Gradient-Based Optimization of Markov Reward Processes: Practical Variants[1]

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Abstract

We consider a discrete time, finite state Markov reward process that depends on a set of parameters. In earlier work, we proposed a class of (stochastic) gradient descent methods that tune the parameters in order to optimize the average reward, using a single (possibly simulated) sample path of the process of interest. The resulting algorithms can be implemented online, and have the property that the gradient of the average reward converges to zero with probability 1. There is a drawback, however, in that the updates can have a high variance, resulting in slow convergence. In this paper, we address this issue and propose two approaches to reduce the variance which, however, introduce an additional bias into the update direction. We derive bounds for the resulting bias term and characterize the asymptotic behavior of the gradient of the average reward. For one of the approaches considered, the magnitude of the bias term exhibits an interesting dependence on the mixing time of the underlying Markov chain. We use a call admission control problem to illustrate the performance of one of the algorithms.

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1 Introduction

We consider a discrete time, finite state Markov reward process in which the transition probabilities and one-stage rewards depend on a parameter vector $\theta \in \Re^K$. In earlier work [MT98], we proposed a method for tuning the parameter θ to optimize the average reward $\lambda(\theta)$. This method relies on simulations to produce an estimate of the gradient of the average reward. It can be implemented on-line, and has the property that the gradient of the average reward converges to zero with probability 1 (which is the strongest possible result for gradient-related stochastic approximation algorithms).

A drawback of the algorithms proposed in [MT98] is that the updates may have a high variance, which can result in slow convergence. This is because they essentially employ a renewal period (interval between visits to a certain recurrent state) to produce an estimate of the gradient. If the length of a typical renewal period is large (as it tends to be the case when the state space is large), then the variance of the corresponding estimate will also be large. In this paper, we address this issue and propose two approaches to reduce the variance: one which estimates the gradient based on trajectories which tend to be shorter than a renewal period, and another which employs a discount factor. However, the resulting algorithms introduce an additional bias into the update direction. As a result, we cannot guarantee the convergence of $\nabla\lambda(\theta)$ to zero. We will nevertheless establish a result of the form

$$\liminf_{m\to\infty} \|\nabla\lambda(\theta_m)\| \le D,$$

where the constant D is an upper bound on the magnitude of the bias. Thus, if the bound D is small, then the gradient $\nabla\lambda(\theta_m)$ is small infinitely often. We interpret the bias bound D in terms of qualitative properties of the underlying process. In particular, for the case where a discount factor α is employed, we show that D is small, as long as the underlying Markov chain reaches steady-state at a rate faster than α^t. As gradient-type methods tend to be robust with respect to small biases, these modified algorithms are expected to perform better in practice. We provide a numerical case study to illustrate this point.

The gradient estimation formula used in [MT98] falls within the class of likelihood-ratio methods [Gly86, Gly87]. Using simulation to estimate the gradient of a performance metric with respect to a parameter vector is in the spirit of infinitesimal perturbation analysis (IPA) [CR94], which can be specialized to Markov reward processes [CC97, CW98, FH94, FH97]. We refer to [MT98] for a more detailed comparison of our approach to this literature.

Related formulas have been used in the reinforcement learning literature, e.g., in [W92]. The introduction of a discount factor, mostly with the purpose of limiting the variance of the gradient estimates, appears in [JSJ95, KMK97], as well as in the more recent reference [BB99]. The results reported here have also been presented in [Mar98] and [MT99].

The rest of the paper is structured as follows. In Sections 2 and 3, we provide a brief summary of the framework and results of [MT98]. In Section 3.2, we propose two approaches to reduce the variance in the update, which we study in more detail in Sections 4 and 5, where we also state our main results. In Section 6, we briefly mention how this methodology can be applied to Markov decision processes. Finally, in Section 7, we provide numerical results from a case study involving an admission control problem.

2 Formulation

Consider a discrete-time, finite-state Markov chain $\{i_n\}$ with state space $S = \{1, \ldots, N\}$, whose transition probabilities depend on a parameter vector $\theta \in \Re^K$. We denote the one-step transition probabilities by $P_{ij}(\theta)$, $i,j \in S$, and the n-step transition probabilities by $P_{ij}^n(\theta)$, i.e.,

$$P_{ij}(\theta) = P(i_1 = j \mid i_0 = i, \theta), \qquad \text{and } P_{ij}^n(\theta) = P(i_n = j \mid i_0 = i, \theta), \quad n = 1, 2, \dots,$$

where i_n stands for the state of the chain at time n. Whenever the state is equal to i, we receive a one-stage reward that also depends on θ, and is denoted by $g_i(\theta)$.

For every $\theta \in \Re^K$, let $P(\theta)$ be the stochastic matrix with entries $P_{ij}(\theta)$. Let $\mathcal{P} = \{P(\theta) \mid \theta \in \Re^K\}$ be the set of all such matrices, and let $\overline{\mathcal{P}}$ be its closure. Note that every element of $\overline{\mathcal{P}}$ is also a stochastic matrix and, therefore, defines a Markov chain on the same state space. We make the following assumptions.

Assumption 1 (Recurrence) *The Markov chain corresponding to every $P \in \overline{\mathcal{P}}$ is aperiodic. Furthermore, there exists a state $i^* \in S$ which is recurrent for every such Markov chain.*

Assumption 2 (Regularity) *For all states $i,j \in S$, the transition probability $P_{ij}(\theta)$, and the one-state reward $g_i(\theta)$, are bounded, twice differentiable, and have bounded first and second derivatives. Furthermore, we have*

$$\nabla P_{ij}(\theta) = P_{ij}(\theta) L_{ij}(\theta), \quad \theta \in \Re^K$$

for some bounded function $L_{ij}(\cdot)$.

Assumption 2 allows us to use the recurrent state i^* as a reference state and to employ results of renewal theory (see for example [Gal95]) for our analysis. Assumption 2 (Regularity) ensures that the transition probabilities $P_{ij}(\theta)$ and the one-stage reward $g_i(\theta)$ depend smoothly on θ, and that the quotient $\nabla P_{ij}(\theta)/P_{ij}(\theta) = L_{ij}(\theta)$ is well behaved.

Under Assumption 1, the balance equations

$$\pi'(\theta) P(\theta) = \pi'(\theta)$$

have a unique solution for every $\theta \in \Re^K$, where $\pi'(\theta)$ is the row vector $\big(\pi_1(\theta), \dots, \pi_N(\theta)\big)$, and $\pi_i(\theta)$ is the steady state probability of state i in the Markov chain with transition probabilities $P_{ij}(\theta)$.

As a performance metric associated with the parameter θ, we use the average reward criterion

$$\lambda(\theta) = \lim_{t\to\infty} \frac{1}{t} E_\theta \left[\sum_{k=0}^{t-1} g_{i_k}(\theta) \right].$$

Here, i_k is the state visited at time k, and the notation $E_\theta[\cdot]$ indicates that the expectation is taken with respect to the distribution of the Markov chain with transition probabilities $P_{ij}(\theta)$. Under Assumption 1 (Recurrence), the average reward $\lambda(\theta)$ is well defined for every θ, and does not depend on the initial state.

We define the the differential reward $v_i(\theta)$ of state $i \in S$, and the mean recurrence time $E_\theta[T]$ by

$$v_i(\theta) = E_\theta \left[\sum_{k=0}^{T-1} \big(g_{i_k}(\theta) - \lambda(\theta) \big) \mid i_0 = i \right],$$
$$E_\theta[T] = E_\theta[T \mid i_0 = i^*],$$

where $T = \min\{k > 0 \mid i_k = i^*\}$ is the first future time that the recurrent state i^* is visited. We have $v_{i^*}(\theta) = 0$. The following lemma, established in [MT98], states that $\lambda(\theta)$, $E_\theta[T]$ and $v_i(\theta)$, $i \in S$, depend smoothly on θ.

Lemma 1 *Let Assumption 1 (Recurrence) and Assumption 2 (Regularity) hold. Then,* $\lambda(\theta)$, $E_\theta[T]$ *and* $v_i(\theta)$, $i \in S$, *are (as functions of* θ*) bounded, twice differentiable, and have bounded first and second derivatives. Furthermore, for every integer* $s > 0$, *there exists a constant* D_s, *such that for all* $\theta \in \Re^K$, *we have*

$$E_\theta[T^s] = E_\theta[T^s \mid i_0 = i^*] \leq D_s,$$

where $T = \min\{k > 0 \mid i_k = i^*\}$ *is the first future time that state* i^* *is visited.*

3 Background

To maximize the average reward $\lambda(\theta)$, we will use a gradient-type method of the form

$$\theta := \theta + \gamma F(\theta),$$

where $F(\theta)$ is a simulation-based estimate of $\nabla\lambda(\theta)$, and γ is a positive step size. In order to construct such an estimate $F(\theta)$, we start with the formula

$$\nabla\lambda(\theta) \quad = \quad \sum_{i\in S} \pi_i(\theta) \left(\nabla g_i(\theta) + \sum_{j\in S} \nabla P_{ij}(\theta) v_j(\theta) \right),$$

(see [CC97], [Gly87], or [MT98], for a derivation) which we rewrite as

$$\nabla\lambda(\theta) \quad = \quad \sum_{i\in S} \pi_i(\theta) \left(\nabla g_i(\theta) + \sum_{j\in S} P_{ij}(\theta) L_{ij}(\theta) v_j(\theta) \right),$$

where $L_{ij}(\theta)$ is as in Assumption 2.

Let the parameter vector θ be fixed to some value, and let $\{i_n\}$ be a sample path of the corresponding Markov chain, possibly obtained through simulation. Furthermore, let t_m be the time of the mth visit at the recurrent state i^*, i.e. $i_{t_m} = i^*$ for $m = 1, 2, \ldots$. Consider the estimate of $\nabla\lambda(\theta)$ given by

$$F_m(\theta, \tilde{\lambda}) = \sum_{n=t_m}^{t_{m+1}-1} \left(\tilde{v}_{i_n}(\theta, \tilde{\lambda}) L_{i_{n-1} i_n}(\theta) + \nabla g_{i_n}(\theta) \right), \tag{1}$$

where

$$\tilde{v}_{i_n}(\theta, \tilde{\lambda}) = \sum_{k=n}^{t_{m+1}-1} \left(g_{i_k}(\theta) - \tilde{\lambda} \right), \quad t_m < n < t_{m+1}, \tag{2}$$

is an estimate of the differential reward $v_{i_n}(\theta)$, and $\tilde{\lambda}$ is some estimate of $\lambda(\theta)$. Noting that $v_{i^*}(\theta) = 0$, we let

$$\tilde{v}_{i_n}(\theta, \tilde{\lambda}) = 0, \qquad \text{if } n = t_m.$$

Assumption 1 (Recurrence) allows us to employ renewal theory (see, for example, [Gal95]) to obtain the following result, which states that the expectation of $F_m(\theta, \tilde{\lambda})$ is aligned with $\nabla\lambda(\theta)$ to the extent that $\tilde{\lambda}$ is close the $\lambda(\theta)$ (see [MT98]).

