Sharp decay estimates of the solutions to a class of nonlinear second order ODE

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Résumé. On étudie l'ordre de convergence vers 0 et les propriétés oscillatoires des solutions de l'EDO scalaire d'ordre 2 : $u'' + c|u'|^\alpha u' + |u|^\beta u = 0$ où c, α, β sont des constantes positives. Diverses généralisations (force extérieure, système) sont considérées.

Abstract. We establish the rate of decay to 0 and we study the oscillation properties of solutions to the scalar second order ODE : $u'' + c|u'|^\alpha u' + |u|^\beta u = 0$ where c, α, β are positive constants. Various extensions (forced equation, system) are considered.

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Introduction and preliminary remarks.

The main object of this work is to obtain sharp decay estimates as t tends to infinity of u and u' where $u \in C^2(\mathbb{R}^+)$ is a solution of the second order scalar ODE

$$u'' + a|u'|^\alpha u' + b|u|^\beta u = f(t) \tag{1}$$

where a, b, α, β are positive constants and f tends to 0 rapidly as t tends to infinity. This equation is a special case of the vector evolution problem

$$u'' + c\|u'\|^\alpha u' + \nabla F(u) = f(t) \tag{2}$$

for which decay estimates have been obtained recently by Chergui [3] and Ben Hassen-Chergui [2] when F satisfies a uniform *Lojasiewicz gradient inequality* (cf. [9] and also [7,8] for related works). Their estimates are optimal for general functions F but in the case of (1) or more generally for equations having a similar structure better results can be proved.

When $f = 0$, (1) becomes

$$u'' + a|u'|^\alpha u' + b|u|^\beta u = 0 \tag{3}$$

which, from the mechanical point of view, represents the motion of an oscillator subject to a nonlinear damping and a nonlinear restoring force. Both damping and restoring forces are weaker than linear when the argument approaches 0, and the comparison with the linear case $\alpha = \beta = 0$ suggests that the global behavior of $u(t)$ will depend on the competition between restoring and damping. If the damping is weak compared to the restoring force, which means α large with respect to β, solutions tend to oscillate in the sense that they will change sign for arbitrary large values of t. In the opposite case, if α is small with respect to β, we expect the dissipation to stop the oscillations for t large, as it happens in the case of a linear restoring force and a comparatively large linear friction term.

A first challenge is to find the relationship between α and β which determines which phenomenon, oscillation or damping, is dominant over the other. Actually this relationship can be easily guessed as follows: an immediate calculation shows that the family of equations depending on the positive parameter c

$$u'' + c|u'|^\alpha u' + |u|^\beta u = 0 \tag{4}$$

where b has been reduced to 1 by a single space renorming is, for α and β fixed, *globally* invariant under the transformations

$$v(t) = \lambda^{\frac{2}{\beta}} u(\lambda t)$$

When λ runs over $\mathbb{R}_*{}^+$, c achieves all positive values except in the special case

$$\alpha = \frac{\beta}{\beta+2}$$

in which case all equations of the form (4) are *individually* invariant. This means that in a sense the equations (4) are then all different, and it is then natural to conjecture that $\alpha = \frac{\beta}{\beta+2}$ is the only value for which the competition between oscillation and damping depends on the size of c. This will be confirmed later even though the critical value of α (which has to be < 1 no matter how large β can be) seems overwhelmingly small. In fact there are previous parallel results for the *backward equation* of (4) showing *oscillatory blow up properties* by a completely different method, cf [1, 10].

The plan of the paper is the following: Section 1 contains basic energy estimates of solutions to (4). Section 2 is devoted to the oscillatory (or non-oscillatory) behavior of these solutions, in particular we show that all non-trivial solutions of (4) are oscillatory for $\alpha > \frac{\beta}{\beta+2}$ and non-oscillatory for $\alpha < \frac{\beta}{\beta+2}$. The object of Section 3 is a detailed study of the non-oscillatory range. In Section 4 we generalize the decay estimates for the full equation (1). Section 5 is devoted to a discussion of optimality properties. Section 6 contains a generalization of the basic estimates of Sections 1 and 4 to a class of vector equations.

1 - Basic energy estimates for equation (4).

A crucial role will be played in this section by the energy of the solution u defined by

$$E(t) = \frac{1}{2}u'^2(t) + \frac{1}{\beta+2}|u(t)|^{\beta+2} \tag{1.1}$$

Indeed an immediate calculation shows that on any open time interval where u is C^2, the energy is non-increasing and more precisely

$$\frac{d}{dt}E(t) = -c|u'(t)|^{\alpha+2} \leq 0 \tag{1.2}$$

In particular u is global to the right and for any $(u_0, u_1) \in \mathbb{R}^2$ there is a unique solution $u \in C^2(\mathbb{R}^+)$ of (4) such that $u(0) = u_0, \quad u'(0) = u_1$.

The main result of this Section is the following

Theorem 1.1. *There exists a positive constant η independent of the initial data such that*

$$\liminf_{t \to +\infty} t^{\frac{2}{\alpha}} E(t) \geq \eta \tag{1.3}$$

Moreover

i) If $\alpha \geq \alpha_0 := \frac{\beta}{\beta+2}$, then there is a constant C depending boundedly on E(0) such that

$$\forall t \geq 1, \quad E(t) \leq C t^{-\frac{2}{\alpha}}$$

ii) If $\alpha < \alpha_0 := \frac{\beta}{\beta+2}$, then there is a constant C depending boundedly on E(0) such that

$$\forall t \geq 1, \quad E(t) \leq C t^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}}$$

Proof. Since $|u'(t)|^{\alpha+2} \leq K E(t)^{\frac{\alpha+2}{2}}$ for some positive constant K, from (1.2) we deduce

$$\frac{d}{dt} E(t) \geq -cK E(t)^{\frac{\alpha+2}{2}}$$

from which we derive

$$\frac{d}{dt} E(t)^{-\frac{\alpha}{2}} = -\frac{\alpha}{2} E(t)^{-\frac{\alpha+2}{2}} E'(t) \leq \frac{\alpha}{2} cK := K_1$$

