INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET AUTOMATIQUE

Dither in Systems with Hysteresis

Pierre-Alexandre Bliman, Alexander M. Krasnosel'skii, Michel Sorine

N° 2690

Octobre 1995

PROGRAMME 5

Traitement du signal,

automatique

et productique

Rapport de recherche

1994

ISSN 0249-6399

Dither in Systems with Hysteresis

Pierre-Alexandre Bliman, Alexander M. Krasnosel'skii, Michel Sorine

Programme 5 — Traitement du signal, automatique et productique
Projet Sosso

Rapport de recherche n° 2690 — Octobre 1995 — 43 pages

Abstract: This paper deals with differential inclusion containing an hysteresis nonlinearity and two inputs: a control input and a dither input of high frequency. Conditions are introduced under which its solution admits asymptotic behavior when the dither frequency goes to infinity. According to asymptotic growth of the dither amplitude, two different behaviors appear: the nonlinearity is smoothed (resp. quenched) if the velocities induced by the dither are asymptotically bounded (resp. unbounded). Convergence results for finite and infinite time intervals are given, and linked with the averaging principle. The case of bounded dithering velocities is of interest in a mechanical context, where hysteresis is used to model dry friction. A very interesting feature is that the averaged hysteresis operator may be linearized for small velocities. The hypotheses on the dither include periodicity, F-repetitiveness and (asymptotic) almost-periodicity.

Key-words: hysteresis operators, differential inclusions, averaging principle, almost-periodic functions, dither, dry friction

(Résumé : tsvp)

The first and third authors are with INRIA Rocquencourt, pierre-alexandre.bliman@inria.fr, michel.sorine@inria.fr. The second author is with Institute for Information Transmission Problems, 19 Bol. Karetny per., 101447 Moscow, Russia, amk@ippi.msk.su. His work was partially supported by Grant # 93-01-00884 of Russian Foundation of Fundamental Researchs.
This work was partly done during the visits of A.M. Krasnosel'skii to INRIA Rocquencourt.
This paper is submitted for publication in SIAM Journal on Applied Mathematics

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
Téléphone : (33 1) 39 63 55 11 – Télécopie : (33 1) 39 63 53 30

Dither dans des systèmes à hystérésis

Résumé : Nous considérons dans cet article une inclusion différentielle contenant une nonlinéarité hystérétique et deux entrées: une entrée de commande et une entrée de fréquence élevée, nommée *dither*. On introduit des conditions sous lesquelles la solution de ce probleme admet a un comportement asymptotique simple quand la fréquence du *dither* tend vers l'infini. Selon la croissance asymptotique de l'amplitude *dither*, deux comportements différents peuvent apparaître: la nonlinéarité est régularisée (resp. annulée) si les vitesses induites par le *dither* sont asymptotiquement bornées (resp. non bornées). Des résultats de convergence sur horizons fini et infini sont donnés et reliés aux résultats classiques de moyennisation. Le cas de vitesses bornées est d'intérêt dans un contexte mécanique, dans lequel l'hystérèsis est utilisé pour modéliser du frottement sec. Une caractéristique très intéressante de ce cas-là est le fait que l'opérateur d'hystérèsis moyenné peut-être linéarisé aux faibles vitesses. Les hypothèses sur le *dither* font intervenir la périodicité, la F-répétitivité ou la presque-périodicité.

Mots-clé : opérateurs d'hystérèsis, inclusions différentielles, moyennisation, fonctions presque-périodiques, dither, frottement sec,

Contents

1 Introduction

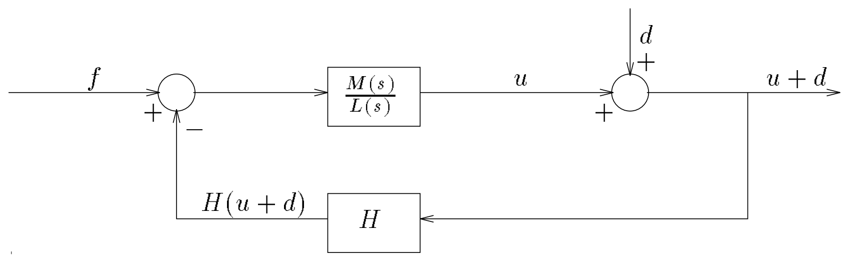

Figure 1

We consider the following differential inclusion in the scalar function $u(t)$:

$$L\left(\frac{d}{dt}\right) u \in M\left(\frac{d}{dt}\right)(-H(u+d)+f), \quad (\, u, \quad u', \quad \dots \quad u^{(l-1)}\,)(0) = (\, u_0, \quad u_1, \quad \dots \quad u_{l-1}\,) \tag{1}$$

where L and M are real coprime polynomials of order l and m respectively, $H(u)$ is a nonlinear operator of hysteresis type proposed to model dry friction [5, 6, 7], f is a control input term and d a so-called *dither function* of "high frequency" (see Figure 1). Due to the differential and possibly multivalued nature of H, equation (1) may indeed be considered as a differential inclusion.

The injection of dither signals to change the characteristics of nonlinear elements is known for a long time. This technique is used to increase stability, in particular to quench limit cycles, or to smoothen or linearize the nonlinearities. Concluding experiments have been conducted in systems with relays [19], with dry friction [4, 14, 15, 16], with rotative amplifiers [1], with DC motors [24]. Rigorous statements have also been proposed, concerning systems with memoryless nonlinearities [27, 28], backlash [12] or a special class of hysteresis [12] (a particular subclass of Duhem model, according to the nomenclature in [25]). For a recent bibliography on experiments (resp. theory), see [24] (resp. [12]). For details on hysteresis operators, see [18, 25].

Qualitatively, the key idea for using dither is the following: if d is e.g. a "fast" periodic function, one expects the input-output map $f \mapsto u$ to be closely related to the corresponding map for the equation (see Figure 2)

$$L\left(\frac{d}{dt}\right) u \in M\left(\frac{d}{dt}\right)(\bar{H}(u)+f), \quad (\, u, \quad u', \quad \dots \quad u^{(l-1)}\,)(0) = (\, u_0, \quad u_1, \quad \dots \quad u_{l-1}\,) \tag{2}$$

where the operator $\bar{H}(u)$ is defined[1] by [12]:

$$\bar{H}(u)(t) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T H(u(t)+d(s))\cdot ds$$

[1] This definition is valid for memoryless nonlinearity H, but may be as well extended to some nonlinearities with memory.

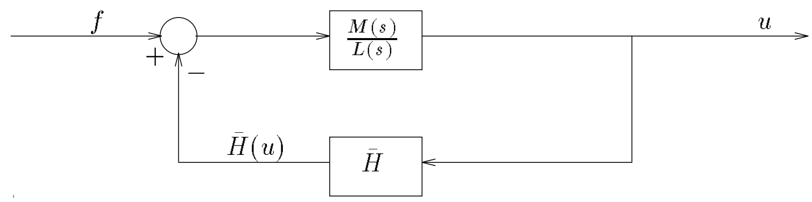

Figure 2

In [27, 28, 12], links between the behavior of systems (1) and (2) are studied. In these papers it is shown that under appropriate hypotheses, stability property (boundedness or Lipschitz continuity of the input-output map $f \mapsto u$ in adequate functional spaces) for (1) may be *deduced* from the same property for (2). The hypotheses state that the period of the dither (or more generally the maximal repetition interval [27]) is small enough, that the amplitude of the dither is large enough and that the tranfer function M/L has a low-pass filter property. Stability is improved, due to the fact that the sectors or incremental sectors for $\bar{H}$ are smaller than those of H.

We are here interested in the asymptotic behaviour of the solution u when the period of the dither goes to zero. We focus on periodic, F-repetitive and almost-periodic (more precisely asymptotic almost-periodic) dither. Related problems are very usual in asymptotic methods and are considered for instance by the averaging principle and its various modifications. The unusual facts here are the presence of a multivalued term and of a nonlinearity with memory.

According to the asymptotic growth of the amplitude dither when the frequency goes to infinity, one may have two different asymptotic behaviors.

When the amplitude growth is such that the velocity[2] $\dot{d}$ remains bounded when its frequency goes to infinity, u tends towards the solution of a nonlinear ordinary differential equation, where $H(u)$ has been replaced by another operator. However, we show that this operator is smoother (in particular, it is singlevalued): the hysteresis operator has been *smoothed*. The convergence holds on every compact set of $\mathbb{R}^+$, (as in [20], where periodic dithers and memoryless nonlinearities are considered), but an additional hypothesis (essentially a persistence hypothesis on the limit solution plus an incremental sector inequality) provides convergence on $\mathbb{R}^+$. This case is solvable with the classical averaging principle [23, 8] when the hysteresis operator is singlevalued.

A very interesting property of the preceding model is that it is *linear* for small velocities, and this linearization is indeed *exact* if the dither is chosen adequately. In the paper [10] by P. Contensou or in the book [21, chapter IV] by Ju.I. Neĭmark and al., same kind of results are presented in the context of gyroscopic motion, where the dither is created by combining sliding and spinning of an axis[3]. It shows linearization of Coulomb friction model at low

[2]Differentiation is denoted by a prime, and by a dot when one wants to emphasize that it is carried out wrt the time variable t.

[3]The authors are indebted to Professor V. Zhuravlev, from Russian Academy of Sciences, for drawing their attention on the paper of P. Contensou.

velocity. In the present paper, if H is chosen to model static friction, the linearized version of the averaged operator still gives rise to overshoots, and the friction may decrease as a function of velocity. This phenomenon, a transient version of the Stribeck effect[4], is hence preserved by the linearization.

In the case where the velocity $\dot{d}$ is unbounded when its frequency goes to infinity, u tends towards the solution of a linear ordinary differential equation: the influence of the nonlinear operator $H(u)$ has been *quenched*. The stability of the linear part of the equation together with a property of uniform convergence of $H(u)$ when the frequency of d tends to infinity, allow us to consider uniform convergence on the half-line $\mathbb{R}^+$.

Also, we are interested in the output $u+d$, not only u. So in our setting, the choice of the dither is made in such a way that 1. d is sufficiently powerful to induce the expected averaging mechanism, but 2. d should stay small in order that $u+d$ remains bounded: it is a "small dither problem". As the operator H we consider is continuous in $W^{1,1}$ but not in C^0, these specifications may be fulfilled if one takes a sequence d_n converging towards 0 in C^0, but not in $W^{1,1}$: $u_n + d_n$ will also converge in C^0. Hence, the "averaging principle" is based on the gap between the regularity of the nonlinearity and the regularity of the expected convergence of the solutions $u+d$.

The paper is organized as follows: in Section 2 we give the description of the hysteresis nonlinearity $H(u)$ and its main properties, and formulate a theorem on well-posedness of (1). In Section 3 the main results are formulated (Theorems 4, 6 and 9). Section 4 shows how the assumptions on the dither may be fulfilled with periodic, F-repetitive or asymptotic almost-periodic functions. Various proofs are sent back to Appendix.

2 Well-posedness of a linear ode with an hysteresis nonlinearity

The hysteresis operator H is defined by

$$H(u) \triangleq Cx + D\text{sgn}\ \dot{u}, \quad \dot{x} = Ax|\dot{u}| + B\dot{u}, \quad x(0) = x_0 \in \mathbb{R}^N \tag{3}$$

where the matrices $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times 1}$, $C \in \mathbb{R}^{1\times N}$ and the real number D are given, and sgn is the multivalued operator defined by

$$\text{sgn}\ x = \begin{cases} 1, & x > 0 \\ [-1,1], & x = 0 \\ -1, & x < 0 \end{cases}$$

The operator H contains two parts: the smooth term Cx and the multivalued term $D\text{sgn}\ \dot{u}$. When modeling dry friction, the latter is the Coulomb friction force, and the former is an elastic regularization of this force. This simple model permits to reproduce various

[4]The usual Stribeck effect corresponds to a decreasing friction with velocity at steady state.

observed phenomena [6]. In particular, if the transfer function $C(sI-A)^{-1}B+D$ is positive real, the map $\dot{u} \mapsto H(u)$ is dissipative. We define $H_{sv}(u) = Cx$ (sv as "singlevalued").

We borrow from [5] the following result[5] (see this reference for a proof):

Theorem 1 (Properties of H) *The map $H_{sv} : u \mapsto Cx$ where x is given by* (3) *is locally Lipschitz in $W^{1,p}_{loc}(0,+\infty)$ for every $1 \le p \le +\infty$. Moreover, if*

(H1) *The matrix A is stable*

then

$$\sup_{u \in W^{1,p}_{loc}(0,+\infty)} \sup\text{ess}_{s \ge 0} |H(u)(s)| < +\infty$$

the precise bound depending upon A, B, C, D and the initial state value x_0.

These properties of the operator H allows to solve the Cauchy problem (1). Here and in the sequel, we shall denote l (resp. m) the degree of L (resp. M).

The following result is proved in Appendix:

Theorem 2 (Well-posedness of (1)**)** *Suppose that* **(H1)** *holds, that*

(H2) *The transfer function $M(s)/L(s)$ is stable with $l - m \ge 2$*

If $f \in L^1_{loc}(0,+\infty)$ and $d \in W^{1,1}_{loc}(0,+\infty)$, then there exists a solution $u \in W^{l-m,1}_{loc}(0,+\infty)$ to equation (1). *Moreover, if*

(H3) $D = 0$, *or* $l - m = 2$ *and* $D \cdot \lim_{s \to +\infty} s^{l-m} \dfrac{M(s)}{L(s)} > 0$

this solution is unique and continuous wrt the initial conditions and to f and d.

(H2) contains a low-pass filter assumption. **(H3)** is needed, in the case $D \neq 0$, to ensure that the multivalued term in the hysteresis operator is monotone (the limit appearing in **(H3)** is the first non zero derivative at the origin of the impulse response of $M(s)/L(s)$: the $l - m - 1$ order derivative).

3 Main results

3.1 Averaged hysteresis operators

We illustrate here the key ideas of the averaging, using a simpler form of dither. The precise assumptions on the dither are given in Subsection 3.2.

[5] We recall the definition of some Sobolev spaces: for any integer n, any $p \ge 1$ (or $+\infty$) and any real interval $\mathcal{J}$,

$$W^{n,p}(\mathcal{J}) \triangleq \{u : u, u' \dots u^{(n)} \in L^p(\mathcal{J})\}$$

and we define:

$$X_{loc}(\mathcal{J}) \triangleq \{u : \forall \mathcal{K} \text{ compact }, u \in X(\mathcal{J} \cap \mathcal{K})\}$$

where X denotes e.g. $L^p, C^0, W^{n,p}$. A property will be said true in $X_{loc}(\mathcal{J})$ if it is true in $X(\mathcal{J} \cap \mathcal{K})$ for any compact set $\mathcal{K}$.

We consider here dithers of the form $d_n = n^{-\gamma} g(nt)$ where g is an auxiliary function, on which typical assumptions will be: periodicity, F-repetitiveness, asymptotic almost-periodicity (see Section 4), n is "large enough" and $\gamma \in \mathbb{R}$. Indeed, one is led to take more restrictively $\gamma \geq 0$ (otherwise, the dither is not bounded), and $\gamma \leq 1$ (otherwise, the dither tends to zero in $W^{1,1}$ and is ineffective, as explained in Introduction).

In order to understand the averaging process, let us examinate briefly the equation

$$\dot{x} = Ax|\dot{u} + n^{1-\gamma} g'(nt)| + B(\dot{u} + n^{1-\gamma} g'(nt))$$

Writing $\bar{x}(t) \triangleq x(t/n)$, one deduces

$$\dot{\bar{x}} = \frac{1}{n} A\bar{x}|\dot{u}(t/n) + n^{1-\gamma} g'(t)| + \frac{1}{n} B(\dot{u}(t/n) + n^{1-\gamma} g'(t))$$

In the case where $\gamma = 1$ (which corresponds to bounded dithering velocity), this writes as

$$\dot{\bar{x}} = \varepsilon f(\bar{x}, \varepsilon, t, \varepsilon t)$$

where $\varepsilon = 1/n$, a form which is classical in the context of averaging [23, 3, 8]. In view of the classical results, we then define the operator $\bar{H}$:

$$\forall u \in W^{1,1}_{loc}(0, +\infty), \quad \bar{H}(u) \triangleq Cx + D\bar{s}(\dot{u}), \quad \dot{x} = Ax\bar{g}(\dot{u}) + B\dot{u}, \quad x(0) = x_0 \in \mathbb{R}^N \tag{4}$$

where we suppose the existence of the following mean values, for any $\alpha \in \mathbb{R}$:

$$\bar{g}(\alpha) \triangleq \lim_{T \to +\infty} \frac{1}{T} \int_0^T |\alpha + g'(s)| \cdot ds, \quad \bar{s}(\alpha) \triangleq \lim_{T \to +\infty} \frac{1}{T} \int_0^T \text{sgn}\,(\alpha + g'(s)) \cdot ds \tag{5}$$

Remark that $\bar{H}$ defined by (4), (5) is not a hysteresis operator in the sense of [25]: it is not rate-independent. However, $\bar{H}$ is single-valued (due to Hypothesis **(D4)** below): the nonlinearity has been smoothed.

