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# POINTWISE CIRCUMSCRIPTION: PRELIMINARY REPORT

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## Abstract

Circumscription is the minimization of predicates subject to restrictions expressed by predicate formulas. We propose a modified notion of circumscription so that, instead of being a single minimality condition, it becomes an "infinite conjunction" of "local" minimality conditions; each of these conditions expresses the impossibility of changing the value of a predicate from *true* to *false* at one point. We argue that this "pointwise" circumscription is conceptually simpler than the traditional "global" approach and, at the same time, leads to generalizations with the additional flexibility needed in applications to the theory of commonsense reasoning.

## 1. Introduction

*Circumscription* (McCarthy 1980, 1986) is *logical minimization*, that is, the minimization of predicates subject to restrictions expressed by predicate formulas.

The interpretation of a predicate symbol in a model can be described in two ways. One is to represent a $k$-ary predicate by a subset of $U^k$, where $U$ is the universe of the model. This approach identifies a predicate with its *extension*. The other possibility is to represent a predicate by a Boolean-valued function on $U^k$. These two approaches are, of course, mathematically equivalent; but the intuitions behind them are somewhat different, and they suggest different views on what "minimizing a predicate" might mean.

If a predicate is a set then predicates are ordered by *set inclusion*, and it is natural to understand the minimality of a predicate as minimality relative to this order. A *smaller* predicate is a *stronger* predicate. A predicate satisfying a given condition is minimal if it cannot be made stronger without violating the condition. This understanding of minimality leads to the usual definition of circumscription.

Let us accept now the view of predicates as Boolean-valued functions, or, in other words, as *families of truth values*. Each predicate is a family of elements of the ordered set $\{false, true\}$. Understanding "smaller" as "stronger" still makes sense; but now we can also think of making a predicate smaller *at a point* $\xi \in U^k$ as changing its value at that point from *true* to *false*. As far as the values at other points are concerned, we can require, in the simplest case, that they remain the same; or we can allow them to change in an arbitrary way; or some of them can be required to remain fixed, and the others allowed to vary.

The new definition of circumscription proposed in this paper expresses, intuitively, the minimality of a predicate "at every point". It can be interpreted as an "infinite conjunction" of "local" minimality conditions; each of these conditions expresses the impossibility of changing the value of a predicate from *true* to *false* at one point. (Formally, this "infinite conjunction" will be represented by a universal quantifier).

We argue that this "pointwise" approach to circumscription is in some ways conceptually simpler than the traditional "global" approach and, at the same time, leads to generalizations with the additional flexibility needed in applications to the theory of commonsense reasoning.

Proofs of the mathematical facts stated below will be published in the full paper.

## 2. The Basic Case of Pointwise Circumscription

Let us start with the simplest case of circumscribing one predicate with all other non-logical constants treated as parameters. Let $A(P)$ be a sentence containing a predicate constant $P$. Recall that the *(global) circumscription of* $P$ *in* $A(P)$ is, by definition, the second-order formula

$$A(P) \wedge \forall p \neg (A(p) \wedge p < P). \qquad (1)$$

Here $p$ is a predicate variable of the same arity as $P$, and $p < P$ stands for

$$\forall x(px \supset Px) \wedge \neg \forall x(Px \supset px)$$

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($x$ is a tuple of object variables). We denote (1) by Circum($A(P); P$).

The *pointwise circumscription* of $P$ in $A(P)$ is

$$A(P) \wedge \forall x \neg[Px \wedge A(\lambda y(Py \wedge x \neq y))]. \qquad (2)$$

Notice that this is a first-order formula. We denote (2) by $C_P(A(P))$. A model of (2) is a model of $A(P)$ which cannot be transformed into another model of $A(P)$ by changing the value of $P$ from *true* to *false* at one point. The quantifier $\forall x$ represents the "infinite conjunction" mentioned above, and the formula following the quantifier can be viewed as a minimality condition: it asserts the minimality of the value of $P$ at point $x$.

It is easy to check that Circum($A; P$) implies $C_P(A)$. If we assume that all occurences of $P$ in $A(P)$ are positive then these two formulas are equivalent. This special case is important, because in standard applications of circumscription (McCarthy 1986) the minimized predicates usually have no negative occurences in the axioms.

