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MINIMAL EDGE-COVERINGS OF PAIRS OF SETS

by

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ABSTRACT. A new min-max theorem concerning bi-supermodular functions on pairs of sets is proved. As a special case, we derive an extension of (A. Lubiw's extension of) E. Győri's theorem on intervals, W. Mader's theorem on splitting off edges in directed graphs, J. Edmonds' theorem on matroid partitions, and an earlier result of the first author on the minimum number of new directed edges whose addition makes a digraph k-edge-connected. As another consequence, we solve the corresponding node-connectivity augmentation problem in directed graphs.

1. INTRODUCTION

In the last two decades a large number of general frameworks concerning sub- and supermodular functions have been developed such as polymatroids, submodular flows, lattice polyhedra, linking systems, polymatroidal flows, independent flows, kernel systems, Δ-matroids, etc. A general account on these models and their relationship can be found in [Schrijver, 1984]. There is a single most important feature of these models in common: each of them is described by a totally dual integral (TDI) linear system. This implies, loosely speaking, that the corresponding primal and dual linear programming problems have integer-valued optima for any primal cost-function.

This central property is an explanation why (apart from results including parity considerations) a great part of min-max theorems in graph theory and combinatorial optimization, especially those involving sub- or supermodular functions, are implied by the models above.

But not all! For example, K.P. Eswaran and R.E. Tarjan [1976] proved a min-max theorem for the minimum number of new edges whose addition makes a given directed graph strongly connected. They also observed that the minimum cost version of this problem is NP-complete as it includes the directed Hamiltonian circuit problem. Therefore, though the proof of this theorem is not difficult at all, it cannot be expected that a general model with a TDI describing system implies the result of Eswaran and Tarjan. Naturally, the same argument holds for a generalization given by Frank [1992] where the problem of optimally making a digraph k-edge-connected was solved.

From this point of view TDI-ness is too strong a property and it is highly desirable to develop general frameworks where integrality results hold only for a restricted class of cost functions while the problem for general costs may well include NP-complete problems.

The main purpose of the present paper is to introduce such a model. We are going to consider bi-supermodular functions and to prove a min-max theorem concerning minimum "coverings" of these functions by edges. Note that bi-submodular functions have been investigated earlier in a different context [A. Schrijver, 1979].

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Several consequences of the main theorem will be discussed. It implies a nice result of W. Mader [1983] on splitting off edges in a digraph that preserves edge-connectivity. A. Frank's [1992] above-mentioned result on edge-connectivity augmentation also follows. As a new result we provide a solution to the node-connectivity augmentation problem in digraphs by deriving a min-max formula on the minimum number of new edges whose addition makes a directed graph k-connected.

Finally, we show that an extension of E. Győri's [1984] famous min-max theorem on intervals is also an easy special case. This beautiful theorem has notoriously resisted so far to every attempt to relate it to other well-cultivated parts of combinatorial optimization. A. Lubiw writes in a paper [1988] generalizing Győri's theorem:

"In [Gy] Győri proved a min-max equality for intervals which is remarkable for the difficulty of the proof . . . and for the lack of similarity to previous min-max results in combinatorial optimization."

Our model does provide the missing link and also a simple proof. Actually, we are going to extend Lubiw's generalization in two senses as it will be explained in Section 6.

To conclude this introductory section we remark that the proof of the main theorem is simple and only the standard, hundred-times-used uncrossing technique is invoked. (A price we must pay for this simplicity is that the proof is not algorithmic. The only algorithm we have at present does use the main theorem, some ideas from its proof and relies on the ellipsoid method.)

The reader may ponder on this phenomenon: how come that a theorem, with a short and routine proof, may have those far from being trivial consequences? An explanation may be based on the new view we took: do not insist on TDI-ness, require integrality only for a restricted class of objective functions. We are hoping that this new look may also be successful in other areas and would like to encourage the readers (and ourselves, as well) to work out other general models in this vein.

The organization of the paper is as follows. In Section 2 we state and prove the main results. Section 3 includes degree-constrained and minimum cost variations as well as the description of a relationship of our model to contra-polymatroids. Algorithmic aspects are the topic of Section 4. Edge- and node-connectivity augmentation problems of directed graphs are discussed in Section 5. The last section is offered to deal with Győri's theorem on intervals and its extensions.

Let V be a set. A family of disjoint subsets of V is called a **sub-partition.** Two subsets X, Y of V is **co-disjoint** if $V - X$ and $V - Y$ are disjoint. X, Y are **intersecting** if none of $X - Y, Y - X, X \cap Y$ is empty. If, in addition, $V - (X \cup Y) \neq \emptyset$, X and Y are **crossing.** A family of subsets of V is called **cross-free** if it contains no two crossing sets.

We extend these notions to ordered pairs of sets and say that two pairs $(A, B), (A', B')$ are **half-disjoint** or **independent** if at least one of $A \cap B$ and $A' \cap B'$ is empty.

Let $\mathcal{A}$ denote the set of all ordered pairs of subsets of V. We define a partial order $P := (\mathcal{A}, \leq)$ on $\mathcal{A}$ as follows. For $(X, Y), (X', Y') \in \mathcal{A}$ let $(X, Y) \leq (X', Y')$ if $X \subseteq X'$ and $Y \supseteq Y'$. We say that two members of $\mathcal{A}$ are **comparable** if they are comparable in P. The restriction of P to a subset $\mathcal{F}$ of $\mathcal{A}$ will be denoted by $P(\mathcal{F})$.

The pairs $(A, B), (A', B')$ are called **non-crossing** if they are half-disjoint or comparable. Otherwise $(A, B), (A', B')$ are said to **cross** or to be **crossing.**

A family $\mathcal{F}$ of pairs of sets is **cross-free** if $\mathcal{F}$ contains no two crossing members. $\mathcal{F}$ is called **crossing** if for any two crossing members $(A, B), (A', B')$ of $\mathcal{F}$ both $(A \cap A', B \cup B')$ and $(A \cup A', B \cap B')$ belong to $\mathcal{F}$.

In a directed graph $G = (V, E)$ for a subset $X \subseteq V$, $\varrho(X)$ (respectively, $\delta(X)$) denote the number of edges entering (leaving) X. For a function $x : E \to \mathbf{R}$, let $\varrho_x(X) := \sum(x(e) : e \in E, e \text{ enters } X)$.

Let S and T be two disjoint sets and let $E := E(S, T)$ denote the set of all edges connecting elements of S and T. Let $\mathcal{E} := \mathcal{E}(S, T)$ denote the set of all pairs (X, Y) with $X \subseteq S, Y \subseteq T$. For an edge $e = st \in E$ and a pair $(X, Y) \in \mathcal{E}$ we say that e **covers** (X, Y) if $s \in X, t \in Y$. For a vector $z : E \to \mathbf{R}$ and a pair $(X, Y) \in \mathcal{E}$ we use the notation $z(X, Y) := \sum(z(xy) : x \in X, y \in Y)$.

Let $\mathcal{F}$ be a sub-family of $\mathcal{E}$. For a positive integer h we say that $\mathcal{F}$ is h**-independent** if every edge $e \in E$ covers at most h members of $\mathcal{F}$. For $h = 1$ we simply say **independent.** This is equivalent to requiring that the members of $\mathcal{F}$ are pairwise half-disjoint. More generally, we say that a function z on $\mathcal{E}$ is h**-independent** if $\sum(z(X, Y) : e \text{ covers } (X, Y)) \leq h$ holds for every $e \in E$.

Throughout the paper we adopt the convention that if a function f is defined explicitely only on some elements of $\mathcal{E}$, then we mean f to be 0 on all other elements of $\mathcal{E}$. For a function f on the subsets of $\mathcal{E}$, we use the notation $f(\mathcal{F}) := \sum f(X,Y) : (X,Y) \in \mathcal{F}$.

We call a set-function p **fully supermodular** (in short, supermodular) if

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y) \tag{1.1}$$

holds for every pair of subsets $X, Y \subseteq V$. If, in addition, p is non-negative and monotone increasing (that is, $p(X) \geq p(Y)$ whenever $Y \subseteq X$), then we speak of a **contra-polymatroid function.** For such a p, a polyhedron $Q := \{x \in \mathbf{R}_+^V : x(A) \geq p(A) \text{ for all } A \subseteq V\}$ is called a **contra-polymatroid.** It is known that Q uniquely determines p. In the next section we are going to extend these notions to functions defined on pairs of sets.

2. MAIN RESULTS

Let p be a non-negative integer-valued function defined on $\mathcal{E}$. p is said to be **crossing bi-supermodular** if

$$p(X,Y) + p(X',Y') \leq p(X \cap X', Y \cup Y') + p(X \cup X', Y \cap Y') \tag{2.1}$$

holds whenever $p(X,Y), p(X',Y') > 0$, $X \cap X', Y \cap Y' \neq \emptyset$.

