of $Sym(n)$ or $CPG(n,k)$, and by symmetry each of these subgroups are isomorphic. Let $Sub(G)$ be the number of connected subgroups of the group generated by G. Thus the size of each subgroup can be computed from $Sub(G)$, and visa-versa.

A set of generators T can be *derived* from S when each member in T can be generated using a subset of generators from S. The number of connected subgroups of T, $Sub(T) = m \cdot Sub(S)$, where m is an integer. If S and T have equivalent transformations, $m = 1$.

Before considering the general case, we resolve the structure of the Cayley graphs for special cases of k and n. We assume that $k > 1$, or no elements can be moved in any reversal. We also assume that $n > k$, although larger reversals can be interpreted as n-reversals that simply reverse the entire permutation. The trivial cases are those of $(n-1)-$, and n-reversals. For $k = n$, any permutation can be transformed only to its reverse permutation. Thus there are $n!/2$ connected components for $Sym(n)$, $k = n > 2$, and $(n-1)!/2$ connected components for $CPG(k,n)$, $k = n > 3$. For smaller values of $k = n$, these graphs are connected. For circular permutations, $(n-1)$-reversals are functionally equivalent to n-reversals, so the discussion above applies. For permutations, $(n-1)$-reversals generates components of either n or $2n$ permutations depending upon parity.

In Section 4.1 we identify the connected components of $Sym(n)$. In Section 4.2 we identify the connected components of $CPG(k,n)$.

4.1 Connected Components in $Sym(n)$

In this section, we consider the connected components of the Cayley graph of the symmetric group under k-reversals. All the equivalence transformations in Theorem 1 can be used, since none depend upon circular permutations. For permutations (as opposed to circular permutations), there is no difference between even and odd n.

By Lemma 4, $2\text{-}reversal \iff (4l+2)\text{-}reversal$, so the Cayley graph $Sym(4l+2,n)$ is connected.

To show that this is the only connected case, we use the permutation *sign* function. In group theory, the sign $F(\pi)$ of permutation $\pi = \{a_1, a_2, ..., a_n\}$ is defined by the polynomial

$$F(\pi) = \prod_{i<j}(a_j - a_i)$$

Any adjacent transposition $\tau_{1,1}(i) = (a_i a_{i+1})$ changes $(a_{i+1} - a_i)$ into its negative $(a_i - a_{i+1})$ and so changes F to $-F$. By restricting permutations so $a_1 = 1$, we still define F for circular permutations such that the sign does not change for $Cyc(i, i+1, i+2)$:

Lemma 15 *The adjacent 3-Cycle transposition $\tau_{2,1}(i)$ does not change the sign of F for permutations and $(2m+1)$-length circular permutation, $\pi = (c_1, c_2, ..., c_{2m+1})$ where $c_1 = 1$.*

Proof: The permutation $\tau_{2,1}(i) = \tau_{1,2}(i)\ \tau_{1,2}(i)$, so we only need to show $\tau_{1,2}$ does not change the sign of $F(\pi)$.

We distinguish two cases of $\tau_{1,2}(i)$. For $1 < i < 2m$, the position of c_1 does not change. For $i \in \{2m, 2m+1, 1\}$ c_1 is involved. Assume $v = \pi\tau_{1,2}(i)$.

In the first case, function $F(v)$ only changes three terms of $F(\pi)$. The sign does not change, because

$$(c_{i+1}-c_i)(c_{i+2}-c_i)(c_{i+2}-c_{i+1}) = (c_{i+2}-c_{i+1})(c_i-c_{i+1})(c_i-c_{i+2})$$

In the second case, c_1 is among any of the c_i, c_{i+1}, and c_{i+2}, For $i=1$,

$$\pi = (c_1, c_2, c_3, c_4, c_5, ..., c_{2m+1})$$

$$v = (c_1, c_4, c_5, ..., c_{2m+1}, c_2, c_3)$$

The difference between $F(\pi)$ and $F(v)$ is the product of these terms which changed signs:

$$(c_4-c_2)(c_5-c_2)...(c_{2m+1}-c_2)(c_4-c_3)(c_5-c_3)...(c_{2m+1}-c_3) = (-1)^{4m-4} = 1$$

so $F(\pi) = F(v)$. If $c_i = c_{2m}$, $c_{i+1} = c_{2m+1}$, and $c_{i+2} = c_1$, this is just the reverse permutation from v to π, so we still have $F(\pi) = F(v)$.