To illustrate the difference between (1) and (2), take $A$ to be $Pa \equiv Pb$. Any model with $P$ identically false satisfies both Circum($A; P$) and $C_P(A)$. In addition, any model of $A$ in which $P$ is true at exactly two points, $a$ and $b$, is a model of the pointwise version, but not of the global one.

### 3. A Generalization

Even in simple applications to formalizing commonsense knowledge we usually need forms of circumscription slightly more general than those defined in the previous section.

Let us start with a formula $A(P, Z)$, where $Z$ is a tuple of predicate and/or function constants. The *(global) circumscription of $P$ in $A(P, Z)$ with $Z$ allowed to vary* is

$$A(P, Z) \wedge \forall pz \neg(A(p, z) \wedge p < P), \qquad (3)$$

where $z$ is a tuple of predicate and/or function variables similar to $Z$. This formula, denoted by Circum($A(P, Z); P; Z$), asserts that the extension of $P$ cannot be made smaller even at the price of changing the interpretations of the symbols included in $Z$.

The corresponding form of *pointwise circumscription* is

$$A(P, Z) \wedge \forall xz \neg[Px \wedge A(\lambda y(Py \wedge x \neq y), z)]. \qquad (4)$$

It will be denoted by $C_P(A(P, Z); Z)$. Because of the variables $z$, (4) is, generally, a second-order formula. (4) is equivalent to (3) if all occurences of $P$ in $A(P, Z)$ are positive.

### 4. Minimizing Several Predicates

In a further generalization of global circumscription (3), $P$ is a *tuple* of predicate constants $P_1, \ldots, P_n$. The meaning of Circum($A; P; Z$) is given again by (3), with $p$ standing this time for a tuple of predicate variables $p_1, \ldots, p_n$, and $p < P$ understood as

$$\bigwedge_{i=1}^{n} \forall x(p_i x \supset P_i x) \wedge \bigvee_{i=1}^{n} \neg \forall x(P_i x \supset p_i x).$$

This form of joint minimization of several predicates is called *parallel circumscription* (to distinguish it from the case when different members of $P$ are minimized with different priorities, which is discussed below).

What is the relationship between circumscribing $P_1, \ldots, P_n$ in parallel and circumscribing each $P_i$? It is easy to show that Circum($A; P; Z$) implies $\bigwedge_i$ Circum($A; P_i; Z$). In the important special case when all occurences of $P_1, \ldots, P_n$ in $A$ are positive, the converse also holds; hence, in this case, Circum($A; P; Z$) is equivalent to $\bigwedge_i C_{P_i}(A; Z)$. This conjunction asserts that, whenever one value of one of the predicates $P_1, \ldots, P_n$ is changed from *true* to *false*, and the interpretation of $Z$ is changed in an arbitrary way, the resulting structure cannot possibly be a model of $A$. This formula is the pointwise counterpart of parallel circumscription; there is no need to introduce a special definition.

Let us turn now to *prioritized* circumscription, and consider, for simplicity, the case of two predicates $P_1$, $P_2$. The circumscription Circum($A; P_1 > P_2; Z$), which assigns a higher priority to the task of minimizing $P_1$, is defined by the same formula (3), but with $p < P$ interpreted lexicographically; for details, see (Lifschitz 1985). This circumscription is equivalent to

$$C_{P_1}(A; P_2, Z) \wedge C_{P_2}(A; Z)$$

whenever $P_1$, $P_2$ have only positive occurences in $A$. This conjunction is the pointwise counterpart of prioritized circumscription: no value of $P_1$ can be changed from *true* to *false* even at the price of changing $P_2$ arbitrarily.

### 5. A Further Generalization

Our next goal is to introduce some more general forms of pointwise circumscription. We start with a motivating example.