Proposition 1 *We have*

$$E_\theta \left[F_m(\theta, \tilde{\lambda}) \right] = E_\theta[T] \nabla\lambda(\theta) + G(\theta)(\lambda(\theta) - \tilde{\lambda}),$$

where

$$G(\theta) = E_\theta \left[\sum_{n=t_m+1}^{t_{m+1}-1} (t_{m+1} - n) L_{i_{n-1} i_n}(\theta) \right],$$

and $E_\theta[T]$ is the mean recurrence time.

3.1 An Algorithm that Updates at Visits to the Recurrent State

Using the estimate of the gradient $\nabla\lambda(\theta)$ given above, we obtain an algorithm which updates the parameter vector θ at visits to the recurrent state i^*. At the same time, the estimate $\tilde{\lambda}$ of the average reward gets updated to drive the bias term $G(\theta)(\lambda(\theta) - \tilde{\lambda})$ to zero.

At the time t_m that state i^* is visited for the mth time, we have available a current vector θ_m and an average reward estimate $\tilde{\lambda}_m$. We then simulate the process according to the transition probabilities $P_{ij}(\theta_m)$ until the next time t_{m+1} that i^* is visited, and update according to

$$\theta_{m+1} = \theta_m + \gamma_m F_m(\theta_m, \tilde{\lambda}_m), \tag{3}$$

$$\tilde{\lambda}_{m+1} = \tilde{\lambda}_m + \eta\gamma_m \sum_{n=t_m}^{t_{m+1}-1} (g_{i_n}(\theta_m) - \tilde{\lambda}_m), \tag{4}$$

where η is a positive scalar and γ_m is a step size sequence which satisfies the following assumption.

Assumption 3 *The step sizes γ_m are nonnegative and satisfy*

$$\sum_{m=1}^{\infty} \gamma_m = \infty, \qquad \sum_{m=1}^{\infty} \gamma_m^2 < \infty.$$

This assumption is satisfied, for example, if we let $\gamma_m = 1/m$. We then have the following convergence result [MT98].

Proposition 2 *Let Assumption 1 (Recurrence), Assumption 2 (Regularity), and Assumption 3, hold and let $\{\theta_m\}$ be the sequence of parameter vectors generated by the above described algorithm. Then, $\lambda(\theta_m)$ converges and*

$$\lim_{m\to\infty} \nabla\lambda(\theta_m) = 0,$$

with probability 1.

3.2 Variance reduction methods

For systems involving a large state space, as it is the case in many applications, the interval between visits to the state i^* can be large. Consequently, the parameter vector θ gets updated only infrequently, and the estimate $F_m(\theta)$ can have a large variance. In [MT98], we extended the method of Section 3.1, so that the parameter vector gets updated at every time step. We will give a summary of the corresponding algorithm in the next subsection. In addition, we will consider two ways of reducing the variance in the updates, by replacing the estimate $\tilde{v}_{i_n}(\theta)$ of the differential reward $v_{i_n}(\theta)$ [cf. Eq. (2)] by alternative estimates.

In the first approach, we replace the (generally large) time until we reach the recurrent state i^*, by the (generally smaller) first time that a *set of states S^**, containing i^*, is reached. Given a simulated trajectory $(i_0, i_1, \ldots)$ under the parameter θ, this leads us to estimate $v_{i_n}(\theta)$ by

$$\tilde{v}_{S^*,i_n}(\theta, \tilde{\lambda}) = \sum_{k=n}^{\hat{T}-1} \Big(g_{i_k}(\theta) - \tilde{\lambda} \Big), \tag{5}$$

where

$$\hat{T} = \min\{k > n \mid i_k \in S^*\}$$

is the first future time that a state in the set S^* is visited.

In the second approach, we introduce a discount factor $\alpha \in (0,1)$ and form the estimate

$$\tilde{v}_{\alpha,i_n}(\theta, \tilde{\lambda}) = \sum_{k=n}^{T-1} \alpha^k \Big(g_{i_k}(\theta) - \tilde{\lambda} \Big), \tag{6}$$

where $T = \min\{k > n \mid i_k = i^*\}$ is the first future time the state i^* is visited.

3.3 An Algorithm that Updates at Every Time Step

By rearranging terms (see [MT98]), we can rewrite the estimate $F_m(\theta, \tilde{\lambda})$ in the following form

$$F_m(\theta, \tilde{\lambda}) \quad = \quad \nabla g_{i^*}(\theta) + \sum_{k=t_m+1}^{t_{m+1}-1} \Big(\nabla g_{i_k}(\theta) + (g_{i_k}(\theta) - \tilde{\lambda}) z_k \Big),$$

where

$$z_k = \sum_{n=t_m+1}^{k} L_{i_{n-1} i_n}(\theta), \qquad k = t_m + 1, \ldots, t_{m+1} - 1,$$

is a vector (of the same dimension as θ) that becomes available at time k.

Using this expression, we obtain the following algorithm which updates the parameter vector at each time step. At a typical time k, the state is i_k, and the values of θ_k, z_k, and $\tilde{\lambda}_k$ are available from the previous iteration. We update θ and $\tilde{\lambda}$ according to

$$\begin{aligned} \theta_{k+1} &= \theta_k + \gamma_k \Big(\nabla g_{i_k}(\theta_k) + \big(g_{i_k}(\theta_k) - \tilde{\lambda}_k \big) z_k \Big), \\ \tilde{\lambda}_{k+1} &= \tilde{\lambda}_k + \eta \gamma_k (g_{i_k}(\theta_k) - \tilde{\lambda}_k), \end{aligned}$$

where η is a positive scalar, and γ_k is a step size parameter. We then simulate a transition to the next state i_{k+1} according to the transition probabilities $P_{ij}(\theta_{k+1})$, and update z by letting

$$z_{k+1} = \begin{cases} 0, & \text{if } i_{k+1} = i^*, \\ z_k + L_{i_k i_{k+1}}(\theta_k), & \text{otherwise.} \end{cases}$$

From a theoretical point of view, the algorithm of this subsection and Subsection 3.1 differ only by certain small terms that are of second order in the stepsize. This is because θ_k moves by $O(\gamma)$ between successive visits to i^*. Under a minor additional assumption on the step size (which again holds for $\gamma_k = 1/k$) and on the recurrence property of the state i^*, it can be shown that such $O(\gamma^2)$ modifications do not affect the asymptotic behavior and that this algorithm converges, i.e., $\lambda(\theta_k)$ converges and

$$\lim_{k \to \infty} \nabla \lambda(\theta_k) = 0,$$

with probability 1 (we refer to [Mar98, MT98] for a detailed proof). On the other hand, there are clear practical advantages when $E_\theta[T]$ is very large.

In the remainder of the paper, we incorporate the variance reducing estimates of $v_{i_n}(\theta)$ of Section 3.2 into the algorithms of Sections 3.1 and 3.3, and study the resulting biases and convergence properties.

4 Using a Set S^* to Reduce the Variance

In this section, we use Eq. (5) to produce an estimate of the gradient $\nabla\lambda(\theta)$, and then proceed to analyze the resulting gradient-like algorithms for tuning θ.

4.1 An Estimate of the Gradient $\nabla\lambda(\theta)$

Let the parameter $\theta \in \Re^K$ be fixed to some value and let $(i_1, i_2, \ldots)$ be a simulated trajectory of the Markov chain with transition probabilities $P_{ij}(\theta)$. Let t_m be the time of the mth visit to the recurrent state i^*. We fix a set $S^* \subset S$ containing i^*. Let $\kappa(m)$ be the number of times a state in the set S^* is visited in the interval $k = t_m + 1, \ldots, t_{m+1} - 1$, let $t_{m,n}$ be the time of the nth visit to such a state, and let $t_{m,0}$ and $t_{m,\kappa(m)+1}$ be equal to t_m and t_{m+1}, respectively. Using these definitions, we consider the estimate $F_{S^*,m}(\theta, \tilde{\lambda})$ of the gradient $\nabla\lambda(\theta)$ given by

$$F_{S^*,m}(\theta, \tilde{\lambda}) = \sum_{k=t_m}^{t_{m+1}-1} \left(\tilde{v}_{S^*,i_k}(\theta, \tilde{\lambda}) L_{i_{k-1}i_k}(\theta) + \nabla g_{i_k}(\theta) \right), \tag{7}$$

where, for $t_{m,n} \le k < t_{m,n+1}$, $n = 0, ..., \kappa(m)$, we set

$$\tilde{v}_{S^*,i_k}(\theta, \tilde{\lambda}) = \sum_{l=k}^{t_{m,n+1}-1} \left(g_{i_l}(\theta) - \tilde{\lambda} \right).$$

For $k = t_m$, we let $\tilde{v}_{i_k}(\theta, \tilde{\lambda}) = 0$.

We define

$$f_{S^*}(\theta, \tilde{\lambda}) = E_\theta[F_{S^*,m}(\theta, \tilde{\lambda})],$$

and we have the following result. The proof parallels the proof of Proposition 1, which is given in [MT98], and is omitted.