The final case is $c_i = c_{2m+1}$, $c_{i+1} = c_1$, and $c_{i+2} = c_2$,

$$v = (c_1, c_2, c_{2m+1}, c_3, ..., c_{2m})$$

where $F(v)$ changes the signs of following $2m-2$ terms,

$$(c_n - c_3)(c_{2m+1}-c_4)...(c_{2m+1}-c_{2m}) = (-1)^{2m-2} = 1$$

so $F(\pi) = F(v)$. ∎

Corollary 2 *Adjacent 3-cycle transposition cannot sort permutation* $\pi = (a_1, ..., \underline{a_{i-1}, a_{i+1}, a_i}, a_{i+2}, ..., a_n)$ *and circular permutation* $v = (c_1, ..., \underline{c_{i-1}, c_{i+1}, c_i}, c_{i+2}, ..., c_{2m+1})$

By Corollary 2, $\tau_{2,1}$ cannot change the sign of an (odd length circular) permutation. The identity (circular) permutation has positive sign, so (circular) permutations with negative signs cannot be sorted.

Lemma 16 $\tau_{2,1}$ *can sort exactly half of* $Sym(n)$.

Proof: It is easily shown that $\tau_{2,1} \Longrightarrow \tau_{2m,1}$. The following algorithm sorts $\pi = \{a_1, a_2, ..., a_n\}$, using $\tau_{2m,1}$:

1. **for** $i = 1$ to $n-2$
2. **find** j such that $a_j = i$ ($j = \pi^{-1}(i)$)
3. **if** $(j-i)$ is even, apply $\tau_{j-i,1}(i)$
4. **else** apply $\tau_{j-i-1,1}(i+1)$ then $\tau_{1,2}(i)$
5. **end** for

The ith iteration places a_j into ith position, where $a_j = i$. At termination, either $\pi = \{1, 2, ..., n-2, n-1, n\}$ or $\pi = \{1, 2, ..., n-2, n, n-1\}$. Because they have different sign, Lemma 15 shows that $\{1, 2, ..., n-2, n, n-1\}$ cannot be transformed into $\{1, 2, ..., n-2, n-1, n\}$ using $\tau_{1,2}$, thus $\tau_{1,2}$ divides the permutation group into two equal-sized subgroups, and $\tau_{1,2}$ sorts just half of $Sym(n)$. ∎

By item 2 in Theorem 1, $4l$-*reversal* $\iff \tau_{1,2}, \tau_{2,1}$ and Lemma 15, sorting $(2m+1)$ circular permutation is the same as sorting a permutation by $\tau_{1,2}$ and

Corollary 3 *The Cayley graphs $PG(4l, n)$ and $CPG(4l, 2m+1)$ both consist of two disjoint subgraphs.*

Lemma 17 *The Cayley Graph $PG(4l+3, n)$ has $\binom{n}{\lfloor n/2 \rfloor}$ connected components.*

Proof: By Theorem 1, $(4l+3)$-*reversal* $\iff$ 3-*reversal*. Applying a 3-reversal $Rev_3(i)$ to a permutation π simply exchanges a_i with a_{i+2}. Thus for all $i = 2j$, $Rev_3(i)$ exchanges adjacent even positions, while for all $i = 2j+1$, $Rev_3(i)$ exchanges adjacent odd positions. There is no way to move elements from even to odd position.

Therefore all elements of π can be partitioned into sets of $\lceil n/2 \rceil$ odd and $\lfloor n/2 \rfloor$ even elements. Any permutation on each partition can be generated using 3-reversals as transpositions, so each connected component contains $V = \lceil n/2 \rceil \times \lfloor n/2 \rfloor$ vertices, and hence $N!/V = \binom{n}{\lfloor n/2 \rfloor}$ connected components. ∎

Lemma 18 *The Cayley Graph $PG(8l+5, n)$ has $2\binom{n}{\lfloor n/2 \rfloor}$ connected components. The Cayley graph $CPG(8l+5, 2m)$ has $2\binom{2m-1}{m}$ connected components.*

Proof: By Theorem 1, $\tau_{2,2}$ has equivalent transformation with $(8l+5)$-reversal, and by Lemma 13, $\tau_{2,2} \Longrightarrow Cyc(i, i+2, i+4)$, the interval 3-cycle. We use the following algorithm to sort permutations, and thus circular permutations:

1. **Sort** $\{c_1, c_3, ..., c_{2m-1}\}$ into $\{1, 3, 5, ..., 2m-1\}$ using $\tau_{2,2}$.
2. **Sort** $\{c_2, c_4, ..., c_{2m}\}$ into $\{2, 4, 6, ..., 2m\}$ using $Cyc(i, i+2, i+4)$.

