

On the adjacency matrix of a block graph

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Abstract

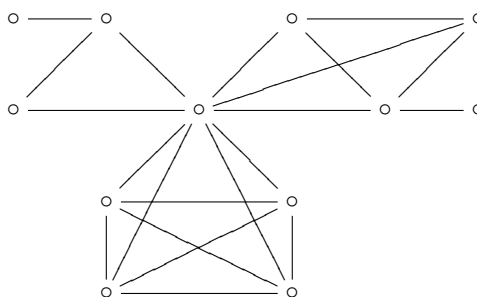
A block graph is a graph in which every block is a complete graph. Let G be a block graph and let A be the adjacency matrix of G . We first obtain a formula for the determinant of A over reals. It is shown that A is nonsingular over \mathbb{F}_2 if and only if the removal of any vertex from G produces a graph with exactly one odd component. A formula for the inverse of A over \mathbb{F}_2 is obtained, whenever it exists. We obtain some results for the adjacency matrices over \mathbb{F}_2 , of claw-free block graphs, which are the same as the line graphs of trees, and for the adjacency matrices of flowers, which are block graphs with only one cut-vertex.

Key words. block graph, adjacency matrix, determinant, tree, line graph

AMS Subject Classifications. 15A15, 05C05.

1 Introduction

We consider undirected graphs with no loops or parallel edges. We refer to [11] for graph theoretic preliminaries. Recall that a block in a graph is a maximal connected subgraph that has no cut-vertex ([11],p.15). The complete graph with n vertices is denoted by K_n . A block graph is a graph in which each block is a complete graph. A block graph whose blocks are K_2, K_2, K_3, K_4 and K_5 is shown below.



We consider the adjacency matrix of a block graph over the field of reals as well as over the two-element field $\{0, 1\}$, which we denote as \mathbb{F}_2 . For basic definitions concerning matrices associated with graphs, see [1].

A graph is said to be nonsingular over reals if its adjacency matrix is nonsingular over reals. Similarly, a graph is said to be nonsingular over \mathbb{F}_2 if its adjacency matrix is nonsingular over \mathbb{F}_2 . It is well-known that a tree is nonsingular over reals if and only if it has a perfect matching. Moreover, when a tree is nonsingular over reals, there is a formula for its inverse in terms of alternating paths, see [3],[4], and the references contained therein. These two results motivated this work. Since a tree is a block graph, it is natural to investigate the adjacency matrix of a general block graph.

We now describe the contents of this paper. Let G be a block graph and let A be the adjacency matrix of G . In Section 2 we obtain a formula for the determinant of A over reals. As a corollary we obtain a sufficient condition for the determinant to be an even integer. In Sections 3,4 and 5 we work over \mathbb{F}_2 . In Section 3 we consider the adjacency matrix of a block graph over \mathbb{F}_2 . It is shown that A is nonsingular over \mathbb{F}_2 if and only if the removal of any vertex from G produces a graph with exactly one odd component. A formula for the inverse of A over \mathbb{F}_2 is obtained, whenever it exists. In the final two sections we illustrate applications of our results to two special classes of block graphs. In Section 4 we obtain some results for the line graph of a tree, which is a block graph in which every cut-vertex is incident to at most two blocks. A flower, which is a block graph with only one cut-vertex is considered in Section 5.

2 Adjacency matrix of a block graph over reals

A pendant vertex is a vertex of degree 1. A block is called a pendant block if it has only one cut-vertex or if it is the only block in that component. A graph is called even (odd) if it has an even (odd) number of vertices. An isolated vertex in a graph is considered to be a block of the graph. The vertex set of the graph G is denoted by $V(G)$. The transpose of the matrix A is denoted A' . Let G be a block graph and let B_1, \dots, B_k be the blocks of G . If $S \subset \{1, \dots, k\}$, then G_S will denote the subgraph of G induced by the blocks $B_i, i \in S$. We first prove a preliminary result.

Lemma 1 *Let G be a block graph with n vertices. Let B_1, \dots, B_k be the blocks of G where B_i is the complete graph with b_i vertices, $i = 1, \dots, k$. Let $(\alpha_1, \dots, \alpha_k)$ be a k -tuple of nonnegative integers satisfying the following conditions:*

$$(i) \sum_{i=1}^k \alpha_i = n$$

(ii) for any nonempty $S \subset \{1, \dots, k\}$,

$$\sum_{i \in S} \alpha_i \leq |V(G_S)|. \quad (1)$$

If B_i is a pendant block, then α_i equals either b_i or $b_i - 1$.

