GLOBAL ANALYSIS OF THIRD-ORDER RELAY FEEDBACK SYSTEMS *

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Abstract
Relays are common in automatic control systems. It is well-known that a linear dynamical system under relay feedback can give complex oscillations. In this paper it is proved that several of these phenomena can actually be captured by third-order systems. It is shown that there exist systems giving arbitrarily fast relay switches similar to sliding modes. A novel method for analyzing linear dynamical systems under relay feedback is also introduced. Trajectories for a class of third order systems are shown to converge in a certain sense.

Keywords: Nonlinear control systems, Limit cycles, Relays, Sliding mode, Oscillation

1. INTRODUCTION

Analysis of relay feedback systems is a classical topic in control theory. The early work was motivated by relays in electromechanical systems and simple models for dry friction. An important property of a linear dynamical system under relay feedback is its tendency to oscillate. Design of simple relay controllers in aerospace applications is described in Flügge-Lotz (1953). This work gave inspiration to the development of the self-oscillating adaptive controller in the 1960's. Recently new interest of relay feedback appeared due to the idea of using relays for tuning of simple controllers, see Åström and Hägglund (1984). By simply replacing the controller by a relay, measure the amplitude and frequency of the possible oscillation, and out of these derive the controller parameters, a robust control design method is given. Even if this method is widely spread and accepted in practice, it has not been theoretically investigated in greater detail. The idea of putting the plant under relay feedback is also used in Smith and Doyle (1993) for estimating perturbation bounds in robust control design. More historical comments on relays in control systems and their applications are given in Tsypkin (1984) and Åström (1993).

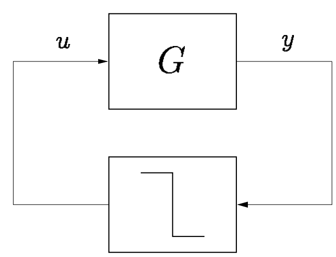

Fig. 1. Relay feedback of linear system G.

Analysis of linear systems under relay feedback is a difficult task. Restrictions about the considered linear dynamics have to be made. The monograph Andronov *et al.* (1965) is an early classical reference (first edition published in Russian in 1937) discussing oscillations in mostly second-order systems using phase-plane analysis. For some systems a fruitful approach to get approximate results is the describing function method, Atherton (1975). In Yakubovich (1973) a frequency condition is used to give sufficient conditions for a certain type of oscillation. The major reference about relay control

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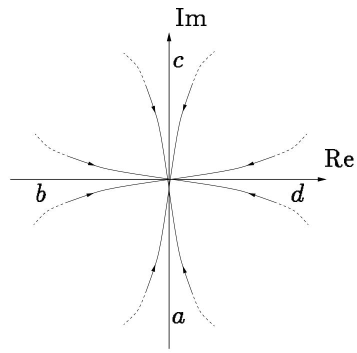

Fig. 2. Sliding modes occur when the high-frequency asymptote of the Nyquist curve is the negative imaginary axis and fast switches if it is the negative real axis.

systems Tsypkin (1984) surveys a number of frequency methods.

Even if relay feedback systems have been studied for a long time, they are far from fully understood. There is, for instance, little known about global attraction of limit cycles. For second-order processes analysis can be done in the phase-plane. Stable second-order nonminimum phase processes can in this way be shown to give a globally attractive limit cycle. In Megretski (1996) it is proved that this also holds for processes having an impulse response sufficiently close in a certain sense to a second-order nonminimum phase process. The main contribution of our work is to analyze third-order relay feedback systems. It is shown that despite the low dimension, these systems have a rich structure. Sliding modes as well as a similar type of fast relay switches can appear. The phenomena can be detected from the high-frequency asymptote of the Nyquist curve for the linear part of the system, see Figure 2. If positive steady-state gain is assumed, the high-frequency asymptote a gives sliding modes, the asymptote at b gives fast switches, while c and d do not. A method for analysing relay feedback systems based on convergence of switch plane intersections is introduced in the latter part of the paper.