By integrating we obtain

$$E(t)^{-\frac{\alpha}{2}} \leq E(0)^{-\frac{\alpha}{2}} + K_1 t$$

and finally

$$\liminf_{t \to +\infty} t^{\frac{2}{\alpha}} E(t) \geq \eta := K_1^{-\frac{2}{\alpha}}$$

hence (1.3) is proved. In order to establish i) and ii) we consider the pertubed energy fonction

$$F(t) := E(t) + \varepsilon |u|^{\gamma} u u' \tag{1.4}$$

where $\gamma > 0$ and $\varepsilon > 0$ shall chosen as follows: assuming first $2(\gamma + 1) \geq \beta + 2$ which reduces to

$$\gamma \geq \frac{\beta}{2}$$

we obtain, as a consequence of Young's inequality, the existence of $M > 0$ for which

$$(1 - M\varepsilon)E(t) \leq F(t) \leq (1 + M\varepsilon)E(t)$$

therefore, assuming $\varepsilon \leq \frac{1}{2M}$ we achieve

$$\forall t \geq 0, \quad \frac{1}{2}E(t) \leq F(t) \leq 2E(t) \tag{1.5}$$

Then by differentiating and dropping t for simplicity we find

$$F' = -c|u'|^{\alpha+2} + \varepsilon(\gamma+1)|u|^{\gamma}u'^2 + \varepsilon|u|^{\gamma}u\left(-c|u'|^{\alpha}u' - |u|^{\beta}u\right)$$

$$F' = -c|u'|^{\alpha+2} - \varepsilon|u|^{\beta+\gamma+2} + \varepsilon(\gamma+1)|u|^{\gamma}u'^2 - c\varepsilon|u|^{\gamma}u|u'|^{\alpha}u' \tag{1.6}$$

In order to control the third term we notice that by Young's inequality applied with the conjugate exponents $\frac{\alpha+2}{2}$ and $\frac{\alpha+2}{\alpha}$

$$|u|^{\gamma}u'^2 \leq \delta|u|^{\frac{(\alpha+2)\gamma}{\alpha}} + C(\delta)|u'|^{\alpha+2}$$

Assuming

$$\frac{(\alpha+2)\gamma}{\alpha} \geq \beta + \gamma + 2$$

which reduces to the condition

$$\gamma \geq \frac{\alpha}{2}(\beta+2) \tag{1.7}$$

and taking δ small enough (depending on the initial energy) yields

$$\varepsilon(\gamma+1)|u|^{\gamma}u'^2 \leq \frac{\varepsilon}{4}|u|^{\beta+\gamma+2} + P\varepsilon|u'|^{\alpha+2} \tag{1.8}$$

and then (1.6) implies

$$F' \leq (-c + P\varepsilon)|u'|^{\alpha+2} - \frac{3\varepsilon}{4}|u|^{\beta+\gamma+2} - c\varepsilon|u|^{\gamma}u|u'|^{\alpha}u' \tag{1.9}$$

In order to control the last term we notice that by Young's inequality applied with the conjugate exponents $\alpha + 2$ and $\frac{\alpha+2}{\alpha+1}$

$$-|u|^{\gamma}u|u'|^{\alpha}u' \leq \delta|u|^{(\alpha+2)(\gamma+1)} + C'(\delta)|u'|^{\alpha+2}$$

This term will be dominated by the negative terms assuming

$$(\alpha+2)(\gamma+1) \geq \beta + \gamma + 2 \Longleftrightarrow (\alpha+1)(\gamma+1) \geq \beta + 1$$

which reduces to

$$\gamma \geq \frac{\beta - \alpha}{\alpha + 1} \tag{1.10}$$

and taking δ small enough (depending on the initial energy) yields

$$-c\varepsilon|u|^{\gamma}u|u'|^{\alpha}u' \leq \frac{\varepsilon}{4}|u|^{\beta+\gamma+2} + P'\varepsilon|u'|^{\alpha+2}$$

By replacing in (1.9) we finally obtain

$$F' \leq (-c + Q\varepsilon)|u'|^{\alpha+2} - \frac{\varepsilon}{2}|u|^{\beta+\gamma+2} \tag{1.11}$$

where $Q = P + P'$. By choosing ε sufficiently small we end up with

$$F' \leq -\frac{\varepsilon}{2}(|u'|^{\alpha+2} + |u|^{\beta+\gamma+2}) \tag{1.12}$$

valid under the condition

$$\gamma \geq \gamma_0 := \max\{\frac{\beta}{2}, \frac{\alpha}{2}(\beta + 2), \frac{\beta - \alpha}{\alpha + 1}\} \tag{1.13}$$

We now distinguish 2 cases.

i) If $\alpha \geq \alpha_0 := \frac{\beta}{\beta+2}$, then clearly $\frac{\alpha}{2}(\beta + 2) \geq \frac{\beta}{2}$ and moreover

$$\frac{\beta - \alpha}{\alpha + 1} = \frac{\beta + 1}{\alpha + 1} - 1 \leq \frac{\beta + 1}{\frac{\beta}{\beta+2} + 1} - 1 = \frac{\beta}{2}$$

In this case $\gamma_0 = \frac{\alpha}{2}(\beta + 2)$ and choosing $\gamma = \gamma_0$ we find

$$\beta + \gamma + 2 = (\frac{\alpha}{2} + 1)(\beta + 2)$$

so that (1.12) now gives, since $\frac{\beta+\gamma+2}{\beta+2} = (\frac{\alpha}{2} + 1) = \frac{\alpha+2}{2}$

$$F' \leq -\rho E^{\frac{\alpha}{2}+1} \leq -\rho' F^{\frac{\alpha}{2}+1} \tag{1.14}$$

for some positive constants ρ, ρ'. Then the result is an easy consequence of (1.14) and (1.5).

ii) If $\alpha < \alpha_0 := \frac{\beta}{\beta+2}$, then clearly $\frac{\alpha}{2}(\beta + 2) < \frac{\beta}{2}$ and moreover

$$\frac{\beta - \alpha}{\alpha + 1} - \frac{\beta}{2} = \frac{2(\beta - \alpha) - \beta(\alpha + 1)}{2(\alpha + 1)} = \frac{\beta - \alpha(\beta + 2)}{2(\alpha + 1)} > 0$$

In this case $\gamma_0 = \frac{\beta-\alpha}{\alpha+1}$ and choosing $\gamma = \gamma_0$ we find

$$\beta+\gamma+2 = (\beta+2)(1+\frac{\gamma}{\beta+2}) = (\beta+2)(1+\frac{\beta-\alpha}{(\alpha+1)(\beta+2)})$$

In addition here since $\gamma > \frac{\alpha}{2}(\beta+2)$ we have

$$\frac{\beta+\gamma+2}{\beta+2} = 1+\frac{\gamma}{\beta+2} > 1+\frac{\alpha}{2} = \frac{\alpha+2}{2}$$

so that (1.12) now gives

$$F' \leq -\rho E^{(1+\frac{\beta-\alpha}{(\alpha+1)(\beta+2)})} \leq -\rho' F^{(1+\frac{\beta-\alpha}{(\alpha+1)(\beta+2)})} \tag{1.15}$$

for some positive constants ρ, ρ'. Then the result is an easy consequence of (1.15) and (1.5).