If the reverse inequality holds in (2.1) we speak of **bi-submodular functions.** Bi-submodular functions were extensively studied by A. Schrijver [1979].

With the help of an edge $e = st$ we define a $0-1$ function on $\mathcal{E}$: $b_e(A,B) := 1$ if e covers (A,B) and 0 otherwise. The following claim is trivial but important.

CLAIM 2.1 *b_e is bi-submodular.* ♠

Let $x : E \to \mathbf{R}_+$ be a non-negative vector. Let us define a function $b_x : \mathcal{E} \to \mathbf{R}_+$ by $b_x(A,B) := x(A,B)$. From Claim 2.1 it follows immediately:

CLAIM 2.2 *b_x is bi-submodular.* ♠

We say that a non-negative vector x on E **covers** p or that x is a **covering** of p if $b_x \geq p$. We will be interested in integer-valued coverings minimizing cx for certain linear objective functions c.

Our main result is:

THEOREM 2.3 *For an integer-valued crossing bi-supermodular function p the following min-max equality holds. $\tau_p := \min(z(E) : z$ an integer-valued covering of $p) = \nu_p := \max(p(\mathcal{F})$: $\mathcal{F}$ independent).*

Proof. Clearly, $\nu_p \leq \tau_p$. To prove the other direction we are going to show that there is an integer-valued covering z and an independent $\mathcal{F}$ for which $z(E) = p(\mathcal{F})$.

We use induction on $\sum p(X,Y)$. If this sum is zero, then $p \equiv 0$ and hence $z = 0$ and $\mathcal{F} := \{(\emptyset, \emptyset)\}$ will do. Let now $p(A,B) > 0$ for a pair (A,B).

For any edge $e = ab \in E$ covering (A,B), define p', as follows. $p'(X,Y) := p(X,Y) - 1$ if $p(X,Y) > 0$ and $a \in X, b \in Y$ and $p'(X,Y) := p(X,Y)$ otherwise. Since b_e is bi-submodular, p' is crossing bi-supermodular. If $\nu_{p'} < \nu_p$, then using the inductive hypothesis, we obtain $\tau_p - 1 \leq \tau_{p'} = \nu_{p'} \leq \nu_p - 1 \leq \tau_p - 1$, from which $\tau_p = \nu_p$.

Therefore we may assume that for every edge $e = ab \in E$ covering (A,B) there is an independent family $\mathcal{F}_e$ for which $p(\mathcal{F}_e) = \nu_p$ and e does not cover any member of $\mathcal{F}_e$. For each $e \in E$ covering (A,B) let w_e be a 0-1 function on $\mathcal{E}$ so that w_e is 1 on the members of $\mathcal{F}_e$ and 0 otherwise. Let w_0 be 1 on (A,B) and 0 otherwise.

Define $w := w_0 + \sum(w_e : e \text{ covers } (A,B))$ and let $\mathcal{F}' := \{(X,Y) : w(X,Y) > 0\}$. Let $m := |A||B|$. Then

$$\sum(w(X,Y)p(X,Y) : (X,Y) \in \mathcal{E}) = \nu_p m + p(A,B) \geq \nu_p m + 1 \tag{2.2a}$$

and

$$w \text{ is } m- \text{ independent.} \tag{2.2b}$$

Assume that a w satisfying (2.2) is chosen such a way that $s(w) := \sum(w(X,Y)|X||Y| : (X,Y) \in \mathcal{E})$ is as small as possible.

CLAIM 2.4 *$\mathcal{F}'$ is cross-free.*

Proof. Suppose to the contrary that $\mathcal{F}'$ has two crossing members (X,Y) and (X',Y'). Revise w by decreasing its value on (X,Y) and (X',Y') by 1 and increasing it on $(X \cap X', Y \cup Y')$ and $(X \cup X', Y \cap Y')$ by 1. By Claim 2.1, the revised w' is also m-independent. w' clearly satisfies (2.2a) and $s(w') < s(w)$ contradicting the minimal choice of w. ♠

Recall the partial order $P' := P(\mathcal{F}')$ defined in the introduction. Because $\mathcal{F}'$ is cross-free, two members of $\mathcal{F}'$ are non-comparable in P' if and only if they are half-disjoint. By (2.2) there is no chain of P' with length greater than m. By a weighted version of the polar-Dilworth theorem (:*given a non-negative integer weighting w of a partially ordered set P', the maximum weight of a chain is equal to the minimum number of anti-chains covering all elements $v \in P'$ by $w(v)$ times*) it follows that there is an anti-chain of weight at least $\nu_p + 1$, contradicting the definition of ν_p. ♠♠♠

We can use the above min-max theorem to derive feasibility results concerning coverings. For a vector $x : E \to \mathbf{Z}_+$ and an element $v \in S \cup T$ we use the notation $d_x(v) := \sum(x(uv) : uv \in E)$.

For a subset Z of S, let $p_\nu^S(Z)$ denote $\max(p(\mathcal{F}) : \mathcal{F}$ an independent sub-family of $\mathcal{E}(Z,T))$. In particular, the value ν_p defined in Theorem 2.3 is equal to $p_\nu^S(S)$. For $Z \subseteq T$, $p_\nu^T(Z)$ is defined analogously. Finally, let $p_\nu(X \cup Y) := p_\nu^S(X) + p_\nu^T(Y)$ for $X \subseteq S$, $Y \subseteq T$.

Let $m : S \cup T \to \mathbf{Z}_+$ be a non-negative integer-valued vector so that

$$m(S) = m(T) = \gamma \tag{2.3}$$

THEOREM 2.5 *There exists a non-negative integer-valued covering x of p for which*

$$d_x(v) = m(v) \tag{2.4}$$

for every $v \in S \cup T$ if and only if

$$p_\nu(Z) \leq m(Z) \tag{2.5}$$

holds for every subset Z of S and of T.

Proof. Necessity. Let x be a covering satisfying (2.4) and Z a subset of, say, S. Then, clearly, $\gamma = x(E) = d_x(S) \geq m(S-Z) + p_\nu(Z) = \gamma - m(Z) + p_\nu(Z)$ and hence $m(Z) \geq p_\nu(Z)$, as required for (2.5).

Sufficiency. Let us define p' by modifying p as follows. $p'(v,T) := m(v)$ for every $v \in S$ and $p'(S,v) := m(v)$ for every $v \in T$, while $p'(X,Y) := p(X,Y)$ whenever $|X|, |Y| \geq 2$.

From (2.5) it follows that $p(v,T) \leq p_\nu(v) \leq \gamma - m(T-v) = m(v)$ for every $v \in S$ and analogously $p(S,v) \leq m(v)$ for every $v \in T$. Therefore $p' \geq p$. It can easily be seen that p' is crossing bi-supermodular. Apply Theorem 2.3 to p' and let x be a minimum covering of p'. Clearly, x is a covering of p, as well, and $\gamma' := x(E) \geq \gamma$.

If $\gamma' = \gamma$, then $\gamma' = \sum(d_x(v) : v \in S) \geq \sum(m(v) : v \in S) = \gamma$ and hence $d_x(v) = m(v)$ follows for every $v \in S$ and, analogously, for every $v \in T$, that is, x is a covering of p satisfying (2.4).

Suppose now that $\gamma' > \gamma$. By Theorem 2.3 there exists an independent family $\mathcal{F}'$ of members of $\mathcal{E}$ for which $p'(\mathcal{F}') = \gamma'$.

Since $\mathcal{F}'$ is independent, it cannot contain pairs of both forms (s,T) and (S,t) where $s \in S$ and $t \in T$. So suppose that $\mathcal{F}'$ does not contain pairs of form, say, (s,T). Let Z' denote the subset of T consisting of those elements t for which (S,t) belongs to $\mathcal{F}'$ and let $Z := T - Z'$. (Z' may be empty.) Let $\mathcal{F}_1 := \{(S,t) : t \in Z'\}$ and $\mathcal{F} := \mathcal{F}' - \mathcal{F}_1$. We have $p(\mathcal{F}) = p'(\mathcal{F}) = \gamma' - p'(\mathcal{F}_1) \geq \gamma' - m(Z') > \gamma - m(Z) = m(Z)$. Since $\mathcal{F}'$ is independent, $X \subseteq Z$ whenever $(X,Y) \in \mathcal{F}$. Hence $p_\nu(Z) \geq p(\mathcal{F}) > m(Z)$, contradicting (2.5). ♠♠♠

The following two corollaries immediately follow from Theorem 2.5.