The rate-dependence of $\bar{H}$ is evident on its steady state behavior[6], corresponding to constant velocities $\dot{u}$:

$$\bar{H}_{ss}(u) = -CA^{-1}B\frac{\dot{u}}{\bar{g}(\dot{u})} + D\bar{s}(\dot{u}) \tag{6}$$

Under the hypothesis that $\bar{g}'(0) = \bar{s}(0) = 0$, an approximation of $\bar{H}$ for small values of $\dot{u}$ is given by the following linear time-invariant operator

[6]Formulas like (6), deriving from a differential model of friction, have been proposed by C. Canudas de Wit and al. [2] to model steady state Stribeck effect. In our case, $\bar{g}(\dot{u}) - \dot{u}\bar{s}(\dot{u}) < 0$ for some $\dot{u}$ would be necessary to have such decreasing friction with increasing velocity: this is impossible (because $\bar{g}(\dot{u}) \geq |\dot{u}|$ and $|\bar{s}(\dot{u})| \leq 1$, see Theorem 7), so that we are unable to model steady state Stribeck effect with this type of dither.

$$\forall u \in W^{1,1}_{loc}(0,+\infty), \quad \bar{H}_{lin}(u) \stackrel{\Delta}{=} Cx + D\bar{s}'(0)\dot{u}, \quad \dot{x} = \bar{g}(0)Ax + B\dot{u}, \quad x(0) = x_0 \qquad (7)$$

This linearization is stated more precisely below.

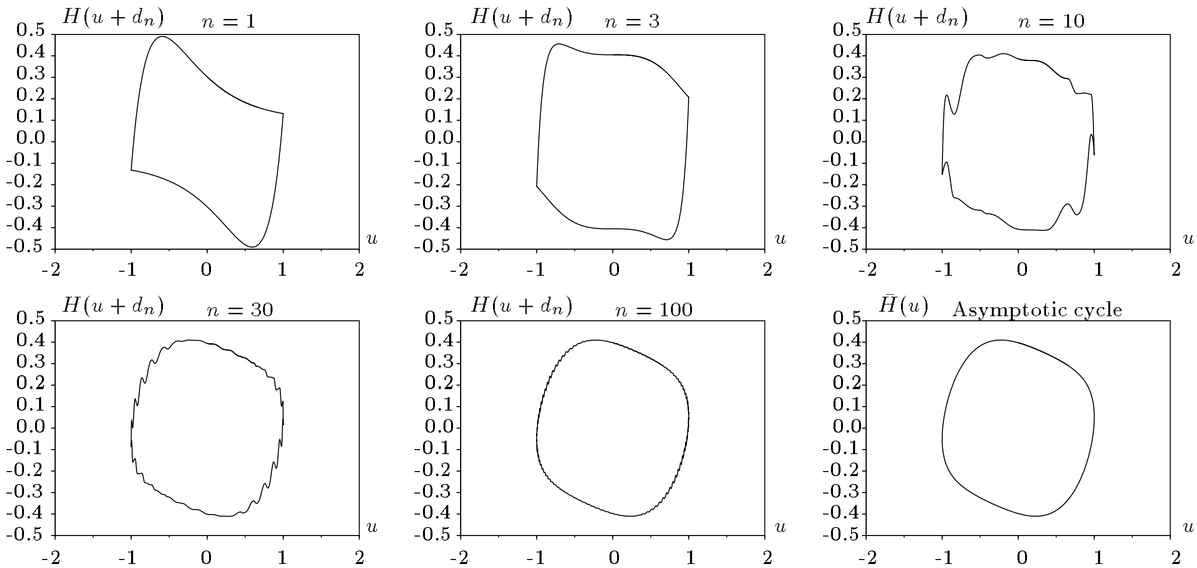

Figure 3 – Asymptotic behavior with bounded velocity

In the case where $0 < \gamma < 1$ (unbounded velocity), one has

$$\dot{\bar{x}} = \frac{1}{n^\gamma} A\bar{x} \left| \frac{\dot{u}(t/n)}{n^{1-\gamma}} + g'(t) \right| + \frac{1}{n} B(\frac{\dot{u}(t/n)}{n^{1-\gamma}} + g'(t))$$

and, taking $\varepsilon = 1/n^\gamma$, this has almost the classical form (the $\dot{u}(t/n)$ term is unusual). Assuming $\dot{u}$ and g bounded, one expects an asymptotic behavior described by

$$\dot{\bar{x}} = \frac{1}{n^\gamma} A\bar{x}\bar{g}(0)$$

that is $\bar{x}(t) = e^{An^{-\gamma}\bar{g}(0)t}x_0$. But in the "natural" time t, this leads to $x(t) = e^{An^{1-\gamma}\bar{g}(0)t}x_0$, and, as A is stable and $\bar{g}(0)$ is supposed to be strictly positive (cf. Hypothesis **(D2')**), a singular perturbation occurs, leading to $x(t) = 0$ if $t > 0$. We are led to define the (constant) operator $\bar{H}_\gamma$:

$$\forall u \in W^{1,1}_{loc}(0,+\infty), \quad \bar{H}_\gamma(u) \equiv D\bar{s}(0) \qquad (8)$$

Notice that all the operators defined here, namely $\bar{H}, \bar{H}_{lin}, \bar{H}_{ss}, \bar{H}_\gamma$, are based upon $\bar{g}$ and $\bar{s}$ (as defined by (5)), hence they depend upon the choice of the dither.

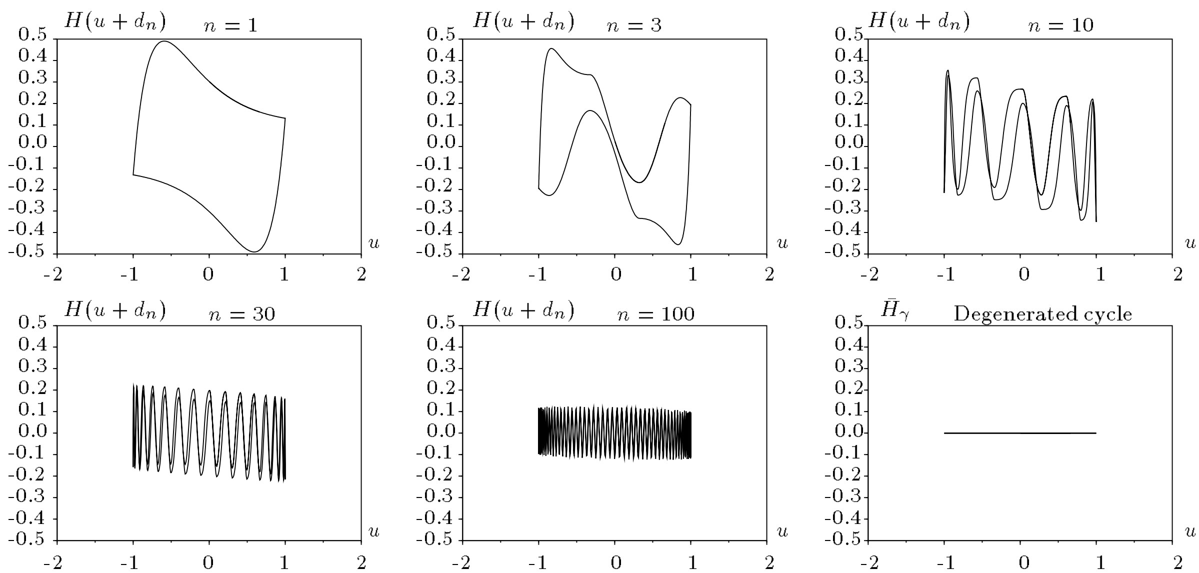

Figure 4 – Asymptotic behavior with unbounded velocity

Figures 3 and 4 intend to illustrate the two different limit behaviors. We choose for the hysteresis model

$$A = -\frac{1}{\varepsilon}\begin{pmatrix} \frac{1}{\eta} & 0 \\ 0 & 1 \end{pmatrix},\ B = \frac{1}{\varepsilon}\begin{pmatrix} \frac{f_1}{\eta} \\ -f_2 \end{pmatrix},\ C = (\ 1 \quad 1\ ),\ D = 0$$

with $\varepsilon =, \eta = 0.5, f_1 = 1, f_2 = 0.9$, a form which is used in [6, 7] to model dry friction force with static coefficient, and we take

$$g(t) \stackrel{\Delta}{=} -\cos t$$

which implies

$$\bar{g}(\alpha) = \frac{2}{\pi}(\alpha \arcsin \alpha + \sqrt{1-\alpha^2}) \text{ if } |\alpha| \leq 1,\ \ |\alpha| \text{ if } |\alpha| > 1$$

$$\bar{s}(\alpha) = \frac{2}{\pi} \arcsin \alpha \text{ if } |\alpha| \leq 1,\ \ \operatorname{sgn} \alpha \text{ if } |\alpha| > 1$$

Figure 3 shows for various values of n the hysteresis cycles $H(u+d_n)$ vs. u with $u = -\cos t$, $d_n = \frac{1}{n} g(nt)$, and the asymptotical cycle $\bar{H}(u)$ vs. u.

Figure 4 is analogous, with $d_n = \frac{1}{\sqrt{n}} g(nt)$ and the degenerated asymptotical cycle $\bar{H}_\gamma$ vs. u.

Figure 5 shows cycles $\bar{H}_{lin}(u)$ vs. u with $u = -\cos\omega t$ for different values of the frequency ω: the enclosed area vanishes with the frequency.

Figure 6 shows examples of periodic dithers, defined by various functions g, and their characteristic functions $\bar{g}$ and $\bar{s}$.

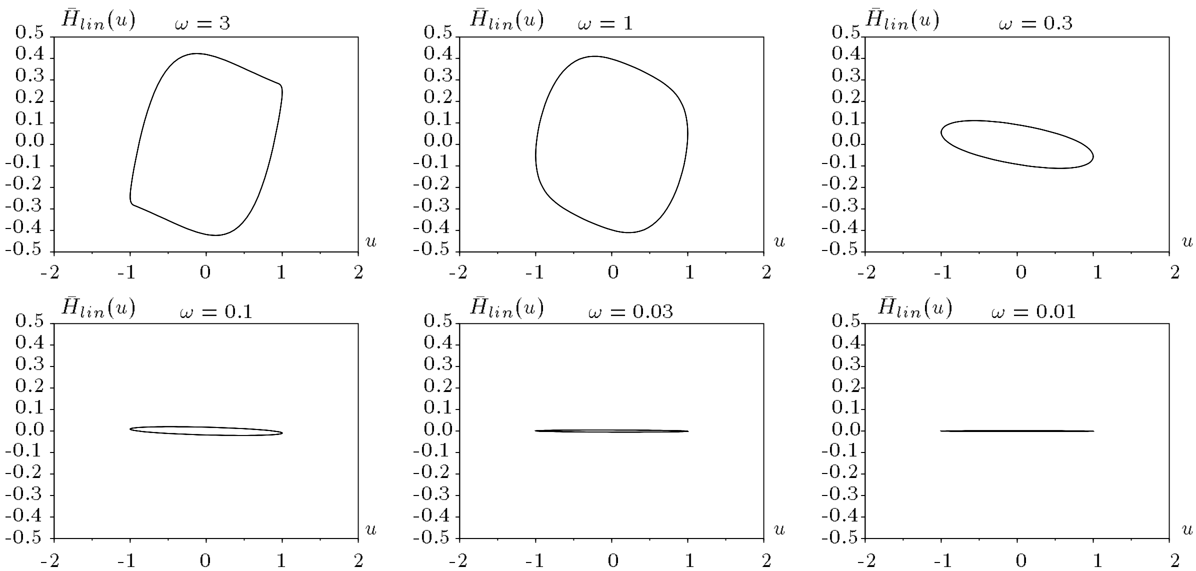

Figure 5 – Response of the linearized operator $\bar{H}_{lin}$ at various frequencies

Form of g'	Saw-tooth	Square	Sinusoidal
$g(t)$	$t(1-\lvert t\rvert)$ for $t \in [-1,+1]$	$1-2\lvert t\rvert$ for $t \in [-1,+1]$	$-\cos t$
$g'(t)$	$1-2\lvert t\rvert$ for $t \in [-1,+1]$	$-2\mathrm{sgn}\ t$ for $t \in [-1,+1]$	$\sin t$
$\bar{g}(\alpha)$	$\frac{\alpha^2+1}{2}$ if $\lvert\alpha\rvert < 1$ $\lvert\alpha\rvert$ otherwise	$\max\{\lvert\alpha\rvert, 1\}$	$\frac{2}{\pi}(\alpha \arcsin\alpha + \sqrt{1-\alpha^2})$ if $\lvert\alpha\rvert < 1$ $\lvert\alpha\rvert$ otherwise
$\bar{s}(\alpha)$	$\min\{1, \max\{\alpha, -1\}\}$	0 if $\lvert\alpha\rvert < 1$ sgn α otherwise	$\frac{2}{\pi}\arcsin\alpha$ if $\lvert\alpha\rvert < 1$ sgn α otherwise

Figure 6 – Examples of periodic dither

3.2 Assumptions on the dither

3.2.1 Dither with bounded velocity

We take dithers under the form $\frac{1}{n}g(nt)$ where g is an auxiliary function and n large enough. Some perturbations e_n of the dither are allowed. More precisely:

(D0) *Let $\bar{\delta}(T) > 0$ be defined on $\mathbb{R}^+$ such that*

$$\lim_{T\to+\infty} \bar{\delta}(T) = 0, \quad \textit{and} \quad \lim_{T\to+\infty} \frac{1}{T\bar{\delta}(T)} = 0 \ \textit{ if } D \neq 0 \tag{9}$$

(D1) *The dither function d is chosen in a sequence of functions d_n such that*

$$d_n(t) = \frac{1}{n}g(nt) + e_n(t)$$

where g is a function of $W^{1,\infty}(0,+\infty)$, and $W^{2,\infty}(0,+\infty)$ if $D \neq 0$, and for some e_n chosen such that ne_n, $\frac{1}{\bar{\delta}(n)}\dot{e}_n$ are uniformly bounded in $L^\infty(0,+\infty)$, together with $\frac{1}{n}\ddot{e}_n$ if $D \neq 0$.

(D2) *There exists a function $\bar{g}(\alpha) > 0$ defined on $\mathbb{R}$ such that*

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T |\alpha + g'(s)| \cdot ds = \bar{g}(\alpha)$$

the convergence being uniform wrt α.

(D3) *For any $\alpha \in \mathbb{R}$,*

$$\lim_{T\to+\infty} \frac{1}{T} mes\{s \in (0,T) : |\alpha + g'(s)| \leq \bar{\delta}(T)\} = 0$$

the convergence being uniform wrt α

(D4) *There exists a continuous function $\bar{s}(\alpha)$ defined on $\mathbb{R}$ such that*

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathrm{sgn}\,(\alpha + g'(s)) \cdot ds = \bar{s}(\alpha)$$

the convergence being uniform wrt α.

3.2.2 Dither with unbounded velocity

This case corresponds to the choice $\frac{1}{n\bar{\delta}(n)}g(nt)$. It necessitates the following weaker assumptions, to be compared with Hypotheses **(D0)** to **(D4)**:

(D0') *Let $\bar{\delta}(T)$ be a strictly positive function defined on $\mathbb{R}^+$ such that*

$$\lim_{T\to+\infty} \bar{\delta}(T) = \lim_{T\to+\infty} \frac{1}{T\bar{\delta}(T)} = 0 \tag{10}$$

(D1') *The dither function d is chosen in a sequence of functions d_n such that*

$$d_n(t) = \frac{1}{n\bar{\delta}(n)} g(nt) + e_n(t)$$

where g is a function of $W^{1,\infty}(0,+\infty)$, and for some e_n chosen such that $n\bar{\delta}(n)e_n$, $\dot{e}_n$ are uniformly bounded in $L^\infty(0,+\infty)$.

(D2') *There exists $\bar{g}(0) > 0$ such that*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T |g'(s)| \cdot ds = \bar{g}(0)$$

(D3') *$\bar{\delta}$ being defined in* **(D0')***, we have*

$$\lim_{T\to+\infty} \frac{1}{T} mes\left\{s \in (0,T) : |g'(s)| \leq \bar{\delta}(T)\right\} = 0$$

(D4') *There exists $\bar{s}(0)$ such that*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T \operatorname{sgn} g'(s) \cdot ds = \bar{s}(0)$$

A simple example for $\bar{\delta}$ satisfying (10) is $\bar{\delta}(T) \triangleq T^{\gamma-1}$ for $0 < \gamma < 1$, which gives $d_n = n^{-\gamma} g(nt)$, as in Subsection 3.1.