Corollary 1.2. *If $\alpha \geq \alpha_0 := \frac{\beta}{\beta+2}$, then there is a constant C depending boundedly on E(0) such that*

$$\forall t \geq 1, \quad |u(t)| \leq Ct^{-\frac{2}{\alpha(\beta+2)}}$$

$$\forall t \geq 1, \quad |u'(t)| \leq Ct^{-\frac{1}{\alpha}}$$

2 - Oscillation of solutions.

In this Section we study the oscillatory behavior of u. This behavior is strongly dependent upon the size of α compared to $\frac{\beta}{\beta+2}$.

Theorem 2.1. *Assume either*

$$\alpha > \alpha_0 := \frac{\beta}{\beta+2} \tag{2.1}$$

or

$$\alpha = \alpha_0 = \frac{\beta}{\beta+2}; \quad c < c_0 := (\beta+2)(\frac{\beta+2}{2\beta+2})^{\frac{\beta+1}{\beta+2}} \tag{2.2}$$

Then any solution $u(t)$ of (4) which is not identically 0 changes sign on each interval (T,∞) and so does $u'(t)$.

Proof. As a consequence of the lifting theorem, since the energy of u is positive for all t we can introduce, as was done in [5], polar coordinates as follows

$$(\frac{2}{\beta+2})^{\frac{1}{2}}|u|^{\frac{\beta}{2}}u = r(t)\cos\theta(t), \quad u'(t) = r(t)\sin\theta(t) \tag{2.3}$$

where r and θ are two C^1 functions and $r(t) = E(t)^{\frac{1}{2}} > 0$. A straighforward calculation shows that θ satisfies the differential equation

$$\theta' + (\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}} r^{\frac{\beta}{\beta+2}} |\cos\theta|^{\frac{\beta}{\beta+2}} + cr^\alpha |\sin\theta|^\alpha \sin\theta\cos\theta = 0 \tag{2.4}$$

On the other hand we know that $r(t)$ tends to 0 exactly like $t^{-\frac{1}{\alpha}}$ as t tends to infinity.

In the case $\alpha > \alpha_0 := \frac{\beta}{\beta+2}$, we find that for t large

$$\theta' \le -\eta t^{-\lambda} |\cos\theta|^{\frac{\beta}{\beta+2}}$$

where $\eta > 0$ and $\lambda := \frac{\beta}{\alpha(\beta+2)} < 1$.

In the case $\alpha = \alpha_0 = \frac{\beta}{\beta+2}$; $\quad c < c_0 := (\beta+2)(\frac{\beta+2}{2\beta+2})^{\frac{\beta+1}{\beta+2}}$ we obtain

$$\theta' = -r^\alpha ((\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}} |\cos\theta|^\alpha + c|\sin\theta|^\alpha \sin\theta\cos\theta)$$

On the other hand it is easy to check that

$$\max_{\theta\in R}(|sin\theta|^{\alpha+1}|cos\theta|^{1-\alpha}) = (\frac{1}{\beta+2})^{\frac{1}{\beta+2}} (\frac{\beta+1}{\beta+2})^{\frac{\beta+1}{\beta+2}}$$

so that the coefficient of $-r^\alpha$ is bounded from below by a positive constant if

$$(\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}} - c(\frac{1}{\beta+2})^{\frac{1}{\beta+2}} (\frac{\beta+1}{\beta+2})^{\frac{\beta+1}{\beta+2}} > 0$$

which reduces to $c < c_0$. Therefore in both cases we find for t large

$$\theta' \le -\eta t^{-1} |\cos\theta|^{\frac{\beta}{\beta+2}}$$

We introduce the function

$$H(s) := \int_a^s \frac{du}{|\cos u|^{\frac{\beta}{\beta+2}}}$$

If u does not vanish for $t \ge t_0$, say, then we may assume, changing if necessary u to $-u$, that

$$\forall t \ge t_0, \quad \theta(t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

Then $H(\theta(t)) := K(t)$ is differentiable for $t \ge t_0$ but also

$$\forall t \ge t_0, \quad K'(t) \le -\eta\, t^{-1}$$

which is impossible since choosing $a = -\frac{\pi}{2}$, $H(\theta(t)))$ is nonnegative for $t \ge t_0$. This contradiction proves that u has a zero on each half-line. Since the derivative u' cannot vanish at the same time, u must change sign. In addition between 2 zeroes of u, there is a zero of u'. Finally if u' and u'' vanish at the same time, the equation shows that u vanishes also, a contradiction which implies that u' changes sign at each zero. This concludes the proof of Theorem 2.1.

Theorem 2.2. *Assume*

$$\alpha < \alpha_0 := \frac{\beta}{\beta+2}$$

Then any solution $u(t)$ of (4) which is not identically 0 has a finite number of zeroes on $(0,\infty)$. Moreover for t large, $u'(t)$ has the opposite sign to that of $u(t)$ and $u''(t)$ has the same sign as $u(t)$.

Proof. We introduce

$$G(s) := \int_0^s |\sin u|^\alpha \sin u \cos u \, du$$

Multiplying (2.4) by $|\sin\theta|^\alpha \sin\theta\cos\theta$ we find, by a simple use of Cauchy-Schwarz inequality

$$[G(\theta(t))]' \le -cr^\alpha |\sin\theta|^{2\alpha} \sin^2\theta\cos^2\theta + C(\beta) r^{\frac{\beta}{\beta+2}} |\cos\theta|^{\frac{\beta}{\beta+2}} |\sin\theta|^\alpha \sin\theta\cos\theta$$

$$\le C'(\beta) r^\mu \le C'' t^{-\lambda}$$

where

$$\mu = \frac{2\beta}{\beta+2} - \alpha$$

and

$$\lambda = \Big(\frac{2\beta}{\beta+2} - \alpha\Big)\frac{(\alpha+1)(\beta+2)}{2(\beta-\alpha)} = \alpha + 1 - \frac{\alpha\beta(\alpha+1)}{2(\beta-\alpha)}$$

$$= 1 + \alpha[1 - \frac{\beta(\alpha+1)}{2(\beta-\alpha)}] = 1 + \alpha[\frac{\beta - \alpha(2+\beta)}{2(\beta-\alpha)}] > 1$$

To finish the proof we shall use the following lemma

Lemma 2.3. *Let $\theta \in C^1(a,+\infty)$ and G be a non constant T-periodic function. Assume that for some $h \in L^1(a,+\infty)$*

$$\forall t \ge t_0, \quad G(\theta(t))' \le h(t)$$

Then for $t \ge t_1$ large enough, $\theta(t)$ remains in some interval of length $\le T$. If, in addition, G' has a finite number of zeroes on [0, T], then $\theta(t)$ has a limit for $t \longrightarrow \infty$.