Proof: Clearly by (1), with $S = \{i\}$, we must have $0 \leq \alpha_i \leq b_i$. Setting $S = \{1, \dots, k\} \setminus \{i\}$ in (1) we see from (i) and (ii) that $\alpha_1 + \dots + \alpha_n - \alpha_i = n - \alpha_i \leq n - b_i + 1$, and hence $\alpha_i \geq b_i - 1$. That completes the proof. \blacksquare

In the next result, which is the main result of this section, we obtain a formula for the determinant of the adjacency matrix of a block graph over reals.

Theorem 2 *Let G be a block graph with n vertices. Let B_1, \dots, B_k be the blocks of G . Let A be the adjacency matrix of G . Then*

$$\det A = (-1)^{n-k} \sum (\alpha_1 - 1) \cdots (\alpha_k - 1) \quad (2)$$

where the summation is over all k -tuples $(\alpha_1, \dots, \alpha_k)$ of nonnegative integers satisfying the following conditions:

$$(i) \sum_{i=1}^k \alpha_i = n$$

(ii) for any nonempty $S \subset \{1, \dots, k\}$,

$$\sum_{i \in S} \alpha_i \leq |V(G_S)|. \quad (3)$$

Proof: We prove the result by induction on k . Note that the result is true for a block graph with one block, since the adjacency matrix of the complete graph K_p has determinant $(-1)^{p-1}(p-1)$. Assume the result to be true for a block graph with at most $k-1$ blocks and proceed. We consider two cases.

Case (i): The graph G has a pendant block with exactly 2 vertices.

Let $B_1 = K_2$ be a pendant block in G . Let the vertices of B_1 be 1 and 2, where 1 is pendant. We assume that the vertex 2 is in blocks B_1, \dots, B_p and we further assume that among these blocks, the first q blocks B_1, \dots, B_q are equal to K_2 . It is possible that $q = 1$. Let $G_1 = G \setminus 1$. The matrix A has the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & & \cdots & \\ 0 & & & & \\ \vdots & \vdots & & A_1 & \\ 0 & & & & \end{pmatrix}, \quad (4)$$

where A_1 is the adjacency matrix of $G_2 = G_1 \setminus 2$. It follows from (4) that $\det A = -\det A_1$.

The graph G_2 has $k-q$ blocks given by $C_i, i = q+1, \dots, k$, defined as follows: $C_i = B_i \setminus 2, i = q+1, \dots, p$ and $C_i = B_i, i = p+1, \dots, k$.

First consider the case when $q > 1$ and that there is at least one pendant block among B_2, \dots, B_q . let us assume, without loss of generality, that B_2 is a pendant block. We also assume that $V(B_2) = \{2, 3\}$. Then the first and the third columns of A are identical and $\det A = 0$. Also if $(\alpha_1, \dots, \alpha_k)$ is a k -tuple of nonnegative integers satisfying the conditions in the Theorem, then at least one of α_1 or α_2 must equal 1 and hence the summation in (2) is zero. Therefore the result is proved in this case. We therefore assume that none of the blocks among B_2, \dots, B_q is pendant.

Consider

$$(-1)^{n-k} \sum (\alpha_1 - 1) \cdots (\alpha_k - 1) \quad (5)$$

where the summation is over all k -tuples $(\alpha_1, \dots, \alpha_k)$ of nonnegative integers satisfying the following conditions:

$$(i) \sum_{i=1}^k \alpha_i = n$$

(ii) for any nonempty $S \subset \{1, \dots, k\}$,

$$\sum_{i \in S} \alpha_i \leq |V(G_S)|. \quad (6)$$

By Lemma 1, α_1 equals either 1 or 2. If $\alpha_1 = 1$, then the corresponding term in (5) is zero, so we assume $\alpha_1 = 2$. Then it follows from (ii) that none of $\alpha_2, \dots, \alpha_q$ can be 2. For

example, if $\alpha_2 = 2$, then $\alpha_1 + \alpha_2 = 4$, whereas the graph induced by blocks B_1 and B_2 has 3 vertices, thus violating (6). Thus each of $\alpha_2, \dots, \alpha_q$ is either 0 or 1, and again we may assume that $\alpha_2 = \dots = \alpha_q = 0$. In view of these observations (5) equals

$$(-1)^{n-k} (-1)^{q-1} \sum (\alpha_{q+1} - 1) \cdots (\alpha_k - 1) = -(-1)^{n-2-(k-q)} \sum (\alpha_{q+1} - 1) \cdots (\alpha_k - 1). \quad (7)$$

The summation in (7) is over all $(k - q)$ -tuples $(\alpha_{q+1}, \dots, \alpha_k)$ of nonnegative integers satisfying the following conditions:

$$(i) \quad \sum_{i=q+1}^k \alpha_i = n - 2$$

(ii) for any nonempty $S \subset \{q + 1, \dots, k\}$,

$$\sum_{i \in S} \alpha_i \leq |V(G_S)|. \quad (8)$$

By the induction assumption, the right side of (7) equals $-\det A_1$ and since $\det A = -\det A_1$, it follows that $\det A$ equals (5). Therefore the proof is complete in this case.