Proposition 3 *We have*

$$f_{S^*}(\theta, \tilde{\lambda}) = E_\theta[T] \sum_{i \in S} \pi_i(\theta) \left(\nabla g_i(\theta) + \sum_{j \in S} \nabla P_{ij}(\theta) v_{S^*,j}(\theta) \right) + G_{S^*}(\theta)(\lambda(\theta) - \tilde{\lambda}),$$

where

$$G_{S^*}(\theta) = E_\theta \left[\sum_{n=t_m+1}^{t_{m,1}-1} (t_{m,1} - n) L_{i_{n-1}j_n}(\theta) + \sum_{k=1}^{\kappa(m)} \sum_{n=t_{m,k}}^{t_{m,k+1}-1} (t_{m,k+1} - n) L_{i_{n-1}j_n}(\theta) \right],$$

and

$$
\begin{aligned}
v_{S^*,j}(\theta) &= E_\theta\left[\sum_{k=0}^{\hat{T}-1}(g_{i_k}(\theta)-\lambda(\theta)) \mid i_0 = j\right], \qquad j \in S\backslash\{i^*\}, \\
v_{S^*,i^*}(\theta) &= 0,
\end{aligned}
$$

with $\hat{T} = \min\{k > 0 \mid i_k \in S^*\}$ *being the first future time that the set* S^* *is visited.*

Note that the expression for $f_{S^*}(\theta, \tilde{\lambda})$ in Proposition 3 is of the same form as the expectation of the original estimate $F_m(\theta, \tilde{\lambda})$ given in Proposition 1. However, the bias term $G(\theta)(\tilde{\lambda} - \lambda(\theta))$ in Proposition 1 is replaced by $G_{S^*}(\theta)(\tilde{\lambda} - \lambda(\theta))$, and the exact value of the differential reward $v_j(\theta)$ is replaced by the approximation $v_{S^*,j}(\theta)$. Replacing $v_j(\theta)$ with $v_{S^*,j}(\theta)$ introduces an additional bias $E_\theta[T]\sigma_{S^*}(\theta)$, where

$$
\sigma_{S^*}(\theta) = \sum_{i \in S}\pi_i(\theta)\left(\sum_{j \in S}\nabla P_{ij}(\theta)(v_{S^*,j}(\theta) - v_j(\theta))\right).
$$

4.2 A Bound on the Bias $\sigma_{S^*}(\theta)$

In this subsection, we derive an upper bound on the magnitude of $\sigma_{S^*}(\theta)$. Clearly, $\sigma_{S^*}(\theta)$ will be small if the difference $v_{S^*,i}(\theta) - v_i(\theta)$ is small for every θ and every i. However, a weaker condition is possible. In particular, using the fact that

$$
\sum_{j \in S}\nabla P_{ij}(\theta) = 0, \quad \text{for all } i \in S \text{ and all } \theta \in \Re^K,
$$

we see that the bias $\sigma_{S^*}(\theta)$ will be zero, if for any i, and all j such that $\nabla P_{ij}(\theta) \neq 0$, $v_{S^*,j}(\theta) - v_j(\theta)$ takes a constant (possibly nonzero) value, that could also depend on i. This leads us to the following definitions. Let

$$
S_i = \left\{j \in S \mid \nabla P_{ij}(\theta) \neq 0 \text{ for some } \theta \in \Re^K\right\},
$$

and

$$
\hat{v}_{S^*,i}(\theta) = v_{S^*,i}(\theta) - v_i(\theta).
$$

Also, let $\bar{N}$ be given by

$$
\bar{N} = \max\{|S_i| \mid i \in S\},
$$

where $|S_i|$ is the number of states in the set S_i. Thus, $\bar{N}$ bounds the number of possible transitions from any state $i \in S$ whose probability depends on θ. We have the following result.

Proposition 4 *Let Assumption 1 (Recurrence) and Assumption 2 (Regularity) hold. Furthermore, let* ϵ *be such that, for all states* $i \in S$*, we have*

$$
\left|\hat{v}_{S^*,j}(\theta) - \hat{v}_{S^*,j'}(\theta)\right| \leq \epsilon, \qquad \text{if } j, j' \in S_i.
$$

Then

$$
\|\sigma_{S^*}(\theta)\| \leq \bar{N}C\epsilon,
$$

where C *is a bound on* $\|\nabla P_{ij}(\theta)\|$.

Note that by Lemma 1 and by Assumption 2 (Regularity) the bound C on $\|\nabla P_{ij}(\theta)\|$ is finite.

Proof: For every state $i \in S$, let $j_r(i)$ be a state in

$$S_i = \big\{ j \in S \mid \nabla P_{ij}(\theta) \neq 0 \text{ for some } \theta \in \Re^K \big\}$$

which we use as a reference state. By assumption, for all states $i \in S$ and for all states $j \in S_i$, we have

$$\Big| \hat{v}_{S^*,j}(\theta) - \hat{v}_{S^*,j_r(i)}(\theta) \Big| \leq \epsilon.$$

Using the fact

$$\sum_{j \in S} \nabla P_{ij}(\theta) = 0,$$

it follows that

$$\begin{aligned} \|\sigma_{S^*}(\theta)\| &= \Big\| \sum_{i \in S} \pi_i(\theta) \sum_{j \in S_i} \nabla P_{ij}(\theta) \hat{v}_{S^*,j}(\theta) \Big\| \\ &= \Big\| \sum_{i \in S} \pi_i(\theta) \sum_{j \in S_i} \nabla P_{ij}(\theta) \Big(\hat{v}_{S^*,j}(\theta) - \hat{v}_{S^*,j_r(i)}(\theta) \Big) \Big\| \\ &\leq \bar{N} C \epsilon, \end{aligned}$$

which completes the proof. □

Proposition 4 suggests that in order to keep the bias $\sigma_{S^*}(\theta)$ small one should choose S^* such that, for all states $i \in S$ and for all states $j, j' \in S_i$, the difference $\big|\hat{v}_{S^*,j}(\theta) - \hat{v}_{S^*,j'}(\theta)\big|$ is small. The following example illustrates this result.

Example 1 *Let ϵ be a scalar in $(0, 1]$, and consider the Markov reward process on the state space $S = \{0, 1, 2, 3\}$, with transition probabilities*

$$\begin{aligned} &P_{01}(\theta) = P_{23}(\theta) = 1, \\ &P_{11}(\theta) = \frac{1}{2}\left[\frac{\exp(\theta)}{1 + \exp(\theta)}\right], \\ &P_{12}(\theta) = 1 - P_{11}(\theta), \\ &P_{30}(\theta) = \epsilon, \qquad P_{31}(\theta) = 1 - \epsilon, \end{aligned}$$

and one-stage rewards

$$g_1(\theta) = 1, \quad \text{and} \quad g_0(\theta) = g_2(\theta) = g_3(\theta) = 0.$$

The structure of this Markov reward process is given in Figure 1. Note that Assumption 1 (Recurrence) and Assumption 2 (Regularity) are satisfied, with

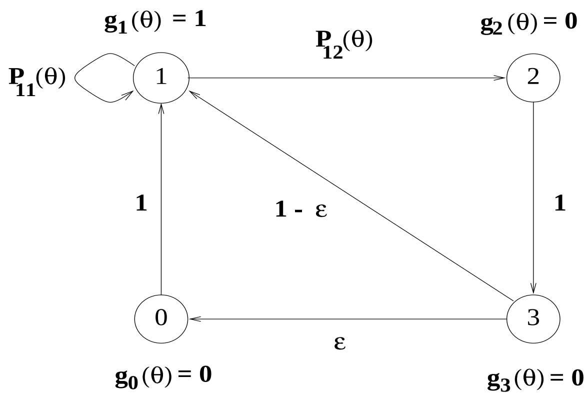

Figure 1: Structure of the Markov reward process in Example 1.

state 0 serving as the recurrent state i^*. Define the set S^* to be $\{0,3\}$ and consider the estimates $\tilde{v}_{S^*,1}(\theta,\tilde{\lambda})$ and $\tilde{v}_1(\theta,\tilde{\lambda})$ of the differential reward of state 1. Note that $\tilde{v}_{S^*,1}(\theta,\tilde{\lambda})$ has the same distribution as the random variable X_θ described by

$$P\Big(X_\theta = n(1-\tilde{\lambda}) - \tilde{\lambda}\Big) = \Big(1 - P_{12}(\theta)\Big)^{n-1} P_{12}(\theta), \quad n = 1, 2, \ldots,$$

and that the estimate $\tilde{v}_1(\theta,\tilde{\lambda})$ has the same distribution as the random variable

$$Y_\theta = \sum_{n=1}^{N} \Big(X_{\theta,n} - \tilde{\lambda}\Big),$$

where $(X_{\theta,n})$ is a sequence of IID random variables with the distribution of X_θ, and N is a random variable, independent of $(X_{\theta,n})$, with distribution

$$P(N = n) = (1-\epsilon)^{n-1}\epsilon, \quad n = 1, 2, \ldots \quad .$$

Using $\tilde{v}_{S^*,1}(\theta,\tilde{\lambda})$, instead of $\tilde{v}_1(\theta,\tilde{\lambda})$, then reduces the variance by the factor

$$\frac{Var\Big(\tilde{v}_1(\theta,\tilde{\lambda})\Big)}{Var\Big(\tilde{v}_{S^*,1}(\theta,\tilde{\lambda})\Big)} = \frac{E[N]\, Var(X_\theta) + E[(X_\theta - \tilde{\lambda})^2]\, Var(N)}{Var(X_\theta)} \geq E[N] = \frac{1}{\epsilon},$$

which becomes large when ϵ becomes small.

Furthermore, note that the set $S_1 = \{j \in S \mid \nabla P_{1j}(\theta) \neq 0 \text{ for some } \theta \in \Re\} = \{1,2\}$, $S_0 = S_2 = S_3 = \emptyset$, and

$$\hat{v}_{S^*,1}(\theta) = \hat{v}_{S^*,2}(\theta) = v_3(\theta).$$

Therefore, by Proposition 4, using $\tilde{v}_{S^*,i}(\theta,\tilde{\lambda})$ can significantly reduce the variance in the estimate of the differential reward $v_1(\theta)$ without introducing a bias into the estimate of the gradient $\nabla\lambda(\theta)$.

An important special case of Proposition 4 arises when the set S^* has the following property:

$$|v_i(\theta) - v_{i^*}(\theta)| \le \frac{\epsilon}{2}, \quad \text{for all } \theta \in \Re^K \text{ and all } i \in S^*; \tag{8}$$

since $v_{i^*}(\theta) = 0$, this is the same as requiring $|v_i(\theta)| \le \epsilon/2$, for all $i \in S^*$ and all θ. It is then easily verified that

$$|\hat{v}_{S^*,j}(\theta)| \le \epsilon/2,$$

for all j, and the assumption in Proposition 4 is satisfied. In practice, condition (8) can be satisfied by picking S^* to be small enough so that $v_i(\theta)$ does not vary much within the set S^*, but should also be large enough so that the set S^* is typically entered much earlier than the state i^* is visited (see Section 7.1.1 for an application).