2. PRELIMINARIES

Consider the relay feedback system in Figure 1. The process G is a stable and strictly proper linear transfer function with scalar input u and scalar output y. In state-space form G is given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ and A is a Hurwitz matrix, that is, all eigenvalues of A lie in the open left half plane. The relay feedback is defined by

$$u = -\operatorname{sgn} y = \begin{cases} -1 & \text{if} \quad y > 0 \\ 1 & \text{if} \quad y < 0 \end{cases} \tag{2}$$

The *switch plane* $\mathcal{S}$ is the hyperplane of dimension $n-1$ where the output vanishes, that is, $\mathcal{S} := \{x : Cx = 0\}$. On either side of $\mathcal{S}$ the feedback system is linear. If $Cx > 0$ the dynamics are given by $\dot{x} = Ax - B$, and if $Cx < 0$ we have $\dot{x} = Ax + B$. We also introduce the notation $\mathcal{S}_+ := \{x \in \mathcal{S} : CAx + CB > 0\}$. If nothing else is mentioned, we assume the process G to have positive steady-state gain. Since the linear dynamics on each side of $\mathcal{S}$ have fixed points equal to $\pm A^{-1}B$, this guarantees that the trajectories do not tend to any of these two fixed points, and thus ensures a relay switch to occur. The differential equation (1)–(2) is only defined in two open subsets of the state-space. By letting $u \in [-1, 1]$ for $x \in \mathcal{S}$, the solution can still be a continuous function which satisfies (1)–(2) everywhere, see Filippov (1988) and Yakubovich (1973).

Let the Poincaré map $g = g(x) : \mathcal{S}_+ \to \mathcal{S}_+$ map a point x to next switch plane intersection of the trajectory starting at x and reflect the intersection in the origin. We have

$$g(x) := -e^{Ah(x)}x + (e^{Ah(x)} - I)A^{-1}B \tag{3}$$

where $h(x)$ is the *switch time*, that is, the unique time it takes between the two intersections x and $g(x)$. Recall that $CB = 0$ if and only if the relative degree of G is greater than one. If the steady-state gain $G(0) = -CA^{-1}B$ is positive, then $CB < 0$ if and only if the relative degree is one and G has an odd number of zeros in the right half plane.

3. SLIDING MODES

It is well-known that *sliding modes* (or *Filippov solutions*) can occur in relay feedback systems. This can easily be understood by studying $\dot{y} = CAx \pm CB$ close to $\mathcal{S}$. We see that depending on the value of CB a classification of the directions of the trajectories divide the switch plane into two or three regions.

EXAMPLE 1
Consider the process

$$G(s) = \frac{\beta s + 1}{(s+1)(s+2)}$$

with state-space representation

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ \beta \end{pmatrix} u \\ y &= \begin{pmatrix} 0 & 1 \end{pmatrix} x \end{aligned}$$

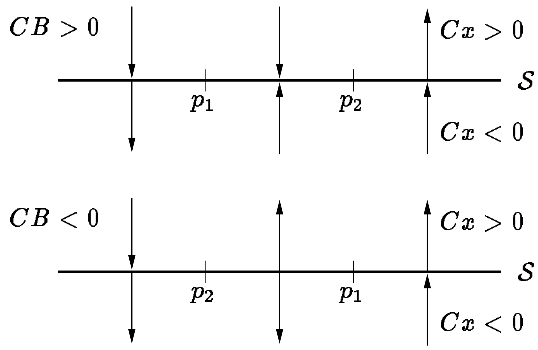

Fig. 3. The switch plane $\mathcal{S}$ and the trajectories for the system in Example 1. If $CB > 0$, then there exist sliding modes.

Then $\mathcal{S}$ equals the x_1-axis, see Figure 3. The points $p_{1,2}$, where the trajectories change directions, are given by the solutions of

$$CAx \pm CB = 0$$

that is, $p_1 = (-\beta, 0)$ and $p_2 = (\beta, 0)$. For $CB = \beta > 0$ there exist sliding modes, while for $CB < 0$ the region between p_1 and p_2 is repelling. The region vanishes if $CB = 0$. □

The statements in the example directly generalize to processes of order $n > 2$. Then p_1 and p_2 denotes hyperplane of order $n-2$, still separating the switch plane into two or three regions. It follows that a sliding mode can occur if and only if $CB > 0$.