Case (ii): The graph G does not have a pendant block with exactly 2 vertices.

Let B_1 be a pendant block of G and let $V(B_1) = \{1, \dots, b_1\}$, $b_1 \geq 3$. Let b_1 be the cut vertex in B_1 . Let $H = G \setminus \{B_1 \setminus b_1\}$ be the block graph with $n - b_1 + 1$ vertices having blocks B_2, \dots, B_k and let A_1 be the adjacency matrix of H .

After a suitable relabeling of the vertices in G we may write

$$A = \begin{pmatrix} D & C \\ C' & A_1 \end{pmatrix} \quad (9)$$

where D is the adjacency matrix of the subgraph induced by the vertex set $V(B_1 \setminus b_1)$, C is a $(b_1 - 1) \times (n - b_1 + 1)$ matrix with $c_{ij} = 1$ if $j = b_1 + 1$, and $c_{ij} = 0$ otherwise. Thus $D = J - I$ where J is the matrix of all ones and C has the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Note that the first row of A_1 corresponds to the vertex b_1 .

By the Schur complement formula ([1],p.4) we have

$$\det A = \det D \cdot \det(A_1 - C'D^{-1}C). \quad (10)$$

We note the following simple facts:

$$(i) \quad \det D = (-1)^{b_1-2} (b_1 - 2)$$

$$(ii) D^{-1} = \frac{1}{b_1-2}J - I$$

(iii) The matrix $C'D^{-1}C$ has all entries zero except the entry in the first row and the first column, which is $(-1)^{b_1} \frac{b_1-1}{\det D}$.

Let M be the adjacency matrix of $H \setminus b_1$. Thus M is the submatrix of A_1 formed by its last $n - b_1$ rows and columns. It follows from (i)-(iii) that

$$\det(A_1 - C'D^{-1}C) = \det A_1 - \frac{b_1 - 1}{\det D} \det M. \quad (11)$$

From (10),(11) we have

$$\begin{aligned} \det A &= (\det D) \left(\det A_1 - (-1)^{b_1} \frac{b_1 - 1}{\det D} \det M \right) \\ &= (\det D)(\det A_1) - (-1)^{b_1} (b_1 - 1) \det M \\ &= (-1)^{b_1-2} (b_1 - 2) \det A_1 - (-1)^{b_1} (b_1 - 1) \det M. \end{aligned} \quad (12)$$

We assume that b_1 is in blocks B_1 and B_2, \dots, B_p . If there are any blocks equal to K_2 containing b_1 , then we enumerate them as B_2, \dots, B_q . The remaining blocks, which do not contain b_1 are B_{p+1}, \dots, B_k . If there are no blocks equal to K_2 containing b_1 , then we set $q = 1$.

The graph $H \setminus b_1$ has $k - q$ blocks given by $C_i, i = q + 1, \dots, k$, defined as follows: $C_i = B_i \setminus b_1, i = q + 1, \dots, p$ and $C_i = B_i, i = p + 1, \dots, k$.

Consider

$$\sum (-1)^{n-k} (\alpha_1 - 1) \cdots (\alpha_k - 1) \quad (13)$$

where the summation is over all k -tuples $(\alpha_1, \dots, \alpha_k)$ of nonnegative integers satisfying the following conditions:

$$(i) \sum_{i=1}^k \alpha_i = n$$

(ii) for any nonempty $S \subset \{1, \dots, k\}$,

$$\sum_{i \in S} \alpha_i \leq |V(G_S)|. \quad (14)$$

By Lemma 1, α_1 equals either b_1 or $b_1 - 1$. If $\alpha_1 = b_1$, then each of $\alpha_2, \dots, \alpha_q$ must be either 0 or 1. For example, if $\alpha_2 = 2$, then $\alpha_1 + \alpha_2 = b_1 + 2$, whereas the graph induced by blocks B_1 and B_2 has $b_1 + 1$ vertices, thus violating (6). If $\alpha_i = 1$ for some $i \in \{2, \dots, q\}$, then the corresponding term in (13) is zero and hence we assume $\alpha_2 = \dots = \alpha_q = 0$. Then it can be seen by the induction hypothesis, that the sum of the terms in (13) corresponding to $\alpha_1 = b_1$ equals $(-1)^{b_1} (b_1 - 1) \det M$.

