Markov Chain Approximations for Deterministic Control Problems with Affine Dynamics and Quadratic Cost in the Control[1]

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June 12, 1997

[1]This research was supported in part by the National Science Foundation (NSF-DMS-9403820), the Army Research Office (DAAH04-96-1-0075) and by the Office of Naval Research (ONR-N00014-96-1-0276). Part of this research took place while the second author was a Visiting Fellow of the Faculty for Engineering and Information Technology at The Australian National University. Support through the ANU 1995 Faculties Research Grants Scheme is gratefully acknowledged.

Abstract

We consider the construction of Markov chain approximations for an important class of deterministic control problems. The emphasis is on the construction of schemes that can be easily implemented and which possess a number of highly desirable qualitative properties. The class of problems covered is that for which the control is affine in the dynamics and with quadratic running cost. This class covers a number of interesting application areas, including problems that arise in large deviations, risk–sensitive and robust control, robust filtering, and certain problems from computer vision. Examples are given, as well as a proof of convergence.

1 Introduction

There are a number of deterministic optimal control problems for which a global approximation to the value function is needed. For example, in small noise risk-sensitive and robust nonlinear filtering [10, 15], the optimal (robust) filter is defined in terms of the value function for a calculus of variations problem in which the variational integrand is a functional of the observation process. Other examples occur in computer vision and related areas [6, 17], where the optimization problem appears as a representation for an unknown shape or other geometric object, and in large deviations, where the solution is used to construct an approximation to an invariant distribution [9]. In these examples the representation usually takes the form of a calculus of variations problem, but in some cases a differential game representation is required [6]. In other problems a global approximation is needed so that an approximation to the optimal control can be defined at all points in the state space. Example from this category include some of the recent formulations of robust nonlinear control. Although for these problems the value function is usually defined in terms of a differential game, in some cases it can also have a representation in terms of an optimal control problem.

For the great majority of these problems one cannot obtain explicit solutions, and thus numerical approximations are required. A standard method for the construction and analysis of approximations in optimal control is the Markov chain approximation method [12, 13, 14]. This method is based on directly approximating the controlled process that appears in the variational problem by a finite state controlled Markov chain. An analogous minimal cost is defined for this controlled chain, and since the state space is finite, implementation on a computer is possible. Since it is the variational problem itself that is approximated, rather than a solution to some intermediate partial differential equation (PDE) characterization, the method is especially useful for problems where the corresponding PDE is not well defined or is difficult to work with. For example, in many of the problems listed above there can be multiple classical and viscosity solutions for the corresponding PDE formulations. When the associated PDEs are well defined (in, say, a viscosity sense [8]) it is often the case that the Markov chain approximation is equivalent to a finite difference approximation of the PDE [14, Chapter 14]. Thus the construction of efficient and implementable Markov chain approximations is of interest even for problems that originate directly from modeling via PDEs (rather than as control problems). Prior work that

considers some of the special features of the Markov chain method in the context of first order problems includes [2, 6, 14]. In some of these cases the probabilistic interpretation of the algorithm is essential, and in all cases the interpretation is useful in understanding the qualitative properties of the algorithm.

There are many standard cost structures for which a proof of convergence of the Markov chain approximation as the discretization tends to zero is well known [12, 14]. The "local consistency" conditions that are assumed of the approximating chain (cf. (3.5)-(3.6) below) are important because they give what can be considered to be the minimal requirements for convergence. A number of techniques for constructing approximating chains that satisfy these conditions can be found in [14]. It is often the case, however, that one is interested in chains that do more than just converge. Within the class of chains that satisfy the local consistency conditions, one might wish to demand certain qualitative or quantitative properties. The purpose of the present paper is to construct approximating chains which possess three such highly desirable properties. We restrict our attention to an important class of deterministic optimal control problems: those with dynamics that are affine in the control and cost that is quadratic in the control.

This paper is organized as follows. In the next section we define a representative optimization problem. Although the approximations we construct have application to a wide variety of cost structures (discounted, average cost per unit time, etc.) and boundary conditions (Neumann, Dirichlet, etc.), it is best to present the approximations in the context of a specific case. The problem introduced in Section 2 will also be used to illustrate convergence issues, which are considered in detail in Section 5. Section 3 presents a list of properties one would like a numerical scheme to satisfy, and Section 4 shows how to construct approximations with these properties. The discussion of convergence in Section 5 is followed by examples of computations in Section 6. The Appendix states and proves a result on existence and uniqueness of a minimizer for the ratio of a convex and a linear function which is used in Section 4.

2 A Representative Problem

To fix the ideas and also to provide a canonical example for the convergence proof of Section 5, it is useful to have a representative control problem or PDE in mind. For these purposes we have chosen an escape time problem.

The chains we will construct are much more broadly applicable, since the only germane part of the problem formulation will be the form of the dynamics and the running cost. In particular, these chains can also be used for discounted problems, infinite time problems that do not involve discounting (e.g., variational problems for large deviation approximation of invariant measures [9]) and constrained dynamics.

The representative problem is as follows. Let $G \subset \mathbb{R}^n$ be open with compact closure. It is assumed that G satisfies uniform interior and exterior cone conditions: there exist $\delta > 0$ and continuous functions $v_1(\cdot)$ and $v_2(\cdot)$ such that for all $x \in \partial G$

$$\left(\cup_{s\in(0,\delta)}\{y : \|x + sv_1(x) - y\| < \delta s\}\right) \subset G$$

and

$$\left(\cup_{s\in(0,\delta)}\{y : \|x + sv_2(x) - y\| < \delta s\}\right) \subset G^c.$$

The only reason for using this condition, rather than the assumption of a C^∞ boundary, is to include cases where G is the intersection of a finite number of sets with smooth boundary (such as a cube).

Given $x \in \mathbb{R}^n$, we consider the dynamics

$$\dot{\phi} = b(\phi) + \sigma(\phi)v, \qquad \phi(0) = x$$

for $t \in [0, \tau]$, where v is square integrable over any interval $[0, T]$, $T < \infty$, and τ is the first exit time of ϕ from G. Associated with these dynamics is the cost

$$\bar{J}(x, v) \doteq \frac{1}{2}\int_0^\tau \left[\|v(t)\|^2 + c(\phi(t))\right] dt + g(\phi(\tau)). \tag{2.1}$$

For the construction of approximations it suffices to assume that $b(\cdot)$, $\sigma(\cdot)$, $c(\cdot)$ and $g(\cdot)$ are continuous on $\mathbb{R}^n$. Further regularity conditions will be stated later when we discuss convergence (see Condition 5.1).

At this point we distinguish two cases. In the first case, which we call the *nondegenerate* case, v takes values in $\mathbb{R}^n$ and $\sigma(\cdot)$ is assumed to be an $n \times n$ matrix with $a(\cdot) \doteq \sigma(\cdot)\sigma'(\cdot)$ uniformly positive definite on $\bar{G}$. By defining

$$L(x, u) \doteq \frac{1}{2}\langle u - b(x), a^{-1}(x)\,(u - b(x))\rangle + c(x),$$

the dynamics and the cost above can be rewritten in the alternative (calculus of variations) form

$$\dot{\phi} = u, \quad \phi(0) = x, \tag{2.2}$$

$$J(x,u) = \int_0^\tau L(\phi(t), u(t))dt + g(\phi(\tau)). \tag{2.3}$$

In the second case, the *degenerate* case, v takes values in $\mathbb{R}^d$, with $1 \leq d < n$. Thus the control directly affects only a subset of the state variables. We can assume that the state components have been ordered so that if ϕ is written in the form (ϕ_1, ϕ_2) with ϕ_1 and ϕ_2 having dimension d and $n-d$ respectively, then the control directly affects the evolution of ϕ_1. This partition is also applied to $b(\cdot)$ and $\sigma(\cdot)$: for all $x \in \mathbb{R}^n$ we write

$$b(x) = \begin{pmatrix} b_1(x) \\ b_2(x) \end{pmatrix}, \quad \sigma(x) = \begin{pmatrix} \sigma_{11}(x) & \sigma_{12}(x) \\ \sigma_{21}(x) & \sigma_{22}(x) \end{pmatrix}.$$

In this degenerate case it is assumed that the $d \times d$ matrix $a_{11}(\cdot) \doteq \sigma_{11}(\cdot)\sigma_{11}'(\cdot)$ is uniformly positive definite on $\bar{G}$, and that σ_{12}, σ_{21} and σ_{22} are all identically zero. Hence the dynamics are given by

$$\begin{aligned} \dot{\phi}_1 &= b_1(\phi) + \sigma_{11}(\phi)v \\ \dot{\phi}_2 &= b_2(\phi). \end{aligned}$$

With the definition

$$L(x,u) \doteq \begin{cases} \frac{1}{2}\langle u_1 - b_1(x), a_{11}^{-1}(x)\,(u_1 - b_1(x))\rangle + c(x), & \text{if } u_2 = b_2(x) \\ \infty & \text{if } u_2 \neq b_2(x), \end{cases}$$

the dynamics and the cost for the degenerate case also take the form (2.2)–(2.3).

Note that a more general cost for the original control problem could also give rise to a calculus of variations problem of the same form. In particular, in place of $\|v\|^2$ in (2.1) one could consider a positive definite quadratic form (e.g. $\langle v - h(x), A(x)(v - h(x))\rangle$) and still obtain a calculus of variations problem from within the class described above.

We are interested in numerically approximating the minimal cost

$$V(x) = \inf_u J(x,u), \tag{2.4}$$

where the infimum is taken over all square integrable $\mathbb{R}^n$-valued controls. To make the connection with the associated class of PDEs, we note here that

under some regularity conditions $V(\cdot)$ can be characterized as the unique viscosity solution [8] to

$$H(x, DV(x)) = 0 \text{ for } x \in G, \qquad V(x) = g(x) \text{ for } x \in \partial G.$$

Here $DV(\cdot)$ denotes the gradient of $V(\cdot)$ and

$$H(x, p) \doteq \inf_u \left[\langle u, p\rangle + L(x, u)\right].$$

3 Numerical Approximation: Desirable Qualitative Properties

The basic idea of the Markov chain approximation method is to construct a discrete time, discrete state controlled Markov chain, such that an appropriate continuous time interpolation of this chain is close in the sense of distributions to the solution to $\dot{\phi} = b(\phi) + \sigma(\phi)v$ or $\dot{\phi} = u$, depending on which description of the problem is used. Once such an approximation has been found, a cost function which is analogous to that of the original control problem is defined for the stochastic model. The original minimal cost is approximated by the minimal cost for the stochastic model. The latter is characterized in terms of its associated dynamic programming equation (DPE), which can then be solved via one of many possible iterative schemes [14, Chapter 6]. Although these methods were originally developed with stochastic control problems as the intended area of application, they also work quite well for deterministic problems when a global approximation to the value function is desired.

