A Generalized Sequential Sign Detector for Binary Hypothesis Testing

R. Chandramouli and N. Ranganathan

Abstract— **It is known that for fixed error probabilities sequential signal detection based on the sequential probability ratio test (SPRT) is optimum in terms of the average number of signal samples for detection. But, often sub-optimal detectors like the sequential sign detector are preferred over the optimal SPRT. When the additive noise statistic is independent and identically distributed (iid), the sign detector is preferred for its simplicity and nonparametric properties. However, in many practical applications such as the usage of high speed sampling devices the noise is correlated. In this paper, a generalized sequential sign detector for detecting binary signals in stationary, first order Markov dependent noise is studied. Under iid assumptions, this reduces to the usual sequential sign detector. The optimal decision thresholds and the average sample number for the test to terminate are derived. Numerical results are given to show that the proposed detector exploits the correlation in the noise and hence results in quicker detection. The method can also be extended to M-th order Markov dependence by converting it to a first order dependence in an extended state space.**

Keywords— **Keywords: SPRT, Markov noise, quantization, sign detector**

I. Introduction

It is well known that the average test length of a sequential signal detector is the minimum among all the tests that achieve the same probability of decision errors [1]. Often, sub-optimal detectors like the sign detector are preferred over the optimal sequential probability ratio test (SPRT) based optimal detector [2]. When the noise statistics are independent and identically distributed (iid) the sign detector possesses good nonparametric properties. But, the received signal samples, in many practical situations, are correlated due to the channel conditions and high speed sampling devices. Therefore, it helps to exploit this correlation when designing detectors. Sequential detection of signals in autoregressive noise has been studied in [3]. In [4] a class of nonparametric detectors based on grouping for data with dependency is introduced. A method using a one-step memory nonlinearity for detection in correlated noise is proposed in [5]. A non-parametric SPRT for additive Markov noise is analyzed in [6]. However, the decision thresholds and the test length are computed using the Wald's approximations.

In this paper, we propose a generalized sequential sign detector for binary signals in stationary, first order Markov dependent noise. Under iid conditions, this reduces to the usual sequential sign detector [7],[8]. The optimum decision thresholds and the average sample number (ASN) for terminal decision are derived. Numerical results are presented to illustrate the performance loss in the independent noise assumption. The proposed method generalizes the results in [7] and [9]. It can be extended to M-th order Markov dependence by converting it to first order dependence in an extended state space.

Let the transmitted signal set be $\{-S, S\}$ where $S > 0$, and the channel noise $\{n_i\}$, a zero mean, stationary, first order Markov process. The problem is defined as a test between the two hypotheses $H : r_i = -S + n_i$ and $K : r_i = S + n_i$, $i = 1, 2, \cdots$. The received signal samples $\{r_i\}$ are quantized to two levels, namely $Z_i = sgn(r_i)$. We assume that $p(Z_i = 0) = 0$. Clearly, $\{Z_i\}$ forms a stationary, first order Markov chain. Further, it is also assumed the this Markov chain is positive regular. The transition probabilities of $\{Z_i\}$ for $i \geq 2$ are denoted by $p_{jk} = p(Z_i = q_k | Z_{i-1} = q_j)$, $1 \leq j, k \leq 2$, where $q_1 = -1$ and $q_2 = 1$. Therefore, the problem can be described as the test between the hypotheses

$$H : \begin{pmatrix} p_{11}^H & p_{12}^H \\ p_{21}^H & p_{21}^H \end{pmatrix} \quad \text{vs} \quad K : \begin{pmatrix} p_{11}^K & p_{12}^K \\ p_{21}^K & p_{21}^K \end{pmatrix} \tag{1}$$

Then, the generalized sequential sign test can be expressed as

$$S_N = S_0 + \sum_{i=1}^{N} Z_i \begin{cases} \geq A & \text{decide } K \\ \leq -B & \text{decide } H \\ \text{else} & N = N + 1 \end{cases} \tag{2}$$

where S_0 is the initial value of the sum which is in general equal to zero. If $\hat{N} = inf\{N : S_N = -B \text{ or } S_N = A\}$ then $\hat{N}$ is a stopping time. As usual, $\hat{N} = \infty$ if the test does not terminate. That, the proposed sequential test is closed with probability one is not shown here due to space constraints. In particular, it can be shown the $p(\hat{N} > N) \to 0$ geometrically as $N \to \infty$. Therefore, from Stein's lemma [10] $E(\hat{N}) < \infty$. Also, all the higher order moments of $\hat{N}$ are finite. Since the test terminates w.p. 1 and $E(\hat{N}) < \infty$ the thresholds B, A and the conditional average sample numbers $\mathbf{E}(\hat{N}|H)$ and $\mathbf{E}(\hat{N}|K)$ for the test to terminate can be derived.