Proof. Let

$$m = \min G, \quad M = \max G, \quad \delta = M - m$$

There is $t_0 > 0$ such that

$$\forall s \geq t_0, \quad \forall t \geq s, \quad G(\theta(t)) - G(\theta(s)) \leq \frac{\delta}{2}$$

We introduce the interval

$$J = \theta[t_0, +\infty)$$

If $|J| \leq T$, we are done. On the other hand if $|J| > T$, there exists $\tau \geq t_0$ such that

$$G(\theta(\tau)) = m$$

Then we have

$$\forall t \geq \tau, \quad m \leq G(\theta(t)) \leq m + \frac{\delta}{2} < M$$

Setting

$$J_1 = \theta[\tau, +\infty)$$

if $|J_1| > T$, there exists $x \in J_1$ such that $G(x) = M$, which means that $M \in G(J_1)$, a contradiction which shows that $|J_1| \leq T$ and gives the first conclusion with $t_1 = \tau$. To establish the second part, we note that under the additional hypothesis, $G^{-1}(x) \cap I$ is finite for any x and any bounded interval J. Introducing

$$\Phi(t) := G(\theta(t)) + \int_t^\infty h(s)ds$$

Φ is bounded and nonincreasing, hence converges to a limit l as $t \longrightarrow +\infty$. Since h is integrable, we deduce

$$\lim_{t \longrightarrow +\infty} G(\theta(t)) = l$$

Then the set of limiting values of $\theta(t)$ as $t \longrightarrow +\infty$ is contained in the finite set $G^{-1}(l) \cap \overline{\theta[t_1, +\infty)}$. By connectedness , this implies that $\theta(t)$ converges to one point of this set as $t \longrightarrow +\infty$.

End of the proof of Theorem 2.2. By Lemma 2.3, $\theta(t)$ has a limit for $t \longrightarrow \infty$. If the limit differs from $\frac{\pi}{2}(mod\, \pi)$, then clearly u has a constant sign for t large. In the opposite case, $|u'(t)|$ is equivalent to $r(t)$ and therefore does not vanish for $t \geq A$, then u can have at most one zero b in $(A, +\infty)$, in this case it has a constant sign on

$(b+1,+\infty)$. Next let t_0 be such that u has a constant sign on $(t_0,+\infty)$. If u' has several zeroes in $(t_0,+\infty)$, then obviously u'' must have different signs at two sucessive zeroes of u', and by the equation the corresponding values of u must have different signs too, a contradiction which shows that u' has at most one zero in $(t_0,+\infty)$ and therefore has a constant sign for t large. Since u tends to 0 at infinity the signs of u and u' must be opposite to each other. Finally, by differentiating (4) it is easy to check that $u''(t)$ has the same sign as $u(t)$. This concludes the proof of Theorem 2.2.

The critical case with c large is quite special. Actually we have

Theorem 2.4. *Assume*

$$\alpha = \alpha_0 = \frac{\beta}{\beta+2}; \quad c \geq c_0 := (\beta+2)(\frac{\beta+2}{2\beta+2})^{\frac{\beta+1}{\beta+2}} \tag{2.2}$$

Then any solution $u(t)$ of (4) which is not identically 0 has at most one zero on $(0,\infty)$.

Proof. In this case

$$\theta' = -r^{\alpha}((\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}}|\cos\theta|^{\alpha} + c|\sin\theta|^{\alpha}\sin\theta\cos\theta)$$

If $c = c_0$, the coefficient of $-r^{\alpha}$ remains nonegative, so that θ is non-increasing. Due to periodicity, the distance of two zeroes of $h(\theta) := (\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}}|\cos\theta|^{\alpha}+c|\sin\theta|^{\alpha}\sin\theta\cos\theta$ other than $\frac{\pi}{2}(mod\,\pi)$ is not more than π and therefore either $\theta(t)$ remains in an interval of length less than π, or it coincides with one of these zeroes for a finite value of t. In the first case $\theta(t)$, being non increasing and bounded, converges to a limit and achieves at most once a value for which u vanishes. In the second case, due to existence and uniqueness for the ODE satisfied by $\theta(t)$ near the non-trivial equilibria, $\theta(t)$ must remain constant and actually u never vanishes. If $c > c_0$, the coefficient of $-r^{\alpha}$ still has non-trivial zeroes. If θ does not take any of the corresponding values, then it remains bounded and since the coefficient of $-r^{\alpha}$ remains positive near the trivial zeroes, θ is monotone, hence convergent. If $\theta(t)$ takes one of the corresponding values, as previously it has to remain constant and u never vanishes. Otherwise as previously u vanishes at most once.

3- A detailed study of the non oscillatory case.

Theorem 3.1. *Assuming* $0 < \alpha < \frac{\beta}{\beta+2}$*, any solution* u *of (4) satifies the following alternative: Either there is a constant* C *for which*

$$\forall t \geq 1, \quad E(t) \leq Ct^{-\frac{2}{\alpha}} \tag{3.1}$$

Or we have

$$\limsup_{t\to+\infty} t^{\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}} E(t) > 0 \tag{3.2}$$

Proof. We use the notation of the proof of Theorem 2.2. As a consequence of Lemma 2.3, we know that $\theta(t)$ tends to a limit l as $t \to \infty$. Moreover if $\sin l \cos l \neq 0$ we find as $t \to \infty$:

$$\theta' \sim -cr^\alpha |\cos l|^\alpha \sin l \cos l$$

and since $r^\alpha(t) \geq \frac{\eta}{t} \notin L^1(0,\infty)$ this contradicts boundedness of $\theta(t)$. Therefore we have only 2 possible cases

Case 1: $\cos l = 0$, then $|\sin l| = 1$. In this case for t large enough

$$\frac{d}{dt}E(t) = -c|u'(t)|^{\alpha+2} \leq -\rho E(t)^{\frac{\alpha+2}{2}}$$

and then (3.1) follows at once.