4.3 An Algorithm that Updates at Visits to the Recurrent State

We will now use the estimate $F_{S^*,m}(\theta)$ of the gradient $\nabla\lambda(\theta)$ to formulate an algorithm which updates the parameter vector θ at visits to the recurrent state i^*. Again, we use the variable m to index the times when the state i^* is visited and the corresponding updates. At the time t_m, we have available a current vector θ_m and an average reward estimate $\tilde{\lambda}_m$. We then simulate the process according to the transition probabilities $P_{ij}(\theta_m)$ until the next time t_{m+1} that i^* is visited and update according to

$$\begin{aligned} \theta_{m+1} &= \theta_m + \gamma_m F_{S^*,m}(\theta_m), \\ \tilde{\lambda}_{m+1} &= \tilde{\lambda}_m + \eta\gamma_m \sum_{n=t_m}^{t_{m+1}-1} (g_{i_n}(\theta_m) - \tilde{\lambda}_m), \end{aligned}$$

where η is a positive scalar, and γ_k is a step size parameter. We have the following result.

Proposition 5 *Let Assumption 1 (Recurrence), Assumption 2 (Regularity), and Assumption 3, hold, and let D be such that, for all $\theta \in \Re^K$, we have*

$$\|\sigma_{S^*}(\theta)\| \le D,$$

where $\sigma_{S^*}(\theta)$ is as in Proposition 4. Furthermore, let $\{\theta_m\}$ be the sequence of parameter vectors generated by the above described algorithm. Then,

$$\liminf_{m\to\infty} \|\nabla\lambda(\theta_m)\| \le D,$$

with probability 1.

We defer the proof of this proposition to Appendix A.

Proposition 5 establishes that if the bias $\|\sigma_{S^*}(\theta)\|$ is small, then the gradient $\nabla\lambda(\theta_m)$ is small at infinitely many visits to the recurrent state i^*.

4.4 An Algorithm that Updates at Every Time Step

Similar to Section 3.2, we can break down the total update $F_{S^*,m}(\theta_m)$ in the algorithm in Section 4.3 into a sum of incremental updates carried out at each time step, and derive the following algorithm which updates the parameter vector at each time step. At a typical time k, the state is i_k, and the values of θ_k, $\tilde{\lambda}_k$ and z_k, are available from the previous iteration. We update θ according to

$$\begin{aligned}
\theta_{k+1} &= \theta_k + \gamma_k\Big(\nabla g_{i_k}(\theta_k) + \Big(g_{i_k}(\theta_k) - \tilde{\lambda}_k\Big)z_k\Big), \\
\tilde{\lambda}_{k+1} &= \tilde{\lambda}_k + \eta\gamma_k(g_{i_k}(\theta_k) - \tilde{\lambda}_k).
\end{aligned}$$

We then simulate a transition to the next state i_{k+1} according to the transition probabilities $P_{ij}(\theta_{k+1})$, and update z by letting

$$z_{k+1} = \begin{cases} 0, & \text{if } i_{k+1} = i^*, \\ L_{i_k i_{k+1}}(\theta_k), & \text{if } i_{k+1} \in S^*, \\ z_k + L_{i_k i_{k+1}}(\theta_k), & \text{otherwise.} \end{cases}$$

Similar to Subsection 3.3, it can be shown that the conclusions of Proposition 5 remain valid for this algorithm as well, under some minor additional assumptions (see [Mar98]).

5 Using a Discount Factor to Reduce the Variance

In this section, we produce an estimate of the gradient $\nabla\lambda(\theta)$ by using another expression for estimating the differential reward of state i, namely

$$\tilde{v}_{\alpha,i}(\theta,\tilde{\lambda}) = \sum_{k=0}^{T-1} \alpha^k \Big(g_{i_k}(\theta) - \tilde{\lambda}\Big), \tag{9}$$

where $T = \min\{k > 0 \mid i_k = i^*\}$ is the first future time that the state i^* is visited and $\alpha \in (0,1)$ is a discount factor.

5.1 An Estimate of the Gradient $\nabla\lambda(\theta)$

Let the parameter $\theta \in \Re^K$ be fixed to some value and let $(i_1, i_2, \ldots)$ be a simulated trajectory of the Markov chain with transition probabilities $P_{ij}(\theta)$. Furthermore, let t_m be the time of the mth visit at the recurrent state i^* and consider the following estimate $F_{\alpha,m}(\theta,\tilde{\lambda})$ of the gradient $\nabla\lambda(\theta)$,

$$F_{\alpha,m}(\theta,\tilde{\lambda}) = \sum_{n=t_m}^{t_{m+1}-1} \Big(\tilde{v}_{\alpha,i_n}(\theta,\tilde{\lambda})L_{i_{n-1}i_n}(\theta) + \nabla g_{i_n}(\theta)\Big), \tag{10}$$

where, for $t_m \le n \le t_{m+1} - 1$, we set

$$\tilde{v}_{\alpha,i_n}(\theta,\tilde{\lambda}) = \sum_{k=n}^{t_{m+1}-1} \alpha^{k-n} \Big(g_{i_k}(\theta) - \tilde{\lambda}\Big).$$

We have the following result for $f_\alpha(\theta,\tilde{\lambda})$, which we define to be the expected value of $F_{\alpha,m}(\theta,\tilde{\lambda})$, namely,

$$f_\alpha(\theta,\tilde{\lambda}) = E_\theta[F_{\alpha,m}(\theta,\tilde{\lambda})].$$

Proposition 6 *We have*

$$f_\alpha(\theta,\tilde{\lambda}) = E_\theta[T]\sum_{i\in S}\pi_i(\theta)\left(\nabla g_i(\theta) + \sum_{j\in S}\nabla P_{ij}(\theta)v_{\alpha,j}(\theta)\right) + G_\alpha(\theta)(\lambda(\theta)-\tilde{\lambda}),$$

where

$$G_\alpha(\theta) = E_\theta\left[\sum_{n=t_m}^{t_{m+1}-1}\sum_{k=n}^{t_{m+1}-1}\alpha^{k-n}L_{i_{n-1}i_n}(\theta)\right],$$

and

$$v_{\alpha,j}(\theta) = E_\theta\left[\sum_{k=0}^{T-1}\alpha^k(g_{i_k}(\theta)-\lambda(\theta)) \mid i_0 = j\right], \qquad j\in S,$$

with $T = \min\{k>0 \mid i_k = i^*\}$ *being the first future time that the state* i^* *is visited.*

As in Section 4, the expression for $f_\alpha(\theta,\tilde{\lambda})$ in Proposition 6 is of the same form as the expectation of the original estimate $F_m(\theta,\tilde{\lambda})$ of the gradient $\nabla\lambda(\theta)$, except that the bias term $G(\theta)(\tilde{\lambda}-\lambda(\theta))$ is replaced by $G_\alpha(\theta)(\tilde{\lambda}-\lambda(\theta))$, and the exact value of the differential reward $v_j(\theta)$ is replaced by the approximation $v_{\alpha,j}(\theta)$.

5.2 A Bound on the Bias $\sigma_\alpha(\theta)$

In this subsection, we analyze the bias

$$E_\theta[T]\sigma_\alpha(\theta) = E_\theta[T]\sum_{i\in S}\pi_i(\theta)\left(\sum_{j\in S}\nabla P_{ij}(\theta)\Big(v_{\alpha,j}(\theta)-v_j(\theta)\Big)\right),$$

which is due to replacing $v_j(\theta)$ with $v_{\alpha,j}(\theta)$, and derive a bound for the magnitude of $\sigma_\alpha(\theta)$.

To do that, we consider the "mixing behavior" of the Markov reward process, i.e. we define scalars A and β, with $0\le\beta<1$, such that, for all states $i\in S$ and all integers $n\ge 0$, we have

$$\Big|\sum_{j\in S}\Big(P^n_{ij}(\theta)-\pi_j(\theta)\Big)g_j(\theta)\Big| \le A\beta^n.$$

Such constants are guaranteed to exist under Assumption 1 (Recurrence). In particular, β can be taken to be an upper bound on the second largest of the magnitudes of the eigenvalues of the stochastic matrices $P(\theta)$. This setting becomes interesting when β is small relative to α, which corresponds to "fast mixing".

Proposition 7 *Let Assumption 1 (Recurrence) and Assumption 2 (Regularity) hold. Furthermore, let the constants* β, $0\le\beta<1$, *and* A, *be such that for all* $\theta\in\Re^K$, *for all* $i\in S$, *and all integers* $n\ge 0$, *we have*

$$\Big|\sum_{j\in S}\Big(P^n_{ij}(\theta)-\pi_j(\theta)\Big)g_j(\theta)\Big| \le A\beta^n.$$

We then have

$$\|\sigma_\alpha(\theta)\| \le \frac{AC\bar{N}}{1-\alpha\beta}\left(\frac{\beta(1-\alpha)}{1-\beta} + \sum_{i\in S}\pi_i(\theta)E_\theta\Big[\alpha^T \mid i_0 = i\Big]\right),$$

where $T=\min\{k>0\mid i_k=i^*\}$, C *is a bound on* $\|\nabla P_{ij}(\theta)\|$, *and* $\bar{N}$ *is the same bound as in Proposition 4.*

Proof: We introduce some additional notation. For any $\alpha \in [0,1]$, we let $v^{\infty}_{\alpha,i}(\theta)$, $i \in S$, be given by

$$v^{\infty}_{\alpha,i}(\theta) = \sum_{k=0}^{\infty} \alpha^k E_\theta \left[(g_{i_k}(\theta) - \lambda(\theta)) \mid i_0 = i \right].$$

In the special case where $\alpha = 1$, the above infinite infinite sum is known to exist, because under our assumptions, the summands converge exponentially fast to zero. Furthermore, the resulting differential reward $v^{\infty}_{1,i}(\theta)$ turns out to be the same as the earlier defined $v_i(\theta)$, modulo an additive constant. That is, there exists a constant c such that

$$v_i(\theta) = v^{\infty}_{1,i}(\theta) + c, \qquad \text{for all } i. \tag{11}$$

We then have

$$\begin{aligned} \sigma_\alpha(\theta) &= \sum_{i\in S} \pi_i(\theta) \left(\sum_{j\in S} \nabla P_{ij}(\theta) \Big(v_{\alpha,j}(\theta) - v^{\infty}_{\alpha,j}(\theta) \Big) \right) \\ &\quad + \sum_{i\in S} \pi_i(\theta) \left(\sum_{j\in S} \nabla P_{ij}(\theta) \Big(v^{\infty}_{\alpha,j}(\theta) - v^{\infty}_{1,j}(\theta) \Big) \right) \\ &\quad + \sum_{i\in S} \pi_i(\theta) \left(\sum_{j\in S} \nabla P_{ij}(\theta) \Big(v^{\infty}_{1,j}(\theta) - v_j(\theta) \Big) \right). \end{aligned} \tag{12}$$

Using Eq. (11) and the property $\sum_j \nabla P_{ij}(\theta) = 0$, the third sum above vanishes.