Consider a simple version of the blocks world, in which a block can be in only one of two places: either on the table or on the floor. We want to describe the effect of one particular action, putting block $B$ on the table. This can be done using two unary predicate constants, $ONTABLE_0$ and $ONTABLE_1$, which represent the configurations of blocks before and after the action. There are two axioms:

$$\neg AB\ x \supset (ONTABLE_0\ x \equiv ONTABLE_1\ x) \qquad (5)$$

and

$$ONTABLE_1\ B. \qquad (6)$$

Here $AB$ is the "abnormality" predicate which will be circumscribed in the conjunction of (5) and (6). The first axiom expresses what John McCarthy calls the "commonsense law of inertia": normally, objects remain where they are. This formula exemplifies the use of circumscription for solving the frame problem (McCarthy 1986). The second axiom expresses the basic property of the action under consideration: in the new configuration of blocks, $B$ is on the table.

What should be varied in the process of minimizing $AB$? The purpose of our axiom set is to characterize the new configuration of blocks; hence it is natural to circumscribe $AB$ with $ONTABLE_1$ allowed to vary. Such a circumscription (global or pointwise) gives

$$AB\ x \equiv (x = B \wedge \neg ONTABLE_0\ B).$$

This is exactly what we would intuitively expect; the only block which changes its location is $B$, and this only happens if it was not on the table prior to the event. It can be shown that circumscription does not lead to the same result if $ONTABLE_0$ and $ONTABLE_1$ are both varied or both fixed; we must treat $ONTABLE_0$ and $ONTABLE_1$ in different ways.

Let us change now slightly the formal language used in this example and move closer to the formalism of the *situation calculus* of (McCarthy and Hayes 1969). In addition to variables for blocks, we introduce a second sort of variables, variables for *situations*. There are two situation constants, $S_0$ and $S_1$, which represent two situations separated by the action of placing $B$ on the table. Instead of two unary predicates $ONTABLE_0$ and $ONTABLE_1$, we have now one binary predicate $ONTABLE$, which is supposed to have a situation term as its second argument. In the new notation, (5) and (6) become

$$\neg AB\ x \supset (ONTABLE(x, S_0) \equiv ONTABLE(x, S_1))$$

and

$$ONTABLE(B, S_1).$$

We also add the axiom $S_0 \neq S_1$.

What corresponds to the circumscription described above in this new notation? We would like to vary the values of $ONTABLE(x, s)$ for $s = S_1$, and have the values corresponding to $s = S_0$ fixed. Definitions (3), (4), do not allow us to do that; for any predicate or function constant in the language, we have to either include it in list $Z$, and then all values of that predicate or function may vary, or not include it in $Z$, and then all of its values must remain fixed in the process of minimization. We would like to be able to specify, for each function and predicate in $Z$, the *part of its domain* on which its values must remain fixed, and allow the other values to be varied.

The following notation will be useful. If $p$, $q$, $r$ are predicate symbols of the same arity then we write $EQ_r(p, q)$ for $\forall x(\neg rx \supset (px \equiv qx))$ ("$p$ and $q$ are equal outside $r$"). If $f$, $g$ are function symbols of the same arity as $r$ then $EQ_r(f, g)$ stands for $\forall x(\neg rx \supset (fx = gx))$.

Assume first that $Z$ consists of only one symbol, a predicate constant or a function constant. Consider a $\lambda$-expression $V$ of the same arity as $Z$, which has no parameters and contains neither $P$ nor $Z$. Intuitively, it specifies the part of the domain of $Z$ on which $Z$ may vary.

For global circumscription, we propose the following formula:

$$A(P, Z) \wedge \forall pz \neg [EQ_V(z, Z) \wedge A(p, z) \wedge p < P]. \qquad (7)$$

If $V$ is identically true then (7) becomes (3). Making $V$ identically false is equivalent to treating $Z$ as a parameter rather than varying it; (7) becomes (1). In the example from the previous section, $Z$ is $ONTABLE$, and we can get the desired effect, for instance, by taking $V$ to be $\lambda ys(s \neq S_0)$; $ONTABLE$ is allowed to vary in situations other than $S_0$.