The odd elements can be sorted in the first step, because $\tau_{2,2}$ transposes c_{2i-1}, c_{2i} with c_{2i+1}, c_{2i+2}, realizing adjacent transposition of neighboring odd elements without changing the order of any other odd numbers. There are $(m-1)!$ arrangements of 3, 5, ..., $2m-1$ (in a circular permutation, we fix the position of 1), which can thus be sorted with $\tau_{2,2}$.

However, only $m!/2$ arrangements of the even elements can be sorted using interval 3-cycle. Consider the permutation $\pi = \{1, \underline{4}, 3, \underline{2}, 5, 6, ..., 2m, 2m+1\}$. It is easily verified that $\tau_{2,2}$ (or the weaker $Cyc(i, i+2, i+4)$) will not change the sign of polynomial sign function F. However, $F(\pi)$ is equal to $-F(I)$, where I is identity permutation. So while π can be sorted with one 3-reversal, it cannot be sorted by $\tau_{2,2}$. Thus, $Cyc(i, i+2, i+4)$ can only sort half of permutations in $\{c_2, c_4, ..., c_{2m}\}$.

The number of vertices in each component is $V = (m-1)! \cdot m!/2.$, so the total number of connected components is $(2m-1)!/V = 2\binom{2m-1}{m}$. The equivalence for circular permutations follows from Lemma 8. ∎

Lemma 19 *$CPG(8l+9, 4m)$ has $2\binom{4m-1}{2m}$ connected components, while $CPG(8l+9, 4m+2)$ has $4\binom{4m+1}{2m}$ connected components. For permutations, the Cayley Graph $PG(8l+9, n)$ has $4\binom{n}{\lfloor n/2 \rfloor}$*

Proof: By Lemma 5 $(8l+9)$-*reversal* $\Longrightarrow \tau_(2q, 4p)$. To generate $\tau_{2,2}$ in $G_{CPG}(8l+9, 4m)$:

$$\{c_1, c_2, \underline{c_3, c_4}, \underline{c_5, ..., c_{4m-1}, c_{4m}}\} \quad \overset{\tau_{2,4m-4}(2)}{\longrightarrow} \quad \{c_1, c_2, \underline{c_5, ..., c_{4m-1}, c_{4m}}, \underline{c_3, c_4}\}$$

$$\{c_1, c_2, c_5, ..., c_{4m-1}, c_{4m}, \underline{c_3, c_4}\} \quad \overset{Shift}{\longrightarrow} \quad \{\underline{c_3, c_4}, c_1, c_2, c_5, ..., c_{4m-1}, c_{4m}\}$$

Thus the number of connected components is $2\binom{4m-1}{2m}$ by the same argument as Lemma 18.

By Lemma 14, $\tau_{2,4}, \tau_{4,2} \Longleftrightarrow Cyc(i, i+2, i+4)$ for circular permutations of $n = 4m+2$. We may sort with the following algorithm:

1. **Sort** $\{c_1, c_3, ..., c_{2m-1}\}$ into $\{1, 3, 5, ..., 2m-1\}$ using $Cyc(i, i+2, i+4)$.
2. **Sort** $\{c_2, c_4, ..., c_{2m}\}$ into $\{2, 4, 6, ..., 2m\}$ using $Cyc(i, i+2, i+4)$.

As shown above, we fix the position of 1, so $(2m)!/2$ permutations of 3, 5, ..., $4m+1$ may be sorted. The second step can sort $(2m+1)!/2$ permutations of 2, 4, ..., $4m+2$, leading to a total of $(4m+1)!/((2m+1)!(2m)!/4) = 4\binom{4m+1}{2m}$ connected components in $G_{CPG}(8l+9, 4m+2)$. For permutations, the Cayley Graph $PG(8l+9, n)$ has $4\binom{n}{\lfloor n/2 \rfloor}$ ∎

In summary:

Theorem 4 *The number of connected components in the Cayley graph of $Sym(n)$ under k-reversals is:*

$k \equiv 0 \bmod 4$	*2*
$k \equiv 5 \bmod 8$	$2\binom{n}{\lfloor n/2 \rfloor}$
$k \equiv 1 \bmod 8$	$4\binom{n}{\lfloor n/2 \rfloor}$
$k \equiv 2 \bmod 4$	*1*
$k \equiv 3 \bmod 4$	$\binom{n}{\lfloor n/2 \rfloor}$

4.2 The Connected Components of $CPG(k, n)$

Several equivalence transformations for circular permutations have been introduced in Section 3.2. These demonstrate that the connectivity of $CPG(k, n)$ depends upon both n and k. In Section 4.2.1, we establish the relationship between k-reversals and $(n-k)$-reversals. In Section 4.2.2, we consider the case where k is even, and in Section 4.2.3 the more complicated case when k is odd.