It is well-known that oscillations can occur in mechanical control systems due to friction. The oscillation may include a stick-slip motion, that is, the mechanical device is moving only a part of each period. A fifth-order system is reported to behave this way in Atherton *et al.* (1985), but the following example shows that third-order dynamics are sufficient.

Example 2
Consider

$$G(s) = \frac{s(s-\zeta)}{(s+1)(s^2+s+1)} \tag{4}$$

with state-space representation

$$\dot{x} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ -\zeta \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x$$

In Figure 4 the (clockwise) limit cycle is shown for $\zeta = 1$. The fixed points of $\dot{x} = Ax - B$ and $\dot{x} = Ax + B$ are marked with asterisks and the sliding mode region $\{x \in \mathcal{S} : |CAx| < CB\}$ with dashed lines. The sliding mode part of each period decreases with increasing ζ, so $\zeta = 10$ gives a smooth limit cycle. The intersection of the Nyquist curve of (4) with the negative real axis does not correspond to the true period of the oscillation. Hence, the describing function method gives an erroneous estimate of the limit cycle period for this example. □

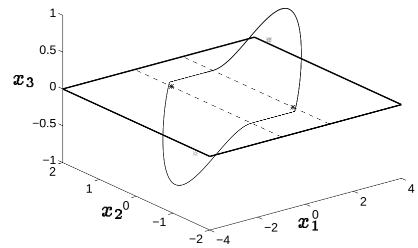

Fig. 4. Limit cycle with sliding mode.

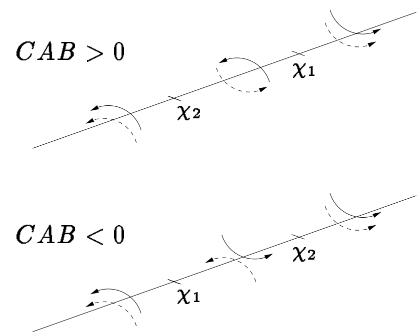

Fig. 5. Illustration of Theorem 1. Fast relay switches occur if $CAB > 0$.

4. FAST SWITCHES

In this section a necessary and sufficient condition for the occurrence of fast relay switches similar to sliding modes is proved. From previous section it is obvious that if $CB > 0$ there exist sliding modes and if $CB < 0$ there cannot exist any arbitrarily fast relay switches. Therefore consider a third-order system with $CB = 0$. Figure 5 shows trajectories close to $\{x \in \mathcal{S} : CAx = 0\}$ for examples with $CAB > 0$ and $CAB < 0$. The tick marks indicate $CA^2\chi_1 - CAB = 0$ and $CA^2\chi_2 + CAB = 0$. Solid trajectories are above the switch plane ($Cx > 0$) and dashed under. The figure suggests that the switch times $h(\cdot)$ can be arbitrarily short if $CAB > 0$. A proof will be given next.

THEOREM 1
Consider the relay feedback system (1)–(2) with $n = 3$, $CB = 0$, and $CAB \neq 0$. Then $\inf_{x \in \mathcal{S}_+} h(x) + h(g(x)) = 0$ if and only if $CAB > 0$.

Proof: Let $\phi_-(t, x)$ denote the trajectory of $\dot{x} = Ax - B$ at time t starting in x at time $t = 0$. Consider $x_0 \in \mathcal{S}$ such that $CAx_0 = 0$ and $CA^2x_0 - CAB < 0$. It follows directly that

$$C\phi_-(t, x_0) = (CA^2x_0 - CAB)\frac{t^2}{2} + \mathcal{O}(t^3) \qquad (5)$$

Hence, $C\phi_-(t_0, x_0) < 0$ for t_0 sufficiently small. Furthermore, for this t_0 we have $C\phi_-(t_0, x) < 0$ for $x \in \mathcal{S}_+$ with $|x - x_0|$ sufficiently small. It follows for $x \in \mathcal{S}_+$ that $h(x) \to 0$ as $x \to x_0$, and also that $g(x) \to -x_0$. A symmetric argument with $g(x)$ gives $h(g(x)) \to 0$ as $x \to x_0$ if x_0 also satisfies $CA^2x_0 + CAB > 0$. Hence, sufficiency follows, since for $CAB > 0$ there exists $x \in \mathcal{S}_+$ such that $|CA^2x| < CAB$.