II. Optimal Decision Boundaries and Average Sample Number

In this section, we derive the optimal decision boundaries and the average sample number of the test for a fixed false alarm (α) and miss probability (β). If $a_{k,N}^{m,h}$ denotes $P(S_N = -B | Z_1 = q_k, h)$, $h = H, K$, and $S_0 = m$, where

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$$\mathbf{E}(\hat{N}|h) = q_1^h \left\{ \left[\frac{(p_{21}^h + p_{12}^h)(m+B) - 2p_{12}^h}{p_{21}^h - p_{12}^h} \right] + p_{12}^h \left[\frac{(A+B-2)(p_{12}^h + p_{21}^h) + 2}{p_{21}^h - p_{12}^h} \right] \left[\frac{p_{21}^h \theta_h^{m+B-1} - p_{21}^h}{p_{12}^h - p_{21}^h \theta_h^{A+B-1}} \right] \right\} + p_1^h \left\{ \left[\frac{(p_{21}^h + p_{12}^h)(m+B) + 2p_{11}^h}{p_{21}^h - p_{12}^h} \right] + \left[\frac{(A+B-2)(p_{12}^h + p_{21}^h) + 2}{p_{21}^h - p_{12}^h} \right] \left[\frac{p_{21}^h \theta_h^{m+B} - p_{12}^h}{p_{12}^h - p_{21}^h \theta_h^{A+B-1}} \right] \right\} \tag{3}$$

$-B+1 \leq m \leq A-1$ then, for k=1,2 we have the following homogeneous system of linear difference equations

$$a_{k,N+1}^{m,h} = \sum_{j=1}^{2} a_{j,N}^{(m+q_k),h} p_{kj}^h \tag{4}$$

with initial conditions, $a_{1,1}^{-B+1,h} = 1$ and $a_{2,1}^{A-1,h} = 0$. Solving this using the method of generating functions and summing the solution from $N=0$ to ∞ we get the probability of deciding H,

$$\begin{aligned} a_1^{m,h} &= \frac{p_{21}^h \theta_h^{A+B-1} - p_{12}^h \theta_h^{m+B-1}}{p_{21}^h \theta_h^{A+B-1} - p_{12}^h} \\ a_2^{m,h} &= \frac{p_{21}^h \theta_h^{A+B-1} - p_{21}^h \theta_h^{m+B}}{p_{21}^h \theta_h^{A+B-1} - p_{12}^h} \end{aligned} \tag{5}$$

when $\theta_h = \frac{p_{11}^h}{p_{22}^h} \neq 1$. Similar results can be derived if $\theta_h = 1$. If $p_1^h = P(Z_1 = -1|h)$ and $p_2^h = P(Z_1 = 1|h)$ then the unconditional probability of S_N reaching $-B$ is $P(-B|h) = p_1^h a_1^{m,h} + p_2^h a_2^{m,h}$. Since reaching $-B$ and A are mutually exclusive and exhaustive we have $P(A|h) = 1 - P(-B|h)$. From Eq. (5) we get $p(-B|K) = p_1^K a_1^{m,K} + p_2^K a_2^{m,K}$ and $p(A|H) = 1 - p_1^H a_1^{m,H} - p_2^H a_2^{m,H}$. Therefore, for fixed α and β, the optimum values of the thresholds B and A can be computed from $P(-B|K) \leq \beta$ and $P(A|H) \leq \alpha$. That is, these two inequalities can be used to solve for B and A.

The average sample number denotes the expected number of samples required by the sequential detector to reach one of the decision boundaries. Let $C^{m,h}$ denote the average sample number when $-B$ is reached given that $Z_1 = -1$ and $D^{m,h}$ when $Z_1 = 1$. Then,

$$\begin{aligned} C^{m,h} &= p_{12}^h D^{m-1,h} + p_{11}^h C^{m-1,h} + a_1^{m,h} \\ D^{m,h} &= p_{22}^h D^{m+1,h} + p_{21}^h C^{m+1,h} + a_2^{m,h} \end{aligned} \tag{6}$$

where $C^{-B+1,h} = 1$ and $D^{A-1,h} = 0$. A similar set of equations hold for the boundary A. The solution to these equations is given in Eq. (3).