Hypotheses **(D2')** to **(D4')** and **(D2)** to **(D4)** imply the convergence of the same expressions considered in intervals $[t, t+T]$, $t \geq 0$ rather than $[0,T]$. As an example, **(D2')** (resp. **(D3')**) is equivalent to: $\forall t \geq 0$,

$$\lim_{T\to+\infty} \frac{1}{T} \int_t^{t+T} |g'(s)| \cdot ds = \bar{g}(0) \quad (\text{resp. } \lim_{T\to+\infty} \frac{1}{T} mes\left\{s \in (t,t+T) : |g'(s)| \leq \bar{\delta}(T)\right\} = 0)$$

Moreover, it is easily seen that in all these statements, the convergence is uniform wrt t in any compact set, and this fact will be extensively used. *When the convergence is indeed uniform for $t \geq 0$, we shall say that Hypotheses* **(D2'u)**, ..., **(D4'u)**, **(D2u)**, ...**(D4u)** *are fulfilled.*

There are some obvious relations between these hypotheses, e.g. **(D*i*u)** $\Rightarrow$ **(D*i*)**, $i = 2, 3, 4$.

3.3 Comments and remarks on the assumptions

• The usual statements on averaging assumes that the right-hand side of the differential equation is Lipschitz (see e.g. [23]). Under this hypothesis, condition similar to **(D2)** or **(D2')** is sufficient to get the desired result. For the sgn term present here, Hypotheses **(D4)** and **(D4')** seem natural, and the supplementary conditions **(D3)** and **(D3')** are counterpart of the Lipschitz one: they set that the set of instants where the discontinuities of the right-hand side occur, is neglectible in an adequate sense.
• Hypothesis **(D3')** implies that $mes\{s \in [0,+\infty) : \ g'(s) = 0\} = 0$, due to the following Lemma, which is a consequence of continuity property of the measure [22, Theorem 1.19]:

Lemma 3 *If q is measurable on $[0,T]$, then*

$$mes\{s \in [0,T] : q(s) = 0\} = 0 \Leftrightarrow mes\{s \in [0,T] : |q(s)| \leq \delta\} \to 0 \text{ when } \delta \to 0$$

Hence, Hypothesis **(D3')** assesses the rarity of zeros of g'. As an example, it may checked that **(D3')** is fulfilled if there exists $N \in \mathbb{N}$ for which g' has no more than N zeros on every compact interval of unitary length, with $\min_{g'(t)=0} |g''(t)|$ defined and strictly positive.

Similarly, Hypothesis **(D3)** implies that

$$\forall \alpha \in \mathbb{R}, \ \ mes\{s \in [0,+\infty) : \alpha + g'(s) = 0\} = 0$$

• It is not difficult to prove that **(D3')** implies that for any positive constant c,

$$\forall t \geq 0, \ \ \lim_{T\to+\infty} \frac{1}{T} mes\{s \in (t,t+T) : |g'(s)| \leq c\bar{\delta}(T)\} = 0$$

uniformly wrt t in compact sets.

Similarly, **(D3)** implies that for any $c > 0$,

$$\forall t \geq 0, \ \ \lim_{T\to+\infty} \frac{1}{T} mes\{s \in (t,t+T) : |\alpha + g'(s)| \leq c\bar{\delta}(T)\} = 0$$

uniformly wrt $\alpha \in \mathbb{R}$ and t in compact sets.
• The preceding remark shows that when **(D3')** (resp. **(D3)**) holds, the integrals in **(D4')** (resp. **(D4)**) are defined in a univocal way. Remark however that this is not necessary for existence of a unique limit.
• If **(D3)** and **(D4)** hold, the following definition is meaningful:

$$F_{g'}(-\alpha) = \lim_{T\to+\infty} \frac{1}{T} mes\{s \in [0,T] : g'(s) + \alpha \leq 0\}$$

$$F_{g'}(-\alpha) = \frac{1 - \bar{s}(\alpha)}{2}$$

Remark that in the definition of $F_{g'}$, one may as well replace $\leq$ by $<$, due to Hypothesis **(D3)**. $F_{g'}(-\alpha)$ generalizes the *amplitude distribution function* of G. Zames and N.A. Shneydor [27, 28] to non F-repetitive functions.

3.4 Asymptotic dither effect

Consider the differential inclusion

$$\left\{ \begin{array}{l} L\left(\frac{d}{dt}\right) u_n \in M\left(\frac{d}{dt}\right)(-H(u_n + d_n) + f), \\ \left(u_n \quad u_n' \quad \ldots \quad u_n^{(l-1)} \right)(0) = \left(u_0 \quad u_1 \quad \ldots \quad u_{l-1} \right) \end{array} \right. \tag{11}$$

Here are our main results, proved in Appendix.

3.4.1 Bounded velocity

Theorem 4 (Asymptotic behavior – Bounded velocity) *Let Hypotheses* **(H1)** *to* **(H3)**, **(D0)** *to* **(D4)** *hold. Let f be such that the solution of the equation $L(\frac{d}{dt})u = M(\frac{d}{dt})f$, $\left(u, \quad u', \quad \ldots \quad u^{(l-1)} \right)(0) = 0_l$ is in $L^\infty_{loc}(0, +\infty)$, together with its two first derivatives (a sufficient condition for this is e.g. $f \in L^\infty_{loc}(0, +\infty)$). Then*

$$u_n \text{ and } u_n + d_n \to u_\infty \text{ in } C^0_{loc}(0, +\infty) \text{ when } n \to +\infty$$

where u_n is the solution of (11) and u_∞ is the solution of the nonlinear differential equation

$$\left\{ \begin{array}{l} L\left(\frac{d}{dt}\right) u_\infty = M\left(\frac{d}{dt}\right)(-\bar{H}(u_\infty) + f), \\ \left(u_\infty \quad u_\infty' \quad \ldots \quad u_\infty^{(l-1)} \right)(0) = \left(u_0 \quad u_1 \quad \ldots \quad u_{l-1} \right) \end{array} \right. \tag{12}$$

and where the operator $\bar{H}$ is defined by (4). Besides, for any $\rho > 0$ and any $T_0 > 0$, the convergence is uniform in the set $\{f : L(\frac{d}{dt})u = M(\frac{d}{dt})f, \left(u, \quad u', \quad \ldots \quad u^{(l-1)} \right)(0) = 0_l \Rightarrow \|u\|_{L^\infty(0,T_0)}, \|\dot{u}\|_{L^\infty(0,T_0)}, \|\ddot{u}\|_{L^\infty(0,T_0)} < \rho\}$.

Moreover, if Hypotheses **(D2u)** *to* **(D4u)** *are fulfilled, if $\bar{s}$ is Lipschitz, if for any $t \geq 0$,*

$$\left\| \int_0^t e^{A \int_s^t |\ddot{u}_\infty|} \cdot ds \right\| < c < +\infty \tag{13}$$

then

$$\sup_{\substack{u \in W^{1,\infty}(0,+\infty) \\ \dot{u} - \dot{u}_\infty \neq 0 \ a.e.}} \frac{\|\bar{H}(u) - \bar{H}(u_\infty)\|_{L^\infty(0,+\infty)}}{\|\dot{u} - \dot{u}_\infty\|_{L^\infty(0,+\infty)}} < +\infty$$

and if, denoting $\mathcal{L}$ the Laplace transform,

$$\sup_{\substack{u \in W^{1,\infty}(0,+\infty) \\ \dot{u} - \dot{u}_\infty \neq 0 \ a.e.}} \frac{\|\bar{H}(u) - \bar{H}(u_\infty)\|_{L^\infty(0,+\infty)}}{\|\dot{u} - \dot{u}_\infty\|_{L^\infty(0,+\infty)}} \cdot \left\| \mathcal{L}^{-1}\left(\frac{sM(s)}{L(s)}\right) \right\|_{L^1(0,+\infty)} < 1$$

then the previous facts remain true when replacing C^0_{loc} (resp. L^∞_{loc}, T_0) by C^0 (resp. L^∞, $+\infty$).

Well-posedness of equation (12) will be proved in Theorem 7 and Corollary 8 below.

Hypothesis (13) is fulfilled if $\dot{u}_\infty$ is a non zero AAPF (see Definition 16 below), as the quantities $\frac{1}{T}\int_t^{t+T} |\dot{u}_\infty| \cdot ds$ converge towards a non zero constant, uniformly wrt $t \geq 0$. Hence, this appears as a *persistence hypothesis* on the variation of u_∞. One may conjecture that it is fulfilled if f is AAPF ...

The second term in the product of Theorem 4 last formula is the L^1 norm of the impulse response of $\frac{sM(s)}{L(s)}$.

The uniform convergence property in the Theorem permits to choose the same dither for a class of ("low frequency") input functions f. Notice that Hypothesis **(D0)** gives a related criterion of robustness wrt perturbations of the (high frequency) dither input.

The proof of Theorem 4 (see Appendix) is based on Lemma 23, which shows that for u with bounded first and second derivatives, $H(u+d_n)(t)$ converges in an adequate way towards $\bar{H}(u)$. This convergence property is uniform wrt u if the bounds are uniform, and remains true for non compact time intervals when Hypotheses **(D2u)** to **(D4u)** hold.

The following Lemma (proof in Appendix) shows how the operator $\bar{H}$ constitutes the limit behavior of the initial operator H and enlightens the link with the interpretation given in the Introduction. It states that any average of the output H tends, when the dither frequency tends to $+\infty$, towards the average of the output of $\bar{H}$. See [27, Lemma 2] for a related result.

Lemma 5 (Link between H and $\bar{H}$) *Let Hypotheses* **(D0)** *to* **(D4)** *hold. Let $u \in W^{2,1}_{loc}(0,+\infty)$ be such that $\dot{u} \in W^{1,\infty}_{loc}(0,+\infty)$. Then, for any $t \geq 0$:*

$$\begin{aligned}\frac{1}{T}\int_t^{t+T} \bar{H}(u)(s)\cdot ds &= \lim_{n\to+\infty} \frac{1}{T}\int_t^{t+T} H(u+d_n)(s)\cdot ds, \quad \forall T>0\\ &= \lim_{n\to+\infty}\limsup_{T\to+\infty} \frac{1}{T}\int_t^{t+T} H(u+d_n)(s)\cdot ds\\ &= \lim_{n\to+\infty}\liminf_{T\to+\infty} \frac{1}{T}\int_t^{t+T} H(u+d_n)(s)\cdot ds\end{aligned}$$

and, for any $\rho>0$ and any $T_0>0$, the convergence is uniform in the set $\{u \in W^{2,1}_{loc}(0,+\infty) : \|\dot{u}\|_{L^\infty(0,T_0)}, \|\ddot{u}_n\|_{L^\infty(0,T_0)} < \rho\}$.

Moreover, if **(D2u)** *to* **(D4u)** *hold, for any $\rho>0$, the convergence is uniform wrt $t\geq 0$ in the set $\{u \in W^{2,1}_{loc}(0,+\infty) : \|\dot{u}\|_{L^\infty(0,+\infty)}, \|\ddot{u}\|_{L^\infty(0,+\infty)} < \rho\}$.*

The linearization result presented above is stated as follows:

Theorem 6 (Linearization of $\bar{H}$) *Suppose $\bar{s}(0)=0$. Let $D=0$ (resp. let $\bar{s}'(0)$ exist).*

- *The operator $\bar{H}_{lin}$ given by (7) is tangent to $\bar{H}$ in the following sense:*

$$\frac{\|\bar{H}(u)-\bar{H}_{lin}(u)\|_{W^{1,\infty}(0,+\infty)}}{\|\dot{u}\|_{L^\infty(0,+\infty)}} \to 0 \quad \left(\textit{resp. } \frac{\|\bar{H}(u)-\bar{H}_{lin}(u)\|_{L^\infty(0,+\infty)}}{\|\dot{u}\|_{L^\infty(0,+\infty)}} \to 0\ \right)$$

when $\|\dot u\|_{L^\infty(0,+\infty)} \to 0$.
• *If moreover there exists a neighborhood of* 0 *disjoint from the range* $g'(\mathbb{R}^+)$, *and if* u *is such that* $|\dot u| < \operatorname{inf\,ess}\{|\alpha| : \alpha \in g'(\mathbb{R}^+)\}$ *a.e., then* $\bar H(u) \equiv \bar H_{lin}(u)$.
• *If* **(H1)** *holds, the operator* $\bar H_{lin}$ *is (bounded-input, bounded-output) stable. If* $D \geq 0$ *and if the operator* $\dot u \mapsto H_{sv}(u)$ *defined in Theorem 1 is dissipative, the same is true for the operator* $\dot u \mapsto \bar H_{lin}(u)$ *(and for* $\dot u \mapsto H(u)$ *also!).*

Remark that $\bar s(0) = 0$ e.g. for a T-periodic function g' with the symmetry condition $g'(t + T/2) = -g'(t)$. According to **(D1)**, g must be bounded, hence g' takes on positive and negative values. If $0 \notin g'(\mathbb{R}^+)$, then g' is discontinuous, thus the regularity condition in **(D1)** implies that $D = 0$.

When $\dot u \mapsto \bar H_{lin}$ is dissipative, as $\bar H_{lin}$ is proportional to the velocity, one may consider it as a *viscous linearization* of H. Notice also that for large values of the input velocity, $\bar H$ behaves like H (as $\bar g(\alpha) = |\alpha|$ and $\bar s(\alpha) = \operatorname{sgn}\,\alpha$ for large $|\alpha|$): this may be linked with [10, 21], where the averaging (which is spatial in these works rather than temporal) gives rise to viscous behavior for low steady state values of the velocity, and Coulomb friction for high values.

Proof:
• Let us denote x (resp. x_l) the state of $\bar H$ (resp. $\bar H_{lin}$), we have

$$\dot x - \dot x_l = A(\bar g(\dot u) - \bar g(0))x + A\bar g(0)(x - x_l), \quad x(0) = x_l(0)$$

Hence

$$x(t) - x_l(t) = \int_0^t e^{A\bar g(0)(t-s)} A(\bar g(\dot u(s)) - \bar g(0))x(s) \cdot ds$$

As $\bar g'(0) = \bar s(0) = 0$, one has $\bar g(\alpha) = \bar g(0) + \alpha\varepsilon(\alpha)$ where $\varepsilon(\alpha) \to 0$ when $\alpha \to 0$, and this, together with the fact that A is stable and $\bar g(0) > 0$, implies:

$$\frac{\|x - x_l\|_{L^\infty}}{\|\dot u\|_{L^\infty}} \to 0 \quad \text{when } \|\dot u\|_{L^\infty} \to 0$$

and the same holds for $\|\dot x - \dot x_l\|_{L^\infty}$. As $\bar H(u) - \bar H_{lin}(u) = C(x - x_l) + D(\bar s(\dot u) - \bar s'(0)\dot u)$, we get the desired tangency property.
• If there exists a neighborhood of 0 disjoint from the range $g'(\mathbb{R}^+)$ and containing (almost) every value of $\dot u$, then $\bar s(\dot u) = \bar s(0) = 0$ a.e., and hence $\bar g(\dot u) = \bar g(0)$ a.e.
• Stability of $\bar H_{lin}$ is clear as the matrix A is stable. Suppose now that $\dot u \mapsto H_{sv}(u)$ is dissipative. When $D \neq 0$, this is equivalent to [5]:

$$\exists P \in \mathbb{R}^{N\times N},\ P = P^T \geq 0 \text{ such that } -(A^T P + PA) \geq 0 \text{ and } C^T = PB$$

which implies (as $\bar g(0) > 0$ and $\bar s'(0) \geq 0$ due to Theorem 7 below)

$$\begin{pmatrix} -(A^T P + PA)\bar g(0) & C^T - PB \\ C - B^T P & 2D\bar s'(0) \end{pmatrix} = \begin{pmatrix} -(A^T P + PA)\bar g(0) & 0 \\ 0 & 2D\bar s'(0) \end{pmatrix} \geq 0$$

and $\dot{u} \mapsto \bar{H}_{lin}(u)$ is dissipative. □

In a mechanical context, u is a position, $H(u)$ a friction and equation (1) is the equation of motion, hence $l - m = 2$. The force necessary to induce the displacement d_n is of order $\ddot{d}_n$. The mean power furnished by the dither may hence be evaluated by the integrals

$$\frac{1}{T}\int_0^T \ddot{d}_n(s)\dot{d}_n(s)\cdot ds = \frac{\dot{d}_n^2(T) - \dot{d}_n^2(0)}{2T},\ \ T \geq 0$$

In the setup of Theorem 4 these expressions are bounded as the velocities $\dot{d}_n$ are uniformly bounded. In the case of unbounded dither velocities (Theorem 9), this will not be the case: this suggests that the first case is the only realizable one in practice – at least for a mechanical device where H represents a friction force.

The following result permits to prove well-posedness of (12), and details in which sense the smoothing has been carried out.