The counterpart of (7) for pointwise circumscription is

$$A(P, Z) \wedge \forall xz \neg [Px \wedge EQ_V(z, Z) \wedge A(\lambda y(Py \wedge x \neq y), z)].$$

We can allow even more flexibility by making it possible for $x$ to affect the choice of the part of the domain on which $Z$ may vary when $P$ is minimized at $x$. (This additional flexibility, not needed in this example, is essential for more complex applications.) Let $V$ be a $\lambda$-expression $\lambda xuV(x, u)$ whose arity equals the sum of the arities of $P$ and $Z$. Intuitively, $V$ represents the function which maps every value of $x$ into the set of all values of $u$ satisfying $V(x, u)$; accordingly, we will write $Vx$ for $\lambda uV(x, u)$. The new form of circumscription is

$$A(P, Z) \wedge \forall xz \neg [Px \wedge EQ_{Vx}(z, Z) \wedge A(\lambda y(Py \wedge x \neq y), z)].$$

We will denote this formula by $C_P(A(P,Z);Z/V)$. If $V$ is identically true then $C_P(A;Z/V)$ becomes $C_P(A;Z)$. If $V$ is identically false then $C_P(A;Z/V)$ is equivalent to $C_P(A)$.

## 6. More on Priorities

Now we know how to perform circumscription with some values of $Z$ allowed to vary. There is another interesting possibility: we may vary some values of the minimized predicate $P$ itself.

We start with the case when $Z$ is empty. The new schema is

$$A(P) \wedge \forall xp \neg[Px \wedge EQ_{Vx}(p,P) \wedge A(\lambda y(py \wedge x \neq y))]. \quad (8)$$

Here $V$ is a $\lambda$-expression $\lambda xyV(x,y)$ which has no parameters and does not contain $P$. The second term of (8) expresses that it is impossible to change arbitrarily the values of $P$ on $\{y : V(x,y)\}$, and then change its value at $x$ from *true* to *false*, without loosing the property $A$. We denote (8) by $C_P(A;P/V)$. It is, generally, stronger than the basic form of pointwise circumscription $C_P(A)$ and turns into it when $V$ is identically false.

The following example shows how we can use the new form of circumscription to create the effect of assigning different priorities to the tasks of minimizing $P$ at different points. Applying the basic forms of global circumscription (1) or pointwise circumscription (2) to $Pa \vee Pb$ gives

$$\forall x(Px \equiv x = a) \vee \forall x(Px \equiv x = b).$$

Using the form of pointwise circumscription introduced in this section, we can express the idea of assigning a higher priority to the task of minimizing $P$ at $b$; this circumscription will lead to the stronger result

$$\forall x(Px \equiv x = a). \quad (9)$$

To this end, introduce a binary predicate constant $V$, and let $A(P)$ be the conjunction of $Pa \vee Pb$ and $V(b,a)$. The second formula shows that $P$ may be varied at point $a$ when it is minimized at $b$. This condition expresses in the language of pointwise circumscription that minimizing $P$ at $b$ is given a higher priority. It is easy to see that, in this case, (8) implies (9).

An interesting feature of this example is that information on priorities is represented by the axiom $V(b,a)$, which is included in the database along with $Pa \vee Pb$. A circumscriptive theory is usually thought of as an axiom set along with a *circumscription policy*, a metamathematical statement describing which predicates are allowed to vary, and what the priorities are. The form of circumscription proposed here allows us to describe circumscription policies by *axioms* rather than metamathematical expressions.

As another example, consider the problem posed in (Hanks and McDermott 1985), Section 7.1. Let $A(P)$ be the conjunction of these axioms:

$$a_i \neq a_j \qquad (0 \leq i < j \leq 3),$$

$$S(x,y) \equiv [(x = a_1 \wedge y = a_0) \vee (x = a_2 \wedge y = a_1) \vee (x = a_3 \wedge y = a_2)],$$

$$\neg Px \wedge S(y,x) \supset Py, \qquad \neg Pa_0.$$

We can think of $a_0, \ldots, a_3$ as instances of time, $S$ as the successor relation, and $P$ as an "abnormality" of some kind. Applying any of circumscriptions (1), (2) to $A(P)$ gives

$$\forall x[Px \equiv (x = a_1 \vee x = a_2)] \vee \forall x[Px \equiv (x = a_1 \vee x = a_3)]. \quad (10)$$

Hanks and McDermott ask what kind of formal nonmonotonic reasoning can capture the idea of preferring "minimization at earlier instants of time", which would lead to selecting the second disjunctive term. Their analysis shows that temporal reasoning of this kind is important but apparently cannot be captured by the existing formalisms.