To cover both cases simultaneously, we temporarily use the more general notation $\dot{\phi} = b(\phi, u)$ for the dynamics and $\Lambda(x, u)$ for the running cost. The approximating chain will always take values in some finite grid, and in this paper we will just consider the case $G^h \doteq h\mathbb{Z}^n \cap G$. One can certainly consider more general classes of grids and still construct chains with all the desired properties [14, Chapters 5 & 6]. In particular, the grid spacing need not be the same in different coordinate directions. However, we will not pursue such generalizations here.

Suppose that the chain is denoted by $\{X_i^h\}$ and its controlled transition probabilities by $p^h(x, y|u)$. A function $\Delta t^h(x, u)$ is used to define the continuous time interpolation in a sense made precise below. The fundamental

requirements on p^h and Δt^h are the "local consistency" conditions [14]

$$\begin{aligned} E\left[X_1^h - X_0^h | X_0^h = x, u_0^h = u\right] &= \sum_{y \in h\mathbb{Z}^n} (y-x)\, p^h(x, y|u) \\ &= b(x,u)\Delta t^h(x,u) + o(\Delta t^h(x,u)), \quad (3.5) \\ \operatorname{cov}\left[X_1^h - X_0^h | X_0^h = x, u_0^h = u\right] &= o(\Delta t^h(x,u)). \quad (3.6) \end{aligned}$$

In terms of the continuous time interpolation to be defined below, (3.5) states that when applying the control u, the average increment per unit of continuous time is approximately $b(x,u)$. Equality (3.6) implies that the variance around this mean tends to zero as $h \to 0$. For all the chains we construct the equality

$$E\left[X_1^h - X_0^h | X_0^h = x, u_0^h = u\right] = b(x,u)\Delta t^h(x,u)$$

will hold, and we will use this stronger version as our definition of local consistency from now on. Because we are considering dynamics that are unbounded, some care must be taken in specifying the convergence properties of the error terms $o(\Delta t^h(x,u))$. The precise conditions that are needed are stated in Section 4.

Given a sequence of feedback controls $\{u_i^h\}$, the distribution of the conditioned and controlled Markov chain $\{X_i^h\}$ is determined by p^h and $X_0^h = x$ (precise definitions are given in Section 5). The continuous time interpolation $X^h(\cdot)$ is defined by

$$X^h(t) = X_i^h \quad \text{if } t \in \left[\sum_{j=0}^{i} \Delta t^h(X_j^h, u_j^h), \sum_{j=0}^{i+1} \Delta t^h(X_j^h, u_j^h)\right).$$

The controls are interpolated in the same way (with interpolation $u^h(\cdot)$), and the approximation of the value function is defined to be

$$\begin{aligned} V^h(x) &\doteq \inf_{\{u_i^h\}} E_x\left[\sum_{i=0}^{N^h-1} \Lambda(X_i^h, u_i^h)\Delta t^h(X_i^h, u_i^h) + g(X_{N^h}^h)\right] \\ &= \inf_{u^h} E_x\left[\int_0^{\tau^h} \Lambda(X^h(t), u^h(t))dt + g(X^h(\tau^h))\right], \end{aligned}$$

where $N^h \doteq \min\{i : X_i^h \notin G\}$, $\tau^h \doteq \inf\{t : X^h(t) \notin G\}$, and E_x denotes expectation conditioned on $X_0^h = x$. The infimum is over all feedback controls.

Under suitable conditions on the rest of the problem data, weak convergence arguments can be used to prove that $V^h \to V$ uniformly on $\bar{G}$ (see Section 5 and [14]). The approximation V^h is characterized by the DPE

$$V^h(x) = \inf_u \left[\sum_{y \in G^h} p^h(x, y|u) V^h(y) + \Lambda(x, u) \Delta t^h(x, u) \right] \tag{3.7}$$

and the boundary condition $V^h(x) = g(x)$ for $x \notin G^h$. By considering the stationary problem as a limit of finite time problems, one is naturally led to the iteration

$$V^h_{n+1}(x) = \inf_u \left[\sum_{y \in G^h} p^h(x, y|u) V^h_n(y) + \Lambda(x, u) \Delta t^h(x, u) \right] \tag{3.8}$$

(plus $V^h_{n+1}(x) = g(x)$ for $x \notin G^h$). In this iteration (known as a Jacobi iteration), only the values of $V^h_n(x)$, $x \in G^h$, are used to compute the value of each $V^h_{n+1}(x)$ for $x \in G^h$. Alternatively, one can use a Gauss-Seidel iteration, where each newly calculated value of $V^h_{n+1}(x)$ is successively substituted in (3.8) in place of $V^h_n(x)$. For further discussion of these methods and other related accelerated procedures we refer the reader to [14, Chapter 6].

The book [14] presents a number of techniques for constructing approximating chains. However, little is known about how to construct chains which simultaneously possess the following highly desirable qualitative properties.

1. The first property is minimal numerical diffusion or dissipativity. This is achieved by requiring that the support of $p^h(x, \cdot|u)$ be contained in a small convex cone for each x and u. In the present setting that basically means that for each x and u, $p^h(x, \cdot|u)$ should be supported in an orthant with corner at x. Local consistency then implies that given any coordinate vector e_i, if $\langle e_i, u \rangle \geq 0$ then $\langle e_i, y - x \rangle \geq 0$ for all y in the support of $p^h(x, \cdot|u)$. We will refer to a chain with this property as "one–sided".

   To understand the behavior of a chain without this property, consider one whose transition function is given by $(1 - h)p^h(x, y) + hq^h(x, y)$, where p^h is as above and q^h is the transition function of a standard random walk on $h\mathbb{Z}^n$. With a small modification of the interpolation interval Δt^h the local consistency conditions remain valid. However, the presence of the random walk component perturbs the behavior,

acting like a small second order term in the PDE and like an approximation to a Wiener process in the stochastic control problem. Although the effect of this term disappears in the limit $h \to 0$, it can significantly degrade the performance of the algorithm. For example, as is well known, it will obscure detail in the neighborhood of discontinuities of $DV(\cdot)$. In addition, the rate of convergence of the iterative schemes used to solve equations such as (3.8) is much slower. For example, the spectral radius of the matrix $\{p^h(x, y|u), x \in G^h, y \in G^h\}$ appearing in (3.8) for each fixed control is often zero if the chain is one–sided, but nonzero otherwise.

2. The second property is $p^h(x, x|u) = 0$ for all x and $u \neq 0$. This corresponds to making $\Delta t^h(x, u)$ large, and if one recalls the interpretation of the iterative schemes for solving the DPE as limits in n of finite time problems, this ensures that the iterative solvers for the DPE converge rapidly. This property is extremely important near points where the minimizing trajectory has derivative zero, because each iteration of the discrete algorithm corresponds to a very large continuous time interval in the interpolated process.

3. The final property involves ease of implementation: all the infima required to evaluate the right hand side of iterative solvers such as (3.8) should have an analytic solution. The motivation here is that it is simply not practical in large problems to numerically solve for the minima in (3.8) for every grid point and every iteration.

It is usually difficult to construct chains which possess all three of these properties plus the local consistency. Note that the feasibility of such a construction is tied to the description of the control problem. In our first description of the representative example of Section 2 the dynamics were relatively complicated but the running cost was simple. The reverse was true for the calculus of variations description. There is an ambiguity present in many control problems which allows one to consider a variety of descriptions, and then select the one that is best suited for numerical approximation. The same is true of the related PDE. For example, the value function of a minimum escape time problem can be represented either in terms of a control problem with a very simple running cost (identically equal to 1) and complicated dynamics (due to the constraint on the control space), or in terms of a problem with a more complicated quadratic running cost but with linear

dynamics. The availability of several descriptions was used to great advantage in [6]. One description was used to define a numerical approximation and construct iterative solvers that were fast and easy to implement, while another description was found to be more convenient for proving the convergence of these approximations. It is the authors' experience that if one is interested in constructing approximations with all the properties listed above, then it is best to use a description in which the dynamics are as simple as possible. For this reason, the calculus of variations problem serves as the starting point for our development of approximations. The choice of simple dynamics rather than simple cost is perhaps not surprising, since it is usually only the dynamics that are approximated, with the running cost left more or less as is.

Remark 3.1 In the actual algorithms we use for the computations of Section 6, $V^h_{n+1}(x)$ is given by

$$\inf_u \left[\sum_{y \in G^h} p^h(x, y|u) V^h_n(y) + \Lambda(x, u) \Delta t^h(x, u) \right] \wedge B,$$

where B is any upper bound for $V(x)$. The additional minimization guarantees that $V^h_n(x)$ is monotonically nonincreasing in n, and may be represented in the control problems (discrete and continuous) by a stopping control with stopping cost B [14]. Since B is larger than $V(\cdot)$ on $\bar{G}$, this control does not alter the value of $V(x)$, since it is never invoked by any optimal trajectory. We omit this extra control and cost in order to keep the exposition simple and since it plays no role in the construction of the approximating chain. The convergence proof in Section 5 can easily be extended to include this case.

In the next section we construct schemes with all the aforementioned properties for deterministic problems with dynamics that are affine in the control and with cost that is quadratic in the control.

4 Construction of the Approximations

We next state in precise terms the local consistency conditions for the particular dynamics $\dot{\phi} = u$: for each $x \in G^h$ (recall $G^h = G \cap h\mathbb{Z}^n$)

$$\sum_{y \in G^h} (y - x) p^h(x, y|u) = u \Delta t^h(x, u), \tag{4.9}$$

$$\sum_{y\in G^h}\Big((y-x)-u\Delta t^h(x,u)\Big)\Big((y-x)-u\Delta t^h(x,u)\Big)' p^h(x,y|u) = o(\|u\|\Delta t^h(x,u)). \tag{4.10}$$

Since the dynamics are independent of x, we can and will assume $\Delta t^h(x,u)$ is also independent of x. The term $o(\|u\|\Delta t^h(u))$ is then assumed to hold uniformly for $u \in I\!R^n$.