Theorem 7 (Properties of $\bar{g}, \bar{s}, \bar{H}$) *Let Hypotheses* **(D0)** *to* **(D4)** *hold.*
• *The function $\bar{g}$ is convex, 1-Lipschitz and continuously differentiable, and $\bar{g}' = \bar{s}$. Moreover, for any $\alpha \in \mathbb{R}$, $|\alpha| \leq \bar{g}(\alpha)$, with equality for large enough $|\alpha|$, and $\inf_{\alpha\in\mathbb{R}} \bar{g}(\alpha) = \min_{\alpha\in\mathbb{R}} \bar{g}(\alpha) > 0$.*

The function $\bar{s}$ is monotone and uniformly continuous. Moreover, for any $\alpha \in \mathbb{R}$, $|\bar{s}(\alpha)| \leq 1$, and $\bar{s}(\alpha) = \operatorname{sgn} \alpha$ for large enough $|\alpha|$.
• *The map $\bar{H}_{sv} : u \mapsto Cx$ where x is given by (4) is locally Lipschitz in $W^{1,p}_{loc}(0,+\infty)$, and continuous in $W^{2,p}_{loc}(0,+\infty)$ for every $1 \leq p \leq +\infty$. Moreover, if* **(H1)** *holds and $x(0) = 0$, then*

$$\sup_{u\in W^{1,p}_{loc}(0,\infty)} \operatorname{sup\,ess}_{s\geq 0} |\bar{H}(u)(s)| \leq \sup_{u\in W^{1,p}_{loc}(0,\infty)} \operatorname{sup\,ess}_{s\geq 0} |H(u)(s)| < +\infty$$

The map $\bar{H} - \bar{H}_{sv}$ is continuous in $W^{1,p}_{loc}(0,+\infty)$ for every $1 \leq p \leq +\infty$.
• *If $\bar{s}$ is Lipschitz (e.g. continuously differentiable) $\bar{H}_{sv}$ (resp. $\bar{H} - \bar{H}_{sv}$) is locally Lipschitz in $W^{2,p}_{loc}(0,+\infty)$ (resp. $W^{1,p}_{loc}(0,+\infty)$) for every $1 \leq p \leq +\infty$.*
• *Suppose $\bar{s}(0) = 0$. If $D \geq 0$ and if the operator $\dot{u} \mapsto H_{sv}(u)$ defined in Theorem 1 is dissipative, the same is true for the operator $\dot{u} \mapsto \bar{H}(u)$.*

The supplementary regularity property of $\bar{H}$ vs. H implies that for C^2 input u, the curvature of the cycles u-$\bar{H}(u)$ is continuous: the smoothed cycles do not present angles, contrary to the initial hysteresis cycles (see Figure 3).

Remark that $\bar{s}$ is Lipschitz e.g. for g' 2-periodic defined by $g'(s) = 1 - 2|s|$ for $s \in [-1,1]$ (centered symmetric saw-tooth signal of unitary amplitude): in this case, $\bar{s}(\alpha) = \min\{+1, \max\{\alpha, -1\}\}$ is 1-Lipschitz.
Proof:

• The inequality verified by $\bar{g}$ results from

$$\pm\frac{1}{T}\int_0^T (\alpha + g'(s)) \cdot ds \leq \frac{1}{T}\int_0^T |\alpha + g'(s)| \cdot ds$$

Convexity and Lipschitz property of $\bar{g}$ are deduced from the same properties of $\alpha \mapsto \frac{1}{T}\int_0^T |\alpha + g'(s)| \cdot ds$ for any $T > 0$.

Differentiability and the fact that $\bar{g}' = \bar{s}$ are consequences of the uniform (wrt α) convergence of the derivatives when $T \to +\infty$. $\inf_{\alpha\in\mathbb{R}} \bar{g}(\alpha)$ is not zero, as it is worth $\min\{\bar{g}(\alpha) : \alpha \in \mathcal{S}\}$ where $\mathcal{S}$ is the compact set $[-\max_{\mathbb{R}} g', -\min_{\mathbb{R}} g']$. The uniform continuity of $\bar{s}$ is deduced from the fact that $\bar{s}(\alpha) = \operatorname{sgn}\alpha$ for $\alpha \notin \mathcal{S}$.

• The regularity properties of $\bar{H}$ are deduced from those of $\bar{g}$ and $\bar{s}$.

If $x(0) = 0$, one has for any $u \in W^{1,p}_{loc}(0, +\infty)$,

$$\begin{aligned}
\sup_{t\geq 0} |\bar{H}(u)(t)| &\leq |D| \sup_t |\bar{s}(\dot{u}(t))| + \sup_t \left| \int_0^t Ce^{A\int_s^t \bar{g}(\dot{u}(\cdot))} B\dot{u}(s) \cdot ds \right| \\
&\leq |D| + \sup_t \int_0^t \left| Ce^{A\int_s^t \bar{g}(\dot{u}(\cdot))} B\dot{u}(s) \right| \cdot ds \\
&\leq |D| + \sup_t \int_0^t \left| Ce^{A\int_s^t \bar{g}(\dot{u}(\cdot))} B\bar{g}(\dot{u}(s)) \right| \cdot ds \\
&\leq |D| + \int_0^{+\infty} \left| Ce^{As} B \right| \cdot ds = \sup_{u\in W^{1,p}_{loc}(0,\infty)} \operatorname{sup\,ess}_{s\geq 0} |H(u)(s)|
\end{aligned}$$

• Take $u \in W^{2,1}_{loc}(0, +\infty)$ such that $\dot{u} \in W^{1,\infty}(0, +\infty)$. H_{sv} being dissipative, there exist a positive semi-definite matrix P such that, for all $t \in \mathbb{R}^+$,

$$\int_0^t H_{sv}(u + d_n)(s)(\dot{u}(s) + \dot{d}_n(s)) \cdot ds \geq [x_n(s)Px_n(s)]_0^t$$

Now, defining x_n and x_∞ by $\dot{x}_n = Ax_n|\dot{u} + \dot{d}_n| + B(\dot{u} + \dot{d}_n)$, $x_n(0) = x_0$ and $\dot{x}_\infty = Ax_\infty\bar{g}(\dot{u}) + B\dot{u}$, $x_\infty(0) = x_0$, one has

$$\begin{aligned}
\int_0^t H_{sv}(u + d_n)(s)(\dot{u}(s) + \dot{d}_n(s)) \cdot ds &= \int_0^t C(x_n(s) - x_\infty(s))(\dot{u}(s) + \dot{d}_n(s)) \cdot ds \\
&\quad + \int_0^t Cx_\infty(s)\dot{u}(s) \cdot ds \\
&\quad + \int_0^t Cx_\infty(s)\dot{d}_n(s) \cdot ds
\end{aligned}$$

The results of Lemma 5 show that the first and third terms tend towards 0 (as $x_n \to x_\infty$ in C^0 and $\dot{u} + \dot{d}_n$ is bounded in L^1, and integrating by parts and using the boundedness of d_n).

We thus obtain, as $\bar{s}(0) = 0$ and $\bar{s}$ increasing imply that $\alpha\bar{s}(\alpha) \geq 0$ for any $\alpha \in \mathbb{R}$:

$$\int_0^t \bar{H}(u)(s)\dot{u}(s) \cdot ds = \int_0^t (Cx_\infty(s) + D\bar{s}(\dot{u}(s)))\dot{u}(s) \cdot ds \geq [x_\infty(s)Px_\infty(s)]_0^t$$

If now $u \in W^{1,1}_{loc}(0, +\infty)$, the same property remains true by continuity, hence $\bar{H}$ is dissipative. □

Corollary 8 (Well-posedness of (12)**)** *If* **(H1)**, **(H2)** *hold, if f is locally summable,* (12) *admits a solution $u \in W^{l-m,1}_{loc}(0, +\infty)$, which, if* **(H3)** *holds, is unique and continuous wrt the initial conditions and f.*

3.4.2 Unbounded velocity and complementary result

The analog of Theorem 4 is the following:

Theorem 9 (Asymptotic behavior – Unbounded velocity) *Let Hypotheses* **(H1)** *to* **(H3)** *and* **(D0')** *to* **(D4')** *hold. Let f be such that the solution of the equation $L(\frac{d}{dt})u = M(\frac{d}{dt})f$, $(\, u, \quad u', \quad \dots \quad u^{(l-1)}\,)(0) = 0_l$ is in $L^\infty_{loc}(0, +\infty)$, together with its derivative (a sufficient condition for this is e.g. $f \in L^\infty_{loc}(0, +\infty)$). Then*

$$u_n \text{ and } u_n + d_n \to u_\infty \text{ in } C^0_{loc}(0, +\infty) \text{ when } n \to +\infty$$

where u_n is the solution of (11) *and u_∞ is the solution of the linear differential equation*

$$L\left(\frac{d}{dt}\right)u_\infty = M\left(\frac{d}{dt}\right)(-\bar{H}_\gamma + f), \quad (\, u_\infty, \quad u'_\infty, \quad \dots \quad u_\infty^{(l-1)}\,)(0) = (\, u_0, \quad u_1, \quad \dots \quad u_{l-1}\,)$$

and where the (constant) operator $\bar{H}_\gamma$ is defined by (8)*. Besides, for any $\rho > 0$ and any $T_0 > 0$, the convergence is uniform in the set $\{f : L(\frac{d}{dt})u = M(\frac{d}{dt})f, \ (\, u, \quad u', \quad \dots \quad u^{(l-1)}\,)(0) = 0_l \Rightarrow \|u\|_{L^\infty(0,T_0)}, \|\dot{u}\|_{L^\infty(0,T_0)} < \rho\}$.*

Moreover, if Hypotheses **(D2'u)** *to* **(D4'u)** *are fulfilled, the previous facts remain true when replacing C^0_{loc} (resp. L^∞_{loc}, T_0) by C^0 (resp. L^∞, $+\infty$).*

Most of the remarks following Theorem 4 are still valid here. The proof is conducted in Appendix with similar steps.

An analog of Lemma 5 is the following result:

Lemma 10 (Link between H and $\bar{H}_\gamma$) *Let Hypotheses* **(D0')** *to* **(D4')** *hold. Let $u \in W^{1,1}_{loc}(0, +\infty)$ be such that $\dot{u} \in L^\infty_{loc}(0, +\infty)$. Then, for any $t \geq 0$:*

$$\bar{H}_\gamma \quad = \quad \lim_{n \to +\infty} \frac{1}{T} \int_t^{t+T} H(u + d_n)(s) \cdot ds, \quad \forall T > 0$$

$$
\begin{aligned}
&= \lim_{n\to+\infty} \limsup_{T\to+\infty} \frac{1}{T} \int_t^{t+T} H(u+d_n)(s)\cdot ds \\
&= \lim_{n\to+\infty} \liminf_{T\to+\infty} \frac{1}{T} \int_t^{t+T} H(u+d_n)(s)\cdot ds
\end{aligned}
$$

and, for any $\rho > 0$ and any $T_0 > 0$, the convergence is uniform in the set $\{u \in W^{1,1}_{loc}(0,+\infty) : \|\dot{u}\|_{L^\infty(0,T_0)} < \rho\}$.

Moreover, if **(D2'u)** *to* **(D4'u)** *hold, for any $\rho > 0$, the convergence is uniform wrt $t \geq 0$ in the set $\{u \in W^{1,1}_{loc}(0,+\infty) : \|\dot{u}\|_{L^\infty(0,+\infty)} < \rho\}$.*

For sake of completeness we also give the following result, which states the case where the amplitude of the dither is not sufficient to modify the nonlinearity, and the case where it is so large that it violates the "small dither" specification:

Theorem 11 (Asymptotic behavior – Inappropriate amplitude growth) *Let Hypotheses* **(H1)** *to* **(H3)**, **(D0')** *to* **(D4')** *hold, except formula* (10)*. Let f verify the same assumptions than in Theorem 9.*

- *If $\bar{\delta}(T) \to +\infty$, then*

$$
u_n \text{ and } u_n + d_n \to u_\infty \text{ in } W^{1,1}_{loc}(0,+\infty) \text{ when } n \to +\infty
$$

  where u_∞ is the solution of the differential inclusion

$$
\begin{cases}
L\left(\dfrac{d}{dt}\right) u_\infty \in M\left(\dfrac{d}{dt}\right)(-H(u_\infty)+f), \\
(\, u_\infty \quad u'_\infty \quad \ldots \quad u_\infty^{(l-1)}\,)(0) = (\, u_0 \quad u_1 \quad \ldots \quad u_{l-1}\,)
\end{cases}
$$

- *If $T\bar{\delta}(T) \to 0$, then $\|u_n + d_n\|_{C^0(0,T_0)}$ tends to infinity for any $T_0 > 0$ when $n \to +\infty$.*

Proof:

If $\bar{\delta}(T) \to +\infty$, the sequence d_n tends to 0 in $W^{1,1}(0,T)$ for any $T > 0$, hence $u_n + d_n$ tends in $W^{l-m,1}(0,T)$ to u_∞ defined in the statement, due to continuity of H in $W^{1,1}(0,T)$ for every $T > 0$. The same is true for $u_n + d_n$.

If $T\bar{\delta}(T) \to 0$, u_n is bounded and d_n unbounded. □

4 Usual classes of dither

4.1 A general result

We first give general sufficient conditions for the hypotheses to be verified:

Proposition 12 (Sufficient conditions for the hypotheses) *Let $g \in W^{1,\infty}(0,+\infty)$.*

- *If*
$$\chi_0(\delta) \stackrel{\Delta}{=} \sup_{t\geq 0} mes\,\{s\in[t,t+1]:\ |g'(s)|\leq\delta\} \to 0 \text{ when } \delta\to 0 \tag{14}$$
holds, then g satisfies **(D3'u)**.
- *If*
$$\chi_{\mathbb{R}}(\delta) \stackrel{\Delta}{=} \sup_{\alpha\in\mathbb{R}}\sup_{t\geq 0} mes\,\{s\in[t,t+1]:\ |\alpha+g'(s)|\leq\delta\} \to 0 \text{ when } \delta\to 0 \tag{15}$$
holds, then g satisfies **(D3u)**.
- *To prove* **(D2)** *(resp.* **(D2u)***), it is sufficient to prove for any $\alpha\in\mathbb{R}$ the convergence of $\frac{1}{T}\int_0^T |\alpha+g'(s)|\cdot ds$ (resp. the convergence of $\frac{1}{T}\int_t^{t+T}|\alpha+g'(s)|\cdot ds$ uniformly wrt $t\geq 0$).*
- *If* (15) *holds, to prove* **(D4)** *(resp.* **(D4u)***), it is sufficient to prove for any $\alpha\in\mathbb{R}$ the convergence of $\frac{1}{T}\int_0^T \operatorname{sgn}(\alpha+g'(s))\cdot ds$ (resp. the convergence of $\frac{1}{T}\int_t^{t+T}\operatorname{sgn}(\alpha+g'(s))\cdot ds$, uniformly wrt $t\geq 0$).*
- *To prove* **(D2)** *(resp.* **(D2u)***), it is sufficient to prove* **(D2')** *and* **(D4)** *(resp.* **(D2'u)** *and* **(D4u)***).*

Proof:
• Hypothesis **(D3'u)** follows from (14), as, denoting int $(\cdot)$ the integer part:

$$\begin{aligned}\frac{1}{T} mes\,\{s\in[t,t+T]:\ |g'(s)|\leq\delta\} &\leq \frac{1}{T}\sum_{k=0}^{\text{int}\,(T)} mes\,\{s\in[t+k,t+k+1]:\ |g'(s)|\leq\delta\}\\ &\leq \frac{1}{T}\sum_{k=0}^{\text{int}\,(T)}\chi_0(\delta)\leq\frac{1+T}{T}\chi_0(\delta)\end{aligned}$$

which tends to 0 with δ.
• **(D3u)** is deduced from (15) as **(D3'u)** is deduced from (14).
• We prove the result under Hypothesis **(D2u)**, the case **(D2)** is simpler. For any $T\geq 0$, $t\geq 0$, the functions $\alpha\mapsto\frac{1}{T}\int_t^{t+T}|\alpha+g'(s)|\cdot ds$, and $\bar{g}(\alpha)$ are 1-Lipschitz, and hence equicontinuous: due to Ascoli theorem applied on the compact set $\mathcal{S}=\{\alpha\in\mathbb{R}:-\max g'\leq\alpha\leq-\min g'\}$, the pointwise convergence when $T\to+\infty$ implies the uniform convergence. Outside this compact, $\frac{1}{T}\int_t^{t+T}|\alpha+g'(s)|\cdot ds=|\alpha|+\operatorname{sgn}\alpha\frac{g(t+T)-g(t)}{T}$, and the convergence is hence uniform wrt $\alpha\in\mathbb{R}$: **(D2u)** holds.
• Suppose that **(D4u)** holds. For $\alpha,\alpha'\in\mathbb{R}$, one has

$$\frac{1}{T}\int_t^{t+T}\operatorname{sgn}(\alpha+g'(s))\cdot ds-\frac{1}{T}\int_t^{t+T}\operatorname{sgn}(\alpha'+g'(s))\cdot ds$$

$$\leq \frac{2}{T} mes\,\{s \in (t, t+T) :\ |\alpha + g'(s)| \leq |\alpha - \alpha'|\}$$
$$\leq 2\frac{1+T}{T}\chi_{\mathbb{R}}(|\alpha - \alpha'|)$$

Hence, for any $T \geq 0$, any $t \geq 0$, the functions $\alpha \mapsto \frac{1}{T}\int_t^{t+T} \text{sgn}\ (\alpha + g'(s)) \cdot ds$ and $\bar{s}$ are indeed equicontinuous in the compact set $\mathcal{S}$ defined above. As $\bar{s}(\alpha) = \text{sgn}\ \alpha$ outside $\mathcal{S}$, $\bar{s}$ is uniformly continuous on $\mathbb{R}$, by the same argument than above.