The problem is clearly similar to the one discussed above. Extend the theory by these "policy" axioms:

$$V(a_i, a_j) \qquad (0 \leq i < j \leq 3).$$

The additional axioms tell us that $P$ may be varied at the points "later than $x$"when minimized at $x$; in this way, it "gives preference to the past". If a model with $a_2 \neq a_3$ satisfies the first term of (10) then a "better" model can be constructed by making $Pa_2$ false and $Pa_3$ true.

This method can be also used to resolve a difficulty uncovered in recent attempts to formalize reasoning about the blocks world using circumscription (see (McCarthy 1986), Section 12). We would like to use the formalism of situation calculus to describe the effect of moving a block. One of the axioms is "the law of motion" expressing that, *normally*, moving a block $x$ to a location $l$ leads to a situation in which $x$ is at $l$. Another axiom tells us that the case when $x$ is not clear is an exception. Imagine now two blocks $A$ and $B$ side by side on the table. We attempt to place $A$ on top of

$B$ and then to move $B$ somewhere else. Intuitively, the second action will be unsuccessful, because after the first action $B$ is not clear. In other words, the first action will be "normal" relative to the law of motion, and the second will be "abnormal". Unfortunately, circumscribing abnormality does not allow us to prove this assertion. It does not eliminate the possibility that the first action leaves the positions of the blocks unchanged, so that $B$ remains clear, and the second action leads to the normal result. In this alternative model, the second action is "normal", and the first is not. Each of the two models corresponds to a minimal value of $AB$; circumscription only gives a disjunction.

The "bad" model can be eliminated by "giving preference to the past", as in the previous example. Details will be given in the full paper. The solution uses also the ideas of the previous section, so that what we need is a definition of circumscription which covers all the forms introduced above. This most general definition of pointwise circumscription is given in the next section.

## 7. The General Case

Let $A(S_1,\dots,S_n)$ be a sentence, where each $S_i$ is a predicate symbol or a function symbol (in particular, it can be a 0-ary function symbol, i.e., an object constant). We want to minimize one of the predicate symbols from this list, say, $S_1$. (Thus $S_1$ corresponds to $P$ and $S_2,\dots,S_n$ correspond to $Z$ in the notation used before). The *pointwise circumscription of $S_1$ in $A$ with $S_i$ allowed to vary on $V_i$* is, by definition,

$$A(S) \wedge \forall x s \neg[S_1 x \wedge \bigwedge_{i=1}^{n} EQ_{V_i x}(s_i, S_i) \wedge A(\lambda y(s_1 y \wedge x \neq y), s_2, \dots)]. \quad (11)$$

Here $S$ stands for $S_1,\dots,S_n$, $s$ is a list $s_1,\dots,s_n$ of predicate and function variables corresponding to the predicate and function constants $S$; $V_i$ $(i = 1,\dots,n)$ is a predicate without parameters which does not contain $S_1,\dots,S_n$ and whose arity is the arity of $S_1$ plus the arity of $S_i$.

We denote (11) by $C_{S_1}(A; S_1/V_1,\dots,S_n/V_n)$. If $V_i$ is identically true then we will drop $/V_i$ in this notation. If $V_i$ is identically false then we can drop the term $S_i/V_i$ altogether.

## 8. Conclusion

The pointwise approach to circumscription has the following advantages over the traditional global approach.

1. The basic case of pointwise circumscription is expressed by a first-order formula.

2. There is no need to define the circumscription of more than one predicate.

3. Circumscription policies become more "modular": a separate policy is defined for each of the minimized predicates.

4. Circumscription policies, including the selection of priorities, can be described by axioms, instead of metamathematical definitions.

5. Circumscription policies may vary from point to point, which provides additional flexibility useful in applications to formalizing reasoning about time and actions.

We hope that the form of circumscription proposed in this paper is sufficiently powerful for formalizing many relatively complex forms of commonsense reasoning. Future work on applications of pointwise circumscription should lead to the discovery of general principles regarding the choice of circumscription policies (such as, for instance, the principle of assigning a higher priority to minimization at earlier instants of time in temporal reasoning). The present situation, when the policy is selected in many cases by trial and error, is clearly unsatisfactory. It is also important to extend the existing methods for determining the result of circumscription to more general forms needed in applications.


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