There are many different approximating chains which satisfy conditions (4.9) and (4.10). We will restrict our attention to the following choices for the transition probabilities and the interpolation interval because they are the simplest which are locally consistent and obviously satisfy the first two properties discussed in the previous section. Let $t^+ = t \vee 0$ and $t^- = -(t \wedge 0)$. We define

$$p^h(x,y|u) = \begin{cases} u_i^{\pm} / \sum_j |u_j| & \text{if } y = x \pm he_i, \\ 0 & \text{otherwise,} \end{cases}$$

and $\Delta t^h(u) = h/\|u\|_1$, where $\|u\|_1 \doteq \sum_j |u_j|$. For the case $u = 0$ we set

$$p^h(x,y|0) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

The sequence $\Delta t^h(0) > 0$ can be chosen arbitrarily (for the purposes of satisfying (4.9) and (4.10)) and for simplicity we take $\Delta t^h(0) = h$. Condition (4.9) holds trivially, and using the bound $h^2 = h\|u\|\Delta t^h(u)$ for $u \neq 0$ for each entry on the left hand side of (4.10), it is easily verified that the second condition is also satisfied.

The given choice of transition probabilities and interpolation interval also satisfies the third property discussed in Section 3, namely, that the infima which appear in iterative solvers for the DPE have explicit solutions. The rest of this section is dedicated to verifying this statement.

When solving the DPE (3.7) by an iterative solver, the infima that must be calculated at each iteration depend on the procedure chosen (e.g. Jacobi, Gauss-Seidel). However, for all the standard iterative schemes the quantities that must be computed are always of the form

$$\inf_{u\in I\!R^n}\left\{\Lambda(x,u)\Delta t^h(u) + \sum_{i\in\{1,\ldots,n\},\pm} p^h(x, x\pm he_i|u) f(x\pm he_i)\right\},$$

where f is a real valued function defined on the grid $hZ\!\!\!Z^n$ which depends on the particular iterative method used. We replace the running cost $\Lambda(x,u)$

by $L(x,u)$, where $L(\cdot,\cdot)$ was defined for the nondegenerate and degenerate cases in Section 2. For any $x \in G^h$ and for the given p^h and Δt^h, the infimization for the nondegenerate case becomes

$$\begin{aligned}\inf_{u\neq 0}\left\{\frac{h}{\|u\|_1}L(x,u)+\sum_{i\in\{1,\ldots,n\},\pm}\frac{u_i^{\pm}}{\|u\|_1}f(x\pm he_i)\right\}\\ \wedge\left\{\frac{h}{2}\langle b(x),a^{-1}(x)b(x)\rangle+hc(x)+f(x)\right\}.\end{aligned}\tag{4.11}$$

The second term in the expression above corresponds to the case $u=0$. In the degenerate case, this term is included only if $b_2(x)=0$, in which case $\langle b(x),a^{-1}(x)b(x)\rangle$ is replaced by $\langle b_1(x),a_{11}^{-1}(x)b_1(x)\rangle$.

The evaluation of (4.11) can be carried out by considering a number of constrained infima. To illustrate this, let us consider the problem in three dimensions. Except at the origin, in each of the 8 octants defined by the coordinate directions $p^h(x,y|u)$ and $\Delta t^h(u)$ are smooth functions of u. Thus it is reasonable to infimize over each octant separately and then take the minimum among the obtained infima. Within each octant, the calculation of the infimum requires that one consider a number of unconstrained and constrained infimization problems. However, it turns out that all of these problems can be formulated in terms of one basic constrained minimization problem. In the next two subsections, we will state and solve this problem for the nondegenerate and degenerate cases, respectively. In the third subsection we describe how the solution to this particular problem is used to compute (4.11).

4.1 The nondegenerate case

Let A be a positive definite, real, symmetric $k\times k$ matrix, let B and β be vectors in $\mathbb{R}^k$, and let $\alpha\in(0,\infty)$, $\gamma\in[0,\infty)$ be given. Define $\mathbf{1}$ to be the transpose of the k-dimensional vector $(1,\ldots,1)$.

Lemma 4.1 *Consider the problem of infimizing*

$$\frac{1}{\langle w,\mathbf{1}\rangle}\left\{\frac{\alpha}{2}\langle w-B,A^{-1}(w-B)\rangle+\langle w,\beta\rangle+\gamma\right\}\tag{4.12}$$

over $w\in\mathbb{R}^k$ satisfying $\langle w,\mathbf{1}\rangle>0$. Define

$$s\doteq\Big(\langle A\mathbf{1},\beta\rangle-\alpha\langle B,\mathbf{1}\rangle\Big)^2-\langle A\mathbf{1},\mathbf{1}\rangle\Big[\langle\beta,A\beta\rangle-2\gamma\alpha-2\alpha\langle B,\beta\rangle\Big].$$

If $s \geq 0$ then the minimum of (4.12) *equals*

$$\Gamma \doteq \frac{1}{\langle A\mathbf{1}, \mathbf{1}\rangle}\Big[-\alpha\langle B, \mathbf{1}\rangle + \langle A\mathbf{1}, \beta\rangle + \sqrt{s}\ \Big], \tag{4.13}$$

and when $s > 0$ it is uniquely achieved at

$$w = B - \frac{A}{\alpha}\left[\beta + \left(\frac{\alpha\langle B, \mathbf{1}\rangle - \langle A\mathbf{1}, \beta\rangle - \sqrt{s}}{\langle A\mathbf{1}, \mathbf{1}\rangle}\right)\mathbf{1}\right].$$

If $s < 0$ then the infimum of (4.12) *equals $-\infty$.*

Proof. We first consider the constrained minimization problem

$$\min_{w \in \mathbb{R}^k} \frac{1}{\mu}\left[\frac{\alpha}{2}\langle w - B, A^{-1}(w - B)\rangle + \langle w, \beta\rangle + \gamma\right] \tag{4.14}$$

subject to $\langle w, \mathbf{1}\rangle = \mu > 0$. We associate a multiplier λ with the constraint, differentiate the Lagrangian, and equate the gradient to zero to obtain the system of equations

$$A^{-1}(w - B)\alpha + \beta + \lambda\mathbf{1} = 0.$$

Using the constraint to identify λ, the solution

$$w = B - \frac{A}{\alpha}\left(\beta + \left(\frac{\alpha\langle B, \mathbf{1}\rangle - \langle A\mathbf{1}, \beta\rangle - \alpha\mu}{\langle A\mathbf{1}, \mathbf{1}\rangle}\right)\mathbf{1}\right)$$

is then substituted into (4.14) to obtain the function

$$\frac{1}{\langle A\mathbf{1}, \mathbf{1}\rangle}\left\{\frac{\alpha\mu}{2} + \frac{s}{2\alpha\mu} + \langle A\mathbf{1}, \beta\rangle - \alpha\langle B, \mathbf{1}\rangle\right\},$$

which is to be infimized over $\mu > 0$. Clearly, if $s < 0$ then the infimum is equal to $-\infty$. If $s \geq 0$ then the derivative with respect to μ has two roots, of which only the positive root, $\mu^* \doteq \sqrt{s}/\alpha$, satisfies the constraint on μ and gives a local minimum, which is in fact a global minimum. Substituting μ^* in the expression for w gives the form of the minimizer, and substituting the minimizer in (4.12) yields the minimum value expressed in (4.13). ■

4.2 The degenerate case

For $k \in \{2,3,\ldots\}$ let $\ell \in \{1,\ldots,k-1\}$. Let A be a positive definite, real, symmetric $\ell \times \ell$ matrix, B and β be vectors in $\mathbb{R}^k$, and $\alpha \in (0,\infty)$, $\gamma \in [0,\infty)$ be given. We partition B, β, $\mathbf{1}$ and w as in the degenerate case described in Section 2, writing $B = (B_1, B_2)$, $\beta = (\beta_1, \beta_2)$, $\mathbf{1} = (\mathbf{1}_1, \mathbf{1}_2)$ and $w = (w_1, w_2)$, where B_1, β_1, $\mathbf{1}_1$ and w_1 are ℓ dimensional and B_2, β_2, $\mathbf{1}_2$ and w_2 are $k-\ell$ dimensional.

Lemma 4.2 *Consider the problem of infimizing*

$$\frac{1}{\langle w, \mathbf{1}\rangle}\left\{\frac{\alpha}{2}\langle w_1 - B_1, A^{-1}(w_1 - B_1)\rangle + \langle w, \beta\rangle + \gamma\right\} \tag{4.15}$$

over $w \in \mathbb{R}^k$ satisfying $\langle w, \mathbf{1}\rangle > 0$ and $w_2 = B_2$. Define

$$s = \Big(\langle A\mathbf{1}_1, \beta_1\rangle - \alpha\langle B, \mathbf{1}\rangle\Big)^2 - \langle A\mathbf{1}_1, \mathbf{1}_1\rangle\Big[\langle \beta_1, A\beta_1\rangle - 2\gamma\alpha - 2\alpha\langle B, \beta\rangle\Big].$$

If $s \geq 0$ then the minimum of (4.15) equals

$$\Gamma \doteq \frac{1}{\langle A\mathbf{1}_1, \mathbf{1}_1\rangle}\Big[-\alpha\langle B, \mathbf{1}\rangle + \langle A\mathbf{1}_1, \beta_1\rangle + \sqrt{s}\,\Big],$$

and when $s > 0$ it is uniquely achieved at

$$w = B - \frac{A}{\alpha}\left[\beta_1 + \left(\frac{\alpha\langle B, \mathbf{1}\rangle - \langle A\mathbf{1}_1, \beta_1\rangle - \sqrt{s}}{\langle A\mathbf{1}_1, \mathbf{1}_1\rangle}\right)\mathbf{1}_1\right].$$

If $s < 0$ then the infimum of (4.15) equals $-\infty$.

Proof. A proof for all $s \in \mathbb{R}$ follows the same steps as the proof for the nondegenerate case and is omitted. Alternatively, for $s \neq 0$ we can view the degenerate problem as the limit of a sequence of nondegenerate problems. To this end, define the $k \times k$ matrix

$$\bar{A}(\delta) \doteq \begin{pmatrix} A & 0 \\ 0 & \delta I \end{pmatrix},$$

where $\delta > 0$ and I is the $k-\ell$ dimensional identity matrix. Define $s(\delta)$, $w(\delta)$ and $\Gamma(\delta, s(\delta))$ as indicated in the statement of Lemma 4.1, but with the matrix A there replaced by $\bar{A}(\delta)$. Note that $s(\delta)$ and $w(\delta)$ are continuous

for $\delta \geq 0$, and $\Gamma(\delta, s(\delta))$ is continuous for $\delta \geq 0$ and $s(\delta) \neq 0$. Using the convergence

$$\bar{A}(\delta) \downarrow \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

it easily follows that if $s > 0$ the minimizing w for the degenerate problem equals the limit $w(0)$, and that the given expressions for w, Γ and s are valid. Similarly, if $s < 0$ then the infimum is equal to the limit $\Gamma(0, s) = -\infty$. ■

4.3 A procedure for solving (4.11)

As noted previously, iterative solvers such as Jacobi and Gauss–Seidel all take the same general form (4.11). The method for solving this infimization problem involves solving it in each orthant of $\mathbb{R}^n$ and then minimizing over all orthants.