4.2.1 Similarity of $CPG(k,n)$ and $CPG(n-k,n)$

For circular n-permutations, k-reversals are similar to $(n-k)$-reversals, which can be exploited to bound the diameter and count the components of related instances.

Lemma 20 *In $CPG(k,n)$, if π can be sorted in $2p$ k-reversals, then it can be sorted with $2p$ $(n-k)$-reversals. If π can be sorted in $2p+1$ k-reversals, it can be transformed into $I^{-1} = \{n, n-1, ..., 2, 1\}$ using $2p+1$ $(n-k)$-reversals.*

Proof: In the sequence of reversals which sorts π, replace each k-reversal $Rev_k(i)$ with the $(n-k)$-reversal $Rev_{n-k}(i+k)$. The result of each substitution is an order-reversal of the desired permutation:

$$\{\underline{c_1, c_2, ..., c_k}, c_{k+1}, ..., c_{n-1}, c_n\} \xrightarrow{Rev_k(1)} \{\underline{c_k, ..., c_2, c_1}, c_{k+1}, ..., c_{n-1}, c_n\}$$

$$\{c_1, c_2, ..., c_k, \underline{c_{k+1}, ..., c_{n-1}, c_n}\} \xrightarrow{Rev_{n-k}(k+1)} \{c_1, c_2, ..., c_k, \underline{c_n, c_{n-1}, ..., c_{k+1}}\}$$

Thus after an even number of substitute $(n-k)$-reversals, we obtain the same result as with the original k-reversals. With an odd number of substitute reversals, the final result is reversed, so the sorting sequence returns I^{-1}. ∎

Theorem 5 *Let V_k be the number of vertices in one component of $CPG(k,n)$ and V_{n-k} be that of $CPG(n-k,n)$. If $V_k \geq V_{n-k}$, then either $V_k = V_{n-k}$ or $V_k = 2V_{n-k}$.*

Proof: Let C_k and C_{n-k} denote the components of $CPG(k,n)$ and $CPG(n-k,n)$ containing the identity permutation. Performing a breadth-first search from I in $CPG(k,n)$ partitions C_k into sets S_e and S_o, denoting the vertices on even and odd levels.

By Lemma 20, $S_e \subseteq C_{n-k}$. In S_o, any circular permutation π can be transformed into any other circular permutation v using an even number of reversals: $\pi \longrightarrow I \longrightarrow v$. Thus either every circular permutation in S_o belongs to C_{n-k} or none belongs to C_{n-k}.

If S_o belongs to C_{n-k}, under the assumption $V_k \geq V_{n-k}$, then $V_k = V_{n-k}$. Otherwise, S_o does not belong to C_{n-k}, and we have following features of S_o and S_e:

1. All the circular permutations in S_e can only be sorted in even steps of k-reversals. If there exists an odd steps sorting, it will reach all the circular permutations in S_o by even steps, which will lead to S_o belonging to C_{n-k}.

2. All the circular permutation in S_o can only be sorted in odd steps of k-reversals.

3. The size of S_o equals to the size of S_e. This can be proven by the symmetric feature of C_k.

Now repeat the analysis on C_{n-k}, partitioning C_{n-k} into T_e and T_o. If T_o belongs to C_k, we have $C_{n-k} = S_e$, or $V_k = 2V_{n-k}$ by the third item above. Otherwise, we have $T_e = S_e$, and $V_k = V_{n-k}$. ∎

The number of connected components is $(n-1)!$ over the number of vertices in each components, so

Corollary 6 *Let N_k be the number of connected components of $CPG(k,n)$ and N_{n-k} be that of $CPG(n-k,n)$. If $N_k \geq N_{n-k}$, either $N_k = N_{n-k}$ or $N_k = 2N_{n-k}$.*

4.2.2 Connected Components of $CPG(k, n)$ for even k

By Lemma 4, 2-*reversal* $\Longleftrightarrow$ $(4l + 2)$-*reversal* for $Sym(n)$, so by Lemma 1 the Cayley graph $CPG(4l + 2, n)$ must be connected. By Lemma 10, $4l$-*reversal* $\Longrightarrow$ 2-*reversal* for $n = 2m$, so $CPG(4l, 2m)$ must also be connected. This completes the argument that we can sort for all even k and even n.