If

$$CA^2x_0 - CAB > 0 \qquad (6)$$

for $x_0 \in \mathcal{S}$ and $CAx_0 = 0$, then (5) gives that $C\phi_-(t_0, x_0) > 0$ for t_0 sufficiently small. Hence, $C\phi_-(t_0, x) > 0$ for $x \in \mathcal{S}_+$ and $|x - x_0|$ sufficiently small, and thus $h(x) > t_0$. If $CA^2x_0 - CAB < 0$ and

$$CA^2x_0 + CAB < 0 \qquad (7)$$

then in the same way there exists $t_1 > 0$ such that $h(g(x)) > t_1$ for $|x - x_0|$ sufficiently small. Hence, if $CAB < 0$ one of the two inequalities (6) and (7) most hold and necessity follows. □

EXAMPLE 3
Consider the third-order process

$$G(s) = \frac{\zeta - s}{\zeta(s+1)^3} \qquad (8)$$

with state-space representation

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix} x + \begin{pmatrix} 1 + 1/\zeta \\ -1 - 2/\zeta \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x$$

Figure 6 illustrates Theorem 1. Two trajectories starting close to the origin are shown for $\zeta = -4$ and $\zeta = 1$, respectively. The speed of convergence in number of switches is much lower for the minimum phase system ($CAB = -1/\zeta > 0$) compared to the nonminimum phase ($CAB < 0$). □

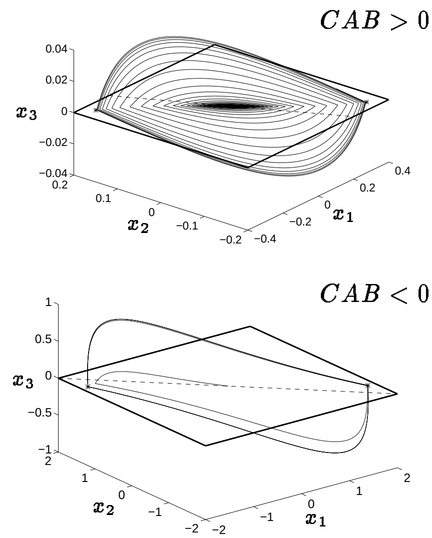

Fig. 6. Two types of convergence to limit cycles.

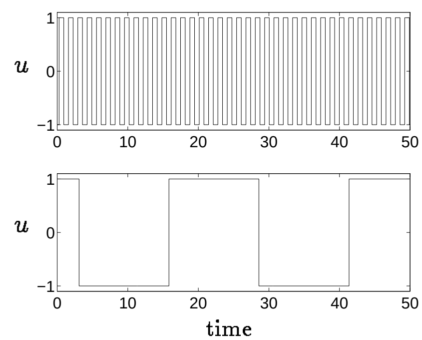

Fig. 7. Two stable limit cycles for the system in Example 4.

5. GLOBAL ANALYSIS OF LIMIT CYCLES

We now derive a method for analyzing relay feedback system using the Poincaré map (3). Let $\phi(t, x_0)$ denote the trajectory of (1)–(2) that starts at x_0. A *closed orbit* is a trajectory such that $\phi(t_1, x_0) = \phi(t_2, x_0)$ for some $t_1 < t_2$. A point p is said to be a *limit point* of the trajectory if there exists a sequence $\{t_k\}$, with $t_k \to \infty$ as $k \to \infty$, such that $\phi(t_k, x_0) \to p$ as $k \to \infty$. The set of all limit points is the *limit set* of the trajectory and is denoted $\mathcal{L}$. A limit set that is a closed orbit is called a *limit cycle*. The limit cycle is called *simple* if it has exactly two intersections with the switch plane $\mathcal{S}$. It is said to be *globally attractive* if it is the limit set of all possible trajectories.

An obvious question is if it exists relay feedback systems not having a unique stable limit cycle. For higher order systems, the answer is yes as shown by the following example.