III. Performance Analysis

The correlation coefficient of $\{Z_i\}$ influences the choice of the decision thresholds and hence the average sample number. Let the correlation coefficient of $\{Z_i\}$ conditioned on hypothesis $h = H, K$ be denoted by $\rho_h(Z_{i+1}, Z_i)$. We analyze the performance of the sequential sign detector when $S_0 = 0$ and $\rho = \rho_H = \rho_K$. For $\rho = 0$, the transition probabilities were chosen to be $p_{11}^H = 0.55$, $p_{22}^H = 0.45$, $p_{11}^K = 0.4$, $p_{22}^K = 0.6$, $p^H = 0.45$ and $p^K = 0.6$, and, when $\rho = 0.1$ the values were $p_{11}^H = 0.7$, $p_{22}^H = 0.4$, $p_{11}^K = 0.4$, $p_{22}^K = 0.7$, $p^H = 0.33$ and $p^K = 0.67$. The optimal decision thresholds are shown in Table I. It is observed that the values of the thresholds decrease when the false alarm (respectively, miss probability) increases. This is due to the relaxation of the constraints on the detector. When the correlation coefficient increases, the bias towards the true hypothesis increases thus decreasing the values of the thresholds.

	$\rho = 0$		$\rho = 0.1$	
$\alpha = \beta$	B	A	B	A
10^{-5}	19	60	14	19
10^{-4}	17	50	10	16
10^{-3}	17	34	9	12
10^{-2}	15	22	8	7

TABLE I
Optimum decision threshold

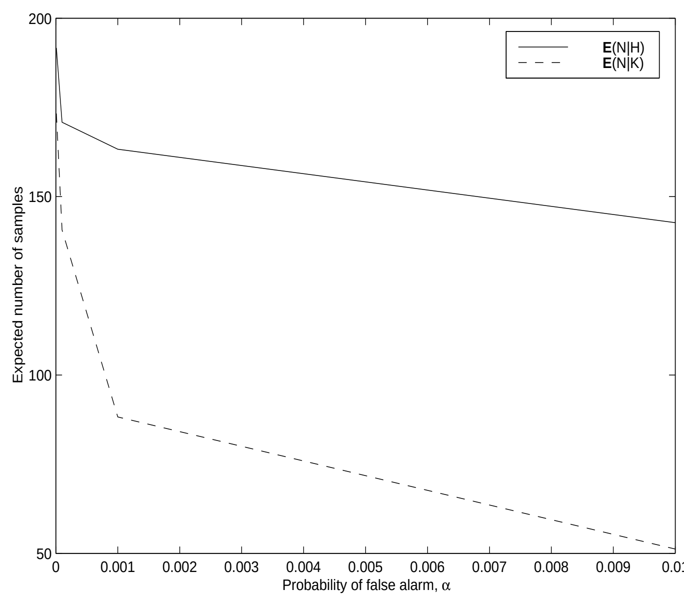

Fig. 1. Expected number of samples for $\rho = 0$

Fig. 1 and Fig. 2 show the ASN for various values of α, when $\alpha = \beta$ and $\rho = 0$ and 0.1 respectively. The false alarm ranges from 10^{-5} to 10^{-1}. Clearly, the ASN is at least five times higher when $\rho = 0$ as compared to $\rho = 0.1$. This indicates that as $\rho \uparrow$ ASN $\downarrow$. Therefore, assuming that the additive noise is iid, when actually it is correlated, leads to considerable loss in the performance of the sequential sign detection in terms of the ASN.

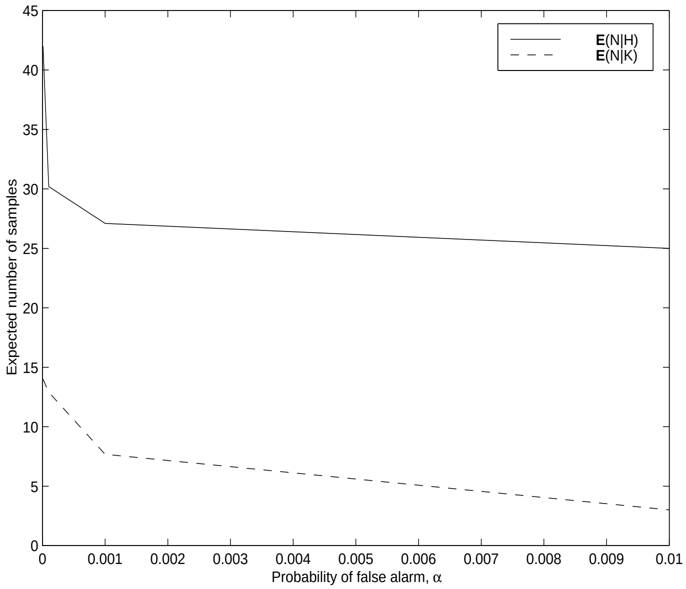

Fig. 2. Expected number of samples for $\rho = 0.1$

IV. Conclusion

A generalized sequential sign detector for stationary, first order additive Markov noise is proposed. The optimum decision thresholds and the average sample number for a fixed false alarm and miss probability are derived. Performance analysis shows that the detector terminates faster for positively correlated noise by exploiting the correlation. It is also observed that there is at least a five times increase in the ASN if the correlation coefficient of the noise is assumed to be zero when it is actually equal to 0.1.

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