Let us consider the second sum. Using also the property $\lambda(\theta) = \sum_{j\in S} \pi_j(\theta) g_j(\theta)$, we obtain

$$\begin{aligned} |v^{\infty}_{\alpha,i}(\theta) - v^{\infty}_{1,i}(\theta)| &\leq \sum_{k=0}^{\infty} |\alpha^k - 1| \left| \sum_{j\in S} P^k_{ij}(\theta) \Big(g_j(\theta) - \lambda(\theta) \Big) \right| \\ &= \sum_{k=0}^{\infty} (1-\alpha^k) \left| \sum_{j\in S} \Big(P^k_{ij}(\theta) - \pi_j(\theta) \Big) g_j(\theta) \right| \\ &\leq A \sum_{k=0}^{\infty} (1-\alpha^k) \beta^k \\ &= A \frac{\beta}{1-\beta} \cdot \frac{1-\alpha}{1-\alpha\beta} \end{aligned}$$

and

$$\left\| \sum_{i\in S} \pi_i(\theta) \left(\sum_{j\in S} \nabla P_{ij}(\theta) \Big(v^{\infty}_{\alpha,j}(\theta) - v^{\infty}_{1,j}(\theta) \Big) \right) \right\| \leq \frac{AC\bar{N}}{1-\alpha\beta} \cdot \frac{\beta(1-\alpha)}{1-\beta}.$$

We finally provide a bound for the first sum. By Assumption 2 (Regularity) and Lemma 1, $|g_{i_k}(\theta) - \lambda(\theta)|$ is bounded and we have, for $\alpha \in (0,1)$,

$$\begin{aligned} v^{\infty}_{\alpha,i}(\theta) &= \sum_{k=0}^{\infty} \alpha^k E_\theta \left[(g_{i_k}(\theta) - \lambda(\theta)) \mid i_0 = i \right] \\ &= E_\theta \left[\sum_{k=0}^{\infty} \alpha^k (g_{i_k}(\theta) - \lambda(\theta)) \mid i_0 = i \right]. \end{aligned}$$

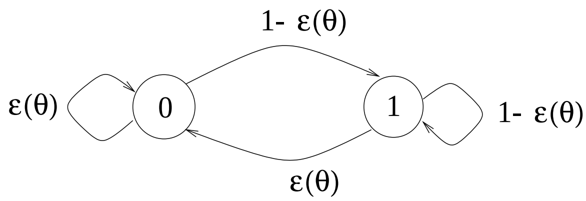

Figure 2: Structure of the Markov reward process in Example 2.

It follows that

$$\begin{aligned}
|v_{\alpha,i}(\theta) - v^{\infty}_{\alpha,i}(\theta)| &= \left| E_\theta \left[\sum_{k=T}^{\infty} \alpha^k \Big(g_{i_k}(\theta) - \lambda(\theta) \Big) \mid i_0 = i \right] \right| \\
&= \left| \sum_{t=1}^{\infty} E_\theta \left[\sum_{k=T}^{\infty} \alpha^k \Big(g_{i_k}(\theta) - \lambda(\theta) \Big) \mid i_0 = i, T = t \right] P_\theta(T = t \mid i_0 = i) \right| \\
&= \left| \sum_{t=1}^{\infty} \alpha^t E_\theta \left[\sum_{k=0}^{\infty} \alpha^k \Big(g_{i_k}(\theta) - \lambda(\theta) \Big) \mid i_0 = i^* \right] P_\theta(T = t \mid i_0 = i) \right| \\
&\leq E_\theta[\alpha^T \mid i_0 = i] \sum_{k=0}^{\infty} \alpha^k \left| \sum_{j \in S} P^k_{i^* j} \Big(g_j(\theta) - \lambda(\theta) \Big) \right| \\
&= E_\theta[\alpha^T \mid i_0 = i] \sum_{k=0}^{\infty} \alpha^k \left| \sum_{j \in S} \Big(P^k_{i^* j} - \pi_j(\theta) \Big) g_j(\theta) \right| \\
&\leq E_\theta[\alpha^T \mid i_0 = i] \cdot A \sum_{k=0}^{\infty} \alpha^k \beta^k \\
&= E_\theta[\alpha^T \mid i_0 = i] \frac{A}{1 - \alpha\beta},
\end{aligned}$$

and we obtain

$$\left\| \sum_{i \in S} \pi_i(\theta) \left(\sum_{j \in S} \nabla P_{ij}(\theta) \Big(v_{\alpha,j}(\theta) - v^{\infty}_{\alpha,j}(\theta) \Big) \right) \right\| \quad \leq \quad \frac{AC\bar{N}}{1 - \alpha\beta} \cdot \sum_{i \in S} \pi_i(\theta) E_\theta[\alpha^T \mid i_0 = i].$$

The result then follows. □

Proposition 7 indicates that the bias will be small as long as β is moderate (not too close to 1, which corresponds to fast mixing), the discount factor is chosen large enough so that $1-\alpha$ is significantly smaller than $1-\beta$, and $\sum_{i \in S} \pi_i(\theta) E_\theta[\alpha^T \mid i_0 = i]$ is also small. Note that the latter term will be small if the time T to reach i^* starting from a random state [drawn according to the steady-state distribution $\pi(\theta)$], is large; this is the typical case in large scale problems. The following example serves as an illustration.

Example 2 *We consider the Markov chain given in Fig. 2, where we set*

$$\epsilon(\theta) = \frac{\exp(\theta)}{1 + \exp(\theta)}.$$

Note that $\|\nabla P_{ij}(\theta)\| < \frac{1}{4}$, for all $\theta \in \Re$. The steady state probabilities are

$$\pi_0(\theta) = \epsilon(\theta) \quad \textit{and} \quad \pi_1(\theta) = 1 - \epsilon(\theta),$$

which implies that

$$\sum_{j \in S} \Big(P_{ij}(\theta) - \pi_j(\theta) \Big) = 0, \qquad \textit{for } i = 0, 1,$$

and the process reaches steady state in a single step. Therefore, we can set the constants β, and A, of Proposition 7 equal to 0, and 1, respectively. Choosing the state $i = 0$ as the recurrent state i^*, we obtain

$$E_\theta[\alpha^T \mid i_0 = 0] = E_\theta[\alpha^T \mid i_0 = 1] = \frac{\epsilon(\theta)\alpha}{1 - \alpha(1 - \epsilon(\theta))},$$

and the bound on the bias becomes

$$\|\sigma_\alpha(\theta)\| \leq \frac{1}{2} \cdot \frac{\epsilon(\theta)\alpha}{1 - \alpha(1 - \epsilon(\theta))}.$$

Thus, the bias can be made arbitrarily small by letting α approach 0.

5.3 An Algorithm that Updates at Visits to the Recurrent State

Using the estimate $F_{\alpha,m}(\theta, \tilde{\lambda})$ of the gradient $\nabla\lambda(\theta)$, we can formulate an algorithm which updates the parameter vector θ at visits to the state i^* according to

$$\begin{aligned} \theta_{m+1} &= \theta_m + \gamma_m F_{\alpha,m}(\theta_m, \tilde{\lambda}), \\ \tilde{\lambda}_{m+1} &= \tilde{\lambda}_m + \gamma_m \eta \sum_{n=t_m}^{t_{m+1}-1} (g_{i_n}(\theta_m) - \tilde{\lambda}_m), \end{aligned}$$

where η is a positive scalar, and where γ_m is a step size parameter for which Assumption 3 holds. This situation is identical to the one in Section 4.3 and Proposition 5 remains valid, except that D must now stand for the bound on $\sigma_\alpha(\theta)$.

With this algorithm, the time between updates, and the resulting variance, would still be large. This problem is alleviated by the variant that we introduce next.

5.4 An Algorithm that Updates at Every Time Step

We now consider a variant of the algorithm of the preceding subsection but in which an update is carried out at each step. At a typical time k, the state is i_k, and the values of θ_k and z_k, are available from the previous iteration. We update θ according to

$$\begin{aligned} \theta_{k+1} &= \theta_k + \gamma_k \Big(\nabla g_{i_k}(\theta_k) + \big(g_{i_k}(\theta_k) - \tilde{\lambda}_k\big) z_k \Big), \\ \tilde{\lambda}_{k+1} &= \tilde{\lambda}_k + \eta\gamma_k (g_{i_k}(\theta_k) - \tilde{\lambda}_k), \end{aligned}$$

We then simulate a transition to the next state i_{k+1} according to the transition probabilities $P_{ij}(\theta_{k+1})$, and update z by letting

$$z_{k+1} = \begin{cases} \alpha L_{i_k i_{k+1}}(\theta_k), & \text{if } i_{k+1} = i^*, \\ z_k + \alpha L_{i_k i_{k+1}}(\theta_k), & \text{otherwise.} \end{cases}$$

Similar to Subsection 3.3 and 4.3, under some minor additional assumptions (see [Mar98]), the algorithm of this and the preceding subsection exhibit the same asymptotic behavior.

5.5 A Modified Estimate

Instead of using the expression given by Eq. (9), we could estimate the differential reward of state i by

$$\tilde{v}^{\infty}_{\alpha,i}(\theta,\tilde{\lambda}) = \sum_{k=0}^{\infty} \alpha^k \left(g_{i_k}(\theta) - \tilde{\lambda} \right).$$

Using this new estimate, one obtains an algorithm that updates at each time step and which is identical to the one in the preceding subsection, except that the state i^* does not play a special role, and the vector z_k is not reset at visits to the recurrent state i^*.