Similarly, by the induction hypothesis, the sum of the terms in (13) corresponding to $\alpha_1 = b_1 - 1$ equals $(-1)^{b_1-2}(b_1 - 2) \det A_1$. It follows that the sum in (13) equals

$$(-1)^{b_1-2}(b_1 - 2) \det A_1 - (-1)^{b_1}(b_1 - 1) \det M,$$

which is $\det A$ by (12). That completes the proof. \blacksquare

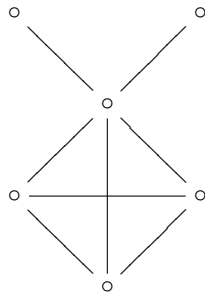
As noted in the introduction, it is well-known that a tree is nonsingular over reals if and only if it has a perfect matching. We now derive this result from Theorem 2.

Corollary 3 *Let T be a tree with n vertices and let A be the adjacency matrix of T . Then A is nonsingular over reals if and only if T has a perfect matching.*

Proof: First suppose that A is nonsingular. Then at least one term in the summation in (2) must be nonzero. If $(\alpha_1, \dots, \alpha_{n-1})$ is an $(n - 1)$ -tuple of nonnegative integers satisfying (i) and (ii) of Theorem 2 and if the corresponding term in (2) is nonzero, then each α_i is either 0 or 2. Moreover if two edges have a common vertex, then the corresponding α 's cannot both be nonzero, in view of (ii). Thus there must be a perfect matching in T and $\alpha_i = 2$ if and only if the corresponding edge is in the matching.

Conversely, suppose T has a perfect matching. As noted in the first part of the proof, a nonzero term in the summation in (2) corresponds to a perfect matching. We invoke the elementary fact, easily proved by induction, that if a tree has a perfect matching then it must be unique. Thus there must be precisely one nonzero term in the summation in (2) which renders $\det A$ nonzero. Thus A is nonsingular over reals and the proof is complete. \blacksquare

Characterizing block graphs, other than trees, which are nonsingular over reals seems to be an interesting open problem. One example of a singular block graph is a tree with no perfect matching. There are other examples. The following block graph is singular since the adjacency matrix has two identical columns corresponding to the pendant vertices.



In the following corollary we identify a class of graphs whose determinant is even. The result will be used in the next section when we consider the adjacency matrix over \mathbb{F}_2 .

Corollary 4 *Let G be a connected block graph with n vertices. Let v be a vertex of G and let H_1, \dots, H_ℓ be the components of $G \setminus v$. Let us suppose that among $|V(H_i)|, i = 1, \dots, \ell$, at least two integers are odd. Then $\det A$ is even.*

Proof: Let us list the blocks of G as follows. Let $B_1^i, \dots, B_{m_i}^i$ be the blocks of $H_i \cup v, i = 1, \dots, \ell$. Let $\alpha_1^i, \dots, \alpha_{m_i}^i, i = 1, \dots, \ell$ be integers satisfying the conditions of Theorem 2. Let B_j^i have p_j^i vertices, $i = 1, \dots, \ell; j = 1, \dots, m_i$. Then

$$\sum_{j=1}^{m_i} \alpha_j^i \leq |V(H_i)| + 1, i = 1, \dots, \ell. \quad (15)$$

It follows that

$$\sum_{i=1}^{\ell} \sum_{j=1}^{m_i} \alpha_j^i \leq \sum_{i=1}^{\ell} (|V(H_i)| + 1), i = 1, \dots, \ell. \quad (16)$$

Also

$$\sum_{i=1}^{\ell} \sum_{j=1}^{m_i} \alpha_j^i = n = \sum_{i=1}^{\ell} (|V(H_i)| + 1) - (\ell - 1). \quad (17)$$

If among the ℓ inequalities in (15) some two are equalities, then it will violate the condition in Theorem 2 (by considering $H_i \cup H_j \cup v$ for those i, j .) If all the ℓ inequalities in (15) are strict, then we get a contradiction as follows. If

$$\sum_{j=1}^{m_i} \alpha_j^i \leq |V(H_i)|, i = 1, \dots, \ell, \quad (18)$$

then

$$\sum_{i=1}^{\ell} \sum_{j=1}^{m_i} \alpha_j^i \leq \sum_{i=1}^{\ell} (|V(H_i)| + 1) - \ell. \quad (19)$$