EXAMPLE 4
Let

$$G(s) = \frac{(s+1)^2}{(s+0.1)^3(s+7)^2}$$

Depending on the initial conditions, the relay feedback system tends to either a slow or a fast limit cycle. In Figure 7 the relay output u is shown for the two cases after the initial transient has disappeared. Analysis shows that the limit cycles are locally stable. □

Denote k successive mappings by $g^k(x)$. If $\phi^*(t, x_0)$ is part of a stable simple limit cycle, and thus $\phi^*(t, x_0) \in \mathcal{L}$ for all $t \geq 0$, then the intersections with $\mathcal{S}$ equals $\pm x^* \in \mathcal{L}$, where x^* is a fixed point of g, $x^* = g(x^*)$. Hence, solving the equation $x = g(x)$ gives candidates for simple limit cycle intersections with $\mathcal{S}_+$. The solution is given by

$$x = (e^{Ah} + I)^{-1}(e^{Ah} - I)A^{-1}B$$

The following proposition gives necessary conditions for existence of a simple stable limit cycle, see Tsypkin (1984) and Åström (1993).

PROPOSITION 1
Consider the relay feedback system (1)–(2). If there exists a simple limit cycle with switch plane intersections $\pm x^*$ and period time $2h^*$, then

$$C(e^{Ah^*} + I)^{-1}(e^{Ah^*} - I)A^{-1}B = 0 \tag{9}$$

The limit cycle is stable if all eigenvalues of

$$\frac{dg}{dx}(x^*) = (I - \frac{(Ax^* + B)C}{C(Ax^* + B)})e^{Ah^*}$$

are in the open unit disc. □

Notice that the trivial solution $h^* = 0$ always satisfies (9). It is easy to show that this is the only solution for first-order and second-order processes with no zeros. Hence, these processes exhibit no simple limit cycles under relay feedback.

By letting

$$X_0 = \{x \in \mathcal{S} : CAx = 0\} \cup \{x \in \mathcal{S}_+ : |x| = R\}$$

for a sufficiently large R and studying the set recursion $X_k = g(X_{k-1})$, we get a numerical method for analyzing convergence to limit cycles. At least for systems of order three and less it is easy to visualize.

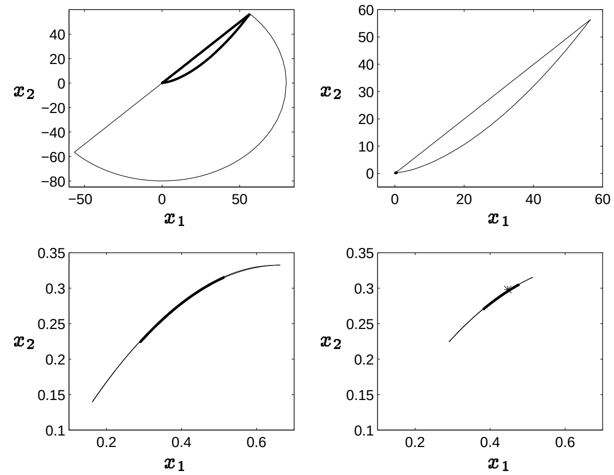

Fig. 8. Area contraction in Example 5.

EXAMPLE 5
Consider the process

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ -1 & -2 & 0 \\ 3 & -3 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x$$

Hence, $\mathcal{S} = \{x : x_3 = 0\}$ and $\mathcal{S}_+ = \{x \in \mathcal{S} : x_1 > x_2\}$. Let X_0 be a semicircle disc with radius 80. Figure 8 shows the contraction under four iterations. The first diagram shows X_0 together with X_1 (drawn with thicker lines), the second X_1 and X_2, etc. In the last diagram the fixed point $x^* = (0.45, 0.30, 0)$ is marked by an asterisk. The contraction is remarkably fast, in particular during the first two iterations. This agrees with the behavior noted also when using relay feedback in practice, see Åström and Hägglund (1984). □

By applying local stability analysis around x^* for a given system, it is possible to prove global convergence to a stable limit cycle with the method described above.

6. AREA CONTRACTION

For a class of processes, we are able to prove global area contraction.