In [BB99], it is shown that, as long as θ is unchanged, the estimate $\tilde{v}^{\infty}_{\alpha,i}(\theta,\tilde{\lambda})$ can be used to produce an estimate of the gradient $\nabla\lambda(\theta)$ for which the bound on the bias is proportional to $(1-\alpha)/(1-\beta)$. The same conclusion is obtained with our approach: with this new estimate, the first sum in Eq. (12) disappears, and the term $\sum_{i\in S}\pi_i(\theta)E_\theta\left[\alpha^T \mid i_0 = i\right]$ is eliminated from the bias bound of Proposition 7. In this respect the variant considered in this subsection has a somewhat better bias bound. However, from a practical point of view, the two algorithms are essentially the same. If the visits to i^* are very rare (as is typical in large problems), the term $\sum_{i\in S}\pi_i(\theta)E_\theta\left[\alpha^T \mid i_0 = i\right]$ in the bias bound is very small. This reflects the fact that the term $\tilde{v}^{\infty}_{\alpha,i}(\theta,\tilde{\lambda}) - \tilde{v}_{\alpha,i}(\theta,\tilde{\lambda})$ which causes the difference between the two algorithms is very small with high probability.

From a mathematical point of view, the convergence analysis of the modified algorithm discussed here is actually much more involved, because the updates during a "renewal cycle" are affected by discounted terms originating in previous renewal cycles. This introduces certain dependencies and martingale-based tools are harder to apply. We feel that the difference between the two algorithms is not significant enough to warrant a long separate proof.

6 Markov Decision Processes

As shown in [MT98], the methodology can be applied to Markov decision processes that are defined on a finite state space $S = \{1,\ldots,N\}$ and a finite action space $U = \{1,\ldots,L\}$. At any state i, the choice of a control action $u \in U$ determines the transition probabilities $P_{ij}(u)$, and the one-stage rewards $g_i(u)$. We consider a parametrized family of randomized policies that associate with each parameter vector $\theta \in \Re^K$ the probability $\mu_u(i,\theta)$ that control action u is applied at state i. The corresponding transition probabilities are given by

$$P_{ij}(\theta) = \sum_{u\in U} \mu_u(i,\theta) P_{ij}(u), \tag{13}$$

and the expected reward per stage is given by

$$g_i(\theta) = \sum_{u\in U} \mu_u(i,\theta) g_i(u). \tag{14}$$

Our original algorithms, as described in Sections 3.1 and 3.3 were shown in [MT98] to have natural counterparts for the case of Markov decision processes. Variance reducing modifications, in the spirit of Sections 4 and 5, are straightforward. Detailed descriptions of the resulting methods can be found in [Mar98] and are omitted from this paper. An illustration is presented, however, in the next section.

Table 1: Call Types.

CALL TYPE m	1	2	3
BANDWIDTH DEMAND $b(m)$	1	1	1
ARRIVAL RATE $\alpha(m)$	1.8	1.6	1.4
AVERAGE HOLDING TIME $1/\beta(m)$	0.6	0.5	0.4
IMMEDIATE REWARD $c(m)$	1	2	4

7 Numerical Results

As a case study, we use an admission control problem. More details on the experiments reported here can be found in [Mar98].

Consider a provider of a communication link with total bandwidth of B units, that supports a finite set $\{1, 2, \ldots, M\}$ of different call types. When a customer requests a new connection for a call, the provider can decide to reject, or, if enough bandwidth is available, to maybe accept the call. Once accepted, a call of class m seizes $b(m)$ units of bandwidth. Whenever a call of class m gets accepted, the provider receives an immediate reward of $c(m)$ units, which is the price the customer pays for using $b(m)$ units of bandwidth of the link for the duration of the call. The goal of the link provider is to exercise call admission control in a way that maximizes the long term revenue.

Assuming that class m calls arrive according to independent Poisson processes (with rate $\alpha(m)$), and that the holding times of class m calls are exponentially (and independently) distributed (with mean $1/\beta(m)$), the problem can be formulated as a discrete-time Markov decision process, where the state i is of the form $i = (s(1), \ldots, s(M), \omega)$. Here $s(m)$, $m = 1, ..., M$, denotes the number of active calls of type m, and ω indicates the type of event that triggers the next transition (a departure or arrival of a call, together with the type of the call).

We a define a randomized policy as a function of $\theta = (\theta(1), ..., \theta(M)) \in \Re^M$, where M is the number of different service types. The provider accepts a new call of class m with probability

$$\mu_{u_a}(i, \theta) = \frac{1}{1 + \exp(s \cdot b - \theta(m))},$$

where $s \cdot b = \sum_m s(m)b(m)$ is the currently occupied bandwidth. Note that

$$\mu_{u_a}(i, \theta) \geq 0.5 \quad \text{if and only if} \quad s \cdot b \leq \theta(m),$$

and $\theta(m)$ can be interpreted as a "fuzzy" threshold on system occupancy, which determines whether type m calls are to be admitted or rejected.

7.1 Experiments

We consider a link with a total bandwidth of $B = 10$ units, which supports three different call types (see Table 1). The number of link configurations (i.e., possible choices of s that do not violate the link capacity constraint) turns out to be 286. Any state (s, ω) in which $s = (0, \ldots, 0)$, and ω corresponds to an arrival of a new call, can serve as the recurrent state i^*.

For this case, we can compute an optimal call admission control policy using methods of dynamic programming [Ber95a]. The policy accepts customers of service type 1 if the currently used bandwidth does not exceed the threshold value of 7 units, while customers of service type 2 and 3 get

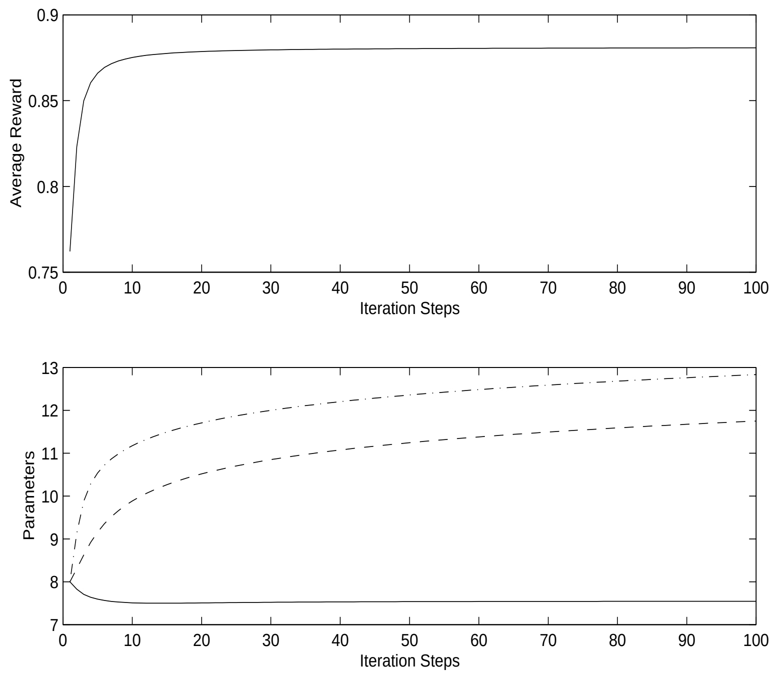

Figure 3: Parameter vectors θ_k and the average rewards $\lambda(\theta_k)$ (computed exactly) of the idealized gradient algorithm. The solid, dashed, and dash-dot line correspond to $\theta_k(1)$, $\theta_k(2)$, and $\theta_k(3)$, respectively.

always accepted (if enough bandwidth is available). The corresponding optimal average reward is equal to 0.8868. In [MT98], we implemented for this problem an idealized gradient algorithm (where we used the exact value of $\nabla\lambda(\theta)$ to update the parameter vector), the simulation-based algorithm of Section 3.3, and the modified algorithm of Section 5.4. As a reference, we give in Figure 3, and 4, the trajectories of the parameter vector and (estimates of the) average reward for the idealized algorithm, and the algorithm of Section 3.3, respectively. Note that the algorithm of Section 3.3 makes rapid progress in the beginning, improving the average reward from 0.78 to 0.87 within the first $1 \cdot 10^6$ iteration steps. After $8 \cdot 10^6$ iterations, the average reward is 0.8789, which is still slightly below 0.8808, the average reward of the idealized gradient algorithm. In [MT98], we showed that the modified algorithm of Section 5.4 converges faster than the algorithm of Section 3.3. Using a discount factor $\alpha = 0.99$, we obtained after $1 \cdot 10^6$ iteration steps an average reward of 0.8785, which is only slightly below the one of the algorithm of Section 3.3 after $8 \cdot 10^6$ iterations.

7.1.1 Modified Algorithm Using the Set S^*

Recall that any state $i = (s, \omega)$ in which $s = (0, \ldots, 0)$, and ω corresponds to an arrival of a new call, can serve as the recurrent state i^*. This leads us to consider a set S^* of the form

$$S^* = \left\{ i = (s, \omega) \in S \mid \sum_{m=1}^{3} s(m)b(m) \leq B_0 \right\}, \qquad B_0 \geq 0,$$

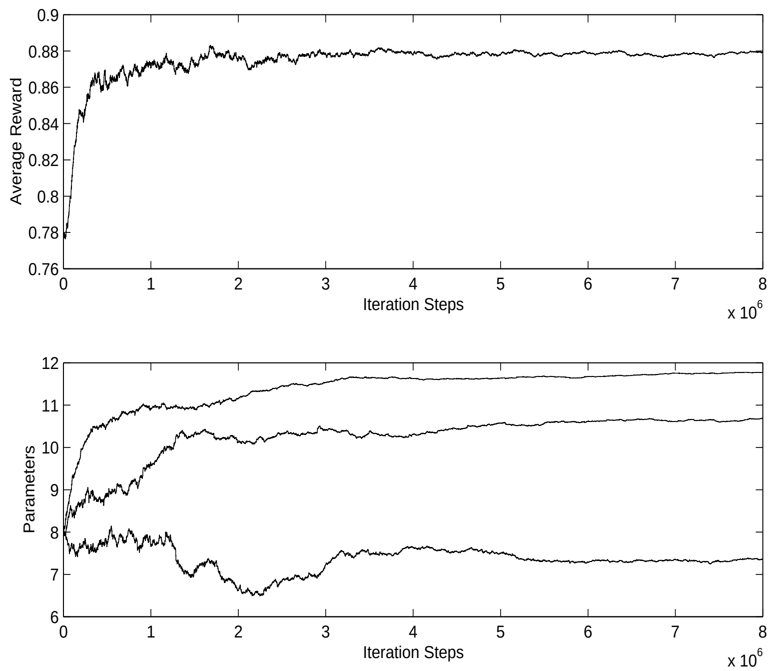

Figure 4: Parameters $\theta_k(1)$, $\theta_k(2)$, and $\theta_k(3)$, and estimates of the average reward $\tilde{\lambda}_k$, obtained by the algorithm of Section 3.3.

for the algorithm of Section 4.4. One would expect that the bias introduced by using the set S^* is small when B_0 is small. Figure 5 gives the trajectories of the parameter vector for the deterministic version of this algorithm (where we replaced the random estimate $F_{S^*}(\theta)$ used to update θ by its mean) for several values of B_0. Comparing Figure 5 with Figure 3 shows that the algorithm is robust in the presence of a small bias, and for B_0 equal to 5 and 7, the effect of the additional bias is negligible.