• Suppose that **(D2'u)** and **(D4u)** hold. The uniform convergence of the derivatives towards $\bar{s}$ obtained *via* **(D4u)**, together with the convergence in $\alpha = 0$ towards $\bar{g}(0)$ due to **(D2'u)**, implies the uniform convergence towards $\bar{g}$ in every compact set. As the convergence is also uniform outside the compact set $\mathcal{S}$, **(D2u)** holds. □

For example, a function that satisfies **(D1)**, **(D2'u)**, (15) and such that for any $\alpha \in \mathbb{R}$, the integrals

$$\frac{1}{T}\int_t^{t+T} \text{sgn}\ (\alpha + g'(s)) \cdot ds$$

converge uniformly wrt $t \geq 0$, verifies Hypotheses **(D3u)** and **(D4u)**.

4.2 Periodic dither

Theorem 13 *Let $g \in W^{1,\infty}(0, +\infty)$ be a T-periodic non constant function, then Hypotheses* **(D2'u)** *and* **(D2u)** *are satisfied. Moreover,*

- *if $g'(t)$ verifies*
$$mes\{s \in (0,T) :\ g'(s) = 0\} = 0 \qquad (16)$$
then Hypotheses **(D3'u)** *and* **(D4'u)** *are satisfied.*

- *if $g'(t)$ verifies*
$$\forall\alpha \in \mathbb{R},\ mes\{s \in (0,T) :\ \alpha + g'(s) = 0\} = 0 \qquad (17)$$
then Hypotheses **(D3u)** *and* **(D4u)** *are satisfied.*

Moreover, for any $\alpha \in \mathbb{R}$, the following relations hold when their left-hand side is defined:

$$\bar{g}(\alpha) = \frac{1}{T}\int_0^T |\alpha + g'(s)| \cdot ds,\ \ \bar{s}(\alpha) = \frac{1}{T}\int_0^T \text{sgn}\ (\alpha + g'(s)) \cdot ds$$

The proof of Theorem 13 is subsumed by the proof of Theorem 15 below.

Remark that (16) (resp. (17)) is necessary to define $\bar{s}(0)$ (resp. $\bar{s}(\alpha)$) in a univocal way.

The functions g which are, up to a constant, finite sum of sinusoidal functions whose least periods admit a finite smaller common multiplier, are examples of periodic functions which satisfy Hypotheses **(D1)** and **(D2u)** to **(D4u)**.

4.3 F-repetitive dither

G. Zames and N.A. Shneydor [27, 28] have considered non-periodic dithers, more precisely F-repetitive ones.

Definition 14 (F-repetitive functions) *Any bounded function $q(t)$, $t \in [0,\infty)$ is called F-repetitive if there exists a strictly increasing unbounded sequence of positive numbers t_i with $t_0 = 0$, such that the following conditions hold:*

- $(t_{i+1} - t_i)^{-1} mes\{s \in (t_i, t_{i+1}) : \ q(s) \le \xi\} = (t_1 - t_0)^{-1} mes\{s \in (t_0, t_1) : \ q(s) \le \xi\}$ *for every $\xi \in \mathbb{R}$ and every $i \in \mathbb{N}$;*
- $\sup(t_{i+1} - t_i) = T < +\infty$. *$T$ is the maximal repetition interval.*

Every T-periodic function is F-repetitive, with $t_i = iT$.

For any F-repetitive function $q(t)$, it follows from the definition of Lebesgue integral that the following identity is valid for any measurable function $h(\cdot)$:

$$(t_{i+1} - t_i)^{-1} \int_{t_i}^{t_{i+1}} h(q(s))\, ds = (t_1 - t_0)^{-1} \int_{t_0}^{t_1} h(q(s))\, ds$$

The following result generalizes Theorem 13:

Theorem 15 *Let $g \in W^{1,\infty}(0,+\infty)$ be a non constant function such that $g'(t)$ is F-repetitive, then Hypothesis* **(D2u)** *is satisfied. Moreover,*

- *if*
$$mes\{s \in (t_0, t_1) : \ g'(s) = 0\} = 0 \tag{18}$$
then Hypotheses **(D3'u)** *and* **(D4'u)** *are satisfied.*
- *if*
$$\forall \alpha \in \mathbb{R}, \ mes\{s \in (t_0, t_1) : \ \alpha + g'(s) = 0\} = 0 \tag{19}$$
then Hypotheses **(D3u)** *and* **(D4u)** *are satisfied.*

Moreover, for any $\alpha \in \mathbb{R}$, any $i \in \mathbb{N}$, the following relations hold when their left-hand side is defined:

$$\bar{g}(\alpha) = \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} |\alpha + g'(s)| \cdot ds, \ \ \bar{s}(\alpha) = \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} \mathrm{sgn}\, (\alpha + g'(s)) \cdot ds$$

Proof:

• The relations below are valid for any locally bounded function h for which $h \circ g'$ is measurable. Let τ, t be positive numbers such that $\tau < t$, and let i_1 and i_2 be positive integers such that $t_{i_1-1} \leq \tau < t_{i_1}$ and $t_{i_2} \leq t < t_{i_2+1}$, for sufficiently large n (in order that $i_1 < i_2$). Then:

$$\begin{aligned}
&\left|\frac{1}{t-\tau}\int_\tau^t h(g'(s))\,ds - \frac{1}{t_1-t_0}\int_{t_0}^{t_1} h(g'(s))\,ds\right| \\
\leq\ & \frac{1}{t-\tau}\left(\left|\int_\tau^{t_{i_1}}\right| + \left|\int_{t_{i_2}}^{t}\right|\right) + \left|\frac{1}{t-\tau}\int_{t_{i_1}}^{t_{i_2}} h(g'(s))\,ds - \frac{1}{t_1-t_0}\int_{t_0}^{t_1} h(g'(s))\,ds\right| \\
\leq\ & \frac{2T\sup|h(g'(\cdot))|}{t-\tau} + \left|\frac{1}{t-\tau}\sum_{k=i_1}^{i_2-1}\frac{t_{k+1}-t_k}{t_1-t_0}\int_{t_0}^{t_1} h(g'(s))ds - \frac{1}{t_1-t_0}\int_{t_0}^{t_1} h(g'(s))\,ds\right| \\
& \text{(using the } F\text{-repetitiveness of } g' \text{ [27])} \\
=\ & \frac{2T\sup|h(g'(\cdot))|}{t-\tau} + \frac{1}{t_1-t_0}\left|\int_{t_0}^{t_1} h(g'(s))\,ds\right| \cdot \left|\frac{t_{i_2}-t_{i_1}}{t-\tau} - 1\right| \leq \frac{4T\sup|h(g'(\cdot))|}{t-\tau}
\end{aligned}$$

We then deduce **(D2u)** taking $h(\cdot) = |\cdot|$. Remark that $\bar{g}(\alpha) \neq 0$, because g' is continuous (as $g \in W_{loc}^{l-m,1}(0,+\infty) \subset W_{loc}^{2,1}(0,+\infty) \subset C_{loc}^1(0,+\infty)$), and the assumption that g is not constant on $\mathbb{R}^+$ together with the F-repetitiveness of g' implies that $g' \not\equiv 0$ in $[t_1, t_0]$.

• For any $i \in \mathbb{N}$, if (18) holds, for any $\delta \geq 0$, one has, using Definition 14,

$$mes\,\{s \in (t_i, t_{i+1}) :\ |g'(s)| \leq \delta\} = \frac{t_{i+1}-t_i}{t_1-t_0} mes\,\{s \in (t_0, t_1) :\ |g'(s)| \leq \delta\}$$

Hence, for any $t \geq 0$, T being the maximal repetition interval,

$$mes\,\{s \in [t, t+1] :\ |g'(s)| \leq \delta\} \leq \frac{1+2T}{t_1-t_0} mes\,\{s \in (t_0, t_1) :\ |g'(s)| \leq \delta\}$$

which tends to zero when $\delta \to 0$, due to (18) and Lemma 3. We conclude that (14) holds and, using Proposition 12, that **(D3'u)** holds. We deduce likewise (15) from (19), and **(D3u)**, again by Proposition 12.

• To prove **(D4'u)** (resp. **(D4u)**), we take $h(\cdot) = \text{sgn}\,(\cdot)$ in the formula given below; the integral is meaningful, as (18) (resp. (19)) implies that sgn g' is well-defined. This implies the existence of $\bar{s}(0)$ (resp. $\bar{s}(\alpha)$) with convergence uniform in $t \geq 0$. We then use Proposition 12 to conclude. □

Let us give an example of function g with F-repetitive derivative verifying Hypotheses **(D1)**, and **(D2u)** to **(D4u)**.

Consider a sequence $0 = t_0 < t_1 < \ldots < t_k < \ldots$ such that

$$0 < \inf(t_{k+1} - t_k) < \sup(t_{k+1} - t_k) < +\infty$$

Theorem 15 shows that the function

$$g(t) \triangleq c + \sin\left(\frac{2\pi(t-t_k)}{t_{k+1}-t_k}\right), \qquad t \in [t_k, t_k+1),\ k \in \mathbb{N},\ c \in \mathbb{R}$$

have F-repetitive derivative and satisfy all the desired properties.

4.4 Asymptotic almost-periodic dither

Following [9, 26], we define:

Definition 16 (Asymptotic almost-periodic functions) *A continuous function $q(t)$ defined on the half-line ($t \geq 0$) is called asymptotic almost-periodic (AAPF for short) if for every $\varepsilon > 0$ there exists a value $\lambda(\varepsilon)$ such that on every interval $\mathcal{J} = [a, a+\lambda(\varepsilon)]$ ($a \geq 0$) there exists $\tau \in \mathcal{J}$, such that:*

$$\forall t \geq 0, \quad |q(t) - q(t+\tau)| < \varepsilon \tag{20}$$

τ is called an ε-almost-period and $\lambda(\varepsilon)$ an inclusion length (corresponding to ε).

The following result generalizes Theorem 13 to asymptotic almost-periodic dithers. The proof uses Lemmas 19 and 20, which are expressed and proved in Subsection 4.5.

Theorem 17 *Let $g \in W^{1,\infty}(0, +\infty)$ be a non constant AAPF such that g' is uniformly continuous (e.g. $g \in W^{2,\infty}(0, +\infty)$), then Hypothesis* **(D2u)** *is satisfied. Moreover,*

- *if g' verifies* (14)*, then Hypotheses* **(D3'u)** *and* **(D4'u)** *are satisfied.*
- *if g' verifies* (15)*, then Hypotheses* **(D3u)** *and* **(D4u)** *are satisfied.*

In particular, this states that if $g' \not\equiv 0$ is AAPF, then $\bar{g}(\alpha) > 0$ for any $\alpha \in \mathbb{R}$. This is not the case in general, even if **(D3u)** holds: consider e.g. $g'(t) = \sum_{n \geq 0} G(t - n^2)$ where $G(t) = 1 - |t|$ for $|t| \leq 1$, 0 otherwise.

Proof:

As g is an AAPF and g' is uniformly continuous, g' is an AAPF, and $|g'(t)|$ also [9]. Hence, the following limit exists (Lemma 19)

$$\bar{g}(0) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T |g'(s)|\, ds$$

and one may prove as in [13] that it is strictly positive if $g' \not\equiv 0$. Moreover, as (14) is satisfied, then $\bar{s}(0)$ defined by (22) exists (Lemma 20). We denote

$$\varphi(T) = \sup_{T \leq T'} \left| \bar{g}(0) - \frac{1}{T'} \int_0^{T'} |g'(s)|\, ds \right|$$

and

$$\psi(T) = \sup_{T \leq T'} \left| \bar{s}(0) - \frac{1}{T'} \int_0^{T'} \operatorname{sgn} g'(s)\, ds \right|$$

and deduce from Lemmas 19 and 20 that $\varphi(T) \to 0$ and $\psi(T) \to 0$ as $T \to +\infty$. Both functions φ and ψ are decreasing.

• To prove **(D2u)**, we use the following relations. Let $T, t, \tau > 0$ and $t - \tau \geq T$, then:

$$\begin{aligned}
\left| \frac{1}{t-\tau} \int_\tau^t |g'(s)|\, ds - \bar{g}(0) \right| &= \left| \frac{1}{t-\tau} \int_\tau^{\tau+(t-\tau)} |g'(s)|\, ds - \bar{g}(0) \right| \\
&\leq \left| \frac{1}{t-\tau} \int_\tau^{\tau+(t-\tau)} |g'(s)|\, ds - \frac{1}{t-\tau} \int_0^{t-\tau} |g'(s)|\, ds \right| \\
&\quad + \left| \frac{1}{t-\tau} \int_0^{t-\tau} |g'(s)|\, ds - \bar{g}(0) \right| \\
&\leq \varepsilon + 2\frac{\lambda(\varepsilon)}{T} \sup |g'(t)| + \varphi(T) \quad \text{(using (23))}
\end{aligned}$$

We then deduce, choosing first ε, then T, that

$$\forall \eta > 0, \exists T,\ \ t - \tau \geq T \Rightarrow |\frac{1}{t-\tau} \int_\tau^t |g'(s)|\, ds - \bar{g}(0)| \leq \eta$$

The same argument holds for $\bar{g}(\alpha)$. This shows the uniformity of the limit wrt t and τ and gives **(D2u)**.
• **(D3'u)** (resp. **(D3u)**) under Hypothesis (14) (resp. (15)) is proved by Proposition 12.
• To prove **(D4'u)** assuming (14), consider

$$\begin{aligned}
&\left| \frac{1}{t-\tau} \int_\tau^t \operatorname{sgn} g'(s)\, ds - \bar{s}(0) \right| \\
&= \left| \frac{1}{t-\tau} \int_\tau^{\tau+(t-\tau)} \operatorname{sgn} g'(s)\, ds - \bar{s}(0) \right| \\
&\leq \left| \frac{1}{t-\tau} \int_\tau^{\tau+(t-\tau)} \operatorname{sgn} g'(s)\, ds - \frac{1}{t-\tau} \int_0^{t-\tau} \operatorname{sgn} g'(s)\, ds \right| \\
&\quad + \left| \frac{1}{t-\tau} \int_0^{t-\tau} \operatorname{sgn} g'(s)\, ds - \bar{s}(0) \right| \\
&\leq 2\frac{T+1}{T} \chi_0(\varepsilon) + 2\frac{\lambda(\varepsilon)}{T} + \psi(T) \quad \text{(using (24) in Lemma 21)}
\end{aligned}$$

and **(D4'u)** is proved.
• Due to Lemma 20, $\bar{s}(\alpha)$ is well-defined for all $\alpha \in \mathbb{R}$, if (15) holds, and the convergence is uniform wrt $t \geq 0$. By Proposition 12 and (15), we conclude that **(D4u)** holds. □

From this, a natural question arises: How to check the validity of (14) and (15)? The authors do not know whether $mes\,\{s \in \mathbb{R}^+ :\ g'(s) = 0\} = 0$ implies (14) or not, as in the periodic case (uniform convergence wrt t is required in (14), so Lemma 3 is not sufficient). We formulate a sufficient condition, useful for various applications.

Lemma 18 *Let an AAPF $q \not\equiv 0$ have the form*

$$q(t) = A_0 + \sum_{i=1}^{k} A_i \sin(\omega_i t + \phi_i)$$

with $k \geq 1$ different frequences ω_i (commensurable or not). Then, the function $g' = q$ satisfies (14) *and* (15)*, and moreover*

$$\exists c > 0, \ \ \chi_0(\delta), \chi_{\mathbb{R}}(\delta) \leq c\,\delta^{\frac{1}{2k}}$$

Proof:
Consider the linear homogeneous ODE:

$$\frac{d}{dt} \prod_{i=1}^{k} \left(\frac{d^2}{dt^2} + \omega_i^2 \right) x = 0 \tag{21}$$

on the interval $[0,1]$. For any real a, the function $q(a+t), \ t \in [0,1]$ is a solution of (21). The set of all these functions is included in a finite-dimensional vector space. The norms of $q(a+t)$ in C^0 are uniformly bounded from below: $\|q(a+t)\|_{C^0[0,1]} \geq \varepsilon_0 > 0$. From [17, Theorem 1.1, Chapter 1], it follows that

$$mes\,\{s : \ s \in [0,1], \ |q(a+s)| \leq \delta\} \leq c\,\delta^{\frac{1}{2k}}$$

which proves the Lemma. □

The simplest examples of asymptotic almost-periodic dither (but not periodic or F-repetitive), satisfying the desired assumptions are functions of the type $e_1(nt) + e_2(n\eta t)$ with T-periodic functions $e_1(t)$ and $e_2(t)$ and irrational η. An example of function satisfying Hypotheses **(D1)** and **(D2u)** to **(D4u)** is $g(t) = c + \cos t + \cos\sqrt{2}t, \ c \in \mathbb{R}$.