From (18) and (19) we get $\ell \leq \ell - 1$, a contradiction. For a similar reason, in any inequality, the deficiency is 1. We conclude that among the ℓ inequalities in (15), $\ell - 1$ must be strict (with deficiency 1) and one must be equality. Without loss of generality, we assume

$$\sum_{j=1}^{m_1} \alpha_j^1 = |V(H_1)| + 1, \quad (20)$$

and

$$\sum_{j=1}^{m_i} \alpha_j^i = |V(H_i)|, i = 2, \dots, \ell. \quad (21)$$

According to our assumption, among $|V(H_i)|, i = 1, \dots, \ell$, at least two integers are odd and hence, in view of (20),(21), there exists $s \in \{2, \dots, \ell\}$ such that

$$\sum_{j=1}^{m_s} \alpha_j^s = |V(H_s)| = 1 \pmod{2}. \quad (22)$$

Therefore at least one integer among $\alpha_1^s, \dots, \alpha_{m_s}^s$ must be odd and the corresponding term in the summation in Theorem 2 is even. Thus each term in the summation in Theorem 2 is even and the result follows. ■

3 Adjacency matrix of a block graph over \mathbb{F}_2

We first prove a preliminary result.

Lemma 5 *Let G be a graph with n vertices and let A be the adjacency matrix of A over reals. If n is odd, then $\det A$ is an even integer. In particular, A is singular over \mathbb{F}_2 .*

Proof: We have

$$\det A = \sum_{\sigma} (-1)^{\epsilon(\sigma)} a_{1\sigma(1)} \cdots a_{n\sigma(n)}, \quad (23)$$

where $\epsilon(\sigma)$ is $1(-1)$ if σ is even(odd), and where the summation is over all permutations σ of $1, \dots, n$. Consider a permutation σ for which $a_{1\sigma(1)} \cdots a_{n\sigma(n)}$ is nonzero. Since the diagonal elements of A are zero and since n is odd, in the cycle decomposition of σ , there must be an odd cycle of length greater than one. Thus if $\tau = \sigma^{-1}$, then $\tau \neq \sigma$ and $a_{1\tau(1)} \cdots a_{n\tau(n)}$ is nonzero as well. Therefore we may group the nonzero terms in the summation in (23) in pairs and hence $\det A$ is even. \blacksquare

Theorem 6 *Let G be a graph with n vertices and let A be the adjacency matrix of G . Then A is nonsingular over \mathbb{F}_2 if and only if n is even and for all $u \in V(G)$, $G \setminus u$ has exactly one odd component.*

Proof: First suppose that A is nonsingular over \mathbb{F}_2 . By Lemma 5, n is an even integer. Also if there exists $u \in V(G)$ such that $G \setminus u$ has at least two odd components, then by Corollary 4, $\det A$, over reals, is an even integer and hence A must be singular over \mathbb{F}_2 , which is a contradiction. This proves the “only if” part of the Theorem.

We turn to the “if” part. The proof will be by induction on the number of blocks. The result is easily verified when G has a single block. Let G be a graph with k blocks. Assume the result to be true for a graph with at most $k-1$ blocks and proceed. Let G have n vertices where n is even and suppose that for all $u \in V(G)$, $G \setminus u$ has exactly one odd component. Let B be a pendant block in G with cut-vertex t . We distinguish two cases.

Case 1. $|V(B)|$ is odd. Note that the graph $G \setminus (B \setminus t)$ also satisfies the conditions of the theorem. After a suitable relabeling of the vertices in G we may write

$$A = \begin{pmatrix} D & C \\ C' & A_1 \end{pmatrix} \quad (24)$$

where D is the adjacency matrix of the subgraph induced by the vertex set $V(B \setminus t)$ and C is a $(t-1) \times (n-t+1)$ matrix with $c_{ij} = 1$ if $j = t+1$, and $c_{ij} = 0$ otherwise. Thus $D = J - I$ where J is the matrix of all ones and C has the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Note that the first row of A_1 corresponds to the vertex t . Perform the following row and column operations on A : Add the first $t - 1$ columns (rows) to the column (row) t . Then the resulting matrix \tilde{A} has the form

$$\tilde{A} = \begin{pmatrix} D & 0 \\ 0 & A_1 \end{pmatrix}. \quad (25)$$

Note that $D = J - I$ being of even order, is nonsingular over \mathbb{F}_2 . Since $G \setminus (B \setminus t)$ also satisfies the conditions of the theorem, by the induction hypothesis, its adjacency matrix A_1 is nonsingular over \mathbb{F}_2 . It follows from (25) that \tilde{A} is nonsingular over \mathbb{F}_2 . Since \tilde{A} and A have the same determinant, it follows that A is nonsingular over \mathbb{F}_2 .

