Computing Dimensionally Parametrized Determinant Formulas

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Abstract

We are interested in dimensionally parametrized determinant formulas for specially structured matrices. Applications of this question occur in the study of arbitrary dimensional geometric predicates. We will investigate determinant formulas for two important matrix classes and discuss the implementation of Maple packages that automatically derive the determinant formula for specified matrices of these classes.

Introduction

Determinants have a long history in mathematics and arise in numerous applications [2]. Here we are not interested in the *value* of a determinant of fixed integer order but rather in the determinant *formula* of a specially structured matrix of symbolic dimension n. It is assumed that a certain simple structure of a matrix yields a corresponding special structure of its determinant formula. Applications of dimensionally parametrized determinant formulas occur in the study of arbitrary dimensional geometric predicates in determinant form: If we want to prove a general statement for a special configuration then we need the determinant formula of the predicate.

In the following sections we will investigate determinant formulas for two important matrix classes, the Frameforms and the Alternants. Moreover, we will discuss the implementation of Maple packages that allow a specification of matrices of these classes and automatically derive its determinant formula.

Frameforms

We will first examine a matrix class where only the bordering rows and columns as well as the main diagonal may contain nonzero entries. Matrices of this class will be called *frameforms*.

Motivation

Geometric predicates such as the in-sphere test – do $d+2$ points of $\mathbb{R}^d$ lie on a common sphere ? – may be written in determinant form [1].

Consider the following example in [1]: We have a configuration of $d+2$ points in $\mathbb{R}^d$, two distinct points $\overline{s} = (s, \ldots, s)$ and $\overline{t} = (t, \ldots, t)$ from the main diagonal and one point $\overline{t_i} = t_i \cdot e_i$ from each axis.

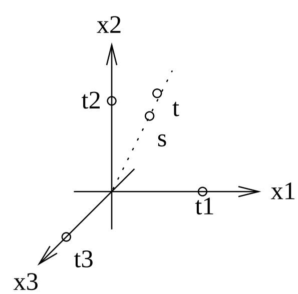

Figure 1: Configuration in $\mathbb{R}^3$

This point configuration results in the following in-sphere determinant:

$$S = \begin{vmatrix} 1 & t & t & \cdots & t & dt^2 \\ 1 & t_1 & 0 & \cdots & 0 & t_1^2 \\ 1 & 0 & t_2 & \ddots & \vdots & t_2^2 \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 1 & 0 & \cdots & 0 & t_d & t_d^2 \\ 1 & s & s & \cdots & s & ds^2 \end{vmatrix}_{d+2} .$$

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†Parts of this work have been supported by the German Research Foundation (DFG)

We are looking for an easy dimensionally parametrized determinant formula such that we can show that the determinant does not vanish for given ranges of the entries which would establish that the $d+2$ chosen points are not cospherical. The determinant S is in frameform and we will show in the sequel how to establish its determinant formula.

Determinant formulas for Frameforms

We will show that it is possible to derive the most general form of frameform matrices

$$\begin{vmatrix} c_1(1) & r_1(2) & \cdots & \cdots & r_1(n-1) & c_n(1) \\ c_1(2) & d(2) & 0 & \cdots & 0 & c_n(2) \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ c_1(n-1) & 0 & \cdots & 0 & d(n-1) & c_n(n-1) \\ c_1(n) & r_n(2) & \cdots & \cdots & r_n(n-1) & c_n(n) \end{vmatrix}$$

from simpler forms.

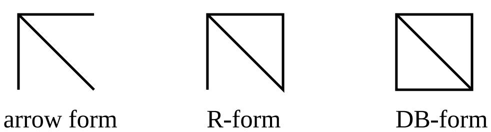

Figure 2: Nonzero shapes of different frameforms

We will proceede as follows: First we will give a dimensionally parametrized determinant formula for arrow forms which will allow us a straightforward generalization to R-forms. The most general form, the DB-form, will be obtained by a combination of R-forms.

Arrow forms

The only nonzero elements of an arrow form determinant are located in the first row, the first column or the main diagonal. It will be denoted as $\text{ARROW}_n(r_1, c_1, d)$.
Expansion of the first row and clever restructuring establishes the following determinant formula:

$$\text{ARROW}_n(r_1, c_1, d) = \prod_{l=1}^{n} d(l) - \sum_{l=2}^{n} r_1(l) c_1(l) \prod_{\substack{k=2 \\ k \neq l}}^{n} d(k).$$

R-forms

In comparison to arrow forms, the last column may also contain nonzero elements in R-forms which will be denoted as $\text{RFORM}_n(r_1, c_1, c_n, d)$.
Expanding the last row it may be seen that an R-form determinant formula can be obtained from two arrow forms of lower order.

$$\begin{aligned} \text{RFORM}_n(r_1, c_1, c_n, d) = \; & c_n(n)\text{ARROW}_{n-1}(r_1, c_1, d) \\ - \; & c_1(n)\text{ARROW}_{n-1}(\widetilde{r}_1, \widetilde{c}_n, \widetilde{d}) \end{aligned}$$

where $\widetilde{r}_1, \widetilde{c}_n, \widetilde{d}$ are new generating functions obtained by swapping column 1 and $n-1$ in the corresponding minor R_{n1} to get arrow form.