What remains is the case for $k \equiv 0 \bmod 4$ and odd n. By Theorem 1 item 2, we know that:

$$\tau_{1,2}, \tau_{2,1} \Longleftrightarrow 4l - reversal$$

where $\tau_{1,2}$, $\tau_{2,1}$ divides $Sym(n)$ into two subgroup. By Lemma 15, $\tau_{1,2}$ and $\tau_{2,1}$ cannot change the sign of F for $(2m+1)$-length circular permutation, thus the Cayley graph $CPG(4l, 2m+1)$ consists of two disjoint connected subgraphs. In summary:

Theorem 7 *The number of disjoint connected components in Circular Permutation Group with even length reversals ($n \geq k + 2$) are:*

$k \setminus n$	n=$2m$	n=$2m$+1
$k \equiv 0 \bmod 4$	*1*	*2*
$k \equiv 2 \bmod 4$	*1*	*1*

4.2.3 Connected components of $G_{CPG}(k, n)$ for odd k

In this section, we complete the analysis of connected components of circular permutations under odd-length reversals. Several subcases have been covered in previous sections.

Lemma 21 *The Cayley Graph $CPG(4l + 3, 2m + 1)$ is connected.*

Proof: By Theorem 1, 3-*reversal* $\Longleftrightarrow$ $(4l + 3)$-*reversal*. We claim that $\tau_{1,1}$ can be generated by 3-reversal for odd circular permutations, since $\tau_{1,2}$, $\tau_{2,1}$ can be generated by Lemma 11:

$$\{c_1, ..., c_{i-2}, \underline{c_{i-1}, c_i, c_{i+1}}, c_{i+2}, ...\} \quad \xrightarrow{Rev_3(i-1)} \quad \{c_1, ..., c_{i-2}, \underline{c_{i+1}, c_i, c_{i-1}}, c_{i+2}, ...\}$$

$$\{c_1, ..., c_{i-2}, \underline{c_{i+1}, c_i}, \underline{c_{i-1}}, c_{i+2}, ...\} \quad \xrightarrow{\tau_{2,1}(i-1)} \quad \{c_1, ..., c_{i-2}, \underline{c_{i-1}}, \underline{c_{i+1}, c_i}, c_{i+2}, ...\}$$

By Lemma 1, $\tau_{1,1}$ is equivalent to sorting. ∎

By Lemma 16, $\tau_{1,2}$ partitions odd-length circular-permutations into 2 connected components, depending upon the sign function. Thus the Cayley Graph $CPG(4l+1, 2m+1)$ has two connected subgraphs.

Lemma 22 *The Cayley Graph $G_{CPG}(4l + 3, 2m)$ has $\binom{2m-1}{m}$ connected components.*

Proof: By Theorem 1, all $(4l+3)$-reversals have equivalent transformations with 3-reversal. Thus every reversal leaves the partition between even and odd-numbered elements unchanged, and the number of components follows from the proof of Lemma 17. ∎

The cases for ($k \equiv 5 \bmod 8$) and ($k \equiv 1 \bmod 8$) follow from Section 4.1. In summary:

Theorem 8 *The number of disjoint connected components in $CPG(k,n)$ (for $n \geq k+2$) are:*

$k \setminus n$	n=2m+1	$n \equiv 0 \bmod 4$	$n \equiv 2 \bmod 4$
$k \equiv 0 \bmod 4$	2	1	1
$k \equiv 5 \bmod 8$	2	$2\binom{n-1}{n/2}$	$2\binom{n-1}{n/2}$
$k \equiv 1 \bmod 8$	2	$2\binom{n-1}{n/2}$	$4\binom{n-1}{n/2}$
$k \equiv 2 \bmod 4$	1	1	1
$k \equiv 3 \bmod 4$	1	$\binom{n-1}{n/2}$	$\binom{n-1}{n/2}$

The result holds for $n = k+2$, since $\tau_{k-1,2} = \tau 1,2$ in $CPG(k,k+2)$, which suffices by Lemma 2.

5 Diameter of $CPG(k,n)$

In this section, we consider the diameter of the Cayley graphs of circular permutations under fixed-length reversals. For the special case $k=2$, bubble sort removes exactly one inversion per reversal, giving an $\Theta(n^2)$ diameter. As we shall see, we can do substantially better for larger reversals.

5.1 An Upper Bound on Diameter

Our upper bound rely on a particularly efficient equivalence relation:

Lemma 23 *The 3-cycle $Cyc(i,i+1,j)$ can be implemented in $9k+2n/k+2$ k-reversals for $CPG(k=2l,n)$, where $k+2<n$.*

$$\{a_1,...,a_{i-1},\underline{a_i,a_{i+1}},a_{i+2},...,a_{j-1},\underline{a_j},a_{j+1},...,a_n\} \xrightarrow{Cyc(i,i+1,j)} \{a_1,...,a_{i-1},\underline{a_j,a_i},a_{i+2},...,a_{j-1},\underline{a_{i+1}},a_{j+1},...,a_n\}$$

Proof: Without loss of generality, we show $Cyc(1,2,j)$. There are two cases, depending upon the position of j.