COROLLARY 2.6 *Let $m : S \cup T \to \mathbf{Z}_+$ satisfy (2.3). There is a covering x of p satisfying (2.4) for every $v \in S \cup T$ if and only if there is one satisfying (2.4) for every $v \in S$ and there is one satisfying (2.4) for every $v \in T$.* ♠

COROLLARY 2.7 *Let $m_S : S \to \mathbf{Z}_+$ be a function on S. There is a covering x of p satisfying (2.4) for every $v \in S$ if and only (2.5) holds for every $Z \subseteq S$.*

Proof. Let x be a covering of p minimizing $x(E)$ and let $m_T := d_x(v)$ for every $v \in T$. Define $m : S \cup T \to \mathbf{Z}_+$ by $m(v) := m_S(v)$ if $v \in S$ and $m(v) := m_T(v)$ if $v \in T$. Now (2.5) holds for every $Z \subseteq S$ by the hypothesis of the corollary and for every $Z \subseteq T$ since x is a minimum covering. ♠♠♠

In applications we will use the following consequence of Theorem 2.3. Let V be a set and let A denote the set of all possible directed edges connecting two (not-necessarily) distinct elements of V. Let $\mathcal{A}$ denote the set of all ordered pairs (X,Y) with $X, Y \subseteq V$.

Let p' be a non-negative integer-valued function p' on $\mathcal{A}$ such that $p'(X,\emptyset) = p'(\emptyset,Y) = 0$ for every $X, Y \subseteq V$. We say that p' is **crossing bi-supermodular** if (2.1) holds whenever $X, Y \subseteq V$, $p'(X,Y), p'(X',Y') > 0$, $X \cap X', Y \cap Y' \neq \emptyset$. We can define the notions of covering and independence analogously to those concerning the bipartite case. Namely, a vector $z : A \to \mathbf{Z}_+$ is called a **covering** of p' if $\sum(z(xy) : x \in X, y \in Y) \geq p'(X,Y)$ holds for every $(X,Y) \in \mathcal{A}$. A subset $\mathcal{F}$ of $\mathcal{A}$ is **independent** if there is no element of A covering more than one member of $\mathcal{F}$. This is equivalent to saying that every two members of $\mathcal{F}$ are half-disjoint.

THEOREM 2.8 $\tau_{p'} := \min(x(E) : x$ *an integer-valued covering of* $p') = \nu_{p'} := \max(p'(\mathcal{F})$: $\mathcal{F}$ *independent).*

Proof. Let S and T be two disjoint copies of V. Define p by $p(X,Y) := p'(X',Y')$ where $X', Y' \subseteq V$, X is a subset of S corresponding to X' and Y is a subset of T corresponding to Y'. Now Theorem 2.3 immediately implies Theorem 2.8. ♠♠♠

Perhaps it is worthwile to re-formulate Theorem 2.8 in the special case when p' is $0-1$-valued. Let $\mathcal{L}$ be a crossing family of ordered pairs of subsets of a ground-set V so that $(X,Y) \in \mathcal{L}$ implies that $X, Y \neq \emptyset$. Clearly, if p' is a $0-1$-valued function on the pairs of subsets of V which is defined to be 1 on a pair (X,Y) precisely when $(X,Y) \in \mathcal{L}$, then p' is crossing bi-supermodular and Theorem 2.8 implies:

THEOREM 2.9 *Given a crossing family $\mathcal{L}$ of pairs of sets, the maximum cardinality of an independent sub-family of $\mathcal{L}$ is equal to the minimum number of directed edges covering all members of $\mathcal{L}$.* ♠

In the following special case of Theorem 2.8 $\nu_{p'}$ can be considerably simplified. Suppose that p' is the same as in the preceding theorem with the restriction that $p'(X,Y)$ may be positive only if X, Y is a bipartition of V. Note that such a function can be identified with a crossing supermodular function p'' defined on the subsets of V and we formulate the theorem so as to concern p''.

THEOREM 2.10 *Let p'' be a crossing supermodular function on the subsets of V with $p''(V) = 0$. Then* $\min(x(A) : x$ *integer-valued,* $\varrho_x(Y) \geq p''(Y)$ *for every* $Y \subseteq V) = \max(p''(\mathcal{F})$*:* $\mathcal{F}$ *is a family of pairwise disjoint or pairwise co-disjoint subsets of* $V)$.

Proof. Let $p'(X, Y) = p''(Y)$ if X, Y is a bipartition of V and zero otherwise. Clearly, p' is crossing bi-supermodular. Apply Theorem 2.8 and let $\mathcal{F}'$ denote an independent subset of $\mathcal{A}$ on which the maximum is attained. We may assume that p' is positive on each member of $\mathcal{F}'$. By the assumption on p'', for each member (X, Y) of $\mathcal{F}'$ is a bipartition of V. Define $\mathcal{F} := \{Y : (X, Y) \in \mathcal{F}'\}$. Then $\mathcal{F}$ is cross-free and no member of $\mathcal{F}$ includes another member. But then $\mathcal{F}$ consists of pairwise disjoint or pairwise co-disjoint sets. Hence Theorem 2.8 implies the result. ♠♠♠

Parallel to the way how Theorem 2.3 implies Theorem 2.5, one can easily derive the corresponding feasibility forms from Theorems 2.8 and 2.10. Let $m_I : V \to \mathbf{Z}_+$ and $m_O : V \to \mathbf{Z}_+$ be two integer-valued function on V for which $\gamma := m_I(V) = m_O(V)$.

THEOREM 2.11 *Let p' be a non-negative integer-valued crossing bi-supermodular function on $\mathcal{A}$. There exists a directed graph $G = (V, E)$ for which*

$$\varrho(v) = m_I(v) \tag{2.6a}$$

and

$$\delta(v) = m_O(v) \tag{2.6b}$$

for every $v \in V$ and for which (*) there are at least $p'(X, Y)$ edges with tail in X and head in Y for every $(X, Y) \in \mathcal{A}$ if and only if for every $Z \subseteq V$

$$m_I(Z) \geq p'(\mathcal{F}) \tag{2.7a}$$

holds for every independent family $\mathcal{F} \subseteq \mathcal{A}$ so that $(X, Y) \in \mathcal{F}$ only if $Y \subseteq Z$ and

$$m_O(Z) \geq p'(\mathcal{F}) \tag{2.7b}$$

holds for every independent family $\mathcal{F} \subseteq \mathcal{A}$ so that $(X, Y) \in \mathcal{F}$ only if $X \subseteq Z$. In particular, if there is a digraph satisfying (*) and (2.6a) and there is a digraph satisfying (*) and (2.6b), then there is one satisfying (*) and both (2.6a) and (2.6b). ♠

THEOREM 2.12 *Let p'' be a non-negative integer-valued crossing supermodular function on the subsets of V with $p''(V) = 0$. There exists a directed graph $G = (V, F)$ for which*

$$\varrho(v) = m_I(v) \tag{2.8a}$$

and

$$\delta(v) = m_O(v) \tag{2.8b}$$

for every $v \in V$ and for which

$$\varrho(X) \geq p''(X) \tag{2.9}$$

for every $X \subseteq V$ if and only if for every $Z \subseteq V$

$$m_I(Z) \geq p''(Z) \tag{2.10a}$$

and

$$m_O(V - Z) \geq p''(Z). \tag{2.10b}$$

In particular, if there is a digraph satisfying (2.9) and (2.10a) and if there is a digraph satisfying (2.9) and (2.10b), then there is one satisfying (2.9) and both (2.10a) and (2.10b). ♠

REMARK Since crossing supermodular functions can be considered as special cases of crossing bi-supermodular functions, the primal side of Theorem 2.12 is a special case of that of Theorem 2.11. The benefit of this speciality is that condition (2.10) in Theorem 2.12 is much simpler than (2.7). Namely, in (2.7) an inequality is required for every subset Z and family $\mathcal{F}$, while (2.10) concerns only subsets Z.

3. DEGREE CONSTRAINTS AND NODE COSTS

At the end of the previous section we have proved theorems concerning coverings satisfying prescriptions on the degrees of nodes. Relying on these results, we study now the extension when the degrees of a covering are required to satisfy lower and upper bound constraints. The ground for the generalization is that the set of degree-vectors of coverings, as we will prove it, forms a contra-polymatroid.

Let $S, T, E, \mathcal{E}, p$ be the same as in Theorem 2.3 and recall the definition of p_ν given before Theorem 2.5. In this section we exhibit a relationship between coverings of p and contra-polymatroids. As a result we will be able to handle degree-constrained coverings as well as minimum-cost coverings provided that the cost-function on E is induced by a cost-function on $S \cup T$.

Let V be a ground-set and q an (integer-valued) contra-polymatroid function. A **contra-polymatroid** in $\mathbf{R}_+^V$ is a polyhedron

$$C(q) := \{x : x(X) \geq q(X) \text{ for every } X \subseteq S\}. \tag{3.1}$$

It is well-known that a contra-polymatroid uniquely determines its defining contra-poly-matroid function q (namely, $q(X) = \min(x(X) : x \in C(q))$. (A more general class of polyhedra, g-polymatroids, was studied in [Frank and Tardos, 1988]. For a relationship of contra-polymatroids and edge-connectivity augmentations, see [Frank, 1992].)