To illustrate the necessary calculations in each orthant, we consider the nondegenerate case with $u_i \geq 0, i = 1, \ldots, n$. All other cases differ only in notation. The resulting constrained minimization is

$$\inf_{\{u: u_i \geq 0, u \neq 0\}} \left[\frac{1}{\langle u, \mathbf{1} \rangle} \left\{ hL(x, u) + \sum_i u_i f(x + he_i) \right\} \right].$$

To simplify notation, we omit the dependence of the functions $b(x)$, $\sigma(x)$ and $c(x)$ on x and write K in place of the n-vector $(f(x+he_1), \ldots, f(x + he_n))$. With these changes in notation the problem takes the form

$$\inf_{\{u: u_i \geq 0, u \neq 0\}} \left[\frac{1}{\langle u, \mathbf{1} \rangle} \left\{ \frac{h}{2} \langle u - b, a^{-1}(u - b) \rangle + hc + \langle u, K \rangle \right\} \right]. \tag{4.16}$$

The solution to (4.16) will depend on whether or not b or c is 0, and thus we consider the two cases separately.

The case $b = 0$ and $c \leq 0$. In this case the infimum in (4.16) is achieved in the limit $u \to 0$. We first note that if $b = 0$ and $c < 0$ then the infimum in (4.16) is $-\infty$. Since for points satisfying this condition we would also have $V(x) = -\infty$, we need not consider this situation. If $c = 0$, the quadratic term in the minimization tends to zero faster than $\langle u, \mathbf{1} \rangle$ as u tends to the origin. Because of this, the infimum will be obtained only in the limit $u \to 0$. In fact, if the minimum of K_j over $j \in \{1, \ldots, n\}$ is attained at i, then the infimum in (4.16) equals K_i and is obtained by letting $u = \delta e_i$ and then sending $\delta \downarrow 0$.

The case $b \neq 0$ or $c > 0$. In this case, the infimum in (4.16) is achieved away from zero. A key fact that is needed is that there is only one local minimum for this constrained problem, which equals the global minimum. This follows from Lemma 3.2 in [4]. To keep the presentation self contained, a proof of this fact is given in the Appendix (Lemma 7.2).

The procedure used to compute (4.16) is as follows. The first step is to consider the relative interior of the orthant, temporarily ignoring the nonnegativity constraints on the u_i (although we still assume $\langle u, \mathbf{1}\rangle > 0$). This allows the application of Lemma 4.1. If the infimum of this problem is finite, the nonnegativity constraints $u_i \geq 0$ are tested for the candidate minimizer. If these constraints are satisfied, then the unique local minimum has been found. If there is no candidate minimizer, or if the nonnegativity constraints are not satisfied, the unique local minimum must be on the boundary of the orthant. Then the next step will be to search the $n-1$ dimensional faces. Fortunately, the minimizations constrained to each of these can also be done by applying Lemma 4.1. For instance, let $u_n = 0$. Just as when the interior of the orthant was under consideration, the first step is to compute the minimum as if the rest of the variables were unconstrained (save the constraint $u_1 + \cdots + u_{n-1} > 0$). The solution of this lower dimensional problem requires a change of variables. Define $\tilde{u} = (u_1, \ldots, u_{n-1})'$, $\tilde{b} = (b_1, \ldots, b_{n-1})'$, $\tilde{K} = (K_1, \ldots, K_{n-1})'$ and $\bar{b} = b_n$. We can then write (with matrices $A_{11}, A_{12}, A_{21}, A_{22}$ of appropriate dimension)

$$\begin{aligned}
&\frac{h}{2\langle u, \mathbf{1}\rangle}\langle u-b, a^{-1}(u-b)\rangle + \frac{hc}{\langle u, \mathbf{1}\rangle} + \frac{\langle u, K\rangle}{\langle u, \mathbf{1}\rangle} \\
&\quad = \frac{h}{2\langle \tilde{u}, \mathbf{1}\rangle}\left\langle \begin{pmatrix} \tilde{u}-\tilde{b} \\ -\bar{b} \end{pmatrix}, \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \tilde{u}-\tilde{b} \\ -\bar{b} \end{pmatrix}\right\rangle + \frac{hc}{\langle \tilde{u}, \mathbf{1}\rangle} + \frac{\langle \tilde{u}, \tilde{K}\rangle}{\langle \tilde{u}, \mathbf{1}\rangle} \\
&\quad = \frac{1}{\langle \tilde{u}, \mathbf{1}\rangle}\left[\frac{h}{2}\langle \tilde{u}-\tilde{b}, A_{11}(\tilde{u}-\tilde{b})\rangle + \langle \tilde{u}, \tilde{K} - hA_{12}\bar{b}\rangle + \frac{h}{2}\langle \bar{b}, A_{22}\bar{b} + 2A_{21}\tilde{b}\rangle + hc\right].
\end{aligned}$$

Lemma 4.1 can be applied once more to compute the minimum of this quantity. If the solution satisfies the constraints $u_i \geq 0$, $i = 1, \ldots, n-1$, and if the gradient of the original objective function points into the n-dimensional orthant, then the search in the orthant is complete. If not, then the search continues through the remaining faces, and if necessary, through the lower dimensional faces. For example, if $n = 3$ and the minimum over the orthant $\{u_1 \geq 0, u_2 \geq 0, u_3 \geq 0\}$ is not found in the interior, the search continues on the faces $\{u_1 \geq 0, u_2 \geq 0, u_3 = 0\}, \{u_1 \geq 0, u_3 \geq 0, u_2 = 0\}, \{u_2 \geq 0, u_3 \geq 0, u_1 = 0\}$. If the search there fails as well, one continues with the faces

$\{u_1 \geq 0, u_2 = u_3 = 0\}$, etc. Lemma 7.2 guarantees that the procedure will find the unique global minimum.

The procedure outlined above is easy to implement on a computer. In one and two dimensions the implementation is quite efficient since one can easily calculate explicit constraints on the data $f(x \pm e_i)$ which automatically locate the face in which the global minimum lies. For higher dimensional problems, the implementation can be made more efficient in various ways. For example, the algorithm can first test the face where the minimum was found the last time the computation was performed at that grid point and for that orthant.

5 Proof of Convergence: The Representative Problem

In this section we give the proof of convergence as $h \to 0$ for the approximations constructed in the last section. The setting is that of the representative problem of Section 2. Although this problem is similar to problems considered in [14], the particular linear/quadratic structure of the present setting allows for a simpler proof. As in the construction of the chain, we will consider the calculus of variations formulation.

We have two goals for the presentation. The first is to show how to handle the unboundedness of the control space and the time interpolations. In particular, we will see how the uniformity assumed in the formulation of local consistency is used. The second is to give the general arguments in the clearest way possible, so that they can be adapted more easily to other problems. To achieve the second goal, we will restrict to the nondegenerate case with $g \equiv 0$. Comments on the additional conditions needed for more general cases are given after the proof.

Although there is some overlap, the assumptions generally fall into two categories: those needed to establish the proper convergence properties of the interpolated controlled chains, and those which are used to guarantee regularity properties of the limit problem. Since the convergence analysis is often very much the same for different cost structures, it is worth noting that the only condition we use in proving convergence of the processes is a lower bound of the form $L(x,u) \geq c_1\|u\|^2/2 - c_2$ on the running cost. While this is implied by the assumptions that we now state, it also holds under much weaker conditions.

Condition 5.1 *The functions $c(\cdot), b(\cdot)$, and $a(\cdot)$ are continuous on $\mathbb{R}^n$, with $c(x) > 0$ for all $x \in \mathbb{R}^n$ and $a(x) > 0$ in the sense of symmetric $n \times n$ positive definite matrices for all $x \in \mathbb{R}^n$. The set G is open with compact closure and satisfies a uniform interior and exterior cone condition. The function g is identically zero.*

In order to simplify the presentation we have assumed that $c(\cdot) > 0$. Under this condition, the nonlinear iteration given in (3.8) (or the corresponding iteration for other iterative methods) is a contraction for each feedback control with finite total cost. In the absence of such a lower bound for the running cost, (3.8) need not be a contraction (even under the optimal control) and therefore the DPE does not uniquely characterize the approximation V^h. Discussion on how to deal with this non-uniqueness can be found in [6] and in [14, Chapter 13].

We next note an important property of $V(\cdot)$ that follows from Condition 5.1. Let $\mathcal{U}$ be the set of all measurable functions $u : [0, \infty) \to \mathbb{R}^n$ such that $\int_0^T \|u(s)\|^2 ds < \infty$ for all $T \in [0, \infty)$. The following lemma is easy to prove under the nondegeneracy assumption.

Lemma 5.2 *Assume Condition* 5.1 *and define $V(x)$ for $x \in \bar{G}$ by* (2.4). *Let $\delta > 0$ be given. There exist $\Delta > 0$ and $\bar{u}^\delta \in \mathcal{U}$, such that if $\bar{\phi}^\delta$ and $\bar{\tau}^\delta$ are defined by $\bar{\phi}^\delta(t) = x + \int_0^t \bar{u}^\delta(s) ds$ and $\bar{\tau}^\delta \doteq \inf\{t : \bar{\phi}^\delta(t) \notin G\}$, then the following properties hold.*

1. *$\bar{u}^\delta$ is continuous on intervals of the form $[i\Delta, i\Delta + \Delta)$, with limits from the left.*
2. *$\bar{\phi}^\delta(t) \notin \bar{G}$ for $t \in (\bar{\tau}^\delta, \bar{\tau}^\delta + \Delta)$.*
3. *The cost for $\bar{u}^\delta$ is within δ of the infimum:*

$$\int_0^{\bar{\tau}^\delta} L(\bar{\phi}^\delta(t), \bar{u}^\delta(t)) dt \leq V(x) + \delta.$$

