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Sub-optimality of Threshold and Constant Proportion Portfolio Insurance Strategies in Defined Contribution Pension Plans

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Sub-optimality of threshold and constant proportion portfolio insurance strategies in defined contribution pension plans

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Abstract

The threshold and constant proportion portfolio insurance (CPPI) strategies are considered for their application in managing defined-contribution (DC) pension plans. The pension plans invest in two types of asset, riskless asset and stocks, or bonds and stocks. When the objective of pension plan is to maximize expected terminal utility that is a function of terminal pension wealth with final wages as numeraire, both threshold and CPPI strategies are suboptimal to the portfolio from inter-temporal optimization. When the objective of pension plan is to minimize expected terminal disutility defined as squared difference between actual wealth and target wealth, the threshold and CPPI strategies are inferior to a corresponding static-to-riskless hybrid strategy. When the objective of pension plans is to maximize expected terminal utility that is a function of terminal wealth over a guaranteed minimum, the threshold and CPPI strategies are inferior to a minimum terminal wealth insurance (MTWI) strategy. Since the threshold strategy is not optimal in minimizing expected terminal disutility and the CPPI strategy not optimal in maximizing utility over a guaranteed minimum, for which they appear to be designed respectively, they are generally suboptimal in managing DC pension plans.

Keywords: Optimal asset allocation; Defined-contribution pension plan; Threshold strategy; Constant proportion portfolio insurance (CPPI); Power utility; Hamilton-Jacobi-Bellman equation.

1. Introduction

With commonly accepted market parameters and investors' risk aversion, the optimal asset allocation strategy from inter-temporal optimization implies a highly leveraged portfolio by short-selling riskless assets (Kim and Omberg 1996; Sorensen 1999; Boulier et al 2001, Deelstra et al 2003, Cairns et al 2006). Since such optimal allocation strategies generally cannot be applied in pension fund management practice because of borrowing and short-sale constraints on pension funds, some simple dynamic allocation strategies are popular in allocating pension fund investment. These simple dynamic strategies include deterministic lifestyle, threshold (funded status) and constant proportion portfolio insurance (CPPI) among others (Blake et al 2001). In theory, these simple dynamic strategies can also have a leveraged portfolio by short-selling riskless assets. The deterministic lifestyle strategy has a simple relationship between portfolio composition and horizon, and a static allocation with same expected return can be found with smaller variance (Ma 2007), therefore it is a suboptimal strategy. This paper will focus on the threshold and CPPI strategies whose portfolio compositions have feedback from the portfolio performance.

The "threshold or funded status" strategy (Derbyshire, 1999; Blake et al 2001) is designed to keep the portfolio wealth around a target level in each period. The strategy uses the current wealth-to-target ratio for allocation decision. The investor starts by setting a lower threshold and an upper threshold. If the current wealth-to-target ratio is lower than the lower threshold, all wealth will be invested in high expected return and high risk equities; if the current wealth-to-target ratio is higher than the upper threshold, all wealth will be invested in low risk or risk-free assets. The proportion in equities decreases linearly from the lower to the upper threshold. It is easy to see that the threshold strategy has a negative feedback from pension wealth on the proportion invested in risky assets.

Keeping the portfolio wealth around a target level in each period appears to maximize utility functions whose value is reduced by deviation from a target. Vigna and Haberman (2001) assume that the objective of pension plans is minimization of cumulated future disutility which is the time discounted sum of squared difference between actual wealth and targeted wealth. They find that the optimal proportion invested in the high risk asset is horizon dependent and consistent with lifestyle strategy (Vigna and Haberman 2001; Haberman and Vigna 2002). The threshold strategy seems suited for minimization of the disutility as defined by Vigna and Haberman (2001). Empirically, outperforming a target will not cause disutility for the majority of investors. Therefore, the threshold strategy is probably appropriate only for a small minority of defined-contribution (DC) pension plan members.

The CPPI strategy (Perold and Sharpe, 1988; Black and Jones, 1988; Black and Perold, 1992) uses a simplified rule to allocation assets dynamically over time. The investor starts by setting a floor which is the lowest acceptable value of the portfolio, and computes the cushion as the excess value of portfolio over the floor. The proportion invested in the risky assets is calculated by multiplying the cushion with a predetermined multiplier. The remaining pension wealth is invested in the riskless assets. The floor and the multiplier are chosen according to investors' risk aversion. Such portfolio insurance strategies give the investors the ability to limit downside risk while allowing some participation in upside market. This property of portfolio insurance strategies appear to suit investors whose utility is a function of excess wealth over a minimum level.

Blake et al (2001) have compared the threshold and CPPI allocation strategies with two static ones for DC pension plans by estimating their value-at-risk with Monte Carlo simulation. The two static strategies are a "50/50" allocation strategy with 50% in T-bills and 50% in bonds which was found to be the minimum-risk strategy for most asset-return models and a 'pension-fund-average' (PFA) strategy (Blake et al, 2001) which might be considered a high-risk strategy on account of its high equity weighting. Blake et al (2001) find that a static asset allocation strategy with a high equity weighting (the 100% PFA strategy) delivers substantially better results than any of the common dynamic strategies investigated over the long term (40 years) of the sample policy. The study of Blake et al (2001) does not examine whether the PFA allocation is the optimal asset allocation strategy and whether a higher risk 100% equity strategy will outperform PFA. As at 31 March 2006, the PFA asset distribution is 35.8% in UK equities, 28.9% in overseas equities, 23.1% in bonds, 7.6% in index-linked gilts, 2.4% in property, 1.8% in cash, and 0.4% in other assets (Mellon Analytical Solutions 2007). If pension funds as a whole follow the optimal asset allocation strategy for pension plans, the results of Blake et al (2001) would only indicate that the optimal allocation outperforms the simple dynamic strategies. If the100% equity strategy outperforms the PFA strategy, the PFA would not be the optimal allocation.