THEOREM 3.1 *p_ν is a contra-polymatroid function.*

Proof. Clearly, p_ν is non-negative, monotone increasing and zero on the empty set. The main content of the theorem is that p_ν is fully supermodular. By the definition of p_ν, it suffices to prove (1.1) only for $X, Y \subseteq S$ and for $X, Y \subseteq T$. Because the two cases are analogous, we may assume that $X, Y \subseteq S$.

For a subset Z of S let $\mathcal{F}_Z$ denote an independent sub-family of $\mathcal{E}(Z,T)$ for which $p(\mathcal{F}_Z) = p_\nu(Z)$. Let X and Y be two subsets of S. Then $\mathcal{F} := \mathcal{F}_X \cup \mathcal{F}_Y$ is a sub-family of $\mathcal{E}$ such that $p(\mathcal{F}) = p_\nu(X) + p_\nu(Y)$ and for every $t \in T$, an edge st (i) covers at most two members of $\mathcal{F}$ whenever $s \in X \cap Y$, (ii) covers at most one member of $\mathcal{F}$ whenever $s \in X \cup Y - X \cap Y$, and (iii) covers no members of $\mathcal{F}$ whenever $s \in S - (X \cup Y)$.

Let us assume that $\mathcal{F}$ is a sub-family of $\mathcal{E}$ for which $p(\mathcal{F}) \geq p_\nu(X) + p_\nu(Y)$, conditions (i), (ii), (iii) are satisfied and $s(\mathcal{F}) := \sum(|A||B| : (A,B) \in \mathcal{F})$ is as small as possible. We claim that $\mathcal{F}$ is cross-free. Indeed, if two members (A,B) and (A',B') of $\mathcal{F}$ are crossing, then replace these two pairs by $(A \cap A', B \cup B')$ and $(A \cup A', B \cap B')$. For the resulting family $\mathcal{F}'$, $p(\mathcal{F}') \geq p(\mathcal{F})$, $\mathcal{F}'$ satisfies (i), (ii), (iii) and $s(\mathcal{F}') < s(\mathcal{F})$, contradicting the minimal choice of $\mathcal{F}$. Because $\mathcal{F}$ is cross-free, any two non-comparable elements of $\mathcal{F}$ are half-disjoint.

Let $\mathcal{F}_1$ consist of the minimal elements of the partial order $P' := P(\mathcal{F} \cap \mathcal{E}(X \cap Y, T))$. Let $\mathcal{F}_2 := \mathcal{F} - \mathcal{F}_1$.

The family $\mathcal{F}_1$ is independent since if it has two comparable elements, then the larger one is not minimal in P', contradicting the definition of $\mathcal{F}_1$. It follows that $p_\nu(X \cap Y) \geq p(\mathcal{F}_1)$.

We claim that $\mathcal{F}_2$ is independent, as well. Suppose indirectly, that for two members of $\mathcal{F}_2$, $(A,B) > (A',B')$. Since $A' \subseteq A$, (ii) implies that $A' \subseteq (X \cap Y)$. Since (A',B') is not in $\mathcal{F}_1$, there is a member (A'',B'') of $\mathcal{F}_1$ for which $(A'',B'') < (A',B')$. But then the existence of these three pairs contradicts (i).

Using (iii), we obtain that $p_\nu(X \cup Y) \geq p(\mathcal{F}_2)$ and hence $p_\nu(X \cap Y) + p_\nu(X \cup Y) \geq p(\mathcal{F}_1) + p(\mathcal{F}_2) = p(\mathcal{F}) \geq p_\nu(X) + p_\nu(Y)$, as required. ♠♠♠

From the definition of p_ν it follows that the contra-polymatroid $C := C(p_\nu)$ is the direct sum of contra-polymatroids $C_S := C(p_\nu^S)$ and $C_T := C(p_\nu^T)$. Our next purpose is to show a relationship between coverings of p and contra-polymatroids.

THEOREM 3.2 (a) *An integer-valued vector $m_S : S \to \mathbf{Z}_+$ belongs to C_S if and only if there is an (integer-valued) covering x of p for which $m_S(s) = d_x(s)$ for every $s \in S$.* (b) *An integer-valued vector $m : S \cup T \to \mathbf{Z}_+$ belongs to C if and only if there is an (integer-valued) covering z of p for which $m(v) = d_z(v)$ for every $v \in S$ and $m(S) = m(T)$.*

Proof. (a) Suppose for a covering z of p that $m_S(s) = d_z(s)$ for every $s \in S$. Then for every $X \subseteq S$, $m_S(X) = z(E(X,T)) \geq p_\nu(X)$ and hence $m_S \in C_S$.
Conversely, let m_S be an integer-valued element of C_S. Then the statement follows from Corollary 2.7.
Part (b) follows from Part (a) and from Corollary 2.6. ♠♠♠

Let q_1 be a fully supermodular function on the subsets of a set V_1 and $C_1 := C(q_1)$ a contra-polymatroid.
We are given two non-negative integer-valued functions $f_1 : V_1 \to \mathbf{Z}_+$ and $g_1 : V_1 \to \mathbf{Z}_+ \cup \{\infty\}$ for which $f_1 \leq g_1$. The following result occurs in a more general form in [Frank and Tardos, 1986] and more concretely in [Frank, 1992, (Proposition 6.9)].

LEMMA 3.3 (a) *C_1 has an integer-valued element m_1 for which $f_1 \leq m_1 \leq g_1$ and $m_1(V_1) = \gamma$ if and only if $\gamma \leq g_1(V_1)$ and $q_1(X) \leq \min(\gamma - f_1(V_1 - X), g_1(X))$ holds for every $X \subseteq V_1$.*
(b) *C_1 has an integer-valued element m_1 for which $f_1 \leq m_1 \leq g_1$ if and only if $q_1(X) \leq \min(g_1(V_1) - f_1(V_1 - X), g_1(X))$ holds for every $X \subseteq V_1$.* ♠

By combining these results, it is possible to handle degree-constrained and minimum-cost versions of the covering problem. First, let us be given two non-negative integer-valued functions $f : S \cup T \to \mathbf{Z}_+$, $g : S \cup T \to \mathbf{Z}_+ \cup \{\infty\}$ for which $f \leq g$. Let γ be a positive integer.

THEOREM 3.4 *There exists a non-negative integer-valued covering z of p for which (a) $z(E) = \gamma$ and*

$$f(v) \leq d_z(v) \leq g(v) \tag{3.2}$$

for every $v \in S \cup T$ if and only if

$$p_\nu(Z) \leq g(Z) \tag{3.3}$$

holds for every subset Z of S and of T,

$$\gamma \leq \min(g(S), g(T)) \tag{3.4}$$

and

$$p_\nu(Z) + f(S - Z) \leq \gamma \text{ for every } Z \subseteq S, \tag{3.5a}$$

$$p_\nu(Z) + f(T - Z) \leq \gamma \text{ for every } Z \subseteq T. \tag{3.5b}$$

(b) *There exists a non-negative integer-valued covering z of p satisfying (3.2) if and only if (3.3) holds for every $Z \subseteq S$ and $Z \subseteq T$ and*

$$p_\nu(Z) + f(S - Z) \leq \min((g(S), g(T)) \text{ for every } Z \subseteq S, \tag{3.6a}$$

$$p_\nu(Z) + f(T - Z) \leq \min(g(S), g(T)) \text{ for every } Z \subseteq T. \tag{3.6b}$$

♠♠♠

We mentioned earlier that there is no hope to obtain min-max results for the general minimum cost version of the covering problem since a special case, finding a minimum cost strongly connected augmentation of a digraph, is NP-complete. However, for a special class of cost functions such a characterization exists.

Let us be given a non-negative cost-function $c : S \cup T \to \mathbf{R}_+$ on the nodes. This function induces a cost function on the edges by the rule $c(st) := c(s) + c(t)$ for $st \in E$. (It will not cause ambiguity that we use c to denote both functions.) Let p be a bi-supermodular function on the subsets of S, T.

THEOREM 3.5 *For a node-induced cost function c the linear program*

$$\min(cz : z : E \to \mathbf{R}_+, z(X,Y) \geq p(X,Y), \text{ for every } (X,Y) \in \mathcal{E}) \tag{3.7a}$$

has an integer-valued optimum. If, in addition, c is integer-valued, the dual linear program

$$\max(\sum(w(X,Y)p(X,Y) : (X,Y) \in \mathcal{E}) : \ w : \mathcal{E} \to \mathbf{R}_+, \tag{3.7b}$$

$$\sum(w(X,Y) : x \in X, y \in Y) \leq c(xy) \text{ for every } xy \in E) \tag{3.7b}$$

also has an integer-valued optimum.