We next introduce the additional notation required to state the convergence properties of controls and associated controlled processes. Let $p^h(x, y|u)$ and $\Delta t^h(u)$ be the transition function and interpolation intervals constructed in Section 4. These satisfy the local consistency conditions (4.9) and (4.10). It will be convenient to have the controls defined in a nice way even after the escape time. To achieve this, we enlarge the class of controls to include controls that do not anticipate the future. More precisely,

the control u_i^h that is applied at time i is allowed to be a Borel measurable function of all the past values $\{X_j^h, j = 0, ..., i\}$. Each control of this class will be called an *admissible control*. It is well known [16] that for the approximating problem the infimum over admissible controls equals the infimum over feedback controls. However, if we use the class of admissible controls then we can assume without loss of generality that the controls are defined even after the escape time N^h in a manner that maintains a bounded running cost. This is done simply by setting $u_i^h \doteq u^*$ if $X_j^h \notin G^h$ for any $j \in \{0, ..., i\}$, where u^* is any value for which $L(x, u^*)$ is uniformly bounded over $x \in I\!R^n$. In this way we can assume that $\{X_i^h\}$ and $\{u_i^h\}$ are well defined for all $i \in I\!N_0$. The continuous time interpolation $X^h(\cdot)$ is defined by

$$X^h(t) \doteq X_i^h \ \text{ if } t \in \left[t_i^h, t_{i+1}^h\right), \ \text{where } t_i^h \doteq \sum_{j=0}^{i-1} \Delta t^h(u_j^h).$$

The controls are interpolated in the same way:

$$u^h(t) \doteq u_i^h \ \text{ if } t \in \left[t_i^h, t_{i+1}^h\right).$$

Finally, the approximation to the value function is defined to be

$$\begin{aligned} V^h(x) &\doteq \inf_{\{u_i^h\}} E_x \left[\sum_{i=0}^{N^h-1} L(X_i^h, u_i^h) \Delta t^h(u_i^h) \right] \\ &= \inf_{u^h} E_x \left[\int_0^{\tau^h} L(X^h(t), u^h(t)) dt \right], \end{aligned} \tag{5.17}$$

where $N^h \doteq \min\{i : X_i^h \notin G\}$, $\tau^h \doteq \inf\{t : X^h(t) \notin G\} = t_{N^h}^h$, E_x denotes expectation conditioned on $X_0^h = x$, and the infimum is over all admissible controls.

Let $\kappa^h \doteq \sum_{j=0}^{\infty} \Delta t^h(u_j^h)$. Owing to our convention regarding the definition of the control after the escape time, $\kappa^h < \infty$ is only possible if $N^h = \infty$. Recall that $L(\cdot, \cdot)$ is uniformly bounded from below away from zero. This implies that if

$$E_x \left[\int_0^{\tau^h} L(X^h(t), u^h(t)) dt \right] < \infty,$$

then $N^h < \infty$ w.p.1 and therefore $\kappa^h = \infty$ w.p.1. Thus without loss of generality we can restrict to controls which satisfy this property and assume $\kappa^h = \infty$. This will be done without explicit mention in the sequel.

To deal with the convergence properties of the controls, it is convenient to introduce the following representation for the approximation. For each $t \in [0,\infty)$ let $\nu_t^h(du) \doteq \delta_{u^h(t)}(du)$, where δ_u is the probability measure that places probability 1 at the point u. We also define a measure on $\mathbb{R}^n \times [0,\infty)$ by $\nu^h(A \times B) \doteq \int_B \nu_t^h(A)dt$ for Borel subsets $A \subset \mathbb{R}^n$ and $B \subset [0,\infty)$. In terms of this measure we can represent the approximation (5.17) as

$$V^h(x) = \inf_u E_x \left[\int_{\mathbb{R}^n \times [0,\tau^h)} L(X^h(t),u)\nu^h(du \times dt) \right]. \tag{5.18}$$

The processes $\{X^h, h \in (0,1)\}$ take values in $\mathcal{D}([0,\infty) : \mathbb{R}^n)$, the space of $\mathbb{R}^n$-valued functions that are continuous from the right and have limits from the left. We consider this space together with the Skorokhod metric, under which it is a complete separable metric space [7]. If a sequence in $\mathcal{D}([0,\infty) : \mathbb{R}^n)$ converges to a continuous function under this metric, then the convergence is actually uniform on each compact subset of $[0,\infty)$. The random sequence $\{\nu^h\}$ takes values in the space $\mathcal{R}(\mathbb{R}^n \times [0,\infty))$, the set of all Borel measures ν on $\mathbb{R}^n \times [0,\infty)$ that satisfy $\nu(\mathbb{R}^n \times [0,T]) = T$ for all $T \in [0,\infty)$. This space can be metrized as a complete separable metric space, with a metric such that $\nu^h \to \nu$ if and only the restriction of ν^h to $[0,T]$ converges weakly to the restriction of ν to $[0,T]$ for all $T \in [0,\infty)$ [3]. Note that the second marginal of any measure $\nu \in \mathcal{R}(\mathbb{R}^n \times [0,\infty))$ is Lebesgue measure, and thus for each measure in this space we have the decomposition $\nu(du \times dt) = \nu_t(du)dt$, where ν_t is a probability measure for each $t \in [0,\infty)$. If ν is a random variable, then this decomposition can be done so that it holds almost surely and so that for all $t \in [0,\infty)$ ν_t is also a random variable.

We note the following property of convergence in $\mathcal{R}(\mathbb{R}^n \times [0,\infty))$. Let $f : \mathbb{R}^n \times [0,\infty)$ be bounded and continuous, and assume that $\nu^h \to \nu$. For any $t \in [0,\infty)$, the set $\{\mathbb{R}^n \times \{t\}\}$ has measure zero under ν, and thus for all such t

$$\int_{\mathbb{R}^n \times [0,t]} f d\nu^h \to \int_{\mathbb{R}^n \times [0,t]} f d\nu. \tag{5.19}$$

The analogous version of Fatou's Lemma holds, as does convergence under the assumption that f is uniformly integrable (rather than bounded) with respect to $\{\nu^h\}$ on each set $\mathbb{R}^n \times [0,T], T \in (0,\infty)$ [3, A.3.11]. These facts will be used without explicit mention in the sequel.

We next state a basic convergence result. For terminology we refer the reader to [3, 14].

Lemma 5.3 *For each $h > 0$ let $x^h \in G^h$, and assume that $x^h \to x \in G$ as $h \to 0$. Assume also that for all $T < \infty$ there exists $K_T < \infty$ such that*

$$\limsup_{h \to 0} E_{x^h} \left[\int_{\mathbb{R}^n \times [0,T]} L(X^h(t), u) \nu^h(du \times dt) \right] \leq K_T. \tag{5.20}$$

Then $\{\nu^h, h \in (0,1)\}$ is tight. Let h index a subsequence that converges in distribution to a limit ν taking values in $\mathcal{R}(\mathbb{R}^n \times [0,T])$. In terms of the almost sure decomposition $\nu(du \times dt) = \nu_t(du)dt$ we have

$$\int_0^T \int_{\mathbb{R}^n} \|u\|^2 \nu_t(du)dt < \infty \tag{5.21}$$

w.p.1 for all $T < \infty$, and thus the process

$$X(t) \doteq x + \int_0^t \int_{\mathbb{R}^n} u \, \nu_s(du)ds \tag{5.22}$$

is well defined. Along the same subsequence for which we have the convergence $\nu^h \to \nu$ in distribution, we also have the convergence $X^h \to X$ in distribution.

Proof. Since the topologies on $\mathcal{D}([0,\infty) : \mathbb{R}^n)$ and $\mathcal{R}(\mathbb{R}^n \times [0,\infty))$ are equivalent to convergence in the appropriate restricted metric on $[0,T]$ for each $T < \infty$, it suffices to prove the conclusion of the lemma restricted to a fixed but arbitrary finite interval $[0,T]$. The standard diagonalization argument then gives the result as stated. Thus we fix $T \in [0,\infty)$, and to simplify the exposition we take $T = 1$. The proof for $T \neq 1$ differs only in notation.

We have assumed that $a(\cdot)$ is continuous on $\mathbb{R}^n$. Since G is bounded, we can assume without loss of generality that $a(\cdot)$ is also uniformly bounded above on $\mathbb{R}^n$. In this case, there exist $c_1 > 0$ and $c_2 < \infty$ such that

$$L(x,u) \geq \frac{c_1}{2}\|u\|^2 - c_2$$

for all $u \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Together with (5.20), this implies

$$\limsup_{h \to 0} E_{x^h} \left[\int_{\mathbb{R}^n \times [0,1]} \|u\|^2 \nu^h(du \times dt) \right] \leq 2(K_1 + c_2)/c_1. \tag{5.23}$$

Consider the collection $\mathcal{A}(M)$ of probability measures ν on $\mathbb{R}^n \times [0,1]$ such that

$$\int_{\mathbb{R}^n \times [0,1]} \|u\|^2 \nu(du \times dt) \leq M.$$

By Chebyshev's inequality

$$\nu(\{u : \|u\| \geq B\} \times [0,1]) \leq M/B^2$$

for all $\nu \in \mathcal{A}(M)$. It follows from Prohorov's Theorem that $\mathcal{A}(M)$ is precompact. Since Fatou's Lemma implies that $\mathcal{A}(M)$ is closed, $\mathcal{A}(M)$ is actually compact for each $M < \infty$.

We now apply Chebyshev's inequality to (5.23). Consequently, there exists $h_0 \in (0,1)$ such that for each $h \in (0, h_0)$

$$P_{x^h}\left\{\nu^h \notin \mathcal{A}(M)\right\} \leq \frac{2(K_1 + c_2)}{c_1 M}.$$

A second application of Prohorov's Theorem shows that $\{\nu^h, h \in (0,1)\}$ is tight.

We must also consider the convergence of $\{X^h, h \in (0,1)\}$. Instead of focusing directly on X^h, we consider the sequence $\{e^h\}$ defined by

$$e^h(t) \doteq X^h(t) - x^h - \int_{I\!R^n \times [0,t_i^h]} u\, \nu^h(du \times ds) \text{ if } t \in [t_i^h, t_{i+1}^h). \tag{5.24}$$

Define the σ-algebra $\mathcal{F}_t^h \doteq \sigma\left(X_i^h, u_i^h : t_i^h \leq t\right) = \sigma\left(X_i^h : t_i^h \leq t\right)$. The local consistency condition (4.9) then implies that $\{e^h\}$ is a vector valued $\mathcal{F}_t^h-$martingale. Let θ^h satisfy $1 \in [t_{\theta^h}^h, t_{\theta^h+1}^h)$. Using (4.10), the submartingale inequality [11, Theorem 3.8], and a standard conditioning argument, we obtain for any $\varepsilon > 0$

$$\begin{aligned}
P_{x^h}\left\{\sup_{t\in[0,1]} \|e^h(t)\| \geq \varepsilon\right\} &\leq \frac{1}{\varepsilon^2} E_{x^h}\left[\|e^h(1)\|^2\right] \\
&= \frac{1}{\varepsilon^2} E_{x^h}\left[\|e^h(t_{\theta^h}^h)\|^2\right] \\
&= \frac{1}{\varepsilon^2} E_{x^h}\left[\left\|\sum_{i=0}^{\theta^h-1}\left(X_{i+1}^h - X_i^h - u_i^h \Delta t^h(u_i^h)\right)\right\|^2\right] \\
&\leq \frac{1}{\varepsilon^2} E_{x^h}\left[\sum_{i=0}^{\theta^h-1}\left\|X_{i+1}^h - X_i^h - u_i^h \Delta t^h(u_i^h)\right\|^2\right] \\
&\leq \frac{h}{\varepsilon^2} E_{x^h}\left[\sum_{i=0}^{\theta^h-1}\|u_i^h\|\Delta t^h(u_i^h)\right] \\
&\leq \frac{h}{\varepsilon^2} E_{x^h}\left[\int_{I\!R^n\times[0,1]}\|u\|\nu^h(du\times dt)\right].
\end{aligned}$$

Since (5.23) implies $E_{x^h}\left[\int_{\mathbb{R}^n\times[0,1]}\|u\|\nu^h(du\times dt)\right]$ is uniformly bounded for all small $h>0$, we conclude that $\sup_{t\in[0,1]}\|e^h(t)\|$ converges in distribution to zero.