DB-forms

Now we turn to the most general case of frameforms which will be denoted as $\text{DBFORM}_n(r_1, r_n, c_1, c_n)$.
Expanding the last row, we see after some restructuring that it is possible to express its formula as a combination of R-forms.

$$\begin{aligned} \text{DBFORM}_n(r_1, r_n, c_1, c_n) &= \text{RFORM}_n(r_1, c_1, c_2, d) \\ &- \sum_{l=2}^{n-1} r_n(l)\text{RFORM}_{n-1}(\widehat{r}_1, \widehat{c}_1, \widehat{c}_n, \widehat{d}) \end{aligned}$$

where $\widehat{r}_1, \widehat{c}_1, \widehat{c}_n, \widehat{d}$ are new generating functions obtained by swapping row l down to the bottom in the corresponding minor DB_{nl} to get R-form.

Generalizations

It is easy to transform similar shapes like arrows pointing to the bottom right corner into the discussed standard shapes.
The assumption that the nonzero elements should reside in bordering rows and columns may also be dropped since it is possible to obtain this bordering form via pairwise swappings. Refer to [3] for details.

A Maple package for Frameforms

We have implemented a Maple package that enables the user to specify a general frameform determinant and automatically computes the corresponding determinant formula using the preceding results. The package works as follows: First, the specification is parsed and tested for correctness, then a transformation into standard form is performed and finally the corresponding formula is applied after trying out simplifications. Features are options that enable the display of the specified dimensionally parametrized determinant (using "o"s as dots for illustration) and that check the computed formula via substitution of integer orders and comparison with the normally computed determinant. Details can be obtained from [3] or the online help pages.

Example

```
> with(FRAMEFORMS):
> DBform(d+2,[[col[1],1],[col[d+2],[[1,
d*t^2],[2..d+1,t[i-1]^2],[d+2,d*s^2]]],
[diag,[2..d+1,t[i-1]]],[row[1],[2..d+1,t]],
[row[d+2],[2..d+1,s]]],print,check);
```

$$\mathit{Matrix}:$$

$$\begin{bmatrix} 1 & t & t & o & o & t & d\,t^2 \\ 1 & t_1 & 0 & 0 & 0 & 0 & {t_1}^2 \\ 1 & 0 & t_2 & 0 & 0 & 0 & {t_2}^2 \\ o & 0 & 0 & o & 0 & 0 & o \\ o & 0 & 0 & 0 & o & 0 & o \\ 1 & 0 & 0 & 0 & 0 & t_d & {t_d}^2 \\ 1 & s & s & o & o & s & d\,s^2 \end{bmatrix}$$

$$\mathit{Determinant}:$$

$$\mathit{formula\ valid\ for}\ \ d > 1$$

$$(1 - \frac{t}{s})\,((\prod_{l=1}^{d} t_l)\,d\,s^2 - s\,(\prod_{l=1}^{d} t_l)\,(\sum_{l=1}^{d} t_l)) \\ -\,(d\,t^2 - d\,s\,t)\left((\prod_{l=1}^{d} t_l) - s\,(\prod_{l=1}^{d} t_l)\left(\sum_{l=1}^{d} \frac{1}{t_l}\right)\right)$$

The preceding formula which is the result of our insphere example, looks rather nasty. If we take a closer look at it, we see that it may be cleaned up a little bit: Factoring out the nonzero term $(\prod_{l=1}^{d} t_l)/(t-s)$ we get the much nicer formula

$$\sum_{l=1}^{d} t_l \;\; - dt + dst \sum_{l=1}^{d} \frac{1}{t_l}.$$

Alternants

Let us turn to another important determinant class, the *alternants*. An alternant of order n is a determinant where the entries of the first row are generated by functions $f_1, \ldots, f_n$ (we assume multivariate polynomials over a ring) in one variable x_1, the entries of the second row by the same functions in another variable x_2 and so on:

$$\begin{vmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ \vdots & \vdots & & \vdots \\ f_1(x_n) & f_2(x_n) & \cdots & f_n(x_n) \end{vmatrix}.$$

The most well known type of an alternant is the Vandermonde determinant, generated by the functions x_i^{j-1} for $i, j = 1, \ldots, n$.

$$V = \begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{vmatrix}.$$

A generally known fact is that the formula of the Vandermonde determinant is

$$V = \prod_{1 \le i < j \le n} (x_j - x_i)$$

which is the *difference product* of the variables.

Indeed it is straightforward to show that this difference product appears as a factor of every alternant and that its cofactor is a symmetric function in the variables. How can we compute this cofactor ?