First, consider $3 \leq j \leq k+2$. If $(j-3)$ is even, there are an even number of elements between 2 and j. By Lemma 9, *k-reversal* $\Longrightarrow \tau_{2,4}, \tau_{4,2}$ using 8 reversals. Thus $\tau_{2p,4}$ and $\tau_{4,2p}$ can be derived by p steps of $\tau_{2,4}$, or $8p$ steps of k-reversals. We let $p=(j-3)/2 < k/2$:

$$\{a_1,a_2,\underline{a_3,...,a_{j-1}},\underline{a_j,a_{j+1},a_{j+2},a_{j+3}},a_{j+4},...\} \xrightarrow{\tau_{2p,4}(3)} \{a_1,a_2,\underline{a_j,a_{j+1},a_{j+2},a_{j+3}},\underline{a_3,...,a_{j-1}},a_{j+4},...\}$$

Then we apply $\tau_{2,1}(1)$ derived from Lemma 3 using k steps of $(k=2l)$-reversals.

$$\{\underline{a_1,a_2},\underline{a_j},a_{j+1},a_{j+2},a_{j+3},a_3,a_4,...,a_{j-1},a_{j+4},...\} \xrightarrow{\tau_{2,1}(1)} \{\underline{a_j},\underline{a_1,a_2},a_{j+1},a_{j+2},a_{j+3},a_3,a_4,...,a_{j-1},a_{j+4},...\}$$

Finally, we apply $\tau_{4,2p}$, moving a_2, a_{j+1}, a_{j+2}, and a_{j+3} back to jth position.

$$\{a_j,a_1,\underline{a_2,a_{j+1},a_{j+2},a_{j+3}},\underline{a_3,a_4,...,a_{j-1}},a_{j+4},...\} \xrightarrow{\tau_{4,2p}(3)} \{a_j,a_1,\underline{a_3,a_4,...,a_{j-1}},\underline{a_2,a_{j+1},a_{j+2},a_{j+3}},a_{j+4},...\}$$

We have realized $Cyc(1, 2, j)$ using at most $4k + k + 4k = 9k$ steps of k-reversals. If $(j - 3)$ is odd, we apply one extra $Rev_k(3)$ in the beginning, such that the 2th and jth position are an even distance apart and use the same method as above, performing one final $Rev_k(3)$ on the last step. If a_1 is one of a_{j+1}, a_{j+2}, or a_{j+3}, we can realize $Cyc(1, 2, j)$ in $9k$ reversals by a similar method. Thus the total cost is at most $9k + 2$ reversals.

The second case is $j > k+2$. We apply $Rev_k(j-(k-1))$, $Rev_k(j-2(k-1))$, $Rev_k(j-3(k-1))$, ..., until a_j is moved to the position between 3 and $k + 2$. Then we proceed as above and apply k-reversals in reverse sequence. This costs $2n/k$ extra reversals, for a total of $9k + 6 + 2n/k$. ∎

Theorem 9 *The diameter of $CPG(k, n)$ is $O(n^2/k + nk)$, if $k = 2l$, $n = 2m$, and $k + 2 \leq n$.*

Proof: To sort a circular permutation $\pi = \{c_1, c_2, ..., c_n\}$ using $Cyc(i, i+1, j)$, consider the following variation of selection sort:

1. **Set** $c_1 = 1$
2. **For** $i = 2$ **to** $n - 2$
3. **Find** j **such that** $c_j = i$
4. **If** $j > i + 1$ **then do 3-cycle** $Cyc(i, i + 1, j)$ **to transform** c_i, c_{i+1}, c_j **into** c_j, c_i, c_{i+1}
5. **If** $j = i + 1$ **then do** $\tau_{1,2}(i)$
6. **End For**

It is apparent that the first $n - 2$ elements are sorted using this algorithm. At the start of Step 5, there remain two possibilities:

$$\{1, 2, ..., n - 2, n - 1, n\}, \{1, 2, ..., n - 2, n, n - 1\}$$

Reaching this point used $(n - 2) \cdot (9k + 2 + 2n/k)$ reversals. From $\{1, 2, ..., n - 2, n, n - 1\}$ we may use $n/2$ $\tau_{2,1}$ to move n to reach $\{n, 1, 2, ..., n - 2, n - 1\}$, which can be shifted into identity permutation. This takes an additional $(n/2) \cdot 4k = 2nk$ steps, for a total of $O(n^2/k + nk)$ reversals. ∎