Case 2: $\sin l = 0$, then $|\cos l| = 1$. In this case as $t \to \infty$

$$r(t) \sim (\frac{2}{\beta+2})^{\frac{1}{2}} |u|^{\frac{\beta}{2}+1}$$

and we obtain

$$\frac{u'}{|u|^{\frac{\beta}{2}+1}} \to 0$$

In particular for t large enough, $|u(t)| \geq t^{-\frac{2}{\beta}}$. By the non-oscillation result we may assume that $u > 0$ and $u' < 0$ for t large. Then by integrating (2.4) on $(t, 2t)$ we find

$$\theta(2t) - \theta(t) + K\int_t^{2t} r^{\frac{\beta}{\beta+2}} |\cos\theta|^{\frac{\beta}{\beta+2}} ds + c\int_t^{2t} r^{\alpha-1} u' |\sin\theta|^\alpha \cos\theta ds = 0$$

hence for t large, since $r(t) \sim (\frac{2}{\beta+2})^{\frac{1}{2}} |u|^{\frac{\beta}{2}+1}$ and u is positive, non increasing

$$-\int_t^{2t} r^{\alpha-1} u' |\sin\theta|^\alpha \cos\theta ds \leq C_1 (1 + \int_t^{2t} r^{\frac{\beta}{\beta+2}} ds) \leq C_2 [1 + tu(t)^{\frac{\beta}{\beta+2}(1+\frac{\beta}{2})}]$$

$$= C_2[1 + tu(t)^{\frac{\beta}{2}}] \le 2C_2 tu(t)^{\frac{\beta}{2}}$$

On the other hand for t large

$$-\int_t^{2t} r^{\alpha-1}u'|\cos\theta|^\alpha \sin\theta ds \ge \eta \int_t^{2t} |u'(s)|^{\alpha+1}r^{-1}ds$$

hence

$$\int_t^{2t} |u'(s)|^{\alpha+1}r^{-1}ds \le C_3 tu(t)^{\frac{\beta}{2}}$$

which gives

$$\int_t^{2t} |u'(s)|^{\alpha+1}|u|^{-(\frac{\beta}{2}+1)}ds \le C_4 tu(t)^{\frac{\beta}{2}}$$

By using Holder's inequality with exponents $\alpha+1$ and $\frac{\alpha+1}{\alpha}$ we deduce

$$\int_t^{2t} |u'(s)||u|^{\frac{-(\beta+2)}{2(\alpha+1)}}ds \le C_5 t\, u(t)^{\frac{\beta}{2(\alpha+1)}}$$

in other words

$$u^{-\delta}(2t) - u^{-\delta}(t) = \int_t^{2t} \frac{d}{ds}[u^{-\delta}(s)]ds \le C_6 t\, u(t)^{\frac{\beta}{2(\alpha+1)}}$$

with $\delta = \frac{\beta-2\alpha}{2(\alpha+1)} > 0$. We claim that there is a set $S \subset (0, +\infty)$ containing arbitrarily large numbers such that for some $\nu > 0$

$$\forall t \in S, \quad u^{-\delta}(2t) - u^{-\delta}(t) \ge \nu u^{-\delta}(t) \tag{3.3}$$

Indeed we have

Lemma 3.2 *Let $u : \mathbb{R}^+ \to \mathbb{R}^+_*$ be such that for some constants T, K, $\lambda > 0$*

$$\forall t \ge T, \quad u(t) \le Kt^{-\lambda} \tag{3.4}$$

Then for any $\gamma < \lambda \ln 2$, there exists a sequence $t_n \to +\infty$ for which

$$\forall n \in \mathbb{N}, \quad u(2t_n) \le e^{-\gamma}u(t_n)$$

Proof of Lemma 3.2. Assuming the contrary, for some $A > 0$ we have

$$\forall t \ge A, \quad u(2t) > e^{-\gamma}u(t)$$

In particular

$$\forall n \in \mathbb{N}, \quad u(2^n A) \geq e^{-n\gamma} u(A) = u(A)(2^n)^{-\frac{\gamma}{\ln 2}}$$

contradicting (3.4) whenever $\frac{\gamma}{\ln 2} < \lambda$. This contradiction proves the claim.

Proof of Theorem 3.1 continued. Applying Lemma 3.2 to $u(t)$ with $\lambda = \frac{\alpha+1}{\beta-\alpha}$ we find for some $\gamma > 0$

$$\forall t \in S, \quad u(2t) \leq e^{-\gamma} u(t)$$

hence

$$\forall t \in S, \quad u^{-\delta}(2t) \geq e^{\delta\gamma} u^{-\delta}(t)$$

and then

$$\forall t \in S, \quad u^{-\delta}(2t) - u^{-\delta}(t) \geq (e^{\delta\gamma} - 1) u^{-\delta}(t)$$

hence (3.3) with $\nu := e^{\delta\gamma} - 1$. Now we have for some $C_7 > 0$ and $\sigma = \frac{1}{C_7} > 0$

$$\forall t \in A, \quad u^{-\delta}(t) \leq C_5 t u(t)^{\frac{\beta}{2(\alpha+1)}} \Longrightarrow \forall t \in A, \quad u^{\delta + \frac{\beta}{2(\alpha+1)}}(t) \geq \sigma t^{-1}$$

with

$$\delta + \frac{\beta}{2(\alpha+1)} = \frac{\beta - 2\alpha}{2(\alpha+1)} + \frac{\beta}{2(\alpha+1)} = \frac{\beta - \alpha}{\alpha + 1}$$

and finally we find for some $\sigma' > 0$

$$\forall t \in A, \quad u(t) \geq \sigma' t^{-\frac{\alpha+1}{\beta-\alpha}}$$

which implies (3.2). This concludes the proof of Theorem 3.1.

Remark 3.2. *If $\alpha < \alpha_0 := \frac{\beta}{\beta+2}$, then for each solution u of (4) which satisfies (3.1) there is a constant C such that*

$$\forall t \geq 1, \quad |u(t)| \leq C t^{-\frac{1-\alpha}{\alpha}}$$

and

$$\forall t \geq 1, \quad |u'(t)| \leq C t^{-\frac{1}{\alpha}}$$

Remark 3.3. Theorem 3.1 generalizes Theorem 1 of [6] corresponding to the case $\alpha = 0$, in which there are exceptional solutions which decay at the maximal rate permitted by the lower bound of the energy decay, which is in that case exponential.

Such a result is also known for $\alpha < 0$, cf. eg. [4]. The following result shows that if $0 < \alpha < \alpha_0 := \frac{\beta}{\beta+2}$ there are indeed non-trivial solutions of (4) which satisfy (3.1).