Case 2. $|V(B)|$ is even. Note that the graph $G \setminus B$ also satisfies the conditions of the theorem. After a suitable relabeling of the vertices in G we may write

$$A = \begin{pmatrix} D & C \\ C' & A_1 \end{pmatrix} \quad (26)$$

where $D = J - I$ is the adjacency matrix of B , A_1 is the adjacency matrix of $G \setminus B$, and C is a $t \times (n - t)$ matrix with $c_{ij} = 0$ if $i = 1, 2, \dots, t - 1$. Perform the following row and column operations on A : Add the first $t - 2$ columns (rows) to the column (row) $t - 1$. Evaluate the determinant of the resulting matrix by Laplace expansion with respect to column t , followed by Laplace expansion with respect to row t . Then it is seen that $\det A = \det A_1$. Since $G \setminus B$ satisfies the conditions of the theorem, by the induction hypothesis, its adjacency matrix A_1 is nonsingular over \mathbb{F}_2 . It follows that A is nonsingular over \mathbb{F}_2 and the proof is complete. ■

We record the following consequence of Theorem 6.

Corollary 7 *Let G be a graph, let B be a pendant block of G , and let t be the cut-vertex in B . If $|V(B)|$ is odd, then G is nonsingular over \mathbb{F}_2 if and only if $G \setminus (B \setminus t)$ is nonsingular. If $|V(B)|$ is even, then G is nonsingular over \mathbb{F}_2 if and only if $G \setminus B$ is nonsingular.*

Proof: We first consider the case when $|V(B)|$ is odd. Let G be nonsingular. By Theorem 6, for any $u \in V(G)$, $G \setminus u$ has exactly one odd component. We claim that for any $u \in V(G \setminus (B \setminus t))$, the graph $(G \setminus (B \setminus t)) \setminus u$ has exactly one odd component. Consider two subcases.

Subcase (i), $u = t$: If $(G \setminus (B \setminus t)) \setminus t$ has more than one odd components, they continue to be odd components of $G \setminus t$, which is a contradiction.

Subcase (ii), $u \neq t$: If $(G \setminus (B \setminus t)) \setminus u$ has more than one odd components, denote two such components by \mathcal{O}_1 and \mathcal{O}_2 . If $t \notin \mathcal{O}_1 \cup \mathcal{O}_2$, then \mathcal{O}_1 and \mathcal{O}_2 are odd components of $G \setminus t$, which is a contradiction. If $t \in \mathcal{O}_1$, then again $\mathcal{O}_1 \cup B$ and \mathcal{O}_2 are odd components in $G \setminus u$, which is a contradiction. A similar conclusion holds if $t \in \mathcal{O}_2$.

Thus the claim is proved and by Theorem 6, $G \setminus (B \setminus t)$ is nonsingular. This completes the proof of the “only if” part. Conversely, suppose $G \setminus (B \setminus t)$ is nonsingular. Then for any $u \in V(G \setminus (B \setminus t))$, the graph $(G \setminus (B \setminus t)) \setminus u$ has exactly one odd component. Using a similar argument as before, we can show that for any $u \in V(G)$, $G \setminus u$ has exactly one odd component, and by Theorem 6, G is nonsingular.

We omit the proof in the case when $|V(B)|$ is even, since it is similar. \blacksquare

As noted in Corollary 3, a tree is nonsingular over reals if and only if it has a perfect matching. Since the determinant over reals of the adjacency matrix of a tree is either 0 or ± 1 , it follows that a tree is nonsingular over \mathbb{F}_2 if and only if it has a perfect matching. It may be instructive to note the connection between existence of a perfect matching in a tree and the hypothesis in Theorem 6. Let T be a tree with a perfect matching. Let $v \in V(T)$ be of degree k , and let it be adjacent to v_1, \dots, v_k . Let T_1, \dots, T_k be the components of $T \setminus v$, containing v_1, \dots, v_k respectively. Since T has a perfect matching, there exists a unique $i \in \{1, \dots, k\}$ such that the edge vv_i is in the matching. Then for any $j \neq i$, T_j has a perfect matching and must be an even component. Thus T_i is the unique odd component in $T \setminus v$ and the hypothesis in Theorem 6 holds.

If A is an $n \times n$ matrix, we denote by $A(i|j)$, the submatrix obtained by deleting row i and column j . Similarly, $A(i, j|i, j)$ will denote the submatrix obtained by deleting rows i, j and columns i, j .

Theorem 8 *Let G be a graph with n vertices and let A be the adjacency matrix of G . Let A be nonsingular over \mathbb{F}_2 and let $B = A^{-1}$. Then $b_{ii} = 0, i = 1, \dots, n$. Moreover, if $i \neq j$ are vertices of G , then the following conditions are equivalent:*

(i) $b_{ij} = 1$

(ii) $\det A(i|j) = 1$

(iii) $\det A(i, j|i, j) = 1$

(iv) $G \setminus \{i, j\}$ is nonsingular over \mathbb{F}_2

(v) For any vertex k , not equal to i, j , $G \setminus \{i, j, k\}$ has exactly one odd component.