Proof. Recall the definition of the contra-polymatroid $C := C(p_\nu)$. It is known from polymatroid theory that the system (3.1) is totally dual integral. Therefore there is an integer vector $x \in C$ and there is a dual variable $y \in \mathbf{R}_+^{S \cup T}$ (integer-valued, if c is integer-valued) so that $\sum(y(Z)p_\nu(Z) : v \in Z \subseteq S \cup T) \leq c(v)$ for every $v \in S \cup T$ and $cx = \sum y(Z) : Z \subseteq S \cup T$.

Since C is the direct sum of C_S and C_T, we may assume that $y(Z)$ is positive only if $Z \subseteq S$ or $Z \subseteq T$. By Theorem 3.2 there is a covering z_0 of p for which $d_{z_0}(v) = x(v)$ for every $v \in S \cup T$.

By definition, for each $Z \subseteq S$ (resp., $Z \subseteq T$) there is an independent sub-family $\mathcal{F}_Z$ of $\mathcal{E}(Z,T)$ (resp., $\mathcal{E}(S,Z)$) so that $\nu_p(Z) = p(\mathcal{F}_Z)$.

Define $w_0(A,B) := \sum(y(Z) : (A,B) \in \mathcal{F}_Z, Z \subseteq S$ or $Z \subseteq T)$. An easy calculation shows that w_0 is a solution to (3.7b) and that $\sum(w_0(X,Y)p(X,Y) : (X,Y) \in \mathcal{F} = cz_0)$ showing that z_0 is a primal optimum and w_0 is a dual optimum. ♠♠♠

4. ALGORITHMIC ASPECTS

How can we construct an optimal (integer-valued) covering of a bi-supermodular function? The proof of the main Theorem 2.3 includes non-constructive parts and at the time being we do not know any other proof (even for the consequence Theorem 2.9) that may give rise to a polynomial-time algorithm. (Note however, that, relying on contra-polymatroids, there is a combinatorial algorithmic approach to Theorems 2.10 and 2.12.) Because of the applications we are going to describe in the next two sections, it would be highly desirable to develop a constructive proof for Theorem 2.3.

In order to indicate the level of difficulties, here we just mention that Edmonds' well-known theorem [1965] on partition of matroids follows from our model. Indeed, let us be given k matroids M_i on a common ground-set S with rank-function r_i and consider the problem of finding k disjoint bases, one from each matroid. This is equivalent to requiring that S can be partitioned into k generators where a **generator** is a set including a bases. Edmonds' theorem asserts that this is possible if and only if

$$\sum p_i(Z) \leq |Z| \tag{4.1}$$

holds for every subset Z of S where $p_i(Z) := |Z| - r_i(Z)$. (That is, $p_i(Z)$ denotes the minimum cardinality of the intersection of Z and a basis of matroid M_i.)

To derive Edmonds' theorem, let $T := \{t_1, \ldots, t_k\}$ be a set of k elements and define a function p as follows. For $X \subseteq S, Y \subseteq T$, $p(X,Y) := p_i(X)$ if $Y = \{t_i\}$ for some $i = 1, \ldots, k$ and $:= 0$ otherwise. This p is a crossing bi-supermodular function. Define $m_S \equiv 1$. Now the k disjoint generators exist if and only if there is an integer-valued covering z of p so that $d_z(s) = 1$ for every $s \in S$. By Corollary 2.7, this is the case if and only if $p_\nu(Z) \leq |Z|$ holds for every $Z \subseteq S$. Since p_i is monotone increasing, $p_\nu(Z) = \sum p_i(Z)$, and Edmonds' theorem follows. ♠

On the other hand, the proof of Theorem 2.3 includes some constructive elements in the sense that it gives rise to an algorithm if certain oracles are available. Here we briefly outline these elements and show that, via the ellipsoid method, a polynomial time algorithm exists to construct a minimum integer-valued covering of p. As for the dual problem ((4.3) below) is concerned, unfortunately we cannot offer any polynomial-time algorithm for the general case. Such an algorithm, however, does exist in the special case when the number of pairs (X, Y) for which $p(X, Y) > 0$ is bounded by a polynomial of $m := |S||T|$. The extension of Győri's theorem, to be discussed in Section 6, is deduced from such a special case.

As is typically the case with combinatorial applications of the ellipsoid method, the purpose is only to prove the existence of a polynomial-time algorithm and we do not think that our algorithm might have any practical use. The existence of such an algorithm may serve as an encouragement to construct purely combinatorial, more efficient polynomial time algorithms for the covering problem.

Actually, we will use the ellipsoid method on two levels. First, we will need an oracle to minimize submodular functions and the only known polynomial time algorithm for such a routine uses the ellipsoid method. (Naturally, there may be special cases when the submodular function minimization in question can be done combinatorially. The applications in the forthcoming sections are of this type.) Second, we use the fact, proved by the ellipsoid method, that the optimization problem is solvable if a separation oracle is available.

Theorem 2.3 asserts that the following pair of dual linear programs have integer-valued optima:

$$\min(z(E) : z : E \to \mathbf{R}_+, z(X,Y) \geq p(X,Y), \text{ for every } (X,Y) \in \mathcal{E}) \tag{4.2}$$

$$\max(\sum(w(X,Y)p(X,Y) : (X,Y) \in \mathcal{E}) : \; w : \mathcal{E} \to \mathbf{R}_+,$$
$$\sum(w(X,Y) : x \in X, y \in Y) \leq 1 \text{ for every } xy \in E). \tag{4.3}$$

SEPARATION PROBLEM

In order to apply the ellipsoid method one has to be able to solve the following separation problem. Given vector $z : E \to \mathbf{R}_+$, decide if z is a covering of p and if not, determine a pair (X, Y) for which $z(X, Y) < p(X, Y)$. To solve this problem, for each edge $e = st \in E(S, T)$ define $p_e(X, Y)$ to be $p(X, Y)$ if e covers (X, Y) and zero otherwise. Clearly, p_e is bi-supermodular and z is a covering of p if and only if z is a covering of each p_e. Therefore it suffices to solve the separation problem separately for each possible edge e.

Let us fix now $e = st$ and define a set-function p'_e on the subsets of $S \cup T$ as follows. For $X \subseteq S, Y \subseteq T$, let $p'_e(X \cup (T - Y)) := p_e(X, Y)$ if $p_e(X, Y) > 0$ and $-\infty$ otherwise. Then p'_e is fully supermodular set-function. Since d_z is a fully submodular, so is $b := d_z - p'_e$. Now z is a covering of p_e if and only if b is non-negative. Therefore if we can minimize the submodular function b, we can decide if z is a covering of p.

We note that the general submodular function minimization problem is solvable via the ellipsoid method [Grötschel, Lovász, Schrijver, 1988].

ALGORITHM FOR THE PRIMAL OPTIMUM

It is well-known (Grötschel, Lovász, Schrijver 1987, Theorem 6.4.1) that with the help of the above separation algorithm the ellipsoid method provides fractional optimal solutions to (4.2) and (4.3), that is, we are able to compute an optimal fractional covering and an optimal fractional solution to the dual problem, both having value ν_p.

How can we determine an integer-valued primal optimum? This consists of two parts. First, we describe an algorithm whose complexity is proportional to the maximum value of p and polynomial in $m := |S||T|$. Therefore this is a polynomial-time algorithm only if the maximum value of p is a bounded by a power of m. In the second part we show how the general covering problem can be reduced in polynomial time to another covering problem where the maximum value of the defining bi-supermodular function is at most m.

The idea behind the first part of the algorithm comes from the proof of Theorem 2.3. We consider all edges in $E(S,T)$ in an arbitrarily specified order and compute $z(e)$ for the currently considered edge e. At the beginning $z \equiv 0$.

Choose the first edge $e = st$. In an elementary step of the procedure we describe how to increase $z(e)$ by 1. Define p' as in the proof of Theorem 2.3, that is, $p'(X,Y) := p(X,Y) - 1$ if $p(X,Y) > 0$, e covers (X,Y) and $p'(X,Y) := p(X,Y)$, otherwise. With the help of the ellipsoid method, compute the optimum values ν_p and $\nu_{p'}$. Clearly, $\nu_p - 1 \leq \nu_{p'} \leq \nu_p$. If (*) $\nu_p - 1 = \nu_{p'}$, then revise $z(e) := z(e) + 1$ and iterate this elementary step re-starting with the same edge e and with the revised function $p := p'$ as long as (*) is true. Then the current $z(e)$ is declared final and we proceed with the subsequent edge in the given ordering of edges by iterating the above procedure.

Since this algorithm is nothing but the repeated application of the elementary steps applied in the proof of Theorem 2.3, the final z will be an optimal integer-valued covering of p.