Theorem 3.4. *Let $\alpha < \frac{\beta}{\beta+2}$, $\quad c > 0$. Then there exists a solution $u > 0$ of (4) such that for some constant $C > 0$*

$$\forall t \geq 0, \quad u(t) \leq C(1+t)^{-\frac{1-\alpha}{\alpha}}, \quad |u'(t)| \leq C(1+t)^{-\frac{1}{\alpha}} \tag{3.5}$$

Proof. Due to the invariance recalled in the introduction it is enough to prove the result for $c = 1$. We introduce two Banach spaces X and Y as follows

$$X = \{z \in C([1,+\infty), \quad t^{\frac{1}{\alpha}} z(t) \in L^\infty([1,+\infty))\}$$

with norm

$$\forall z \in X \quad \|z\|_X = \|t^{\frac{1}{\alpha}} z(t)\|_{L^\infty([1,+\infty))} \tag{3.6}$$

and

$$Y = \{z \in C([1,+\infty), \quad t^{1+\frac{1}{\alpha}} z(t) \in L^\infty([1,+\infty))\}$$

with norm

$$\forall z \in Y, \quad \|z\|_Y = \|t^{1+\frac{1}{\alpha}} z(t)\|_{L^\infty([1,+\infty))} \tag{3.7}$$

The proof proceeds in 3 steps.

Step 1: a preliminary estimate. Let $f \in Y$, $\varphi \in \mathbb{R}$ and consider the problem

$$v' + |v|^\alpha v = f; \quad v(1) = \varphi \tag{3.8}$$

Lemma 3.5. *Under the conditions*

$$|\varphi| \leq (\frac{2}{\alpha})^{\frac{1}{\alpha}}; \quad \|f\|_Y \leq \frac{1}{\alpha}(\frac{2}{\alpha})^{\frac{1}{\alpha}}$$

the unique solution v of (3.8) is in X with

$$\|v\|_X \leq (\frac{2}{\alpha})^{\frac{1}{\alpha}}$$

Proof. It is sufficient to establish the result for $f, \varphi \geq 0$. Let

$$w(t) = (\frac{2}{\alpha})^{\frac{1}{\alpha}} t^{-\frac{1}{\alpha}}$$

Then we have $w(1) = \left(\frac{2}{\alpha}\right)^{\frac{1}{\alpha}}$ and

$$w' + |w|^\alpha w = (\frac{2}{\alpha} - \frac{1}{\alpha})(\frac{2}{\alpha})^{\frac{1}{\alpha}} t^{-\frac{1}{\alpha}-1} = \frac{1}{\alpha}(\frac{2}{\alpha})^{\frac{1}{\alpha}} t^{-\frac{1}{\alpha}-1}$$

Hence the result is an immediate consequence of the standard comparison principle.

Step 2: an integrodifferential problem. We introduce the integral operator $\mathcal{K}$ defined on $L^1([1, +\infty)))$ by

$$\forall v \in L^1([1, +\infty)), \quad \forall t \in [1, +\infty)), \quad \mathcal{K}(v)(t) = \left| \int_t^\infty v(s)ds \right|^\beta \int_t^\infty v(s)ds \qquad (3.9)$$

We claim that $\mathcal{K}(X) \subset Y$ with

$$\forall v \in X, \quad \|\mathcal{K}(v)\|_Y \leq C\|v\|_X^{\beta+1} \qquad (3.10)$$

Indeed since $\alpha < 1$, it is clear that $X \subset L^1([1, +\infty))$; Moreover

$$\forall v \in X, \quad \forall t \in [1, +\infty)), \quad |\mathcal{K}(v)(t)| \leq \left(C(\alpha)\|v\|_X t^{1-\frac{1}{\alpha}}\right)^{\beta+1}$$

$$\leq C'(\alpha)\|v\|_X^{\beta+1} t^{(1-\frac{1}{\alpha})(\beta+1)}$$

But $\alpha < \frac{\beta}{\beta+2} \Longrightarrow (\frac{1}{\alpha} - 1)(\beta + 1) > 1 + \frac{1}{\alpha}$, and (3.10) follows easily.

Now we consider for ε small enough the solution $z = \mathcal{T}(v)$ of the perturbed problem

$$z' + |z|^\alpha z = \varepsilon \mathcal{K} v; \quad z(1) = \varphi \qquad (3.11)$$

Let

$$B := \{z \in X, \ \|z\|_X \leq (\frac{2}{\alpha})^{\frac{1}{\alpha}}\}$$

Under the condition

$$\varepsilon C(\frac{2}{\alpha})^{\frac{\beta+1}{\alpha}} \leq \frac{1}{\alpha}(\frac{2}{\alpha})^{\frac{1}{\alpha}} \qquad (3.12)$$

as a consequence of lemma 3.5 we have $\mathcal{T}(B) \subset B$.

Step 3. An iterative scheme. We consider the sequence $v_n = \mathcal{T}^n(0)$ defined inductively as follows: v_1 is the solution of

$$v' + |v|^\alpha v = 0; \quad v(1) = \varphi$$

Clearly v_1 is nonegative, nonincreasing and belongs to B. When v_n is known we define v_{n+1} as the solution of

$$v' + |v|^\alpha v = \varepsilon \mathcal{K} v_n; \quad v(1) = \varphi$$

Since $\mathcal{K}$ is an *increasing* operator, it is easy to see that the sequence v_n is increasing, nonnegative and bounded by a fixed positive element of X. Hence v_n is bounded by a fixed integrable function and, since v_n' is uniformly bounded, v_n converges locally uniformly and in $L^1([1, +\infty))$. The limit v is a solution of

$$v' + |v|^\alpha v = \varepsilon \mathcal{K} v; \quad v(1) = \varphi \tag{3.13}$$

Step 4. Conclusion. Therefore v is a positive solution of

$$v' + |v|^\alpha v = \varepsilon \Big| \int_t^\infty v(s) ds \Big|^\beta \int_t^\infty v(s) ds; \quad v(1) = \varphi \tag{3.14}$$

Let

$$\forall t \geq 0, \quad u(t) = \int_{1+t}^\infty v(s) ds \tag{3.15}$$

Then $u \geq 0$ and $u' = -v(. + 1); \quad u'' = -v'(. + 1)$, hence (3.14) rewrites as

$$-u'' - |u'|^\alpha u' = \varepsilon |u|^\beta u \tag{3.16}$$

Since $v \in X$, we have finally

$$|u(t)| \leq C_1 (1 + t)^{-(\frac{1}{\alpha} - 1)}; \quad |u'(t)| \leq C_2 (1 + t)^{-\frac{1}{\alpha}}$$

hence (3.5). Finally, replacing $u(t)$ by $ku(mt)$ for some $k, m > 0$ suitably chosen we obtain the solution we were looking for.

Remark 3.5. In the sequel we shall call *fast* solutions the non-trivial soltions of (4) satisfying (3.1). In the case $\alpha = 0$, the fast solutions are exceptional, cf [6, Theorem 1]. When $0 < \alpha < \alpha_0 := \frac{\beta}{\beta+2}$ we conjecture the same property. However for the moment we have been unable to exhibit even a single *slow* solution. A fortiori the detailed asymptotic behavior of such solutions remains for the moment obscure.