Using $B_0 = 7$, we implement the algorithm of Section 4.4 (see Figure 6). As expected, it makes much faster progress than the algorithm of Section 3.3. After 150,000 iterations steps the average reward is roughly equal to 0.87, and after $1 \cdot 10^6$ iterations the average reward is 0.8792 (which is even slightly higher than the one obtained with the algorithm of Section 3.3).

8 Conclusions

We have proposed two approaches to reduce the variance of the updates in a simulation-based method for optimizing Markov reward processes that depend on a parameter vector. The resulting algorithms introduce an additional bias into the update direction, for which certain bounds were derived. In addition, we carried out a convergence analysis and showed that when the bias is small, then the algorithms will infinitely often lead to policies at which the gradient of average reward is small, and can therefore be expected to be close to a local optimum. The numerical results for an admission control problem are encouraging: compared with the original algorithm, the modified algorithms obtain essentially the same average reward, but converge much faster.

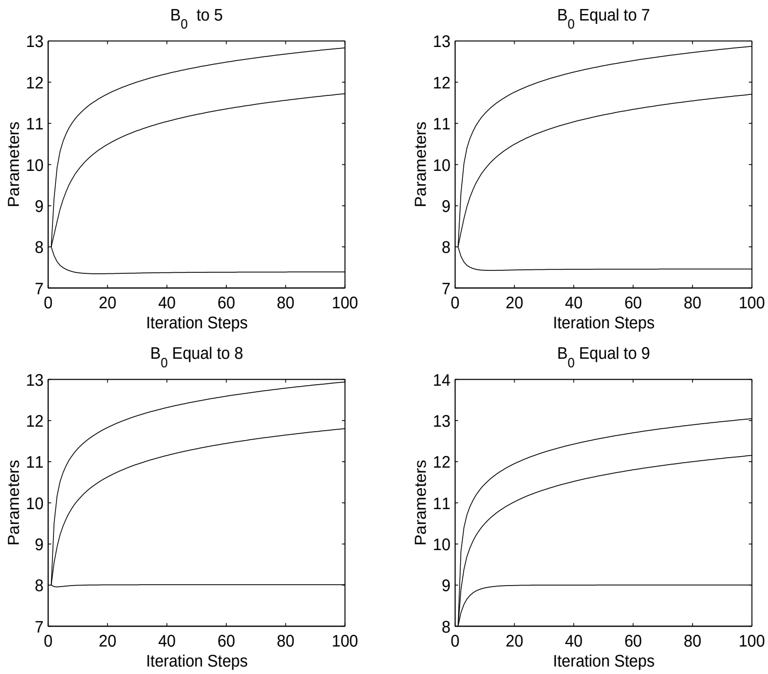

Figure 5: Parameters $\theta_k(1)$, $\theta_k(2)$, and $\theta_k(3)$, of the deterministic version of the algorithm of Section 4.4 for B_0 equal to 5, 7, 8 and 9.

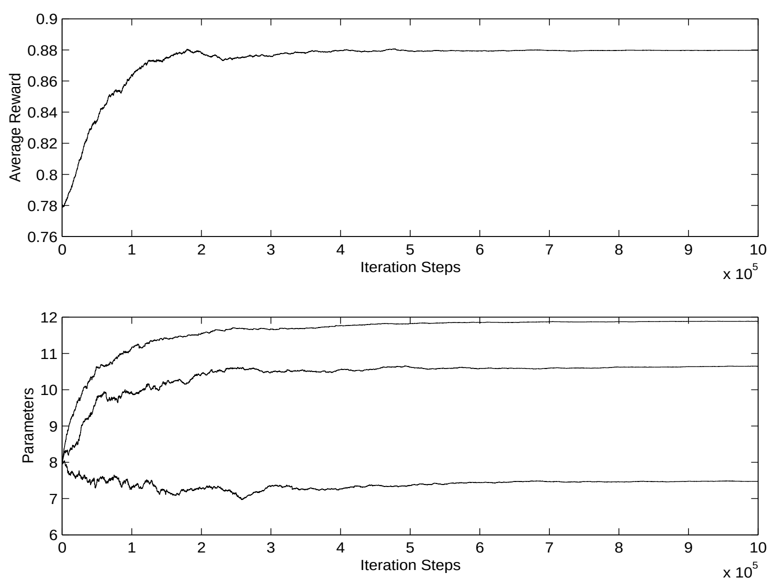

Figure 6: Parameters $\theta_k(1)$, $\theta_k(2)$, and $\theta_k(3)$, and estimates of the average reward $\tilde{\lambda}_k$, of the algorithm of Section 4.4 for $B_0 = 7$.

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A Proof of Proposition 5

In this section, we analyze the algorithm proposed in Section 4 (the same analysis applies verbatim to the algorithm of Section 5). We will take the same approach as in [MT98] (and accordingly omit those parts that are identical to the proof in [MT98]). In particular, we will use a few different Lyapunov functions to analyze the algorithm in different regions.

Before we start with the proof of Proposition 5, we introduce some additional notation and definitions. Recall the update equations

$$\begin{aligned} \theta_{m+1} &= \theta_m + \gamma_m F_{S^*,m}(\theta_m, \tilde{\lambda}_m), & (15) \\ \tilde{\lambda}_{m+1} &= \tilde{\lambda}_m + \eta\gamma_m \sum_{n=t_m}^{t_{m+1}-1} (g_{i_n}(\theta_m) - \tilde{\lambda}_m), & (16) \end{aligned}$$

where the estimate $F_{S^*,m}(\theta_m, \tilde{\lambda}_m)$ of the gradient $\nabla\lambda(\theta_m)$ is defined by Eq. (7) in Section 4. We rewrite (15) and (16) as

$$\begin{aligned} \theta_{m+1} &= \theta_m + \gamma_m \Big(E_{\theta_m}[T]\nabla\lambda(\theta_m) + E_{\theta_m}[T]\sigma_{S^*}(\theta_m) + G_{S^*}(\theta_m)(\lambda(\theta_m) - \tilde{\lambda}_m) \Big) + \varepsilon_{\theta,m}, \\ \tilde{\lambda}_{m+1} &= \tilde{\lambda}_m + \eta\gamma_m E_{\theta_m}[T](\lambda(\theta_m) - \tilde{\lambda}_m) + \varepsilon_{\lambda,m}, \end{aligned}$$

where $G_{S^*}(\theta)$ and $\sigma_{S^*}(\theta)$ are defined in Propositions 3 and 4 in Section 4, and

$$\begin{aligned} \varepsilon_{\theta,m} &= \gamma_m \left(F_{S^*,m}(\theta_m, \tilde{\lambda}_m) - E_{\theta_m}\left[F_{S^*,m}(\theta_m, \tilde{\lambda}_m) \right] \right), \\ \varepsilon_{\lambda,m} &= \eta\gamma_m \left(\sum_{n=t_m}^{t_{m+1}-1} (g_{i_n}(\theta_m) - \tilde{\lambda}_m) - E_{\theta_m}[T](\lambda(\theta_m) - \tilde{\lambda}_m) \right). \end{aligned}$$

Let us establish some properties of $G_{S^*}(\theta)$, $\varepsilon_{\theta,m}$, and $\varepsilon_{\lambda,m}$. The following lemma states that $G_{S^*}(\theta)$ is a bounded function of θ.

Lemma 2 *Let Assumption 1 (Recurrence) and Assumption 2 (Regularity) hold, then there exists a constant L such that, for all $\theta \in \Re^K$, we have*

$$\|G_{S^*}(\theta)\| \leq L.$$

Proof: Recall the definition of $G_{S^*}(\theta)$, namely

$$G_{S^*}(\theta) = E_\theta \left[\sum_{n=t_m+1}^{t_{m,1}-1} (t_{m,k+1} - n) L_{i_{n-1}j_n}(\theta) + \sum_{k=1}^{\kappa(m)} \sum_{n=t_{m,k}}^{t_{m,k+1}-1} (t_{m,k+1} - n) L_{i_{n-1}j_n}(\theta) \right].$$

Let C be a bound on $\left\| L_{ij}(\theta) \right\|$, which exists by Assumption 2 (Regularity). Then we obtain

$$\|G_{S^*}(\theta)\| \quad \leq \quad C E_\theta \left[(t_{m+1} - t_m)^2 \right].$$

By Lemma 1, the expectation $E_\theta \left[(t_{m+1} - t_m)^2 \right]$ is bounded, and the result follows. □

Similar to [MT98], we define the augmented parameter vector $r_m = (\theta_m, \tilde{\lambda}_m)$, and write the update equations in the form

$$r_{m+1} = r_m + \gamma_m (h(r_m) + \rho(r_m)) + \varepsilon_m,$$

where

$$\begin{aligned} h(r_m) &= = \begin{bmatrix} E_{\theta_m}[T] \nabla \lambda(\theta_m) + G_{S^*}(\theta_m)(\lambda(\theta_m) - \tilde{\lambda}_m) \\ \eta E_{\theta_m}[T](\lambda(\theta_m) - \tilde{\lambda}_m) \end{bmatrix}, \\ \rho(r_m) &= \begin{bmatrix} E_{\theta_m}[T] \sigma_{S^*}(\theta_m) \\ 0 \end{bmatrix}, \\ \varepsilon_m &= \begin{bmatrix} \varepsilon_{\theta,m} \\ \varepsilon_{\lambda,m} \end{bmatrix}. \end{aligned}$$

Also, we define the set $\mathcal{D}_c = \{(\theta, \tilde{\lambda}) \in \Re^{K+1} \mid |\tilde{\lambda}| \leq c\}$, and the set Φ, which contains all functions $\phi : \Re^{K+1} \mapsto \Re$ that are twice differentiable and which have the property that, for every $c \geq 0$, $\nabla \phi$ and $\nabla^2 \phi$ are bounded on $\mathcal{D}_c$. For $\phi \in \Phi$, let $\varepsilon_m(\phi)$ be given by

$$\varepsilon_m(\phi) = \phi(r_{m+1}) - \phi(r_m) - \gamma_m \nabla \phi(r_m) \cdot (h(r_m) + \rho(r_m)).$$

By slightly adapting the martingale argument given in [MT98], we obtain the following lemma.