Note that each elementary step requires the computation of the optimal value of the current covering problem (which is done by the ellipsoid method, as outlined above). How many elementary steps are required at worst case? Let M denote the maximum value of p. For a fixed edge e we need at most M elementary steps, the number of total elementary steps is at most Mm.

Finally, let us describe how this algorithm can be used to get rid of the maximum value M of p in the complexity. Let x be an optimal fractional solution to the covering problem, provided by the ellipsoid method. Let z_i denote the componentwise integer-part of x, that is, for every $e \in E(S,T)$, $z_i(e) := \lfloor x(e) \rfloor$ and let $x' := x - z_i$.

Define p' as follows. $p'(X,Y) := \max(0, p(X,Y) - z_i(X,Y))$. Now p' is crossing bi-supermodular and its maximum value M' is at most m. Applying the method described above, we can find an optimal integer-valued covering z' of p' in no more than $mM' \leq m^2$ elementary steps. Since x' is a fractional covering of p', Theorem 2.3 ensures that $z'(E) \leq x'(E)$. Moreover, $z := z_i + z'$ is clearly an integer-valued covering of p and $z(E) \leq x(e)$. Since x is an optimal covering, $z(E) = x(E)$ and hence z is an optimal integer-valued covering of p.

ALGORITHM FOR THE DUAL OPTIMUM

As we have mentioned we do not have any polynomial-time algorithm to compute an integer-valued optimum to the dual program in (4.3). A natural approach would be the following.

Using the uncrossing technique described in the proof of Theorem 2.3, we can construct an optimal dual solution in which pairs of subsets whose dual variable is positive form a cross-free family $\mathcal{F}'$.

Let us consider the partial order $P' = P(\mathcal{F}')$ mentioned in the proof of Theorem 2.3. The available dual solution shows that there is an assignment of non-negative values $y(X,Y)$ to the elements (X,Y) of P' so that the y-weight of every chain of P' is at most 1 and so that $\sum(y(X,Y)p(X,Y) : (X,Y) \in \mathcal{F}') = \nu_p$. Since a partially ordered set determines a perfect graph, there is an antichain of P' for which the total p-weight is at least ν_p. Such an antichain corresponds to an integer-valued dual solution of our original covering problem. It is well-known that, using network flow techniques, the maximum weight antichain of a partially ordered set can be computed in polynomial time.

The bottleneck in this approach is that, though the uncrossing procedure can be shown to terminate in finite time, we are not able to prove that this procedure is of polynomial complexity. In other words, one can construct an optimal integer-valued dual solution in polynomial time if one is able to find a cross-free optimal dual solution.

In the special case when the number f of pairs (X,Y) for which $p(X,Y) > 0$ is bounded by a power of m the following primitive brute force algorithm is of polynomial complexity.

For any subset $F \subseteq E(S,T)$ we can consider a reduced problem defined by p_F where $p_F(X,Y) := p(X,Y)$ if F does not cover (X,Y) and :=0, otherwise. By Claim 2.1, p_F is crossing bi-supermodular and hence with the PRIMAL ALGORITHM we can compute ν_{p_F} in polynomial time.

The idea behind the algorithm is the easy observation that a pair (X, Y) belongs to an optimal independent family $\mathcal{F}$ of pairs of subsets (in Theorem 2.3) if and only if $\nu_p = p(X, Y) + \nu_{p_F}$ where $F := E(X, Y)$.

Therefore we test this equality for each (X, Y) with positive $p(X, Y)$. Since there are f such pairs, after at most f applications of the PRIMAL ALGORITHM we find a pair (X_1, Y_1) for which the equality holds. We then iterate the same procedure by starting with the reduced problem defined by p_F where $F := E(X_1, Y_1)$. Since an independent family may have at most m members, altogether we need the PRIMAL ALGORITHM no more than m^2 times.

5. AUGMENTING CONNECTIVITY OF DIGRAPHS

In this section we show how the general framework developed in the preceding sections can be applied to solve the connectivity augmentation problem in directed graphs.

Let $D = (V, E)$ be a digraph with possible parallel edges. The **local edge-connect-ivity** (respectively, **local node-connectivity)** from x to y, denoted by $\lambda(x, y; D)$ (respectively, $\kappa(x, y; D)$), is the maximum number of pairwise edge-disjoint (internally node-disjoint) paths from x to y. We say that D is **k-edge-connected** (respectively, **k-node-connected**) if $\lambda(x, y; D) \geq k$ ($\kappa(x, yD) \geq k$) holds for every ordered pair of nodes (x, y) of D, that is, if each local edge-connectivity (node-connectivity) is at least k. When $k = 1$, the term **strongly connected** is used.

By the directed edge-version of Menger's theorem [in: Ford and Fulkerson, 1962] $\lambda(x, y) \geq k$ if and only if $\delta(X) \geq k$ holds for every subset X with $x \in X \subseteq V - y$. It follows that a digraph is k-edge-connected if and only if $\delta_G(X) \geq k$ holds for every non-empty proper subset X of V.

The feasibility form of the edge-connectivity augmentation problem consists of finding a set of new edges that satisfies in-degree and out-degree prescriptions at the nodes and whose addition to D leaves a k-edge-connected digraph. W. Mader [1982] answered this question showing that a natural necessary condition is sufficient as well. (Actually, Mader's original theorem is formulated in terms of splitting off edges but his result can easily be reformulated so as to answer the feasibility problem, see below.)

The minimization form of the edge-connectivity augmentation problem for digraphs consists of determining the minimum number of new directed edges whose addition to D leaves a k-edge-connected graph. This problem was solved in [Frank, 1992] by invoking Mader's theorem and the theory of polymatroids. The solution includes a min-max theorem as well as a (combinatorial) strongly polynomial time algorithm to find the extrema in question.

One of our purposes here is to show that the above-mentioned theorems concerning both forms of the edge-connectivity augmentation problem in digraphs easily follow from the general framework.

An even more important goal of the present section is to show that the corresponding node-connectivity augmentation problems can also be solved with the help of our general model. When $k = 1$, k-edge-connectivity and k-node-connectivity coincide. The minimization problem for this case was solved by K.P. Eswaran and R.E. Tarjan [1976] while the feasibility form follows from Mader's result. For larger k only very little was known. The minimization form was solved by [Masuzawa et al., 1987] when the starting digraph D is an arborescence (that is, a directed tree in which every node is reachable from a root.) T. Jordán [1993b] described a (combinatorial) polynomial time approximation algorithm to augment the node-connectivity of a digraph from k to $k + 1$ and proved that the augmentation of his algorithm uses at most k more edges than the optimum.

Naturally, the analogous questions can be posed to concern undirected graphs. As far as edge-connectivity is concerned, the minimization problem was solved first by T. Nakamura and A. Watanabe [1987]. In [Frank, 1992] a generalization was solved when the desired local edge-connectivities are arbitrary prescribed. This was even further generalized in [Bang-Jensen, Frank, Jackson, 1993] where a generalization concerning mixed graphs was described.

One of the major open questions of the area is to decide if the node-connectivity augmentation problem for undirected graphs belongs to co-**NP** or even to **P** or else it is **NP** -complete. The problem is polynomially solvable for $k = 1$ (trivial), for $k = 2$ [Eswaran, Tarjan, 1976], for $k = 3$ [Watanabe and Nakamura, 1988], [Hsu and Ramachandran, 1991] and for $k = 4$ [Hsu, 1992]. For higher k, the **NP** -completeness status is not known even for the special case when we want to increase the node-connectivity only by one. For that problem T. Jordán [1993a] developed an approximation algorithm that provides an augmenting set of edges whose cardinality is at most $k-2$ larger than the optimum. We only mention these developments just for providing a more general picture: the present paper has nothing to say concerning node-connectivity augmentations in undirected graphs.

Let us turn to edge-connectivity augmentations of digraphs. Let D be a digraph. **Splitting off** a pair of edges $e = us, f = st$ of D means that we replace e and f by a new edge ut. The resulting digraph will be denoted by D^{ef}. The following important result concerning splittings is due to W. Mader [1982]:

THEOREM 5.1 *Let $D = (V+s, A)$ be a directed graph for which $\lambda(x,y;D) \geq k$ for every $x,y \in V$ and $\varrho(s) = \delta(s)$. Then for every edge $f = st$ there is an edge $e = us$ so that $\lambda(x,y;D^{ef}) \geq k$ for every $x,y \in V$.* ♠

By repeated applications one gets:

THEOREM 5.1A *Let $D = (V+s, A)$ be a directed graph for which $\lambda(x,y;D) \geq k$ for every $x,y \in V$ and $\varrho(s) = \delta(s)$. Then the edges entering and leaving s can be partitioned into $\varrho(s)$ pairs so that splitting off all these pairs leaves a k-edge-connected digraph.*

An equivalent form is the following:

THEOREM 5.2 *A directed graph $D = (V,E)$ can be made k-edge-connected by adding a set F of new edges satisfying*

$$\varrho_F(v) = m_{in}(v) \text{ and } \delta_F(v) = m_{out}(v) \tag{5.1}$$

for every node $v \in V$ if and only if both

$$\varrho(X) + m_{in}(X) \geq k \text{ and } \delta(X) + m_{out}(X) \geq k \tag{5.2}$$

hold for every $X \subseteq V$.