4.5 Lemmas on ergodicity

Lemma 19 (Ergodicity for AAPFs) *The following limit exists for any AAPF $q(t)$:*

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T q(s)\,ds$$

This fact may be found in [9, 26], for sake of completeness, we shall prove it together with the following Lemma.

For an AAPF $q(t)$ the function sgn $q(t)$ may not be AAPF, but we have the following result:

Lemma 20 (Ergodicity of the sgn **of AAPFs)** *Let q be an AAPF such that* (15) *holds. Then the following limit exists:*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T \operatorname{sgn} q(s)\, ds \tag{22}$$

To prove Lemmas 19 and 20 we need the following statement.

Lemma 21 *Let q be an AAPF, $\varepsilon > 0$ and $\lambda(\varepsilon)$ be a corresponding inclusion length. Then for any $t \geq 0$ and any $T \geq 0$, the following uniform estimates hold:*

$$\left| \int_0^T q(s)\, ds - \int_t^{t+T} q(s)\, ds \right| \leq T\varepsilon + 2\lambda(\varepsilon) \sup |q(s)| \tag{23}$$

(in which $\sup |q(s)| < +\infty$*, due to asymptotic almost-periodicity* [9]*) and, if* (14) *holds,*

$$\left| \int_0^T \operatorname{sgn} q(s)\, ds - \int_t^{t+T} \operatorname{sgn} q(s)\, ds \right| \leq 2(T+1)\chi_0(\varepsilon) + 2\lambda(\varepsilon) \tag{24}$$

Proof:

Let τ be an ε-almost-period from the interval $[t, t+\lambda(\varepsilon)]$. Then

$$\begin{aligned} \left| \int_0^T q(s)\, ds - \int_t^{t+T} q(s)\, ds \right| &= \left| \int_0^T - \int_t^\tau - \int_\tau^{\tau+T} - \int_{\tau+T}^{t+T} \right| \\ &\leq \left| \int_0^T - \int_\tau^{\tau+T} \right| + 2\lambda(\varepsilon) \sup |q(s)| \leq \int_0^T |q(s) - q(s+\tau)|\, ds + 2\lambda(\varepsilon) \sup |q(s)| \\ &\leq T\varepsilon + 2\lambda(\varepsilon) \sup |q(s)| \end{aligned}$$

which proves (23). The estimate (24) is proved analogously: instead of (20), one uses the inequalities

$$\int_0^T |\operatorname{sgn} q(s) - \operatorname{sgn} q(s+\tau)|\, ds \leq 2 \mathrm{mes}\, \{s \in [0,T] : \ |q(s)| \leq \varepsilon\} \leq 2(T+1)\chi_0(\varepsilon)$$

The last estimate is trivial, as $|q(s)| > \varepsilon$ implies $\operatorname{sgn} q(s) = \operatorname{sgn} q(s+\tau)$. Lemma 21 is proved. □

Proof of Lemmas 19 and 20:

We shall see that Lemma 21 implies the following estimates (we suppose $R \gg r$)

$$\left| \frac{1}{R} \int_0^R q(s)\, ds - \frac{1}{r} \int_0^r q(s)\, ds \right| \leq 2 \left(\frac{r}{R} + \frac{\lambda(\varepsilon)}{r} \right) \sup |q(s)| + \varepsilon \tag{25}$$

and

$$\left|\frac{1}{R}\int_0^R \operatorname{sgn} q(s)\,ds - \frac{1}{r}\int_0^r \operatorname{sgn} q(s)\,ds\right| \leq 2\left(\frac{r}{R}+\frac{\lambda(\varepsilon)}{r}\right) + 2\frac{r+1}{r}\chi_0(\varepsilon) \tag{26}$$

Estimates (25) and (26) are indeed sufficient to prove the ergodicity of $q(s)$ and of sgn $q(s)$. We give a proof of this fact e.g. for sgn $q(s)$: suppose that (26) is proved and that (22) does not hold. Then, as sgn q is bounded uniformly on $\mathbb{R}^+$, there exist two real numbers $\bar{s}_r$, $\bar{s}_R$ and two sequences $r_n, R_n \to +\infty$ such that

$$\lim_{n\to+\infty}\frac{1}{r_n}\int_0^{r_n} \operatorname{sgn} q(s)\,ds = \bar{s}_r \neq \bar{s}_R = \lim_{N\to+\infty}\frac{1}{R_N}\int_0^{R_N} \operatorname{sgn} q(s)\,ds$$

Choose $\varepsilon > 0$ such that $\chi_0(\varepsilon) < 1/20|\bar{s}_r - \bar{s}_R|$ (for such an ε, one has $[2(r_n+1)/r_n]\chi_0(\varepsilon) < 1/5|\bar{s}_r - \bar{s}_R|$ if $r_n > 1$). Then choose n large enough, in order that

$$\left|\bar{s}_r - \frac{1}{r_n}\int_0^{r_n} \operatorname{sgn} q(s)\,ds\right| < 1/5|\bar{s}_r - \bar{s}_R|,\quad 2\frac{\lambda(\varepsilon)}{r_n} < 1/5|\bar{s}_r - \bar{s}_R| \text{ and } r_n > 1$$

Finally, choose N sufficiently large, in such a way that

$$\left|\bar{s}_R - \frac{1}{R_N}\int_0^{R_N} \operatorname{sgn} q(s)\,ds\right| < 1/5|\bar{s}_r - \bar{s}_R| \text{ and } 2\frac{r_n}{R_N} < 1/5|\bar{s}_r - \bar{s}_R|$$

Then

$$\begin{aligned}
|\bar{s}_r - \bar{s}_R| &\leq \left|\bar{s}_R - \frac{1}{R_N}\int_0^{R_N} \operatorname{sgn} q(s)\,ds\right| + \left|\frac{1}{r_n}\int_0^{r_n} \operatorname{sgn} q(s)\,ds - \frac{1}{R_N}\int_0^{R_N} \operatorname{sgn} q(s)\,ds\right| \\
&\quad + \left|\bar{s}_r - \frac{1}{r_n}\int_0^{r_n} \operatorname{sgn} q(s)\,ds\right| \\
&< 1/5|\bar{s}_r - \bar{s}_R| + 3/5|\bar{s}_r - \bar{s}_R| + 1/5|\bar{s}_r - \bar{s}_R| \quad \text{(using (26))}
\end{aligned}$$

which contradicts the fact that $\bar{s}_r \neq \bar{s}_R$.

So, in order to prove Lemmas 19 and 20, it remains to prove the estimates (25) and (26). As an example, the estimate (25) follows from the following relations, where k is the integer part of R/r (i.e. $kr \leq R < (k+1)r$):

$$\begin{aligned}
&\left|\frac{1}{R}\int_0^R q(s)\,ds - \frac{1}{r}\int_0^r q(s)\,ds\right| = \left|\frac{1}{R}\left(\int_0^r + \int_r^{2r} + \ldots + \int_{(k-1)r}^{kr}\right) + \frac{1}{R}\int_{kr}^R - \frac{1}{r}\int_0^r\right| \\
&\quad\leq \left|\frac{1}{R}\left(\int_0^r + \int_r^{2r} + \ldots + \int_{(k-1)r}^{kr}\right) - \frac{1}{r}\int_0^r\right| + \frac{1}{R}\left|\int_{kr}^R\right|
\end{aligned}$$

$$\leq \left| \frac{1}{R} k \int_0^r + \frac{1}{R} \left(\int_r^{2r} - \int_0^r + \int_{2r}^{3r} - \int_0^r + \ldots + \int_{(k-1)r}^{kr} - \int_0^r \right) - \frac{1}{r} \int_0^r \right| + \frac{r}{R} \sup |q(s)|$$

(as $0 \leq R - kr < r$)

$$\leq \left| \frac{kr - R}{Rr} \int_0^r \right| + \frac{k-1}{R} (r\varepsilon + 2\lambda(\varepsilon) \sup |q(s)|) + \frac{r}{R} \sup |q(s)|$$

(using $(k-1)$ times formula (23))

$$\leq 2 \left(\frac{r}{R} + \frac{\lambda(\varepsilon)}{r} \right) \sup |q(s)| + \varepsilon \quad (\text{as } kr \leq R < (k+1)r)$$

Formula (26) may be proved analogously. □

Appendix

A Proof of Theorem 2 : Well-posedness

We shall use a realization of the transfert function $\frac{M(s)}{L(s)}$

$$\dot{\tilde{u}} \in \mathcal{A}\tilde{u} + \mathcal{B}(-H(\mathcal{C}\tilde{u} + d) + f) \quad \text{a.e.}, \mathcal{C}\tilde{u} = u, \;\; \tilde{u}(0) = \tilde{u}_0 \tag{27}$$

where $\mathcal{A} \in \mathbb{R}^{l \times l}$ is stable, $\mathcal{B} \in \mathbb{R}^{l \times 1}$, $\mathcal{C} \in \mathbb{R}^{1 \times l}$ and $\mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B} = \frac{M(s)}{L(s)}$.

From Hypothesis **(H2)**, we deduce

$$\begin{cases} \mathcal{C}\mathcal{A}^i\mathcal{B} = 0, \quad 0 \leq i \leq l - m - 2 \\ \mathcal{C}\mathcal{A}^{l-m-1}\mathcal{B} = \lim_{s \to +\infty} s^{l-m} \frac{M(s)}{L(s)} \neq 0 \end{cases}$$

as for any $i \in \mathbb{N}$, $\mathcal{C}\mathcal{A}^i\mathcal{B} = \lim_{s \to +\infty} s^{i+1} \frac{M(s)}{L(s)}$ holds, both term denoting the value of the i-th derivative at the origin of the impulse response $Ce^{\mathcal{A}t}\mathcal{B}$. In particular, one has $\dot{u} = \mathcal{C}(\mathcal{A}\tilde{u} + \mathcal{B}(-H(\mathcal{C}\tilde{u} + d) + f)) = \mathcal{C}\mathcal{A}\tilde{u}$, as $l - m \geq 2$.

We now prove the existence of solutions.

Replace H by H_{sv} as defined after formula (3) (this is equivalent to suppose $D = 0$). We are led to a differential equation (and no more inclusion). One then considers the sequence of functions $\tilde{u}^k$, $k \in \mathbb{N}$, defined recursively by

$$\tilde{u}^0 \equiv \tilde{u}_0, \;\; \dot{\tilde{u}}^k = \mathcal{A}\tilde{u}^k + \mathcal{B}(-H_{sv}(\mathcal{C}\tilde{u}^{k-1} + d) + f) \quad \text{a.e.}, \tilde{u}^k(0) = \tilde{u}_0 \quad \text{for } k \geq 1$$

This sequence is uniquely defined in $W^{l-m,p}_{loc}(0, +\infty)$ and, due to the (local) Lipschitz property for H_{sv} (Theorem 1), the usual invocation of the Contraction Principle shows that u^k converges in $W^{l-m,p}_{loc}(0, +\infty)$ (i.e. in $W^{l-m,p}(0, T_0)$ for any $T_0 > 0$), towards the (unique) solution of the equation

$$\dot{\tilde{u}} = \mathcal{A}\tilde{u} + \mathcal{B}(-H_{sv}(\mathcal{C}\tilde{u} + d) + f) \quad \text{a.e.}, \quad \tilde{u}(0) = \tilde{u}_0$$

Now, if $D \neq 0$ (and $l-m=2$, due to **(D3')**), one uses the possibility to regularize the sgn operator by smooth operators, with single-valued output, pertaining to the class we consider [5]. Let a new sequence $\tilde{u}^k \in W^{l-m,p}_{loc}(0,+\infty)$ be defined as the sequence of solutions of the equations

$$\dot{\tilde{u}}^k \in \mathcal{A}\tilde{u}^k + \mathcal{B}(-H^k(\mathcal{C}\tilde{u}^k + d) + f) \quad \text{a.e.,} \quad \tilde{u}^k(0) = \tilde{u}_0 \quad \text{for } k \geq 1$$

where

$$H^k \text{ is defined by (3) with the matrices } \begin{pmatrix} A & 0 \\ 0 & -\frac{1}{k} \end{pmatrix}, \begin{pmatrix} B \\ \frac{1}{k} \end{pmatrix}, \begin{pmatrix} C & D \end{pmatrix}, 0$$
$$\text{instead of } A, B, C, D \text{ resp.}$$

These operators are indeed the sum of two operators of the class we consider, namely H_{sv} and the operator defined by $-\frac{1}{k}, \frac{1}{k}, D, 0$.

Suppose first that $f \in L^2_{loc}(0,+\infty)$. A compactness argument shows that the sequence $\dot{u}^k$ admits a cluster point $\dot{u}$ in $L^2(0,T_0)$ weak star for any $T_0 > 0$. In particular, one may suppose that u^k converges in $W^{1,2}(0,T_0)$ for any $T_0 > 0$, and hence $W^{1,1}(0,T_0)$, so the term due to H_{sv} converges in $W^{1,2}(0,T_0)$ towards $H_{sv}(u)$. The supplementary term may be proved to behave as $D\text{sgn } \dot{u}$ asymptotically[7], so the cluster point u verifies

$$\dot{\tilde{u}} \in \mathcal{A}\tilde{u} + \mathcal{B}(-H(\mathcal{C}\tilde{u} + d) + f) \quad \text{a.e.,} \quad \tilde{u}(0) = \tilde{u}_0$$

and existence holds.

We now have to show the uniqueness of the cluster point. This will show the uniqueness of solutions for (1).

Suppose that there exists a symmetric positive definite matrix P such that $\mathcal{A}^T\mathcal{C}^T = DP\mathcal{B}$. Then, for any solutions $\tilde{u}, \tilde{u}'$ of (1), we have

$$\begin{aligned} &(\tilde{u}-\tilde{u}')^T P(\dot{\tilde{u}} - \dot{\tilde{u}}') \in \\ &\in \quad (\tilde{u}-\tilde{u}')^T P\mathcal{A}(\tilde{u}-\tilde{u}') + (\tilde{u}-\tilde{u}')^T P\mathcal{B}(H_{sv}(u) - H_{sv}(u')) \\ &\qquad -D(\tilde{u}-\tilde{u}')^T P\mathcal{B}(\text{sgn } \dot{u} - \text{sgn } \dot{u}') \\ &= \quad (\tilde{u}-\tilde{u}')^T P\mathcal{A}(\tilde{u}-\tilde{u}') + (\tilde{u}-\tilde{u}')^T P\mathcal{B}(H_{sv}(u) - H_{sv}(u')) \\ &\qquad -(\mathcal{C}\mathcal{A}\tilde{u} - \mathcal{C}\mathcal{A}\tilde{u}')(\text{sgn } \mathcal{C}\mathcal{A}\tilde{u} - \text{sgn } \mathcal{C}\mathcal{A}\tilde{u}') \end{aligned}$$

The last term is negative and the Lipschitz property for H_{sv} (Theorem 1) permits to show uniqueness by use of Gronwall Lemma. Analogous argument permits to show the continuity of the solution wrt the initial conditions and to f in the L^p norm. Now, if $f \in L^p_{loc}(0,+\infty)$, this last continuity property permits to show the existence, considering e.g. the sequence $\max\{n, f(t)\}$ in $L^2_{loc}(0,+\infty)$.

The existence of P is guaranteed by the following result applied to $M_1 = \mathcal{C}\mathcal{A}$ and $M_2 = D\mathcal{B}$, using the fact that **(H3)** implies $D\mathcal{C}\mathcal{A}\mathcal{B} > 0$ when $l - m = 2$.

[7] More precisely [6], $(-H_{sv}(u^k) + H^k(u^k))\dot{u}^k \to D|\dot{u}|$ in $L^2_{loc}(0,+\infty)$ strong.

Lemma 22 *Let M_1 (resp. M_2) be a $1 \times n$ (resp. $n \times 1$) real matrix. Then*

$$M_1 M_2 > 0 \Leftrightarrow \exists P = P^T > 0 \text{ such that } M_1^T = P M_2$$

The direct part is straightforward. For the converse part, consider for $\lambda > 0$:

$$P = \lambda \left(I - \frac{M_2 M_2^T}{M_2^T M_2} \right) - \frac{M_1 M_2}{(M_2^T M_2)^2} M_2 M_2^T + \frac{1}{M_2^T M_2} (M_2 M_1 + M_1^T M_2^T)$$

Then, $M_1^T = PM_2$ and $P = P^T$. Moreover, $M_2^T P M_2 = M_1 M_2 > 0$, and for any vector v such that $M_2^T v = 0$, we have $v^T P v = \lambda v^T v > 0$, and hence $P > 0$.

B Proofs of Theorem 4 and Lemma 5 : Bounded velocity

B.1 Proof of Theorem 4

We prove the Theorem under the Hypotheses **(D2u)** to **(D4u)**. The convergence on compact time intervals when only **(D2)** to **(D4)** hold is proved in the same way. The main tool is the following Lemma.