4- The case of equation (1) .

In this section we generalize the main result of Theorem 1.1 to the general equation (1) when f tends to 0 sufficiently fast as $t \to +\infty$. We shall rely on the following simple lemma

Lemma 4.1. *Let $\delta > 0$, $\mu \geq 1 + \frac{1}{\delta}$ and let $\varphi \in C^1([1, +\infty)$ be such that*

$$\varphi \geq 0; \quad \forall t \geq 1, \quad \varphi'(t) \leq -\eta\varphi(t)^{1+\delta} + Kt^{-\mu} \tag{4.1}$$

where K, η are positive constants. Then there is $C \geq 0$ for which

$$\forall t \geq 1, \quad \varphi(t) \leq Ct^{-\frac{1}{\delta}} \tag{4.2}$$

Proof. We introduce $\Psi_C(t) := Ct^{-\frac{1}{\delta}}$ It is immediate to check that

$$\forall t \geq 1, \quad \Psi_C'(t) + \eta\Psi_C(t)^{1+\delta} = (\eta C^{1+\delta} - \frac{C}{\delta})t^{-(1+\frac{1}{\delta})}$$

Selecting C such that $C \geq \varphi(1), \quad \eta C^{1+\delta} - \frac{C}{\delta} \geq K$ it is then classical , from the standard comparison principle, that

$$\forall t \geq 1, \quad \Psi_C(t) \geq \varphi(t)$$

The main results of this Section are the following

Theorem 4.2. *If $\alpha \geq \frac{\beta}{\beta+2}$ and*

$$f \in C(\mathbb{R}^+); \quad |f(t)| \leq Kt^{-\lambda}; \quad \lambda \geq 1 + \frac{1}{\alpha}$$

then there is a constant C depending boundedly on E(0) such that

$$\forall t \geq 1, \quad E(t) \leq Ct^{-\frac{2}{\alpha}}$$

Proof. By following the steps of the proof of Theorem 1.1, we find here

$$F' \leq -\frac{\varepsilon}{2}(|u'|^{\alpha+2} + |u|^{\beta+\gamma+2}) + f(u' + \varepsilon|u|^{\gamma}u)$$

$$F' \leq -\frac{\varepsilon}{4}(|u'|^{\alpha+2} + |u|^{\beta+\gamma+2}) + C_1(\varepsilon)|f|^{\frac{\alpha+2}{\alpha+1}} + C_2(\varepsilon)|f|^{\frac{\beta+\gamma+2}{\beta+1}} \tag{4.3}$$

Here we have

$$\frac{\beta+\gamma+2}{\beta+1} \geq \frac{\alpha+2}{\alpha+1} \Longleftrightarrow \frac{\gamma+1}{\beta+1} \geq \frac{1}{\alpha+1}$$

In order to chech that we compute for $\gamma = \frac{\alpha}{2}(\beta+2)$

$$(2\gamma+2)(\alpha+1)-2(\beta+1) = (\alpha+1)(2\alpha+\alpha\beta+2)-2(\beta+1) \geq (\alpha+1)(\beta+2)-2(\beta+1)$$

$$= \alpha(\beta+2) - \beta \geq 0$$

Therefore in order to apply Lemma 4.1 we need only to assume

$$\frac{\alpha+2}{\alpha+1}\lambda \geq 1 + \frac{2}{\alpha} = \frac{\alpha+2}{\alpha}$$

Then the result follows immediately.

Theorem 4.3. *If* $\alpha < \frac{\beta}{\beta+2}$ *and*

$$f \in C(\mathbb{R}^+); \quad |f(t)| \leq Kt^{-\lambda}; \quad \lambda \geq \frac{(\alpha+1)(\beta+1)}{\beta-\alpha}$$

then there is a constant C depending boundedly on E(0) such that

$$\forall t \geq 1, \quad E(t) \leq Ct^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}}$$

Proof. In this case in (4.3) we take $\gamma = \frac{\beta-\alpha}{\alpha+1}$, so that

$$\gamma+1 = \frac{\beta+1}{\alpha+1} \Longrightarrow \frac{\gamma+1}{\beta+1} = \frac{1}{\alpha+1} \Longrightarrow \frac{\beta+\gamma+2}{\beta+1} = \frac{\alpha+2}{\alpha+1}$$

In order to apply Lemma 3.1 we now need

$$\frac{\alpha+2}{\alpha+1}\lambda \geq 1 + \frac{\beta+2}{\gamma}$$

and here

$$1 + \frac{\beta+2}{\gamma} = 1 + \frac{(\beta+2)(\alpha+1)}{\beta-\alpha} = \frac{\beta(\alpha+2)+2\alpha+2-\alpha}{\beta-\alpha} = \frac{(\beta+1)(\alpha+2)}{\beta-\alpha}$$

Hence the condition reduces to

$$\lambda \geq \frac{(\beta+1)(\alpha+1)}{\beta-\alpha}$$

5- Optimality results .

We start by an optimality result in the oscillatory range. It turns out that Corollary 1.2 gives an exact decay for both u and u', more precisely we have

Proposition 5.1. *Assume either (2.1) or (2.2). Then the results of Corollary 3.3 are optimal. More precisely any solution* $u \not\equiv 0$ *of (4) satisfies*

$$\limsup_{t\to+\infty} t^{\frac{2}{\alpha(\beta+2)}} u(t) > 0 \tag{5.1}$$

$$\limsup_{t\to+\infty} t^{\frac{1}{\alpha}} u'(t) > 0 \tag{5.2}$$

Proof. As a consequence of Theorem 2.1, there is a sequence $t_n \to +\infty$ such that

$$u(t_n) > 0, \quad u'(t_n) = 0$$

Then by the definition of the energy we find

$$u(t_n) = \{(\beta+2)E(t_n)\}^{\frac{1}{\beta+2}}$$

and (5.1) is a consequence of (1.3). Similarly there is a sequence $\tau_n \to +\infty$ such that

$$u(\tau_n) = 0, \quad u'(\tau_n) > 0$$

Then by the definition of the energy we find

$$u'(\tau_n) = \{2E(t_n)\}^{\frac{1}{2}}$$

and (5.2) is a consequence of (1.3).

In the non-oscillatory range, we have an equivalent valid for all positive *fast* solutions.