(To see that this theorem is nothing but a reformulation of Theorem 5.1A, extend D by a new node s and for each $v \in V$ adjoin $m_{in}(v)$ (respectively, $m_{out}(v)$) parallel edges from s to v (from v to s). Now by (5.2) the hypotheses of Theorem 5.1A are satisfied and hence we can split off γ pairs of edges to obtain a k-edge-connected digraph. The resulting set of γ new edges (connecting original nodes) satisfies the requirement.)

Proof. Define $p''(X) = \max(0, k - \varrho_D(X))$ if $X \neq V, \emptyset$ and $p''(V) = p''(\emptyset) = 0$. This function is crossing-supermodular so Theorem 2.12 applies and it just specializes to Theorem 5.2. ♠

The following theorem of A. Frank [1992] provides an answer to the minimization form of the edge-connectivity augmentation problem.

THEOREM 5.3 *Given a directed graph $D = (V,E)$, the minimum number of edges whose addition to D leaves a k-edge-connected digraph is equal to the maximum of γ_i and γ_o where*

$$\gamma_i = \max(\sum(k - \varrho(X_i))) \text{ and } \gamma_o = \max(\sum(k - \delta(X_i))) \tag{5.3}$$

where both maxima are taken over all sub-partitions $\{X_1, \ldots, X_t\}$ of V.

Proof. If we define p'' as before, then Theorem 2.10 specializes to Theorem 5.3. ♠

REMARK Note that the original proof of Theorem 5.3 consists of showing how the minimization problem can be reduced from the feasibility problem, while in the present model Theorem 2.12 (a generalization of the feasibility theorem) was derived from Theorem 2.10 (a generalization of the minimization theorem).

Let us turn to node-connectivity augmentation problems. For a pair (X,Y) of disjoint non-empty subsets of V, let $\delta(X,Y) := \delta_E(X,Y)$ denote the number of edges of digraph $D=(V,E)$ having tail in X and head in Y. Let $h(X,Y) := |V-(X\cup Y)|$. The directed node-version of Menger's theorem [in: Ford and Fulkerson, 1962] asserts that, given a pair of nodes x,y with no edges from x to y, $\kappa(x,y)\geq k$ if and only if it is not possible to cover each directed path from x to y by less than k nodes distinct from x and y. For our purposes the following, obviously equivalent, version will prove more useful.

CLAIM 5.4 *For an arbitrary pair of nodes x,y, $\kappa(x,y)\geq k$ if and only if for every pair (X,Y) of disjoint subsets of V with $x\in X, y\in Y$, $\delta(X,Y)+h(X,Y)\geq k$.* ♠

Menger's theorem implies that a digraph with at least $k+1$ nodes is k-connected if and only if the deletion of at most $k-1$ nodes yields a strongly connected digraph. This is equivalent to requiring that for every non-empty subset X of V for which $|V-X|\geq k$ there are at least k nodes v in $V-X$ for which there is an edge uv with $u\in X$.

For solving the augmentation problem we will need yet another equivalent form of k-connectivity. We call a pair (X,Y) of non-empty disjoint subsets a **one-way pair** if there is no edge from X to Y, that is, $\delta(X,Y)=0$. (It is easy to see that the family of one-way pairs is crossing.)

CLAIM 5.5 *A digraph $D'=(V,E')$ is k-connected if and only if*

$$\delta_{E'}(X,Y)\geq k-h(X,Y) \tag{5.4}$$

holds for every pair (X,Y) of disjoint non-empty subsets. If $|V|\geq k+1$, then in order for D' to be k-connected it suffices to require (5.4) only for one-way pairs.

Proof. The equivalence of k-connectivity and (5.4) directly follows from Claim 5.4. To see the second part, suppose that $|V|\geq k+1$ and assume that there is a pair (X,Y) of subsets violating (5.4). We claim that there are nodes $x\in X, y\in Y$ with no edge from x to y. For otherwise $h(X,Y)+\delta_{E'}(X,Y)\geq h(X,Y)+|X||Y|\geq h(X,Y)+|X|+|Y|-1=|V|-1\geq k$, a contradiction. There are no k openly disjoint paths from x to y and hence by the directed node-version of Menger's theorem there is a subset C of less than k nodes covering all paths from x to y. Now the set X' of nodes reachable from x in D' and $Y':=V-X'-C$ violates (5.4) and $\delta_{E'}(X',Y')=0$. ♠

Let $D=(V,E)$ be a directed graph. The **deficiency** of a pair (X,Y) is defined to be $p'_{def}(X,Y):=\max(0,k-\delta_E(X,Y)-h(X,Y))$ when both X and Y are non-empty and $:=0$ otherwise. It follows from Claim 2.2 that $\delta_E(X,Y)$ is bi-submodular and trivially so is $h(X,Y)$. Hence the p'_{def} is crossing bi-supermodular.

The following statement easily follows from Claim 5.5:

CLAIM 5.6 *The addition of a set F of new edges to D leaves a k-connected digraph $D':=(V,E+F)$ if and only if*

$$\delta_F(X,Y)\geq p'_{def}(X,Y) \tag{5.5}$$

holds for every pair (X,Y) of disjoint non-empty subsets. If $|V|\geq k+1$, then in order for D' to be k-connected it suffices to require (5.5) only for pairs for which $\delta_E(X,Y)=0$. ♠

The main result of this whole section is the following. Recall that a family $\mathcal{F}$ of pairs of disjoint subsets of V was called independent if every two members $(X,Y),(X',Y')$ of $\mathcal{F}$ were half-disjoint, that is, at least one of $X\cap Y$ and $X'\cap Y'$ is empty.

THEOREM 5.7 *A digraph $D=(V,E)$ can be made k node-connected by adding at most γ new edges if and only if*

$$\sum(p'_{def}(X,Y):(X,Y)\in\mathcal{F})\leq\gamma \tag{5.6}$$

holds for every choice of independent families $\mathcal{F}$ of pairs of disjoint non-empty subsets of V. If $|V|\geq k+1$, then we may restrict $\mathcal{F}$ to consist of one-way pairs.

Proof. The theorem immediately follows from Claim 5.6 and Theorem 2.8 when $p' := p'_{def}$. ♠♠♠

We only mention, without formulating the details, that the results of Section 3 may also be specialized to node-connectivity augmentation problems.

6. GENERALIZING GYŐRI'S THEOREM

In 1984 E. Győri proved a deep min-max theorem concerning intervals of a straight line. For our purposes it is more convenient to use a terminology slightly different from Győri's and work with a system of subpaths of a path. To be more specific, let $P = (v_0, e_1, v_1, e_2, v_2, \dots, e_n, v_n)$ be a directed path or circuit where the nodes $v_i's$ of P are distinct, except that $v_0 = v_n$ in case P is a circuit, and each directed edge e_i of P has tail v_{i-1} and head v_i. We denote the node-set of P by V. Let $\mathcal{F} := \{F_1, \dots, F_k\}$ be a system of subpaths of P. In what follows, a path will mean the set of its edges.

We say that a system $\mathcal{B}$ of subpaths of P **generates** $\mathcal{F}$ or that $\mathcal{B}$ is a **generator** of $\mathcal{F}$ if each member of $\mathcal{F}$ is the union of some members of $\mathcal{B}$. For example, $\mathcal{F}$ is a generator of itself, or, the system $\{e_1, \dots, e_n\}$ of one-element paths is also a generator of $\mathcal{F}$. Let $\gamma(\mathcal{F})$ denote the minimum cardinality of a generator of $\mathcal{F}$.

We call a pair (F, e) of a path F and an element e of F a **represented path** and denote by $\mathcal{F}_r$ the system of all represented paths (F, e) with $F \in \mathcal{F}$.

Let $\mathcal{I} := \{I_1, \dots, I_t\}$ be a family of subpaths of P and $\mathcal{R} := \{f_1, f_2, \dots, f_t\}$ a system of distinct representatives of $\mathcal{I}$, that is, $f_i's$ are distinct edges of P so that $f_i \in I_i$ for $i = 1, \dots, t$. We call $\mathcal{R}$ a **strong system of representatives** if $I_i \cap I_j$ does not contain both f_i and f_j for i, j, $1 \le i < j \le t$, and in this case we say that a family $\{(I_1, f_1), (I_2, f_2), \dots, (I_t, f_t)\}$ of represented paths is **independent.** $\mathcal{I} := \{I_1, \dots, I_t\}$ is called **strongly representable** if it has a strong system of representatives.