We now put these facts together to finish the proof of the lemma. The tightness of $\{\nu^h, h\in(0,1)\}$ implies that there is a subsequence along which we have the convergence in distribution $\nu^h\to\nu$. Fix any $t\in[0,1]$. It follows from the convergence in distribution and the uniform integrability implied by (5.23) that [cf. (5.19)]

$$\int_{\mathbb{R}^n\times[0,t]} u\,\nu^h(du\times ds)\to\int_{\mathbb{R}^n\times[0,t]} u\,\nu(du\times ds)$$

in distribution. We claim that more holds, and that in fact

$$\int_{\mathbb{R}^n\times[0,\cdot]} u\,\nu^h(du\times ds)\to\int_{\mathbb{R}^n\times[0,\cdot]} u\,\nu(du\times ds) \tag{5.25}$$

in distribution in the space $\mathcal{C}([0,1]:\mathbb{R}^n)$. This will follow immediately from Prohorov's Theorem once we verify that $\{\int_{\mathbb{R}^n\times[0,\cdot]} u\,\nu^h(du\times ds), h\in(0,1)\}$ is tight in $\mathcal{C}([0,1]:\mathbb{R}^n)$. The required tightness follows from Ascoli's Theorem and the following bound: for all sufficiently small $h\in(0,1)$

$$\begin{aligned}
&P_{x^h}\left\{\sup_{\{0\le s<r\le 1: r-s\le\delta\}}\left\|\int_{\mathbb{R}^n\times[s,r]} u\,\nu^h(du\times dt)\right\|\ge\varepsilon\right\}\\
&\le\ \frac{1}{\varepsilon}E_{x^h}\left[\sup_{\{0\le s<r\le 1: r-s\le\delta\}}\left\|\int_{\mathbb{R}^n\times[s,r]} u\,\nu^h(du\times dt)\right\|\right]\\
&\le\ \frac{\delta^{1/2}}{\varepsilon}E_{x^h}\left[\int_{\mathbb{R}^n\times[0,1]}\|u\|^2\nu^h(du\times dt)\right]\\
&\le\ \frac{2\delta^{1/2}(K_1+c_2)}{\varepsilon c_1}.
\end{aligned}$$

Now Fatou's Lemma and (5.23) imply (5.21), and thus the process X as given by (5.22) is well defined. Along the subsequence for which ν^h converges to ν in distribution we also have the convergence in distribution of e^h to the zero process. Finally, we observe that the bound

$$\begin{aligned}
&\left\| \int_{I\!R^n \times [0,t]} u\, \nu^h(du \times ds) - \int_{I\!R^n \times [0,t_i^h]} u\, \nu^h(du \times ds) \right\| \\
&\quad \le \left\| \int_{I\!R^n \times [t_i^h,t]} u \nu^h(du \times ds) \right\| \\
&\quad \le \|u_i^h\| \Delta t^h(u_i^h) \\
&\quad \le h
\end{aligned}$$

holds for $t \in [t_i^h, t_{i+1}^h)$. Thus by (5.24)

$$X^h(t) = x^h + \int_{I\!R^n \times [0,t]} u\, \nu^h(du \times ds) + e^h(t) + O(h),$$

where $O(h)$ is uniform w.p.1. We take the limit as $h \to 0$ in the last display and use (5.25) to obtain the convergence in distribution $\sup_{t\in[0,1]} \|X^h(t) - X(t)\| \to 0$. ■

We next state the convergence theorem. When we assert that $V^h(x) \to V(x)$, what is really meant is

$$\liminf_{\eta \downarrow 0} \liminf_{h \to 0} \inf_{y \in G^h : \|y-x\| \le \eta} V^h(y) \ge V(x)$$

and

$$\limsup_{\eta \downarrow 0} \limsup_{h \to 0} \sup_{y \in G^h : \|y-x\| \le \eta} V^h(y) \le V(x).$$

Theorem 5.4 *Assume Condition* 5.1 *and define* $V(\cdot)$ *and* $V^h(\cdot)$ *by* (2.4) *and* (5.17) *respectively. Then for all* $x \in G$, $V^h(x) \to V(x)$ *as* $h \to 0$.

Proof. We split the proof into the upper and lower bounds.

Proof of the lower bound. For $x \in G$ consider any sequence $x^h \to x$ as $h \to 0$, with each $x^h \in G^h$. We can extract a subsequence such that $V^h(x^h) \to A$ for some value A in the extended reals. To prove the lower bound

$$\liminf_{\eta \downarrow 0} \liminf_{h \to 0} \inf_{y \in G^h : \|y-x\| \le \eta} V^h(y) \ge V(x),$$

it suffices to show that $A \ge V(x)$. We can assume $A < \infty$ since otherwise there is nothing to prove. Fix $\delta > 0$, and for each h let $\{u_i^h\}$ be a sequence of controls that is within δ of the infimum in (5.17). We recall our convention

that the controls are extended past the exit time in such a manner that the running cost per unit of continuous time is uniformly bounded. Thus $A < \infty$ and the representation for V^h given in (5.18) implies that the assumption (5.20) of Lemma 5.3 is satisfied. By extracting a further subsequence we can assume the convergence properties given in Lemma 5.3 apply, and also that τ^h converges in distribution to a random variable $\bar{\tau}$ that takes values in the compactified space $[0, \infty]$. It is easy to verify from the fact that the convergence in distribution $X^h \to X$ is with respect to the uniform metric that w.p.1 $\bar{\tau} \geq \tau$, where $\tau \doteq \inf\{t : X(t) \notin G\}$. We obtain the following series of inequalities, each line of which is explained after the display:

$$\begin{aligned}
A + \delta &= \lim_{h \to 0} V^h(x^h) + \delta \\
&\geq \liminf_{h \to 0} E_{x^h} \int_0^{\tau^h} L(X^h(t), u^h(t)) dt \\
&= \liminf_{h \to 0} E_{x^h} \int_{\mathbb{R}^n \times [0, \tau^h)} L(X^h(t), u) \nu^h(du \times dt) \\
&\geq E_x \int_{\mathbb{R}^n \times [0, \bar{\tau})} L(X(t), u) \nu(du \times dt) \\
&\geq E_x \int_{\mathbb{R}^n \times [0, \tau)} L(X(t), u) \nu(du \times dt) \\
&\geq E_x \int_0^{\tau} L(X(t), \dot{X}(t)) dt \\
&\geq V(x).
\end{aligned}$$

The first equality is due to the definition of A; the second line follows from the representation (5.17) and the fact that the controls are $\delta-$optimal; the third line is obtained by representing the cost in terms of ν^h; the fourth uses the convergence along the subsequence asserted in Lemma 5.3 and a version of Fatou's Lemma that is appropriate for convergence in distribution [1, Theorem 5.3]; the fifth line follows from the nonnegativity of $L(\cdot, \cdot)$ and the fact that $\bar{\tau} \geq \tau$ w.p.1; the sixth line uses Jensen's inequality and the relation (5.22); finally, the definition of $V(x)$ gives the last line. Sending $\delta \to 0$ completes the proof of the lower bound.

Proof of the upper bound. For $\delta > 0$ let $\Delta > 0, \bar{u}^\delta, \bar{\phi}^\delta$ and $\bar{\tau}^\delta$ be defined as in Lemma 5.2. The control $\bar{u}^\delta$ must be adapted for use on the approximating chain. The definition is given recursively as follows. Suppose that the control $\{u_j^h\}$ has been defined for all $j < i$ as a function of j only (i.e., there is no

direct dependence on X_j^h). Let $t_i^h \doteq \sum_{j=0}^{i-1} \Delta t^h(u_j^h)$. If $t_i^h \le \bar{\tau}^\delta + \Delta$ then we define $u_i^h \doteq \bar{u}^\delta(t_i^h)$. If $t_i^h > \bar{\tau}^\delta + \Delta$ then we define all such controls in such a manner that the running costs per unit of continuous time are all uniformly bounded.

Consider again $x \in G$ and any sequence $x^h \to x$ with each $x^h \in G^h$. We wish to show

$$\limsup_{\eta \downarrow 0} \limsup_{h \to 0} \sup_{y \in G^h : \|y-x\| \le \eta} V^h(y) \le V(x).$$

Using the definition of $V^h(x^h)$ as an infimum over controls and the representation (5.17), it suffices to show that for the controls constructed above

$$\limsup_{h \to 0} E_{x^h} \int_0^{\tau^h} L(X^h(t), u^h(t))dt \le V(x) + \delta. \tag{5.26}$$

Once again, we rewrite these costs in terms of the measures ν^h.