Lemma 3 *For every function $\phi \in \Phi$, the series $\sum_m \varepsilon_m(\phi)$ converges with probability 1.*

We now proceed with the main body of the proof of Proposition 5. We will concentrate on a single sample path for which the sequence $\varepsilon_m(\phi)$ (for the Lyapunov functions to be considered) is summable. Accordingly, we will be omitting the "with probability 1" qualification.

We show in the next lemma that when $\|\nabla \lambda(\theta_m)\|$ is nonzero, and the two quantities $\|\sigma_{S^*}(\theta_m)\|$ and $|\lambda(\theta_m) - \tilde{\lambda}_m|$ are small enough, then the difference $\lambda(\theta_m) - \tilde{\lambda}_m$ increases. Remember that there exists a constant L such that, for all $\theta \in \Re^K$,

$$\|G_{S^*}(\theta)\| \leq L.$$

Lemma 4 *Let L be such that $\|G_{S^*}(\theta)\| \leq L$, for all $\theta \in \Re^K$. For $\kappa \geq 0$, let*

$$B(\theta, \kappa) = \frac{E_\theta[T] \|\nabla \lambda(\theta)\| \Big(\|\nabla \lambda(\theta)\| - \|\sigma_{S^*}(\theta)\| \Big) - \kappa}{\eta E_\theta[T] + \|\nabla \lambda(\theta)\| L}$$

and let

$$\phi(r) = \phi(\theta, \tilde{\lambda}) = \lambda(\theta) - \tilde{\lambda}.$$

We have $\phi \in \Phi$. Furthermore, if $B(\theta, \kappa) > 0$ and $|\tilde{\lambda} - \lambda(\theta)| \leq B(\theta, \kappa)$, then

$$\nabla \phi(r) \cdot \Big(h(r) + \rho(r) \Big) \geq \kappa.$$

Proof: The fact that $\phi \in \Phi$ is a consequence of Lemma 1. We have

$$\begin{aligned}
\nabla\phi(r)\cdot\big(h(r)+\rho(r)\big) &= \begin{pmatrix} \nabla\lambda(\theta) \\ -1 \end{pmatrix} \cdot \begin{pmatrix} E_\theta[T]\nabla\lambda(\theta) + G_{S^*}(\theta)(\lambda(\theta)-\tilde{\lambda}) + E_\theta[T]\sigma_{S^*}(\theta) \\ \eta E_\theta[T](\lambda(\theta)-\tilde{\lambda}) \end{pmatrix} \\
&= -\eta E_\theta[T]\Big(\lambda(\theta)-\tilde{\lambda}\Big) + E_\theta[T]\|\nabla\lambda(\theta)\|^2 \\
&\quad +\Big(\lambda(\theta)-\tilde{\lambda}\Big)\nabla\lambda(\theta)\cdot G_{S^*}(\theta) + E_\theta[T]\nabla\lambda(\theta)\cdot\sigma_{S^*}(\theta) \\
&\geq -\eta E_\theta[T]|\lambda(\theta)-\tilde{\lambda}| + E_\theta[T]\|\nabla\lambda(\theta)\|^2 \\
&\quad -L|\lambda(\theta)-\tilde{\lambda}|\|\nabla\lambda(\theta)\| - E_\theta[T]\|\nabla\lambda(\theta)\|\|\sigma_{S^*}(\theta)\| \\
&= -|\lambda(\theta)-\tilde{\lambda}|\Big(\eta E_\theta[T] + L\|\nabla\lambda(\theta)\|\Big) \\
&\quad +E_\theta[T]\|\nabla\lambda(\theta)\|\Big(\|\nabla\lambda(\theta)\| - \|\sigma_{S^*}(\theta)\|\Big).
\end{aligned}$$

Note that when $B(\theta,\kappa) > 0$ and $|\tilde{\lambda}-\lambda(\theta)| \leq B(\theta,\kappa)$, then we have

$$\nabla\phi(r)\cdot\Big(h(r)+\rho(r)\Big) \geq \kappa.$$

□

By the same argument as in [MT98], we obtain the following lemma.

Lemma 5 *We have* $\liminf_{m\to\infty} |\lambda(\theta_m) - \tilde{\lambda}_m| = 0.$

We are now ready to prove Proposition 5.

Proof of Proposition 5: We assume that Proposition 5 is not true and proceed in two steps as follows.

(1) We show that when the proposition is not true, then there exists a constant $\beta > 0$ (which we define below), such that

$$\limsup_{m\to\infty} |\lambda(\theta_m) - \tilde{\lambda}_m| \geq \beta. \tag{17}$$

(2) Using the result of Step (1), we derive a contradiction.

We introduce some notation. When Proposition 5 is not true, then there exists a constant $\epsilon > 0$, and an integer M, such that

$$\|\nabla\lambda(\theta_m)\| > D + \epsilon, \qquad \text{for all } m \geq M.$$

Let B be a bound on $\|\nabla\lambda(\theta)\|$, and let $T_{\min}$ and $T_{\max}$ be such that, for all $\theta \in \Re^K$, we have

$$1 \leq T_{\min} \leq E_\theta[T] \leq T_{\max}.$$

Such constants exist by Lemma 1. Furthermore, let $D' = D + \epsilon$, and let

$$\beta = T_{min} D' \epsilon / (\eta T_{max} + LB),$$

where L is the bound on $\|G_{S^*}(\theta)\|$ used in the statement of the proposition.

Step (1): Suppose that the proposition is not true and, furthermore, that the condition given by Eq.(17) does not hold. Then there exists an integer M_0, and a scalar $\kappa > 0$, such that, for all $m > M_0$,

$$|\lambda(\theta_m)-\tilde{\lambda}_m| \leq \beta - \frac{\kappa}{\eta T_{\max} + BL} \leq \frac{T_{min} D' \epsilon - \kappa}{\eta T_{\max} + BL} \leq \frac{E_{\theta_m}[T] \|\nabla\lambda(\theta_m)\| \Big(\|\nabla\lambda(\theta_m)\| - \|\sigma_{S^*}(\theta_m)\| \Big) - \kappa}{\eta E_{\theta_m}[T] + \|\nabla\lambda(\theta_m)\| L}.$$

Therefore, Lemma 4 applies and we obtain, for $m > M_0$, that

$$\begin{aligned} \phi(r_{m+1}) &= \phi(r_m) + \gamma_m \nabla\phi(r_m) \cdot \Big(h(r_m) + \rho(r_m) \Big) + \varepsilon_m(\phi) \\ &\geq \phi(r_m) + \gamma_m \kappa + \varepsilon_m(\phi). \end{aligned}$$

As $\lim_{m\to\infty} \varepsilon_m(\phi) = 0$ and $\sum_{m=1}^{\infty} \gamma_m = \infty$, it follows that $\phi(r_m) = \lambda(\theta_m) - \tilde{\lambda}_m$ diverges. This is a contradiction to Lemma 5, which states that

$$\liminf_{m\to\infty} |\lambda(\theta_m) - \tilde{\lambda}_m| = 0.$$

Step (2): Suppose that Proposition 5 is not true. Using the result of Step (1), together with Lemma 5, it follows that there are infinitely many pairs n, n', with $n' > n$, such that

$$\phi(r_{n'}) - \phi(r_n) = \Big(\lambda(\theta_{n'}) - \tilde{\lambda}_{n'} \Big) - \Big(\lambda(\theta_n) - \tilde{\lambda}_n \Big) < -\frac{1}{2}\beta,$$

and, for $m = n, \ldots, n'-1$,

$$\begin{aligned} \|\nabla\lambda(\theta_m)\| &> D', \\ |\lambda(\theta_m) - \tilde{\lambda}_m| &< \beta. \end{aligned}$$

This implies that, for $m = n, \ldots, n'-1$, we have

$$\|\nabla\lambda(\theta_m)\| - \|\sigma_{S^*}(\theta_m)\| \geq \epsilon > 0$$

and

$$\begin{aligned} |\lambda(\theta_m) - \tilde{\lambda}_m| &< \beta \\ &= \frac{T_{min} D' \epsilon}{\eta T_{max} + BL} \\ &\leq \frac{E_{\theta_m}[T] \|\nabla\lambda(\theta_m)\| \Big(\|\nabla\lambda(\theta_m)\| - \|\sigma_{S^*}(\theta_m)\| \Big)}{\eta E_{\theta_m}[T] + \|\nabla\lambda(\theta_m)\| L}. \end{aligned}$$

Therefore, Lemma 4 applies, and we obtain, for $m = n, \ldots, n'-1$, that

$$\nabla\phi(r_m) \cdot \big(h(r_m) + \rho(r_m) \big) \geq 0.$$

Combining these results, we have

$$\begin{aligned} -\frac{1}{2}\beta &> \phi(r_{n'}) - \phi(r_n) \\ &= \sum_{m=n}^{n'-1} \Big(\gamma_m \nabla\phi(r_m) \cdot \big(h(r_m) + \rho(r_m) \big) + \varepsilon_m(\phi) \Big) \geq \sum_{m=n}^{n'-1} \varepsilon_m(\phi). \end{aligned} \tag{18}$$

By Lemma 3, the series $\sum_m \varepsilon_m(\phi)$ converges and the term $\left\|\sum_{m=n}^{n'-1} \varepsilon_m(\phi)\right\|$ becomes arbitrarily small. This leads to a contradiction in Eq. (18) and completes the proof. $\square$