Theorem 5.2. *Assume* $\alpha < \frac{\beta}{\beta+2}$. *Then any solution* $u \not\equiv 0$ *of (4) satisfying (3.1) fulfill the following properties*

$$\lim_{t\to+\infty} t^{\frac{1-\alpha}{\alpha}} |u(t)| = \frac{\alpha}{1-\alpha}\left(\frac{1}{c\alpha}\right)^{\frac{1}{\alpha}} \tag{5.3}$$

$$\lim_{t\to+\infty} t^{\frac{1}{\alpha}} |u'(t)| = \left(\frac{1}{c\alpha}\right)^{\frac{1}{\alpha}} \tag{5.4}$$

Proof. It is sufficient to consider the case $u > 0, u' < 0$. Then equation (4) implies

$$u'' + c(-u')^\alpha u' = -u^{\beta+1} < 0$$

so that if $v := -u'$ we find

$$v' \geq -cv^{1+\alpha}$$

In addition for t large enough, using the fact that $\cos\theta(t)$ tends to 0, we have

$$|u(t)| \leq |u'(t)|^{\frac{2}{\beta+2}}$$

Hence

$$|u(t)|^{\beta+1} \leq |u'(t)|^{\frac{2(\beta+1)}{\beta+2}}$$

Now since $\frac{2(\beta+1)}{\beta+2} - (\alpha+1) = \frac{\beta}{\beta+2} - \alpha > 0$, we find

$$v' \leq (-c + \varepsilon(t))v^{1+\alpha}$$

with $\lim_{t\to+\infty} \varepsilon(t) = 0$. Then (5.4) follows easily. Finally (5.3) is immediate from (5.4) using the formula

$$u(t) = \int_t^\infty v(s)ds$$

Finally the result of Theorem 4.3 is also optimal. More precisely we have

Proposition 5.3. *Let us assume $\alpha < \frac{\beta}{\beta+2}$ and consider the two functions*

$$u_1(t) = \frac{1-\alpha}{\alpha}\Big(\frac{1}{\alpha}\Big)^{\frac{1}{\alpha}} t^{-\frac{1-\alpha}{\alpha}} \tag{5.5}$$

$$u_2(t) = \Big(\frac{\alpha+1}{\beta-\alpha}\Big)^{\frac{\alpha+1}{\beta+1}} t^{-\frac{\alpha+1}{\beta-\alpha}} \tag{5.6}$$

Then u_1 is a solution of (1) with $f(t) = K_1 t^{-\frac{(\beta+1)(1-\alpha)}{\alpha}}$ for some positive constant K displaying the maximum possible decay of the energy for solutions of (4), while u_2 is a solution of (1) with $f(t) = K_2 t^{-(\frac{\alpha+1}{\beta-\alpha}+2)}$ showing the optimality of the energy estimate of Theorem 4.3.

Proof. Actually the constants in (5.5) and (5.6) are ajusted in such a way that u_1 is a solution of

$$u_1'' + |u_1'|^\alpha u_1' = 0$$

therefore

$$u_1'' + |u_1'|^\alpha u_1' + |u_1|^\beta u_1 = |u_1|^\beta u_1 = K_1 t^{-\frac{(\beta+1)(1-\alpha)}{\alpha}}$$

while

$$|u_2'|^\alpha u_2' + |u_2|^\beta u_2 = 0$$

therefore

$$u_2'' + |u_2'|^\alpha u_2' + |u_2|^\beta u_2 = u_2'' = K_2 t^{-(\frac{\alpha+1}{\beta-\alpha}+2)}$$

Showing that f satisfies the decay hypothesis of Theorem 4.3 under the condition $\alpha < \frac{\beta}{\beta+2}$ is an easy matter and we skip the calculation.

6- Generalization .

In this Section we consider the problem

$$u'' + g(u') + \nabla F(u) = f(t) \tag{6.1}$$

where H is a Hilbert space , $u \in C^2(\mathbb{R}^+, H)$ and F, g fulfill the following conditions

$$g \in W^{1,\infty}(B_1, H); \quad F \in W^{2,\infty}(B_2, H) \tag{6.2}$$

where B_1, B_2 are two closed balls of H centered at 0. We denote by $\|u\|$ the norm of a vector $u \in H$ and by $\langle u, v\rangle$ the inner product of two vectors (u, v) of H. We assume that F, g satisfy the following properties for some positive constants $\alpha, \beta, \eta, \rho, M, P$.

$$\forall v \in B_1, \quad \langle g(v), v\rangle \geq \eta \|v\|^{\alpha+2} \tag{6.3}$$

$$\forall v \in B_1, \quad \|g(v)\| \leq M \|v\|^{\alpha+1} \tag{6.4}$$

$$\forall u \in B_2, \quad \langle \nabla F(u), u\rangle \geq \rho \|u\|^{\beta+2} \tag{6.5}$$

$$\forall u \in B_2, \quad |F(u)| \leq P \|u\|^{\beta+2} \tag{6.6}$$

Let $u \in C^2(\mathbb{R}^+, H)$ be a solution of (6.1) such that

$$\forall t > 0, \quad (u(t), u'(t)) \in B_2 \times B_1 \tag{6.7}$$

We introduce

$$E(t) = \frac{1}{2}u'^2(t) + F(u(t)) \tag{6.8}$$

We have the following generalizations of Theorems 4.2 and 4.3.

Theorem 6.1. *If* $\alpha \geq \frac{\beta}{\beta+2}$ *and*

$$f \in C(\mathbb{R}^+, H); \quad \|f(t)\| \leq Kt^{-\lambda}; \quad \lambda \geq 1 + \frac{1}{\alpha}$$

then there is a constant C depending boundedly on E(0) such that

$$\forall t \geq 1, \quad E(t) \leq Ct^{-\frac{2}{\alpha}}$$

Theorem 6.2. *If* $\alpha < \frac{\beta}{\beta+2}$ *and*

$$f \in C(\mathbb{R}^+, H); \quad \|f(t)\| \leq Kt^{-\lambda}; \quad \lambda \geq \frac{(\alpha+1)(\beta+1)}{\beta-\alpha}$$

then there is a constant C depending boundedly on E(0) such that

$$\forall t \geq 1, \quad E(t) \leq Ct^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}}$$

Sketch of the Proof. We introduce

$$F(t) := E(t) + \varepsilon\|u(t)\|^{\gamma}\langle u(t), u'(t)\rangle$$

By following the steps of the proof of Theorem 1.1, we find here by the proper choice of γ

$$F' \leq -\frac{\varepsilon}{2}(\|u'\|^{\alpha+2} + \|u\|^{\beta+\gamma+2}) + \langle f, u' + \varepsilon\|u\|^{\gamma}u\rangle$$

Then we follow the estimates of the proof of Theorem 4.2 to obtain the conclusion of Theorem 6.1, and those of the proof of Theorem 4.3 to obtain the conclusion of Theorem 6.2.

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