If $V(x) = \infty$ there is nothing to prove. If $V(x) < \infty$, then the assumption (5.20) of Lemma 5.3 is satisfied. It follows from the definition of the control sequence that

$$\delta_{u^h(t)}(du) \Rightarrow \delta_{\bar{u}^\delta(t)}(du)$$

for almost every $t \in [0, \bar{\tau}^\delta + \Delta)$. Using the properties of $\bar{\phi}^\delta$ stated in Lemma 5.2, and in particular the property that $\bar{\phi}^\delta(t) \notin G$ for $t \in (\bar{\tau}^\delta, \bar{\tau}^\delta + \Delta)$, the convergence proved in Lemma 5.3 implies the subsequential convergence in distribution

$$(X^h, \nu^h, \tau^h) \to (X, \nu, \bar{\tau}^\delta),$$

where $\nu(du \times dt) = \delta_{\bar{u}^\delta(t)}(du)dt$ and hence $X(t) = \bar{\phi}^\delta(t)$ for $t \in [0, \bar{\tau}^\delta + \Delta)$. We now apply a version of the Lebesgue Dominated Convergence Theorem that is appropriate for convergence in distribution [1, Theorem 5.4] to obtain

$$\begin{aligned}
&\limsup_{h \to 0} E_{x^h} \int_0^{\tau^h} L(X^h(t), u^h(t))dt \\
&= \limsup_{h \to 0} E_{x^h} \int_{\mathbb{R}^n \times [0, \tau^h)} L(X^h(t), u)\nu^h(du \times dt) \\
&= \int_0^{\bar{\tau}^\delta} L(\bar{\phi}^\delta(t), u)\delta_{\bar{u}^\delta(t)}(du)dt \\
&= \int_0^{\bar{\tau}^\delta} L(\bar{\phi}^\delta(t), \bar{u}^\delta(t))dt \\
&\le V(x) + \delta.
\end{aligned}$$

The last line is due to the near optimality of $\bar{u}^\delta$ stated in Lemma 5.2. Since this is precisely (5.26), the proof is complete. ■

Remarks on the proof for other cases. We first consider the nondegenerate case, where the only generalization to consider is $g \neq 0$. In this case there will be an additional term $E_{x^h} g(X^h(\tau^h))$ in the prelimit cost and a term $g(\phi(\tau))$ in the limit cost. The only modification that is needed appears in the proof of the lower bound. In general, the escape time of the process X in (5.22) (denoted by τ) is no greater than the limit of the escape times of the approximations (denoted by $\bar{\tau}$) w.p.1. However, this inequality can be strict and hence we can have

$$E_{x^h} g(X^h(\tau^h)) \to E_x g(X(\bar{\tau})) \neq E_x g(X(\tau)).$$

More precisely, when $\bar{\tau} > \tau$ one cannot conclude from the definition of $V(x)$ that

$$\int_0^{\bar{\tau}} L(X(t), \dot{X}(t))dt + g(X(\bar{\tau})) \geq V(x), \tag{5.27}$$

and this is the source of the difficulty. However, if the last inequality can be shown to be valid w.p.1, then the lower bound will still hold. One way to prove (5.27) when $\bar{\tau} > \tau$ is to construct a trajectory Y near X whose total running cost is close to that of X and for which $\tau^* \doteq \inf\{t : Y(t) \notin G\}$ is close to $\bar{\tau}$ (rather than τ). Since (5.27) will be valid when (X, τ) is replaced by (Y, τ^*), the closeness of the total costs for X and Y will imply (5.27) as stated. One set of sufficient conditions for a perturbed trajectory to exist is that G be star shaped with respect to some point and that an interior cone condition hold. In this situation, the required perturbation of X will be towards the interior of G until the time of exit from $\bar{G}$.

Next we consider the degenerate case with $g = 0$. The necessary modification now comes in the proof of the upper bound. Without the nondegeneracy assumption, Lemma 5.2 does not apply, and therefore the existence of a smooth nearly optimal trajectory that exits the domain in a nontangential fashion is not guaranteed. In fact, if all nearly optimal trajectories exit in a tangential fashion then the value function is usually discontinuous at the starting point x, and one cannot get uniform convergence in a neighborhood of such a point. Thus, convergence will hold only for the points for which the conclusion of Lemma 5.2 holds. However, these points usually make up most of the domain, and for such points, the proof of the upper bound under the non-degeneracy assumption remains valid. Alternatively, if the exact lo-

cation of ∂G is not so critical, then one may use an alternative stopping criteria which guarantees the existence of a suitable nearly optimal control.

6 Numerical Experiments

In this section we present approximations obtained using the algorithm described in Section 4. It is appropriate at this point to remark on the role of the initial condition chosen for implementation.

The positive lower bound on the running cost implies that the DPE (3.7) has a unique solution. Although there is convergence to the unique solution regardless of the initial condition chosen for the algorithm, the speed of convergence depends heavily on this initial condition. To see why this is so, recall once more the interpretation of V_n^h as the minimal cost for an n-step optimal control problem with terminal cost V_0^h. Viewed as a terminal cost, a large value of V_0^h encourages the optimal control to move the chain towards the boundary (in order to avoid this cost). Thus, the boundary data can be learned and propagated back into the interior quickly. On the other hand, a small value of V_0^h gives no incentive for the chain to seek the boundary. The accumulated running cost eventually directs the process towards the boundary, but it may take a large number of iterations to do so and the convergence will be especially slow if the running cost can be near zero. With this in mind, we use large initial conditions when implementing the algorithm for all the examples that follow.

We note here two important features that are common to all of the examples in this section. First, regardless of the procedure chosen (Jacobi or Gauss-Seidel), convergence of the iterative scheme occurs after a low number of iterations, Moreover, when using a Gauss-Seidel procedure, the number of iterations needed for convergence is essentially independent of the number of grid points.

Example 1: Our first example considers a minimum escape time problem, which is a nondegenerate problem with zero drift. We consider escape from an open set $G \subset \mathbb{R}^n$.

As discussed earlier, there are different representations for the value function for this problem. On the one hand, one can take the running cost to be identically equal to 1, in which case the constraint on the control space gives a complicated description of the dynamics. On the other hand,

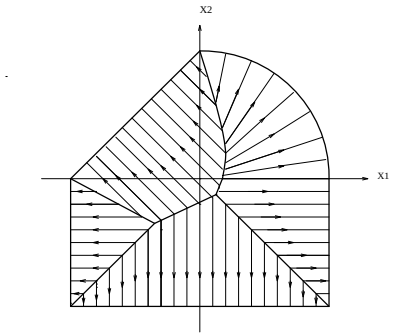

Figure 1: True controls

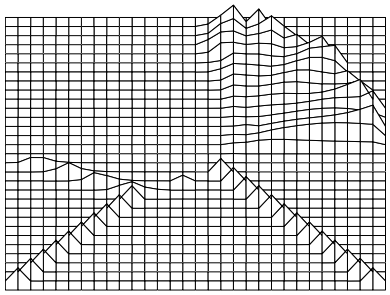

Figure 2: Errors

one can consider a quadratic running cost of the form

$$L(x,u) \doteq \frac{1}{4}\|u\|^2 + 1,$$

in which case the dynamics are simply $\dot{\phi} = u$. We adopt the latter representation since it is the one that is best suited for the numerical approximations described in Section 4. The PDE associated with this problem is

$$\begin{aligned} \|DV(x)\|^2 &= 1 \text{ for } x \in G, \\ V(x) &= 0 \text{ for } x \in \partial G. \end{aligned}$$

We first analyze a two-dimensional problem on the set

$$G \doteq \left\{ (x_1, x_2) \mid -1 < x_1 < 1,\ x_2 > -1,\ x_2 < \sqrt{1 - x_1^2},\ x_2 < 1 + x_1 \right\}. \quad (6.28)$$

The value function for this problem is defined over 5 different regions as shown in Figure 1. The derivative of the value function has discontinuities over the boundaries that separate these regions, resulting in sharp edges in the graph of the value function. As can be seen in Figure 3, the numerical approximation preserves the sharp corners of the figure. Nevertheless, the error in the approximation is highest at points where these sharp edges occur (Figure 2).

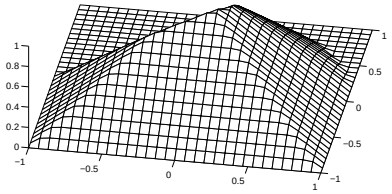

Figure 3: Approximated Value Function for (6.28)(n=30)

n	Iterations	Maximum error
5	8	0.6739E-01
10	14	0.6066E-01
15	20	0.3821E-01
20	26	0.3160E-01
25	32	0.2828E-01
30	38	0.2246E-01

Table 1: Minimum Distance on the set (6.28)
Jacobi Iteration

The approximation results are provided in Table 1. The leftmost column corresponds to the number of grid points on each half-axis. The same maximum errors were obtained with a Gauss-Seidel procedure with only 5 iterations irrespective of n. The iterative scheme was applied until the maximum difference between successive iterates was less than .001. The same stopping criterion is also used for the other examples discussed in this section.

The results for a Gauss-Seidel approximation to the same problem in three dimensions considered on the unit cube are given in Table 2. The number of iterations and the maximum error for the Jacobi iteration are very similar to those obtained in the previous two-dimensional problem. Because of the symmetry of the problem, the maximum error was always

obtained at the origin.

n	Iterations	Maximum error
5	7	.999084E-01
10	7	.499875E-01
15	7	.333250E-01
20	7	.249938E-01
25	7	.199950E-01
30	7	.166625E-01

Table 2: Minimum Distance on the unit cube
Gauss-Seidel Iteration

Example 2: [1] Let G be circle with radius 0.9 in $\mathbb{R}^2$ and consider the DPE

$$\inf_u \left[\langle u, DV(x) \rangle + \frac{1}{4}\|u\|^2 + c(x) \right] = 0 \text{ for } x \in G,$$

where

$$\begin{aligned} c(x) \doteq & \gamma^2(x_1,x_2)\left(\phi^2_{x_1}(x_1,x_2) + \phi^2_{x_2}(x_1,x_2)\right) \\ & + \phi^2(x_1,x_2)\left(\gamma^2_{x_1}(x_1,x_2) + \gamma^2_{x_2}(x_1,x_2)\right) \\ & + 2\gamma(x_1,x_2)\phi(x_1,x_2)\left(\gamma_{x_1}(x_1,x_2)\phi_{x_1}(x_1,x_2) + \gamma_{x_2}(x_1,x_2)\phi_{x_2}(x_1,x_2)\right). \end{aligned}$$

Here the functions $\gamma(\cdot,\cdot)$ and $\phi(\cdot,\cdot)$ are taken to be

$$\begin{aligned} \phi(x_1,x_2) &= \cos\left(\pi(x_1\cos\theta - x_2\sin\theta)/2\right)\cos\left(\pi(x_1\sin\theta - x_2\cos\theta)/2\right) \\ \gamma(x_1,x_2) &= (0.9 - |x_1\cos\theta - x_2\sin\theta|)(0.9 - |x_1\sin\theta + x_2\cos\theta|) \end{aligned}$$

with $\theta \in (0, \pi/2)$. The boundary conditions correspond to the true values of $V(x_1,x_2)$ which are known on the boundary of the circle. This problem is equivalent to solving (2.2) and (2.3) with running cost

$$L(x,u) \doteq \frac{1}{4}\|u\|^2 + c(x).$$

The results obtained are given in Table 3. The approximation is compared with the true solution (a product of cosines multiplied by a mollifier which kills the function on the boundary of the square):

$$V(x_1,x_2) = \gamma(x_1,x_2)\cdot\phi(x_1,x_2).$$

[1] M.B. wishes to acknowledge Adam Szpiro for suggesting this example.