Theorem 10 *The diameter of each connected component of $CPG(k, n)$ is $O(n^2/k + nk)$, if $k = 2l$ and $n > k + 2$.*

Proof: We use the same algorithm as above, sorting permutations into

$$\{1, 2, ..., n - 2, n - 1, n\}, \{1, 2, ..., n - 2, n, n - 1\}$$

Now consider all $(n - 1)!$ permutations in $CPG(k, n)$. Those which can be sorted into $\{1, 2, ..., n - 2, n-1, n\}$ by above algorithm form a set S_1, while the others form set S_2. Any pair of permutations from the same set can be transformed to each other within $(n - 2)$ $Cyc(i, i + 1, j)$ operations, or $O(n^2/k + nk)$ k-reversals, since all operations are symmetric.

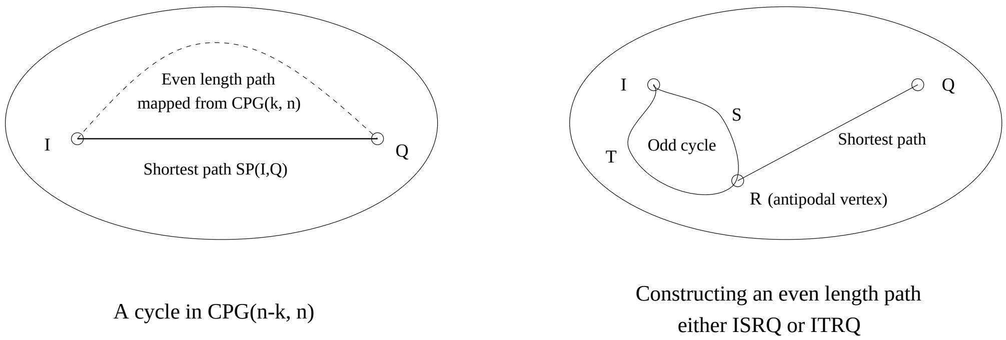

Figure 2: Constructing a short even-length path.

If $CPG(k,n)$ is not connected, we have two connected components in $CPG(k,n)$, each with diameter bounded by $O(n^2/k + nk)$. Otherwise $CPG(k,n)$ is connected by Theorem 8, so there must exist a single k-reversal taking π_{s1} to π_{s2}, where $\pi_{s1} \in S_1$ and $\pi_{s2} \in S_2$.

Thus we can take any permutation from S_2 and transform it to an element of S_1 in $(n-2)$ $Cyc(i,i+1,j)$ operations plus one k-reversal, which can be sorted in an additional $(n-2)$ $Cyc(i,i+1,j)$ operations. The total number of reversals is at most $2(2n^2/k + 9nk + 2n)) + 1$, so the diameter of $G_{CPG}(k,n)$ is $O(n^2/k + nk)$. ∎

Thus, surprisingly, $O(n^{3/2})$ reversals suffice when $k \approx \sqrt{n}$.

To establish bounds for $CPG(4l+3, 2m+1)$, we show that the diameter of $CPG(4l+3, 2m+1)$ is within a factor of two of the diameter of $CPG(2p, 2m+1)$, where $2p = 2m - 4l - 2$:

Lemma 24 *Suppose a graph G contains an odd-length cycle c. Then G has an odd-length cycle of length $\leq 2d+1$, where d is the diameter of G.*

Proof: Let x and y be two antipodal (maximally separated) vertices of c. Thus there are even- and odd-length paths P_e and P_o from x to y of length $\lfloor c/2 \rfloor$ and $\lceil c/2 \rceil$. Now consider the shortest path $P(x,y)$. This path forms a shorter length odd cycle with either P_e or P_o unless it is either P_e or P_o. Thus we can shrink c to length at most $2|P(x,y)|+1$, where $d \geq |P(x,y)|$. ∎

Lemma 25 *$diam(CPG(n-k,n)) \leq 2diam(CPG(k,n)) + 2$, where $diam(G)$ denotes the diameter the connected components of graph G.*

Proof: Let $SP(I,Q)$ denote the length of the shortest path from I to Q in $CPG(n-k,n)$, and let $diam_e(G)$ denote the length of the longest even-length shortest path in G. Note that $diam(G) \leq diam_e(G) + 1$, since the diameter may be even or odd.

Consider the pair of vertices I, Q defining the even diameter of $CPG(n-k,n)$. By Lemma 20, there is an equal length path in $CPG(k,n)$. Now consider the shortest path P from I to Q in $CPG(k,n)$. If P is of even length, then $|P| = diam_e(CPG(n-k,n))$ and $diam(CPG(n-k,n)) \leq diam(CPG(k,n)) + 1$.