Proof: Since A is nonsingular over \mathbb{F}_2 , by Lemma 5, n is even. For any i , $G \setminus \{i\}$ has an odd number of vertices and hence it is singular over \mathbb{F}_2 by Lemma 5. Therefore $\det A(i|i) = 0$ and hence $b_{ii} = 0, i = 1, \dots, n$.

The equivalence of (i), (ii) is obvious, and so is the equivalence of (iii) and (iv). By Theorem 6, (iv) and (v) are equivalent. It remains to show the equivalence of (ii) and (iii). We recall the well-known identity (which is a special case of Sylvester’s identity, see [9],p.22): If A is an $n \times n$ matrix, then for any $i \neq j$,

$$\det A(i|i) \det A(j|j) - \det A(i|j) \det A(j|i) = (\det A) \det A(i, j|i, j). \quad (27)$$

In particular, in the present situation, A is a matrix over \mathbb{F}_2 , $\det A(i|i) = 0$ for all i , and $\det A = 1$. Hence it follows from (27) that $\det A(i|j) = \det A(i, j|i, j)$ and therefore $\det A(i|j) = 1$ if and only if $\det A(i, j|i, j) = 1$. Thus (ii) and (iii) are equivalent. That completes the proof. ■

4 Line graph of a tree

Let T be a tree with $V(T) = \{1, \dots, n\}$ and $E(T) = \{e_1, \dots, e_{n-1}\}$. Let $L(T)$ be the line graph of T . Thus $L(T)$ has vertex set $E(T)$ and two vertices are adjacent if the corresponding edges of T have a vertex in common. It can be seen that $L(T)$ is a block graph with the additional property that any cut-vertex is in at most two blocks. Conversely, a connected block graph in which any cut-vertex is in at most two blocks is the line graph of a tree. Let $K_{1,3}$ be the complete bipartite graph with the two parts having 1 and 3 vertices, respectively. Recall that a graph is said to be claw-free if it does not contain $K_{1,3}$ as an induced subgraph. Then it can be seen that a block graph is the line graph of a tree if and only if it is claw-free. The next result is a simple consequence of Theorems 6 and 8.

Theorem 9 *Let G be a claw-free block graph with n vertices and let A be the adjacency matrix of G . Then A is nonsingular over \mathbb{F}_2 if and only if n is even. Moreover let $B = A^{-1}$ in case A is nonsingular. Then $b_{ii} = 0, i = 1, \dots, n$, and for $1 \leq i \neq j \leq n$, $b_{ij} = 1$ if and only if $L(T) \setminus \{i, j\}$ has no odd component.*

Proof: If n is odd, then by Lemma 5, A is singular. So suppose n is even. Since G is a block graph in which any cut-vertex is in at most two blocks, $G \setminus \{i\}$ has at most two components for any vertex i of G . Since G has an even number of vertices, $G \setminus \{i\}$ must have exactly one odd component, and by Theorem 6, A is nonsingular over \mathbb{F}_2 .

If $B = A^{-1}$, then by Theorem 8, $b_{ii} = 0, i = 1, \dots, n$. Moreover, $b_{ij} = 1$ if and only if $G \setminus \{i, j\}$ is a nonsingular graph over \mathbb{F}_2 . Since each component of $G \setminus \{i, j\}$ is the line graph of a tree, by the first part, $G \setminus \{i, j\}$ is nonsingular over \mathbb{F}_2 if and only if it has no odd component. That completes the proof. ■

Theorem 10 *Let G be a claw-free block graph with n vertices and let A be the adjacency matrix of G . If n is even, then A is nonsingular over \mathbb{F}_2 . If n is odd, then A has nullity 1 over \mathbb{F}_2 .*

Proof: If n is even, then since G is claw-free, it readily follows from Theorem 6 that A is nonsingular.

Now assume that n is odd. Let x be the incidence vector of the vertices i such that $G \setminus \{i\}$ has no odd component. We first show that x satisfies $Ax = 0$. Let i be a vertex of G with degree r and let i be adjacent to vertices i_1, \dots, i_r . We further assume that $G \setminus \{i_j\}$

has no odd component if and only if $j = 1, \dots, s$. By the hypothesis, $G \setminus \{i_j\}$ has at most two components and we set \mathcal{C}_j to be the component, which may be vacuous, that does not contain $i, j = 1, \dots, r$. Note that \mathcal{C}_j and \mathcal{C}_ℓ are disjoint if $j \neq \ell$. Therefore

$$n = |\mathcal{C}_1| + \dots + |\mathcal{C}_r| + r + 1. \quad (28)$$

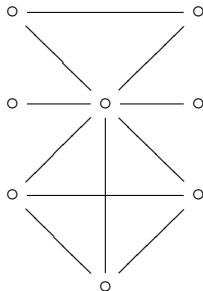
It follows from (28) that n has the same parity as $s + 1$. Since n is odd, s must be even. Thus i is adjacent to an even number of vertices j such that $G \setminus \{j\}$ has no odd component. It follows that the coordinate of Ax indexed by i is zero. Since i is arbitrary we conclude that $Ax = 0$.