Lemma 23 *Let Hypotheses* **(D0)**, **(D1)**, **(D2u)** *to* **(D4u)** *hold. Let $u \in W^{2,1}_{loc}(0, +\infty)$ with $\dot{u}$ and $\ddot{u}$ bounded in $L^\infty(0, +\infty)$. Then, defining $\bar{H}$ as in (4), one has $\forall c > 0$,*

$$\int_0^t e^{-c(t-s)} (H(u + d_n)(s) - \bar{H}(u)) \cdot ds \to 0 \text{ uniformly wrt } t \geq 0$$

and, for any $\rho > 0$, the convergence is uniform in the set $\{u \in W^{1,1}_{loc}(0, +\infty) : \|\dot{u}\|_{L^\infty(0,+\infty)}, \|\ddot{u}\|_{L^\infty(0,+\infty)} < \rho\}$.

More precisely,

- **(D3u)**, **(D4u)** *imply that $\forall c > 0$,*

$$\int_0^t e^{-c(t-s)} (\text{sgn } (\dot{u} + \dot{d}_n)(s) - \bar{s}(\dot{u})) \cdot ds \to 0 \text{ uniformly wrt } t \geq 0$$

- **(D2u)** *implies that*

$$x_n \to x_\infty \text{ in } C^0(0, +\infty)$$

  where

$$\dot{x}_n = A x_n |\dot{u} + \dot{d}_n| + B(\dot{u} + \dot{d}_n), \quad x_n(0) = x_0 \text{ and } \dot{x}_\infty = A x_\infty \bar{g}(\dot{u}) + B\dot{u}, \quad x_\infty(0) = x_0$$

Let us show that Lemma 23 leads to the proof of Theorem 4. One has

$$\begin{aligned} u_n - u_\infty &\in \frac{M(s)}{L(s)}(-H(u_n+d_n)+\bar{H}(u_\infty)) \\ &= \frac{M(s)}{L(s)}(-H(u_n+d_n)+\bar{H}(u_n)) + \frac{M(s)}{L(s)}(-\bar{H}(u_n)+\bar{H}(u_\infty)) \end{aligned}$$

As there exists $\rho < +\infty$ such that for any n, $\|\dot{u}_n\|_{L^\infty(0,+\infty)}, \|\ddot{u}_n\|_{L^\infty(0,+\infty)} \leq \rho$, Lemma 23 shows that the first term on the right-hand side of the last equality tends to zero when $n \to +\infty$, uniformly wrt $t \geq 0$.

Now, as the integrals $\int_0^t e^{A\int_s^t |\dot{u}_\infty|} \cdot ds$ are bounded uniformly wrt $t \geq 0$, for any $u \in W^{1,\infty}(0,+\infty)$ with $\dot{u} - \dot{u}_\infty \neq 0$ a.e., denoting x (resp. x_∞) the space variable of the operator $\bar{H}$ corresponding to the input u (resp. u_∞), one has:

$$\dot{x} - \dot{x}_\infty = Ax(\bar{g}(\dot{u}) - \bar{g}(\dot{u}_\infty)) + A(x - x_\infty)\bar{g}(\dot{u}_\infty) + B(\dot{u} - \dot{u}_\infty), \quad x(0) = x_\infty(0)$$

which implies

$$x(t) - x_\infty(t) = \int_0^t e^{A\int_s^t \bar{g}(\dot{u}_\infty)}[Ax(\bar{g}(\dot{u}) - \bar{g}(\dot{u}_\infty)) + B(\dot{u} - \dot{u}_\infty)](s) \cdot ds$$

so, using Lipschitz property for $\bar{g}$ (and for $\bar{s}$ if $D \neq 0$) and the fact that for any $\alpha \in \mathbb{R}$, $\bar{g}(\alpha) \geq |\alpha|$, we get for a certain $c > 0$ independent of u:

$$\frac{\|\bar{H}(u) - \bar{H}(u_\infty)\|_{L^\infty(0,+\infty)}}{\|\dot{u} - \dot{u}_\infty\|_{L^\infty(0,+\infty)}} \leq c < +\infty$$

If now the inequality given in the statement is true, then

$$\begin{aligned} \left(1 - \sup_{\substack{u \in W^{1,\infty}(0,+\infty) \\ \dot{u} - \dot{u}_\infty \neq 0 \text{ a.e.}}} \frac{\|\bar{H}(u) - \bar{H}(u_\infty)\|_{L^\infty(0,+\infty)}}{\|\dot{u} - \dot{u}_\infty\|_{L^\infty(0,+\infty)}} \cdot \left\|\mathcal{L}^{-1}\left(\frac{sM(s)}{L(s)}\right)\right\|_{L^1(0,+\infty)}\right) \|\dot{u}_n - \dot{u}_\infty\|_{C^0(0,+\infty)} \\ \leq \left\|\mathcal{L}^{-1}\left(\frac{sM(s)}{L(s)}\mathcal{L}(H(u_n+d_n) - \bar{H}(u_n))\right)\right\|_{C^0(0,+\infty)} \end{aligned}$$

which tends to 0, as $\frac{sM(s)}{L(s)}$ is strictly proper and stable, so $\dot{u}_n - \dot{u}_\infty$ tends to zero in $C^0(0,+\infty)$, and the same property is deduced for $u_n - u_\infty$ and $u_n + d_n - u_\infty$. This concludes the proof of Theorem 4.

B.2 Proof of Lemma 23

• It is not difficult to prove that Hypotheses **(D0)**, **(D1)**, **(D2u)** to **(D4u)** imply the following equalities, where the convergences are uniform wrt $\alpha \in \mathbb{R}$ and $t \geq 0$:

$$\lim_{n\to+\infty} \frac{1}{\bar{\delta}(n)} \int_t^{t+\bar{\delta}(n)} |\alpha + \dot{d}_n(s)| \cdot ds = \bar{g}(\alpha) \qquad (28)$$

$$\forall c > 0, \quad \lim_{n\to+\infty} \frac{1}{\bar\delta(n)} mes\{s \in (t, t+\bar\delta(n)) : |\alpha + \dot d_n(s)| \le c\bar\delta(n)\} = 0 \tag{29}$$

$$\lim_{n\to+\infty} \frac{1}{\bar\delta(n)} \int_t^{t+\bar\delta(n)} \text{sgn } (\alpha + \dot d_n(s)) \cdot ds = \bar s(\alpha) \tag{30}$$

The proof uses the identities (9), and (to prove (29)) the fact that $\frac{1}{\bar\delta(n)}(\dot d_n(t) - g'(nt))$ is uniformly bounded. Remark that $\bar\delta$ in (28) is not linked with the $\bar\delta$ defined in **(D0)**. This implies that the assumption $n\bar\delta(n) \to +\infty$ is useless when $D = 0$. The other part, $\bar\delta(n) \to 0$, remains however useful to bound $\dot d_n(t) - g'(nt)$ in **(D0)**.

• We use a subdivision $t_k, 0 \le k \le K$ of $[0, t]$ with $K = \text{int } (\frac{t}{\bar\delta(n)}) + 1$, $\bar\delta$ being defined in Hypothesis **(D0)**. It implies that $t_{k+1} - t_k = \frac{t}{K} \sim \bar\delta(n)$. We have

$$\begin{aligned}
&\left|\int_0^t e^{-c(t-s)}(\text{sgn } (\dot u(s) + \dot d_n(s)) - \bar s(\dot u(s))) \cdot ds\right| \\
&\le \sum_{k=0}^{K-1} \left|\int_{t_k}^{t_{k+1}} e^{-c(t-s)}(\text{sgn } (\dot u(s) + \dot d_n(s)) - \text{sgn } (\dot u(t_k) + \dot d_n(s))) \cdot ds\right| \\
&\quad + \sum_{k=0}^{K-1} \left|\int_{t_k}^{t_{k+1}} e^{-c(t-s)}(\text{sgn } (\dot u(t_k) + \dot d_n(s)) - \bar s(\dot u(t_k))) \cdot ds\right| \\
&\quad + \sum_{k=0}^{K-1} \left|\int_{t_k}^{t_{k+1}} e^{-c(t-s)}(\bar s(\dot u(t_k)) - \bar s(\dot u(s))) \cdot ds\right| \triangleq \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3
\end{aligned}$$

Here and in the sequel, we denote by ε various positive functions tending to 0 with their argument, and by c various strictly positive constants, both independent of t and n. We have:

$$\begin{aligned}
\sum\nolimits_1 &\le 2\sum_{k=0}^{K-1} e^{-c(t-t_{k+1})} mes\{s \in (t_k, t_{k+1}) : |\dot u(t_k) + \dot d_n(s)| \le |\dot u(s) - \dot u(t_k)|\} \\
&\le c\sum_{k=0}^{K-1} e^{-c(t-t_{k+1})} mes\{s \in (t_k, t_{k+1}) : |\dot u(t_k) + \dot d_n(s)| \le \|\ddot u\|_{L^\infty}(t_{k+1} - t_k)\} \\
&\quad \text{(as } \ddot u \text{ is bounded in } L^\infty(0, +\infty)) \\
&\le c\sum_{k=0}^{K-1} e^{-c(t-t_{k+1})} mes\{s \in (t_k, t_k + \bar\delta(n)) : |\dot u(t_k) + \dot d_n(s)| \le c\bar\delta(n)\} \\
&\le c\bar\delta(n)\varepsilon(\frac{1}{n}) \sum_{k=0}^{K-1} e^{-c\frac{K-k-1}{K}t} \quad \text{(due to (29))} \\
&= c\bar\delta(n)\varepsilon(\frac{1}{n}) e^{c\frac{t}{K}} \frac{1 - e^{-ct}}{e^{c\frac{t}{K}} - 1} = c\bar\delta(n)\varepsilon(\frac{1}{n}) \frac{1 - e^{-ct}}{1 - e^{-c\bar\delta(n)t}} = \varepsilon(\frac{1}{n}) \to 0
\end{aligned}$$

as $\bar\delta(n)$ tends to 0. $\sum_2$ is estimated as follows:

$$\begin{aligned}\textstyle\sum_2 \quad &\le \quad \sum_{k=0}^{K-1} e^{-c(t-t_k)}\left|\int_{t_k}^{t_{k+1}}(\text{sgn }(\dot u(t_k)+\dot d_n(s))-\bar s(\dot u(t_k)))\cdot ds\right| \\ &\quad +\sum_{k=0}^{K-1}\left|\int_{t_k}^{t_{k+1}} e^{-c(t-s)}\int_s^{t_{k+1}}(\text{sgn }(\dot u(t_k)+\dot d_n(s'))-\bar s(\dot u(t_k)))\cdot ds'\cdot ds\right| \\ &\quad \text{(integrating by parts)}\end{aligned}$$

As $\int_s^{t_{k+1}}(\text{sgn }(\dot u(t_k)+\dot d_n(s'))-\bar s(\dot u(t_k)))\cdot ds'$ is smaller than $2(t_{k+1}-s)$ and $2(s-t_k)+\bar\delta(n)\varepsilon_n$, where $\varepsilon_n \stackrel{\Delta}{=} \max\limits_{0\le k\le K-1} \frac{1}{\bar\delta(n)}\int_{t_k}^{t_{k+1}}(\text{sgn }(\dot u(t_k)+\dot d_n(s))-\bar s(\dot u(t_k)))\cdot ds$ tends towards zero (by (30)), one gets:

$$\begin{aligned}\textstyle\sum_2 \quad &\le \quad \sum_{k=0}^{K-1} e^{-c(t-t_{k+1})}\bar\delta(n)\varepsilon_n+\sum_{k=0}^{K-1}\int_{t_k}^{\frac{t_k+t_{k+1}-\bar\delta(n)\varepsilon_n}{2}} e^{-c(t-s)}(2(s-t_k)+\bar\delta(n)\varepsilon_n)\cdot ds \\ &\quad +\sum_{k=0}^{K-1}\int_{\frac{t_k+t_{k+1}-\bar\delta(n)\varepsilon_n}{2}}^{t_{k+1}} 2e^{-c(t-s)}(t_{k+1}-s)\cdot ds \\ &\le \quad ce^{-ct}\sum_{k=0}^{K-1}\left(e^{ct_{k+1}}\bar\delta(n)\varepsilon_n+e^{ct_{k+1}}-(c\bar\delta(n)\varepsilon_n-1)e^{ct_k}-(2+\bar\delta(n))e^{\frac{t_k+t_{k+1}-\bar\delta(n)\varepsilon_n}{2}}\right) \\ &\le \quad c(\bar\delta(n)+\varepsilon_n) \quad \text{(after computation and simplification)}\end{aligned}$$

The last term is estimated as follows, using the uniform continuity of $\bar s$ (cf. Theorem 7):

$$\textstyle\sum_3 \le \displaystyle\sum_{k=0}^{K-1}\varepsilon(t_{k+1}-t_k)\int_{t_k}^{t_{k+1}} e^{-c(t-s)}ds=\varepsilon(\bar\delta(n))$$

Hence, $\int_0^t e^{-c(t-s)}(\text{sgn }(\dot u+\dot d_n)(s)-\bar s(\dot u))\cdot ds\to 0$, uniformly wrt $t\ge 0$. The convergence is also uniform in sets as defined in Lemma 23, as indeed the choice of c and ε depend only upon $\|\dot u\|_{L^\infty(0,+\infty)}, \|\ddot u\|_{L^\infty(0,+\infty)}$.

• We shall first prove the following estimate: $\forall h\in C^0(0,+\infty)\cap L^\infty(0,+\infty), \forall t,\tau\ge 0$,

$$\left|\int_\tau^t(|\dot u(s)+\dot d_n(s)|-\bar g(\dot u(s)))h(s)\cdot ds\right|\le\varepsilon(1/n)(t-\tau) \tag{31}$$

First, suppose that $h\equiv 1$. For any $\tau,t\in\mathbb{R}$ with $t>\tau$, let us define $t_k=\tau+\frac{k}{K}(t-\tau)$ for $k=0,1,\dots,K$, with $K=\text{int }(\frac{t-\tau}{\bar\delta(n)})+1$, in such a way that $t_{k+1}-t_k=\frac{t}{K}\sim\bar\delta(n)$. Then

$$\left|\int_\tau^t|\dot u(s)+\dot d_n(s)|\cdot ds-\int_\tau^t\bar g(\dot u(s))\cdot ds\right|$$

$$
\begin{aligned}
&\leq \sum_{k=0}^{K-1}\left|\int_{t_k}^{t_{k+1}}(|\dot u(s)+\dot d_n(s)|-|\dot u(t_k)+\dot d_n(s)|)\cdot ds\right| \\
&\quad+\sum_{k=0}^{K-1}\left|\int_{t_k}^{t_{k+1}}(|\dot u(t_k)+\dot d_n(s)|-\bar g(\dot u(t_k)))\cdot ds\right| \\
&\quad+\sum_{k=0}^{K-1}\left|\int_{t_k}^{t_{k+1}}(\bar g(\dot u(t_k))-\bar g(\dot u(s)))\cdot ds\right| \\
&\leq \sum_{k=0}^{K-1} c\int_{t_k}^{t_{k+1}}(s-t_k)\cdot ds+\sum_{k=0}^{K-1}\left|\int_{t_k}^{t_{k+1}}(|\dot u(t_k)+\dot d_n(s)|-\bar g(\dot u(t_k)))\cdot ds\right| \\
&\quad\text{(using the Lipschitz property of } \dot u \text{ and } \bar g) \\
&= c\frac{(t-\tau)^2}{K}+\sum_{k=0}^{K-1}\bar\delta(n)\varepsilon(\frac{1}{n}) \quad \text{(by (28))} \\
&= (t-\tau)(\bar\delta(n)+\varepsilon(\frac{1}{n})) = (t-\tau)\varepsilon(\frac{1}{n})
\end{aligned}
$$

and by continuity, (31) holds also for $t \geq \tau \geq 0$. The result is then deduced for any h using piecewise constant approximations.

• We now prove the convergence property for x_n. One has

$$\dot x_n - \dot x_\infty = Ax_n|\dot u+\dot d_n| - Ax_\infty \bar g(\dot u) + B\dot d_n, \quad x_n(0)=x_\infty(0)$$

Hence, for any $t \geq 0$,

$$
\begin{aligned}
|x_n(t)-x_\infty(t)| &= \left|\int_0^t e^{A\int_s^t \bar g(\dot u)}(Ax_n(s)(|\dot u(s)+\dot d_n(s)|-\bar g(\dot u(s)))+B\dot d_n(s))\cdot ds\right| \\
&= \left|e^{A\int_0^t \bar g(\dot u(s))\cdot ds}\int_0^t (Ax_n(s)(|\dot u(s)+\dot d_n(s)|-\bar g(\dot u(s)))+B\dot d_n(s))\cdot ds\right| \\
&\quad+\left|A\bar g(\dot u(t))\int_0^t e^{A\int_s^t \bar g(\dot u)}\int_s^t (Ax_n(|\dot u+\dot d_n|-\bar g(\dot u))+B\dot d_n)\cdot ds'\cdot ds\right| \\
&\quad\text{(integrating by parts)} \\
&\leq ce^{-ct}\varepsilon(1/n)t + c\int_0^t e^{-c(t-s)}\varepsilon(1/n)(t-s)\cdot ds = \varepsilon(1/n)
\end{aligned}
$$

using (31) and the fact that $d_n \to 0$ in $L^\infty(0,+\infty)$. We conclude that $x_n \to x_\infty$ in $C^0(0,+\infty)$ (uniformly in the sets $\{u \in W^{1,1}_{loc}(0,+\infty) : \|\dot u\|_{L^\infty(0,+\infty)}, < \rho\}$) as c and ε depend only upon $\|\dot u\|_{L^\infty}$), and this achieves the proof.