If P is of odd-length, we show that there is a short odd-length cycle which, appended to P, gives a short even-length path from I to Q in $CPG(k,n)$. The union of P and the even-length path from I to Q gives an odd length cycle c, which by Lemma 24 can be shrunk to length $\leq 2diam(CPG(k,n)) + 1$. Now consider R, an antipodal vertex from I on c (see Figure 2). Since

$SP(R,Q) \leq diam(CPG(k,n))$ and there are both even and odd length paths from I to R of length at most $diam(CPG(k,n))+1$, there must be an even-length path from I to Q in $CPG(n-k,n)$ of length $\leq 2diam(CPG(k,n))+1$, and so $diam(CPG(n-k,n)) \leq 2diam(CPG(k,n))+2$ ∎

Moreover, $diam(CPG(n-2,n)) \leq 2diam(CPG(2,n))+2$, so

Theorem 11 *The diameter of all connected $CPG(k,n)$ is bounded by $O(n^2/k+nk)$, for $n \geq k+2$.*

5.2 A Lower Bound on Diameter

Our lower bound is based on counting the equivalent of inversions in circular permutations, which is more complicated than for linear permutations since there are two distinct distances separating each pair of elements.

Theorem 12 *The diameter of $CPG(k,n)$ is $\Omega(n^2/k^2+n)$.*

Proof: First, we prove that $n/2$ is a lower bound on the diameter for any size reversal. In a sorted permutation, each element i is flanked by the elements $i-1$ and $i+1$. Any single reversal creates at most two such adjacencies in the permutation, because adjacencies can only be created at the endpoints of the reversal. Any permutation with no adjacencies, such as $\{1,3,\ldots,n,2,4,\ldots,n-1\}$ requires at least $n/2$ reversals to create the necessary adjacencies.

Now consider a circular permutation $\pi = \{c_1, c_2, ..., c_n\}$, where $c_1 = 1$ and $\pi^{-1}(i) = s$ iff $c_s = i$. Let $d_\pi(i,j)$ denote the *distance* between $\pi^{-1}(i)$ and $\pi^{-1}(j)$. If $\pi^{-1}(i) < \pi^{-1}(j)$,

$$d_\pi(i,j) = min(\pi^{-1}(j) - \pi^{-1}(i), \pi^{-1}(i) + n - \pi^{-1}(j))$$

Let $D(\pi)$ denote the total distance of a permutation π, ie.

$$D = \sum_{i=1}^{n-1}(d_\pi(i,i+1)) = d_\pi(1,2) + d_\pi(2,3) + ... + d_\pi(n-1,n)$$

For the identity permutation $I = \{1,2,...,n\}$, $d_I(i,i+1) = 1$, so $D(I) = n-1$. Sorting π reduces D to $n-1$.

We claim any k-reversal can reduce D by at most

$$\Delta D = 2\sum_{i=1}^{\lfloor k/2 \rfloor} 2(2i-1) \leq k^2$$

since each element we move might reduce the distance with both of its neighbors.

Since $D(\{1,3,5,...,n-1,2,4,...,n\}) = n(n-1)/2$, reducing it down to $n-1$ requires at least $(n-1)(n-2)/2k^2$ k-reversals, so the diameter of $G_{CPG}(k,n)$ is $\Omega(n^2/k^2+n)$. ∎

An inherent weakness of this lower bound technique is that for $k > 2$, the number of inversions in a permutation will not decrease monotonically during sorting, as certain elements get moved in the wrong direction to accommodate other elements. For this reason, we believe the lower bound is not tight.

6 Conclusions and Open Problems

We have completely resolved the question of the connectedness of the Cayley graphs of permutations and circular permutations under fixed-length reversals, and given upper and lower bounds on the diameter of the Cayley graphs of circular permutations. Several open problems remain:

- Tighten our bounds on the diameter of these graphs. We believe our upper bound is better than our lower bound.
- What is the complexity of determining the exact k-reversal distance for sorting a permutation $\pi\Gamma$ Although approximation algorithms are known for arbitrary-length reversal distance, we anticipate that the problem is much more difficult for fixed-length reversals.
- What is the diameter and connectedness under fixed-length *signed* reversals, where each element of the permutation has two possible orientations, reversed or unreversed. Signed reversals are important in reconstructing the history of evolution of the genome, because the orientation of each gene can be determined from its sequence.
- It is conjectured that every connected Cayley graph is Hamiltonian. Is there a way to sequence (circular) permutations so that each differs from its predecessor by exactly one k-reversalΓ

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