We now turn to the uniqueness of the null vector. Let y be a vector over \mathbb{F}_2 such that $Ay = 0$. If $y \neq x$, then $z = x + y$ is a nonzero vector such that $Az = 0$. Moreover, there exists i such that $x_i = 1$ and $z_i = 0$. Then $A(i|i)$ has a nonzero null vector and hence it must be singular. However $G \setminus \{i\}$ has at most two components, both of which must be nonsingular by Theorem 9. This contradiction shows that $y = x$ and hence x is the unique null vector of A . ■

It was shown by Gutman and Sciriha [8] that over the reals, the line graph of a tree is either singular or has zero as a simple eigenvalue. Thus the nullity of the line graph of a tree over the reals is at most one. An alternative proof of the result and some related results were proved in [10]. An easier proof of the same result and some extensions were recently proved in [2],[6]. It may be remarked that in view of Theorem 10, the nullity of the line graph of a tree is at most one over \mathbb{F}_2 as well. This observation is a special case of a result obtained by Doob [5, Theorem 2.9] and can also be derived easily from some results in [7]. Also, the first part (n even) of Theorem 10 is contained in [7, Fact 22].

5 Flower

A block graph is called a flower if it has at most one cut vertex. By $F(b_1, \dots, b_k)$ we denote a flower with k blocks of sizes b_1, \dots, b_k . If $k = 1$, then F is a complete graph. Although the results in this section apply to this case, we implicitly assume in the proofs that $k \geq 2$, so that F has exactly one cut-vertex. A flower with blocks of sizes 2, 2, 3, 4 is shown below.



Theorem 11 *Let F be a flower with n vertices and with k blocks of sizes b_1, \dots, b_k . Let A be the adjacency matrix of F . Then*

$$\det A = (-1)^{n-k} \left(k \prod_{i=1}^k (b_i - 2) + \sum_{i=1}^k \prod_{j \neq i} (b_j - 2) \right). \quad (29)$$

Proof: Note that $n = \sum_{i=1}^k b_i - k + 1 = \sum_{i=1}^k (b_i - 1) + 1$. Thus if $(\alpha_1, \dots, \alpha_k)$ is a k -tuple of nonnegative integers satisfying the conditions (i), (ii) of Theorem 2, then $\alpha_i = b_i$ for some i and $\alpha_j = b_j - 1, j \neq i$. The result now follows easily from Theorem 2. ■

Theorem 12 *A flower is nonsingular over \mathbb{F}_2 if and only if it has exactly one even block.*

Proof: The result may be proved using Theorem 11. It is also an easy consequence of Theorem 6. ■

In the final result we determine the nullity of a flower.

Theorem 13 *Let F be a flower with ℓ even blocks. Let A be the adjacency matrix of F . Then the nullity of A over \mathbb{F}_2 is $|\ell - 1|$.*

Proof: Let F have n vertices and k blocks of sizes b_1, \dots, b_k . Then $n = \sum_{i=1}^k (b_i - 1) + 1$. First suppose $\ell = 0$. Then n is odd and by Lemma 5, A is singular. If we delete a vertex, other than the cut-vertex, from one of the blocks, then the resulting graph is a flower with exactly one even block, and is nonsingular by Theorem 12. Thus $\text{rank } A = n - 1$ and hence the nullity of A is 1. Now let $\ell \geq 1$. Remove one vertex from any $\ell - 1$ of the even blocks. The resulting graph is nonsingular by Lemma 5 and hence the rank of A is at least $n - \ell + 1$. Since A is symmetric, its rank is r if and only if it has a nonsingular principal submatrix of order r . Thus in order to complete the proof we need to show that if we remove any $\ell - 2$ vertices from F , the resulting graph is singular. Let H be the graph obtained from G by removing some $\ell - 2$ vertices. If the cut-vertex is not among the deleted vertices, then H is a flower with at least two even blocks and is singular by Theorem 12. If the cut-vertex is among the deleted vertices, then H must have at least one odd component and again H is singular by Lemma 5. That completes the proof. ■

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