DEFINITION 1
A subset $\mathcal{V} \subset \mathcal{S}_+$ is called *globally attractive*, if for all $x \in \mathcal{S}_+$ there exists a k such that $g^k(x) \in \mathcal{V}$.

If $g(x) \in \mathcal{V}$ for all $x \in \mathcal{V}$, then $\mathcal{V}$ is *invariant.*

The area of a set X is denoted by $\mathcal{A}(X) = \int dX$. The map $g = g(x) : \mathcal{S}_+ \to \mathcal{S}_+$ is called *area contractive* in

$\mathcal{U} \subset \mathcal{S}_+$, if there exists a constant $\rho \in [0, 1)$ such that

$$\mathcal{A}(g(X)) \leq \rho \mathcal{A}(X)$$

for all $X \subset \mathcal{U}$. □

Area contraction is possible to show for a class of third-order processes including the one in Example 5.

THEOREM 2
Let A, B, C satisfy

$$C(sI - A)^{-1}B = \frac{k}{(s+\lambda_1)(s+\lambda_2)(s+\lambda_3)}, k, \lambda_i > 0 \quad (10)$$

Then the switch plane intersections of all solutions of (1)–(2) converge to a region with vanishing area. □

The main step in the proof of Theorem 2 is the following lemma, which gives area contraction a geometric interpretation. The complete proofs are given in Johansson and Rantzer (1996).

LEMMA 1
Assume $A + A^T < 0$. If $CB = 0$, then the map g in (3) is area contractive in every invariant compact subset of

$$\mathcal{U} := \{x \in \mathcal{S}_+ : B^T Ax \leq 0\}$$

Proof sketch: Consider the switch plane intersection $x \in \mathcal{U}$ with surrounding ball

$$\mathcal{B}_\varepsilon(x) := \{z \in \mathcal{U} : |z - x| \leq \varepsilon\}$$

Let $\Phi(t, \mathcal{B})$ denote the set $\mathcal{B}$ after time t following the dynamics $\dot{x} = Ax - B$. The trajectories intersecting $\mathcal{B}_\varepsilon(x)$ pass through the hyperplanes $\mathcal{N}_v(x) := \{z : v^T(z - x) = 0\}$. In particular, define

$$H_-(x, \mathcal{B}_\varepsilon(x)) := \{\mathcal{N}_{Ax-B}(x) \cap \Phi(t, \mathcal{B}_\varepsilon(x)), t \in \mathcal{I}\}$$
$$H_+(x, \mathcal{B}_\varepsilon(x)) := \{\mathcal{N}_{Ax+B}(x) \cap \Phi(t, \mathcal{B}_\varepsilon(x)), t \in \mathcal{I}\}$$

where $\mathcal{I}$ is a small interval around zero. By introducing the angle of incidence $\alpha(x)$ and the angle of refraction $\theta(x)$ by

$$\cos\alpha(x) = \frac{C(Ax + B)}{|C||Ax + B|}, \qquad \cos\theta(x) = \frac{C(Ax - B)}{|C||Ax - B|}$$

we have

$$\mathcal{A}(H_-(x, \mathcal{B}_\varepsilon(x))) = \frac{\cos\theta(x)}{\cos\alpha(x)}\mathcal{A}(H_+(x, \mathcal{B}_\varepsilon(x))) + \mathcal{O}(\varepsilon)$$

The area is thus decreasing when the trajectories pass the switch plane $\mathcal{S}$. Furthermore, since $A + A^T < 0$ implies $\|\exp At\| < 1$, this area decreases even more outside $\mathcal{S}$. Finally, we notice that it is possible to cover any compact set by a sufficiently large number of balls $\mathcal{B}_\varepsilon$. □

7. CONCLUSIONS

The problem of oscillations in linear systems under relay feedback has been addressed. Some heuristics were given, and it was shown that also for third-order systems several important phenomena can arise. It was shown that the existence of arbitrarily fast switches and sliding modes can be detected from the high-frequency asymptote of the Nyquist curve. The second part of the paper introduced a method for global analysis of relay feedback systems based on the Poincaré map between switch plane intersections. In particular, for a certain class of processes it was proved that the intersection points converge to a region with vanishing area.

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