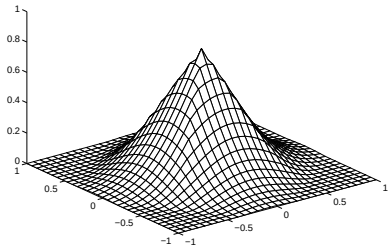

Figure 4: Killed Cosine Value Function

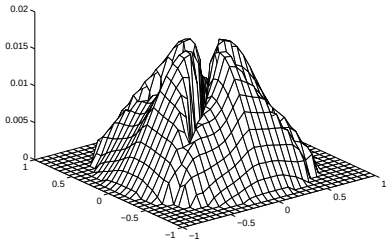

Figure 5: Errors for n=30

	Jacobi		Gauss-Seidel	
n	Iterations	Maximum error	Iterations	Maximum error
5	8	0.84439388E-01	5	0.84439388E-01
10	14	0.44254893E-01	5	0.44254893E-01
15	21	0.31896820E-01	5	0.31896820E-01
20	27	0.24439471E-01	5	0.24439471E-01
25	33	0.20843406E-01	5	0.20843406E-01
30	39	0.17684583E-01	5	0.17692354E-01
35	46	0.15333274E-01	5	0.15354953E-01

Table 3: Killed Cosine Value Function

Example 3: We now discuss a nondegenerate problem with non-zero drift. Let G be the open unit cube on $\mathbb{R}^3$ and consider the PDE

$$\|DV(x)\|^2 - \langle DV(x), b\rangle = c(x) \text{ for } x \in G$$

and boundary conditions

$$
\begin{aligned}
V(1,x_2,x_3) &= \tfrac{11+x_2^2+x_3^2}{2} - x_2x_3 + x_2 \\
V(-1,x_2,x_3) &= \tfrac{7+x_2^2+x_3^2}{2} - x_2x_3 + 3x_2 - 2x_3 \\
V(x_1,1,x_3) &= \tfrac{13+x_1^2+x_3^2}{2} + x_1x_3 - 2x_3 \\
V(x_1,-1,x_3) &= \tfrac{5+x_1^2+x_3^2}{2} + x_1x_3 + 2x_1 \\
V(x_1,x_2,1) &= \tfrac{7+x_1^2+x_2^2}{2} - x_1x_2 + 2x_1 + x_2 \\
V(x_1,x_2,-1) &= \tfrac{11+x_1^2+x_2^2}{2} - x_1x_2 + 3x_2.
\end{aligned}
$$

This is equivalent to solving (2.2) and (2.3) with running cost

$$L(x,u) \doteq \frac{1}{2}\langle u-b, a^{-1}(u-b)\rangle + c(x),$$

where

$$a = 2I, \qquad b = (-2,-2,-4)' \qquad \text{and} \qquad c(x) = 3(x_1 - x_2 + x_3)^2 + 8.$$

Results for a Gauss-Seidel iteration are given in Table 4. The maximum error is obtained by comparing the approximation with the true solution given by

$$V(x) = \frac{x_1^2 + x_2^2 + x_3^2}{2} - x_1x_2 + x_1x_3 - x_2x_3 + x_1 + 2x_2 - x_3 + 4.$$

n	Iterations	Maximum error
5	8	.384591E+00
10	8	.216769E+00
15	8	.150467E+00
20	8	.115211E+00
25	8	.933445E-01
30	8	.784566E-01
35	8	.676655E-01

Table 4: Example 3
Gauss-Seidel Iteration

Example 4: Our last example a concerns a variational problem which arises when considering the large deviation properties of a diffusion approximation to a phase locked loop model [5]. It is an example of a two-dimensional degenerate problem.

On the unit square in $\mathbb{R}^2$, consider the running cost:

$$L(x,u) \doteq \begin{cases} \frac{1}{2\beta^2}(u_1 - b_1(x))^2 + \gamma & \text{if } u_2 = -\pi x_1 \\ \infty & \text{otherwise,} \end{cases}$$

with

$$b_1(x) = -2\pi x_1 + \beta \sin \pi x_2$$

and $\gamma > 0$. This running cost corresponds to the dynamics

$$\begin{aligned} \dot{x}_1 &= -2\pi x_1 + \beta(\sin \pi x_2 + u) \\ \dot{x}_2 &= -\pi x_1. \end{aligned}$$

The "π" scaling above is convenient so that the set of interest can be taken to be the unit square. The associated PDE is given by

$$\begin{aligned} \inf_u \left[\langle u, DV(x)\rangle + L(x,u)\right] &= 0 \quad \text{for } x \in G \\ V(x) &= 0 \quad \text{for } x \in \partial G. \end{aligned}$$

Table 5 includes the results for a Gauss-Seidel approximation with $\gamma = 0.001$ and $\beta = 1$. In the table, we also record the successive differences for the approximations as a function of n in the rightmost column.

n	Iterations	V(0,0)	Successive Differences
60	10	0.25635858E+01	0.00000000
70	10	0.25620889E+01	0.14968709E-02
80	10	0.25609476E+01	0.11412458E-02
90	10	0.25600438E+01	0.90388511E-03
100	10	0.25592926E+01	0.75118958E-03
110	10	0.25586792E+01	0.61332968E-03
120	10	0.25581709E+01	0.50837769E-03
130	10	0.25577435E+01	0.42738899E-03
140	10	0.25573828E+01	0.36069404E-03

Table 5: Example 4 with $\beta = 1$, $\gamma = .001$
Gauss-Seidel Iteration.

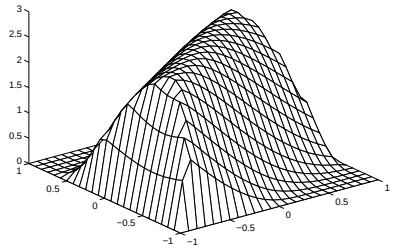

Figure 6: Approximated Value Function for $\beta = 1$ (n=60)

7 Appendix

All of the calculations of Section 4 turn on the solution to (4.16). Although this constrained minimization involves the ratio of a convex function and a linear function, it nonetheless retains some of the nice qualitative properties of constrained minima of convex functions. The following results are taken from [4].

For our purposes a function $f : \mathbb{R}^d \to \mathbb{R}$ is convex if f never takes the value $-\infty$, the set $\{x : f(x) < \infty\}$ is convex, f is lower semicontinuous and for any points $x, y \in \mathbb{R}^d$ at which f is finite and any $\alpha \in (0,1)$, $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$. A convex function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be *strictly convex* if for any points $x, y \in \mathbb{R}^d$ at which f is finite and any $\alpha \in (0,1)$, $f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha)f(y)$. All of the convex functions we consider will be strictly convex.

Lemma 7.1 *Let $g : \mathbb{R} \to \mathbb{R}$ be strictly convex and satisfy $g(x) = \infty$ for $x < 0$, $g(0) > 0$, and $\lim_{x\to\infty} g(x)/x = \infty$. Then there exists a point x which achieves the infimum in $\inf_{x\geq 0} g(x)/x$. This point is the unique solution to $g(x)/x \in \partial g(x)$, where $\partial g(x)$ denotes the set of subdifferentials of g at x, and thus is the unique local minimum of the map $x \to g(x)/x$.*

Proof. Since g is a convex function, there exists at least one point x at which $g(x) < \infty$. Consider the map $x \to g(x)/x$. Using the lower semicontinuity g and the conditions $\lim_{x\to\infty} g(x)/x = \infty$ and $g(0) > 0$ we see that $g(x)/x$

achieves its infimum on $[0,\infty)$ at some point in $(0,\infty)$. Standard calculus for convex functions shows that $g(x)/x \in \partial g(x)$ if and only if x is a local minimizer of $g(x)/x$.

Since we have shown there is at least one local minimizer (namely the global minimizer), the proof will be complete as soon as we show there is only one local minimizer. We argue by contradiction. Assume there are distinct x and y that satisfy $g(x)/x \in \partial g(x)$ and $g(y)/y \in \partial g(y)$. Without loss of generality we can assume $x < y$, and since $g(0) > 0$ we know that $x > 0$. There exists a point $z \in [x,y]$ and $w \in \partial g(z)$ such that

$$g(y) - g(x) = w(y - x).$$

We next use the fact that the subdifferentials of a strictly convex function are monotonic in the sense that $x < y$, $v_x \in \partial g(x)$ and $v_y \in \partial g(y)$ imply $v_x < v_y$. Hence there exist $A \geq 0$ and $B \geq 0$ with $A \wedge B > 0$ such that

$$\frac{g(x)}{x} = w - A, \frac{g(y)}{y} = w + B.$$

In terms of w, A and B, we can write

$$x(w - A) = g(x), \quad y(w + B) = g(y).$$

Subtracting these equalities and using $g(y) - g(x) = w(y - x)$, we obtain

$$(y - x)w + yB + xA = (y - x)w.$$

Since $yB + xA > 0$ this last equality is impossible, and so we conclude $x = y$. ■

We next apply this result to study (4.16) when either $c > 0$ or $b \neq 0$.

Lemma 7.2 *Consider the infimization problem* (4.16) *and assume that either* $c > 0$ *or* $b \neq 0$. *For any vector* K *there exists a unique local minimizer of* (4.16), *which is also the global minimizer.*

Proof. We define functions

$$K(u) \doteq h\left\{\frac{1}{2}(u - b)'a^{-1}(u - b) + c\right\} + \langle u, K\rangle$$

and

$$\tilde{K}(u) \doteq \begin{cases} K(u) & \text{if } u_i \geq 0 \text{ for } i = 1, ..., n, \\ \infty & \text{else.} \end{cases}$$

Since a^{-1} is positive definite, these functions are strictly convex. We also define a function k on $\mathbb{R}$ by $k(x) \doteq \infty$ if $x < 0$ and

$$k(x) \doteq \inf_{u:\langle u,\mathbf{1}\rangle = x} \tilde{K}(u).$$

It is easy to check that because k is defined as the minimum of a strictly convex function subject to a linear constraint, k is itself a strictly convex function. In addition, for any x satisfying $k(x) < \infty$ there is a unique u such that $\langle u, \mathbf{1}\rangle = x, u_i \geq 0, i = 1, ..., n$, and $k(x) = \tilde{K}(u) = K(u)$. The assumption $c > 0$ or $b \neq 0$ implies that $k(0) > 0$, and the superlinear growth any positive definite quadratic function implies $k(x)/x \to \infty$ as $x \to \infty$. The proof is now completed by applying the preceding lemma. ■

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