B.3 Proof of Lemma 5

It is possible, using the same argument than for (31), to prove that

$$\forall T \geq 0, \quad \int_0^T (\text{sgn } (\dot{u} + \dot{d}_n)(s) - \bar{s}(\dot{u})) \cdot ds \leq \varepsilon(\frac{1}{n})T$$

This, together with the convergence of x_n towards x_∞ (see Lemma 23), shows the expected result.

C Proofs of Theorem 9 and Lemma 10 : Unbounded velocity

C.1 Proof of Theorem 9

Again, we show only the infinite time interval result. The same steps than for the proof of Theorem 4 will be made.

We use the following Lemma, counterpart of Lemma 24:

Lemma 24 *Let Hypotheses* **(D0')**, **(D1')**, **(D2'u)** *to* **(D4'u)** *hold. Let* $w \in W^{1,1}_{loc}(0, +\infty)$ *with* $\dot{w}$ *bounded in* $L^\infty(0, +\infty)$*. Then, defining* $\bar{H}_\gamma$ *as in (8), one has* $\forall c > 0$,

$$\int_0^t e^{-c(t-s)}(H(u + d_n)(s) - \bar{H}_\gamma) \cdot ds \to 0 \textit{ uniformly in } t \in [0, +\infty)$$

and, for any $\rho > 0$*, the convergence is uniform in the set* $\{u \in W^{1,1}_{loc}(0, +\infty) : \|\dot{u}\|_{L^\infty(0,+\infty)} < \rho\}$.

More precisely,

- **(D3'u)**, **(D4'u)** *imply that* $\forall c > 0$,

$$\int_0^t e^{-c(t-s)}(D\text{sgn } (\dot{u} + \dot{d}_n)(s) - \bar{H}_\gamma) \cdot ds \to 0 \textit{ uniformly wrt } t \in [0, +\infty)$$

- **(D2'u)** *implies that for all* $T_0 > 0$,

$$x_n \to 0 \textit{ in } C^0(T_0, +\infty)$$

where

$$\dot{x}_n = Ax_n|\dot{u} + \dot{d}_n| + B(\dot{u} + \dot{d}_n), \quad x_n(0) = x_0$$

Let us show that Lemma 24 leads to the proof of Theorem 9. One has

$$u_n - u_\infty \in \frac{M(s)}{L(s)}(-H(u_n + d_n) + \bar{H}_\gamma)$$

Using realization (27), we have:

$$\dot{\tilde{u}}_n \in \mathcal{A}\tilde{u}_n + \mathcal{B}(-H(\mathcal{C}\tilde{u}_n + d_n) + f), \quad \tilde{u}_n(0) = \tilde{u}(0)$$

Hence, for every $t > 0$,

$$\tilde{u}_n(t) \in e^{\mathcal{A}t}\tilde{u}(0) + \int_0^t e^{\mathcal{A}(t-s)}\mathcal{B}(-H(\mathcal{C}\tilde{u}_n + d_n)(s) + f(s)) \cdot ds$$

With the hypotheses on f and due to Hypothesis **(H1)** and Theorem 1, we deduce that the sequences u_n and $\dot{u}_n$ are bounded in $C^0(0, +\infty)$, uniformly wrt f in the sets defined in the statement of Theorem 9.

It is then deduced by the stability of the transfer function $\frac{M(s)}{L(s)}$ and Lemma 24, that $\tilde{u}_n - \tilde{u}_\infty$ tends to zero in $C^0(0, +\infty)$, and the same is true for $u_n - u_\infty$ and hence $u_n + d_n - u_\infty$ (by Hypothesis **(D0')**). This concludes the proof of Theorem 9.

C.2 Proof of Lemma 24

• From Hypotheses **(D0')**, **(D1')**, **(D2'u)** to **(D4'u)** are deduced the following equalities, with uniform convergence wrt $t \geq 0$:

$$\lim_{n\to+\infty} \int_t^{t+\bar{\delta}(n)} |\dot{d}_n(s)| \cdot ds = \bar{g}(0) \tag{32}$$

$$\forall c > 0, \quad \lim_{n\to+\infty} \frac{1}{\bar{\delta}(n)} mes\{s \in (t, t+\bar{\delta}(n)) : |\dot{d}_n(s)| \leq c\} = 0 \tag{33}$$

$$\lim_{n\to+\infty} \frac{1}{\bar{\delta}(n)} \int_t^{t+\bar{\delta}(n)} \text{sgn}\,(\dot{d}_n(s)) \cdot ds = \bar{s}(0) \tag{34}$$

• We have, taking as in the proof of Lemma 23 a subdivision of $[0, t]$ with step $\frac{t}{K} \sim \bar{\delta}(n)$:

$$\begin{aligned}
&\left|\int_0^t e^{-c(t-s)}(\text{sgn}\,(\dot{u}(s) + \dot{d}_n(s)) - \bar{s}(0)) \cdot ds\right| \\
&\leq \sum_{k=0}^{K-1} \left|\int_{t_k}^{t_{k+1}} e^{-c(t-s)}(\text{sgn}\,(\dot{u}(s) + \dot{d}_n(s)) - \text{sgn}\,\dot{d}_n(s)) \cdot ds\right| \\
&\quad + \sum_{k=0}^{K-1} \left|\int_{t_k}^{t_{k+1}} e^{-c(t-s)}(\text{sgn}\,\dot{d}_n(s) - \bar{s}(0)) \cdot ds\right| \\
&\leq c \sum_{k=0}^{K-1} mes\{s \in (t_k, t_{k+1}) : |\dot{d}_n(s)| \leq \|\dot{u}\|_{L^\infty}\} e^{-c(t-t_{k+1})} \\
&\quad + \sum_{k=0}^{K-1} e^{-c(t-t_k)} \left|\int_{t_k}^{t_{k+1}} (\text{sgn}\,\dot{d}_n(s) - \bar{s}(0)) \cdot ds\right|
\end{aligned}$$

$$
\begin{aligned}
& +c\sum_{k=0}^{K-1}\left|\int_{t_k}^{t_{k+1}} e^{-c(t-s)}\int_s^{t_{k+1}}(\text{sgn }\dot{d}_n-\bar{s}(0))\cdot ds'\cdot ds\right| \quad \text{(integrating by parts)}\\
\leq\ & \bar{\delta}(n)\varepsilon(\frac{1}{n})\sum_{k=0}^{K-1} e^{-c(t-t_{k+1})}+\bar{\delta}(n)\varepsilon_n\sum_{k=0}^{K-1} e^{-c(t-t_k)}\\
& +c\sum_{k=0}^{K-1}\int_{t_k}^{\frac{t_k+t_{k+1}-\bar{\delta}(n)\varepsilon_n}{2}} e^{-c(t-s)}(2(s-t_k)+\bar{\delta}(n)\varepsilon_n)\cdot ds\\
& +\sum_{k=0}^{K-1}\int_{\frac{t_k+t_{k+1}-\bar{\delta}(n)\varepsilon_n)}{2}}^{t_{k+1}} 2e^{-c(t-s)}(t_{k+1}-s)\cdot ds \quad \text{(using (33) and the fact}\\
& \text{that } \varepsilon_n \triangleq \max_{0\leq k\leq K-1}\frac{1}{\bar{\delta}(n)}\int_{t_k}^{t_{k+1}} \text{sgn }\dot{d}_n(s)\cdot ds \text{ tends to 0, due to (34))}\\
\leq\ & c(\varepsilon(\frac{1}{n})+\bar{\delta}(n)+\varepsilon_n)
\end{aligned}
$$

In conclusion,

$$\forall c>0,\quad \int_0^t e^{-c(t-s)}(\text{sgn }(\dot{u}+\dot{d}_n)(s)-\bar{H}_\gamma)\cdot ds\to 0 \text{ uniformly in } t\in[0,+\infty)$$

and the convergence is uniform in the sets $\{u\in W^{1,1}_{loc}(0,+\infty):\|\dot{u}\|_{L^\infty(0,+\infty)}<\rho\}$.
• We shall use the following estimate: $\forall h\in C^0(0,+\infty)\cap L^\infty(0,+\infty), \forall t,\tau>0$,

$$\left|\int_\tau^t(|\dot{u}(s)+\dot{d}_n(s)|-\frac{1}{\bar{\delta}(n)}\bar{g}(0))h(s)\cdot ds\right|\leq\frac{1}{\bar{\delta}(n)}\varepsilon(\frac{1}{n})(t-\tau) \tag{35}$$

Formula (35) is deduced from (32) as (31) was deduced from (28).
• Defining x'_n by $\dot{x}'_n=Ax'_n\frac{\bar{g}(0)}{\bar{\delta}(n)}+B\dot{u}$, $x'_n(0)=x_0$, one has clearly $x'_n\to 0$ uniformly on every set $[T_0,+\infty)$. As

$$\dot{x}_n-\dot{x}'_n=Ax_n|\dot{u}+\dot{d}_n|-Ax'_n\frac{\bar{g}(0)}{\bar{\delta}(n)}+B\dot{d}_n,\quad x_n(0)=x'_n(0)$$

one gets, for any $t>0$:

$$
\begin{aligned}
x_n(t)-x'_n(t) &= \int_0^t e^{A\frac{\bar{g}(0)}{\bar{\delta}(n)}(t-s)}\left(Ax_n(s)\left(|\dot{u}(s)+\dot{d}_n(s)|-\frac{\bar{g}(0)}{\bar{\delta}(n)}\right)+B\dot{d}_n(s)\right)\cdot ds\\
&\leq e^{A\frac{\bar{g}(0)}{\bar{\delta}(n)}t}\int_0^t\left(Ax_n(s)\left(|\dot{u}(s)+\dot{d}_n(s)|-\frac{\bar{g}(0)}{\bar{\delta}(n)}\right)+B\dot{d}_n(s)\right)\cdot ds\\
&\quad +A\frac{\bar{g}(0)}{\bar{\delta}(n)}\int_0^t e^{A\frac{\bar{g}(0)}{\bar{\delta}(n)}(t-s)}\int_s^t\left(Ax_n\left(|\dot{u}+\dot{d}_n|-\frac{\bar{g}(0)}{\bar{\delta}(n)}\right)+B\dot{d}_n\right)\cdot ds'\cdot ds
\end{aligned}
$$

(integrating by parts)

$$\leq \quad e^{-c\frac{t}{\bar{\delta}(n)}}\left(\|d_n\|_{L^\infty} + ct\frac{1}{\bar{\delta}(n)}\varepsilon(\frac{1}{n})\right)$$
$$+\frac{1}{\bar{\delta}(n)}\int_0^t e^{-c\frac{t-s}{\bar{\delta}(n)}}\left(\|d_n\|_{L^\infty} + c(t-s)\frac{1}{\bar{\delta}(n)}\varepsilon(\frac{1}{n})\right)\cdot ds$$

using (35). We deduce that for any $T_0 > 0$, $x_n \to 0$ uniformly wrt $t \in [T_0, +\infty)$. As the above relation is true uniformly in the sets $\{u \in W^{1,1}_{loc}(0,+\infty) : \|\dot{u}\|_{L^\infty(0,+\infty)} < \rho\}$, the convergence is uniform, so Lemma 24 is proved.

C.3 Proof of Lemma 10

Using similar procedure than the one used to prove (35), one proves that

$$\int_0^T \text{sgn } (\dot{u} + \dot{d}_n) - \bar{s}(0)) \cdot ds \leq \varepsilon(\frac{1}{n})T$$

On the other hand, the second point in the proof of Lemma 24 shows that $\|x_n\|_{C^0(T_0,+\infty)} \to 0$ for any $T_0 > 0$. Hence, the conclusions of the Lemma are proved.

References

[1] Alexandrovitz A., Rootenberg J., Dithering as a factor in hysteresis elimination in rotating amplifiers, *IEEE Trans. on Aut. Control*, **13**, 170-173, 1968

[2] Canudas de Wit C., Olsson H., Åström K.J., Lischinsky P., A new model for control of systems with friction, *IEEE Trans. on Aut. Control*, **40**, no 3, 419-425, 1995

[3] Balachandra M., Sethna P.R., A generalization of the method of averaging for systems with two time scales, *Arch. Rat. Mech. Anal.*, 58, 261-283, 1975

[4] Bersekerskii V.A., Applying vibrators to eliminate nonlinearities in automatic regulators, *Avtomat. i Telemekh.*, **8**, 411-417, 1947 (in Russian)

[5] Bliman P.-A., Sorine M., A system-theoretic approach of systems with hysteresis. Application to friction modelling and compensation, *Proc. of the 2nd Eur. Cont. Conf.*, Groningen, The Netherlands, 1993, 1844-1849

[6] Bliman P.-A., Sorine M., Easy-to-use realistic dry friction models for automatic control, *Proc. of the 3rd Eur. Cont. Conf.*, Roma, Italy, 1995, 3788-3794

[7] Bliman P.-A., Bonald T., Sorine M., Hysteresis Operators and Tyre Friction Models. Application to Vehicle Dynamic Simulation, to appear in the *Proc. of ICIAM*, Hamburg, Germany, 1995

[8] Bogoliubov N.N., Mitropolsky Y.A., *Asymptotic methods in the theory of non-linear oscillations*, Gordon and Breach, New-York, 1961

[9] Bohr G., *Almost Periodic Functions*, Chelsea Publ. Company, New York, N.Y., 1947

[10] Contensou P., Couplage entre frottement de glissement et frottement de pivotement dans la théorie de la toupie, *Kreiselprobleme: Gyrodynamics*, ed. H. Ziegler, Springer-Verlag, Berlin, 201–216, 1963

[11] Deimling K., *Multivalued differential equations*, Walter de Gruyter, Berlin New-York, 1992

[12] Desoer C.A., Shahruz S.M., Stability of non-linear systems with backlash or hysteresis, *Int. J. of Control*, **43**, no. 4, 1045–1060, 1986

[13] Fink A.M., *Almost periodic differential equations*, Springer-Verlag, Berlin Heidelberg New-York, 1974

[14] Friedman H.D., Levesque P., Reduction of static friction by sonic vibrations, *J. of Applied Physics*, **30**, 10, 1959

[15] Godfrey D., Vibration reduces metal to metal contact and causes an apparent reduction in friction, *ASLE Trans.*, **10**, 183-192, 1967

[16] Hess D.P., Soom A., Normal vibrations and friction under harmonic loads: Part I - Hertzian contacts, *Trans. of the ASME*, **113**, 80-86, 1991

[17] Krasnosel'skii A.M., *Asymptotics of nonlinearities and operator equations*, Birkhäuser, 1995

[18] Krasnosel'skii M.A., Pokrovskiĭ A.V., *Systems with hysteresis*, Springer-Verlag, Berlin Heidelberg, 1989

[19] MacColl L.A., *Fundamentals of servomechanisms*, Van Nostrand, New-York, 1945

[20] Mossaheb S., Application of a method of averaging to the study of dithers in non-linear systems, *Int. J. of Control*, **38**, 3, 557-576, 1983

[21] Neĭmark Ju.I., Fufaev N.A., *Dynamics of nonholonomic systems*, Translations of Mathematical Monographs, vol. 33, AMS, Providence (RI), 1972

[22] Rudin W., *Real and complex analysis*, Mc Graw-Hill, 3rd ed. 1987

[23] Sánchez-Palencia E., *Non-homogeneous media and vibration theory*, Lecture Notes in Physics 127, Springer-Verlag, Berlin Heidelberg New-York, 1980

[24] Tung P.-C., Chen S.-C., Experimental and analytical studies of the sinusoidal dither signal in a DC motor system, *Dynamics and control*, **3**, 53-69, 1993

[25] Visintin A., *Differential models of hysteresis*, Applied Mathematical Sciences 111, Springer-Verlag, Berlin Heidelberg New-York, 1994

[26] Zaidman S., *Almost-periodic functions in abstract spaces*, Research Notes in Math. 126, Pitman Advanced Publishing Program, Boston London Melbourne, 1985

[27] Zames G., Shneydor N.A., Dither in Nonlinear Systems, *IEEE Trans. on Aut. Control*, **AC-21**, 5, October 1976, 660-667

[28] Zames G., Shneydor N.A., Structural Stabilization and Quenching by Dither in Nonlinear Systems, *IEEE Trans. on Aut. Control*, **AC-22**, 3, June 1977, 352-361

Unité de recherche Inria Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 Villers Lès Nancy
Unité de recherche Inria Rennes, Irisa, Campus universitaire de Beaulieu, 35042 Rennes Cedex
Unité de recherche Inria Rhône-Alpes, 46 avenue Félix Viallet, 38031 Grenoble Cedex 1
Unité de recherche Inria Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex
Unité de recherche Inria Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 Sophia-Antipolis Cedex

Éditeur
Inria, Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex (France)
ISSN 0249-6399