It is not difficult to see, as was pointed out by Győri, that if P is a path, then $\mathcal{I}$ is strongly representable if and only if there is an ordering of the elements of $\mathcal{I}$ so that no member I of $\mathcal{I}$ is a subset of the union of the members of $\mathcal{I}$ preceding I in the given order. However, we will not use this second property since the equivalence is no longer true if P is a circuit, while Győri's theorem will turn out to hold in this case as well.

Let $\sigma(\mathcal{F})$ denote the maximum cardinality of a strongly representable sub-family of $\mathcal{F}$. It is rather straightforward to see that for any family $\mathcal{F}$ of subpaths of a path, one has $\sigma(\mathcal{F}) \le \gamma(\mathcal{F})$. Győri's theorem asserts that, in fact, always equality holds:

THEOREM 6.0 [Győri, 1984] *If $\mathcal{F}$ is a family of subpaths of a path P, then $\sigma(\mathcal{F}) = \gamma(\mathcal{F})$.*

Győri's original proof is a long, sophisticated argument and is not algorithmic. Later, D.S. Franzblau and D.J. Kleitman [1984] gave an algorithmic proof which gives rise to a polynomial-time algorithm to compute the extrema in the theorem. This proof is not short or simple either. Further extending the proof-technique of Franzblau and Kleitman, A. Lubiw [1991] was able to find a weighted generalization of Győri's theorem. Our goal here is to show that Theorem 2.8 easily implies Lubiw's result even in the more general case when P is a circuit. (Our proof is not algorithmic as it invokes Theorem 2.8 whose proof in Section 2 was not algorithmic.)

To make the exposition clearer, first we derive Győri's theorem in the more general form when P is a circuit. We then show how the same idea carries over to the weighted case. Henceforth we assume that the underlying P is a circuit.

THEOREM 6.1 *If $\mathcal{F}$ is a system of subpaths of a directed circuit P, the maximum cardinality of a strongly representable sub-family of $\mathcal{F}$ is equal to the minimum cardinality of a generator of $\mathcal{F}$, that is, $\sigma(\mathcal{F}) = \gamma(\mathcal{F})$.*

Proof. We are going to prove only the non-trivial direction $\sigma \ge \gamma$. Let us recall that $\mathcal{F}_r$ denotes the set of all represented paths (F, f) where $f \in F \in \mathcal{F}$. Call a member (F, f) of $\mathcal{F}_r$ **essential** if there is no member F' $(\neq F)$ of $\mathcal{F}$ for which $f \in F' \subset F$. With each essential member (F, f) of $\mathcal{F}_r$ we associate a pair (A, B) of disjoint subsets of V where (A, B) is a partition of the node-set $V(F)$ of path F so that A (respectively, B)

consists of the nodes of F preceding edge f (following f). Let $\mathcal{L}_{\mathcal{F}}$ denote the family of pairs obtained this way.

LEMMA 6.2 *$\mathcal{L}_{\mathcal{F}}$ is crossing.*

Proof. Let (A, B) and (A', B') be two crossing members of $\mathcal{L}_{\mathcal{F}}$. Let (F, f) and (F', f') denote the corresponding essential members of $\mathcal{F}_r$. Since (A, B) and (A', B') are crossing, $f \in F'$ and $f' \in F$.

We claim that neither A and A' nor B and B' are comparable as sets. Indeed, suppose to the contrary that, say, A includes A'. Since (F, f) is essential, B' properly includes B. But this means that (A, B) and (A', B') are comparable pairs contradicting the assumption that they are crossing.

It follows that $(A \cap A', B \cup B')$ is a pair associated with the represented path (F', f) and $(A \cup A', B \cap B')$ is a pair associated with the represented path (F, f').

In order to show that (F, f') and (F', f) belong to $\mathcal{L}_{\mathcal{F}}$, we have to prove that they are essential. We prove this only for (F, f'), the proof is analogous for (F', f). If, indirectly, there were a member X of $\mathcal{F}$ so that $f' \in X \subseteq F$, then $f \notin X$ since (F, f) is essential. But then $X \subseteq F'$ contradicting that (F', f') is essential. ♠

Clearly, an independent subfamily of $\mathcal{L}_{\mathcal{F}}$ corresponds to a strongly representable sub-family of $\mathcal{F}$.

Moreover, let $C := \{c_1, \ldots, c_t\}$ be a covering of $\mathcal{L}_{\mathcal{F}}$, where $c_1, \ldots, c_t$ are directed edges on the ground-set V. Let B_i be a subpath of P whose first node (resp., last node) is the tail (head) of c_i and let $\mathcal{B} := \{B_1, \ldots, B_t\}$. We claim that $\mathcal{B}$ is a generator of $\mathcal{F}$. For otherwise there is a minimal member F in $\mathcal{F}$ that is not the union of some members of $\mathcal{B}$. Hence there is an edge f of F so that (*) there is no member B of $\mathcal{B}$ for which $f \in B \subseteq F$. Because C is a covering of pairs associated with essential pairs, (F, f) cannot be essential. That is there is a member F' of $\mathcal{F}$ so that $f \in F' \subsetneq F$. By the minimal choice of F, F' is the union of some members of $\mathcal{B}$ contradicting (*).

Now the theorem immediately follows from Theorem 2.9. ♠♠♠

REMARK Another natural possibility to generalize Győri's theorem is that we consider a family of subpaths of an arborescence. However, as A. Lubiw [1991] showed by an example, the min-max relation is not necessarily true in this case.

Suppose that we are given a non-negative integer-valued weight-function w on the edges of P. With the help of w we define a weight function on the set of represented paths, namely, $w(F, f) := w(f)$. (It will not cause any ambiguity that the same term w is used for both functions.)

We say that a family $\mathcal{B}$ of (not-necessarily distinct) subpaths of P is a **w-generator** of $\mathcal{F}$ if for each pair (F, e) with $e \in F \in \mathcal{F}$ the family $\mathcal{B}$ contains at least $w(e)$ subpaths of F each containing e. Let $\gamma_w(\mathcal{F})$ denote the minimum cardinality of a w-generator of $\mathcal{F}$. Clearly, when $w \equiv 1$, we are back at the notion of generator. Let $\sigma_w(\mathcal{F})$ denote the maximum weight of an independent sub-family of $\mathcal{F}_r$. The following theorem was proved by A. Lubiw [1991] in the special case when no member of $\mathcal{F}$ contains v_n as an inner node, or equivalently, the underlying P is a simple path.

THEOREM 6.3 *Given a family $\mathcal{F}$ of subpath of a directed circuit P, one has: $\sigma_w(\mathcal{F}) = \gamma_w(\mathcal{F})$.*

Note that this theorem is of some interest even in the special case when w is $0-1$-valued. We are back at Győri's theorem when $w \equiv 1$ and P is a path.

Proof. We define a function p' on the set of all pairs (A, B) of subsets A, B of V as follows. Let $p'(A, B) := w(f)$ if (A, B) is a member of $\mathcal{L}_{\mathcal{F}}$ associated with an essential member (F, f) of $\mathcal{F}_r$ and zero otherwise. It follows from the definition of w and from Lemma 6.2 that p' is crossing bi-supermodular.

Now the theorem follows from Theorem 2.8 precisely the same way how Győri's theorem was derived in the preceding proof from Theorem 2.9. (That is, one observes that an independent sub-family of $\mathcal{L}_{\mathcal{F}}$ corresponds to an independent sub-family of $\mathcal{F}_r$ and hence $\sigma_w(\mathcal{F})$ is at least as big as $\nu_{p'}$ in Theorem 2.8.

Furthermore, a covering of p' corresponds to a w-generator of $\mathcal{F}_r$ and hence $\gamma_w(\mathcal{F})$ is at most $\tau_{p'}$ in Theorem 2.8.) ♠♠♠

Of course, the reduction above makes it possible to use degree-constrained and/or minimum node-cost versions of Theorem 2.8 and therefore one can handle variations of Győri's theorem. For example, given two cost-functions on the nodes, each possible generating path has a cost defined by the sum of the first cost of its first node and the second cost of its last node, one can derive a formula for the minimum weight of a generator of $\mathcal{F}$.

Finally we remark that the path problem in Theorem 6.3 was reduced to such a special case of the problem of covering bi-supermodular functions when the number of pairs with positive $p(X, Y)$ is bounded by $|P|^3$, a power of the size of the ground-set, therefore the general algorithms described in Section 4 may be applied and hence both the primal and the dual optima in Theorem 6.3 can be computed in polynomial time. In the special case when P is a path, A. Lubiw [1991] designed a purely combinatorial algorithm, which provides a proof of the theorem, as well. Is there an analogous combinatorial algorithm for the general case when P is a circuit?

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