We will present a theorem of [2] that determines the cofactor as a combination of *elementary symmetric functions* which form a basis of the symmetric polynomials. The elementary symmetric function σ_r is the sum of all monomials that are products of r distinct variables (for $0 \le r \le n$ and $\sigma_0 := 1$):

$$\sigma_r = \sum_{1 \le i_1 < \cdots < i_r \le n} x_{i_1} \cdots x_{i_r}$$

Theorem

Let $f_j(x_i) = a_{0j} + a_{1j}x_i + a_{2j}x_i^2 + \cdots + a_{rj}x_i^r$ be the column generating functions with $r \ge n-1$ and let $S_k = (-1)^k \sigma_k$.

The cofactor of the difference product of the generated alternant of order n is

$$\begin{vmatrix} a_{01} & a_{11} & \cdots & a_{n1} & a_{n+1,1} & \cdots & a_{r1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{0n} & a_{1n} & \cdots & a_{nn} & a_{n+1,n} & \cdots & a_{rn} \\ S_n & S_{n-1} & \cdots & S_0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & S_n & S_{n-1} & \cdots & S_0 \end{vmatrix}_{r+1}.$$

Proof. See [2] or [3].

At first it seems that we didn't gain anything since we traded an order n determinant for an order $r+1$ determinant involving coefficients and elementary symmetric functions. However, if we assume $r = n + d$ with $d \in \mathbb{N}$ and only consider monomials as column generating functions, it becomes obvious that only one entry in each of the first n rows is nonzero. This allows easy expansion of the first n rows yielding a minor of order $d+1$ involving elementary symmetric functions. This minor is of integer order and can be computed by standard minor expansion.

Example

Consider the following alternant:

$$A = \begin{vmatrix} 1 & x_1^2 & x_1^3 & \cdots & x_1^n \\ 1 & x_2^2 & x_2^3 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n-1}^2 & x_{n-1}^3 & \cdots & x_{n-1}^n \\ 1 & x_n^2 & x_n^3 & \cdots & x_n^n \end{vmatrix}_n .$$

The theorem gives us a cofactor determinant

$$\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ S_n & S_{n-1} & \cdots & S_1 & S_0 \end{vmatrix} = (-1)^{n-1} S_{n-1} = \sigma_{n-1}$$

and hence the determinant formula

$$A = \sigma_{n-1} \prod_{1 \le i < j \le n} (x_j - x_i).$$

Using the multilinearity of the determinant we can also handle the case of polynomial generating functions provided that there are only a fixed integer number of them.

A Maple package for Alternants

We have implemented this approach such that it is possible to obtain the dimensionally parametrized formula of specified alternants meeting the restrictions above. After parsing the specification, the determinant is broken into a combination of monomial alternants whose formulas are determined by computing the cofactor determinant. Since only a fixed integer number of columns may be piecewisely defined, we may simulate the minor expansion of the dimensionally parametrized cofactor determinant. Display and checking facilities are provided as in the frameforms package.

Example

```
> with(ALTERNANT):
> Alternant(n,x,[ [1..n-1,x[i]^(j-1)],
[n..n,2*x[i]^j-x[i]^(j+1)] ],esf,print);
```

Matrix :

$$\begin{bmatrix} 1 & x_1 & o & o & {x_1}^{(n-2)} & 2\,{x_1}^n - {x_1}^{(n+1)} \\ 1 & x_2 & o & o & {x_2}^{(n-2)} & 2\,{x_2}^n - {x_2}^{(n+1)} \\ o & o & o & o & o & o \\ o & o & o & o & o & o \\ 1 & x_{n-1} & o & o & {x_{n-1}}^{(n-2)} & 2\,{x_{n-1}}^n - {x_{n-1}}^{(n+1)} \\ 1 & x_n & o & o & {x_n}^{(n-2)} & 2\,{x_n}^n - {x_n}^{(n+1)} \end{bmatrix}$$

Determinant :

formula valid for $n > 1$

$$(-\mathrm{S}(1,\,n,\,x)^2 + \mathrm{S}(0,\,n,\,x)\,\mathrm{S}(2,\,n,\,x) - 2\,\mathrm{S}(1,\,n,\,x))\; DP(n,\,x,\,x_i)$$

Here, $S(k,n,x)$ denotes $(-1)^k \sigma_k(x_1, \ldots, x_n)$ and $DP(n,x,x_i)$ the difference product $\prod_{1 \le i < j \le n}(x_j - x_i)$.

Conclusion

We derived dimensionally parametrized determinant formulas for two determinant classes, the frameforms and the alternants. We described Maple packages that enabled computing the formula of specified determinants of these classes.

This work is an excerpt of the author's M.Sc thesis. The Maple packages including online documentation and the thesis offering a more detailed treatise of the topic can be downloaded from the WWW page: `http://www-tcs.cs.uni-sb.de/mark/det.html`

References

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