In the study of Blake et al (2001), the threshold strategy uses "current pension ratio" as indicator for allocation decision. The 'current pension ratio' at time t is an approximate immediate pension [$F(t)/a(t,r(t))$] divided by 2/3 of the member's current salary, where $F(t)$ is the current fund size and $a(t,r(t))$ is an annuity factor (the price for one currency unit of pension). The asset allocation rule is: 1) 100% invested in the PFA portfolio if the current pension ratio is below a lower threshold (T_L); 2) 100% in the 50/50 portfolio if the current pension ratio is above an upper threshold (T_U), and 3) linearly increases in the 50/50 portfolio as the current pension ratio rises from T_L to T_U. Blake et al (2001) look at the thresholds $T_L = 0.4$ and $T_U = 0.8$. Such allocation is to hold the pension income (life annuity) at retirement around a targeted level, which will not appeal to DC pension plan members who are happy to have a higher pension income.

In the CPPI strategy investigated by Blake et al (2001), the weight in the high-risk portfolio is given by

$$\text{Weight in the PFA portfolio} = C_M(1-C_F(\text{Floor/Fund})$$
$$= C_M(1-C_F(\text{Liability/Fund})).$$

Where C_F is a parameter measuring the significance attached to the fund being above a floor, and C_M is the multiplier attached to quasi-surplus ratio and values of C_M exceeds unity. The remaining proportion of the fund is invested in the low risk 50/50 portfolio, short selling was not allowed and the portfolio weights were restricted to lie in the range 0-100%. In the study by Blake et al (2001), the floor is set at the level of the liabilities in a comparable DB plan (the ratio of the liabilities to the fund value at retirement is equal to the inverse of the pension ratio), and the values $C_F = 0.5$ and $C_M = 2$ are used. The objective of such CPPI strategy will appeal to risk-averse DC plan members who on one hand want a minimum guaranteed pension income, on the other hand, would like a higher pension income. Boulier et al (2001) and Deelstra et al (2003) also assume that terminal utility is a function of cash surplus over guaranteed benefits in solving the optimal asset allocation problem for DC pension plans.

Since the threshold and CPPI strategies appear to be designed for particular forms of utility function, their sub-optimality cannot be determined solely on the probability of terminal wealth level achieved. The comparison between those allocation strategies and the inter-temporal optimization should be conducted with the utility function for which those simple dynamic strategies perform best. Moreover, those simple dynamic strategies provide simplified dynamic asset allocation rules for the cases where borrowing and short-sale are constrained. When portfolio constraints exist, more than one set of prices will be consistent with no arbitrage, and each price system corresponds to a complete auxiliary market determined by the nature of the constraints (Cvitanić and Karatzas 1992; Teplá 2000). The investor's optimal constrained portfolio is equivalent to the optimal unconstrained portfolio in the corresponding auxiliary market (Cvitanić and Karatzas 1992). The comparison with the inter-temporal optimization needs also to take borrowing and short-sale constraints into account. The present study will compare those simple dynamic strategies with the optimal allocation strategy for both the usual utility assumptions and the utility functions implied by those simple dynamic strategies, under both the usual unconstrained conditions and the borrowing and short-sale constraints.

Three utility functions are considered in this paper: 1) the expected terminal utility as a function of wealth-to-wage ratio (Cairns et al); 2) the expected terminal disutility as squared difference of actual and targeted terminal wealth (Vigna and Haberman 2001; Haberman and Vigna 2002); and 3) the expected utility as a function of excess wealth over a guaranteed minimum (Boulier et al 2001; Deelstra et al 2003). The first utility (as a function of wealth-to-wage ratio) is used in this paper for inter-temporal optimization, the other two utility functions are used to illustrate that the threshold and CPPI strategies are not optimal even for those they are designed respectively. Cairns et al (2006) use replacement ratio or wealth-to-wage ratio, which take the current standard of living into account, as the argument of expected terminal utility. Taking current standard of living into account suggests a role of habit formation in the utility function (Spinnewyn 1981; Becker and Murphy 1988). The use of wealth-to-wage ratio as the argument of the terminal utility function incorporates the wage risk and its correlation with the interest rate and stock returns into the optimal asset allocation decision. It also leads to a computational advantage that the optimal portfolio composition with commonly assumed stochastic interest rate, stock return, and wage income models (Battocchio and Menoncin 2004; Cairns et al 2006) is no longer horizon dependent.

Since dynamic allocation strategies tend to use two assets or two mutual funds and the switch is usually between the riskless asset/low-risk mutual and high-risk equities/mutual fund, two assets (either cash and equity or bond and equity) inter-temporal optimization models are used for the present investigation. This paper shows that the inter-temporal optimization outperforms the threshold and CPPI strategy no matter whether short-sales are allowed or not, when terminal utility is a function of terminal wealth with final wage as numeraire. When the objective of pension plan is minimization of terminal disutility which is squared difference between actual fund and targeted fund, inter-temporal optimization still outperforms the threshold strategy (and the CPPI strategy). When terminal utility is a function of wealth over a guaranteed minimum, inter-temporal optimization outperforms the CPPI strategy (and the threshold strategy).

This paper is organized as follows. Section 2 derives the optimal asset allocation for two assets, cash (or bond) and stock, when terminal utility is a function of the wealth-to-wage ratio, and presents the parameters used later in numerical simulations. Section 3 derives a mathematic presentation of the threshold strategy and compares its terminal utility with portfolios from inter-temporal optimization and a static-riskless hybrid by numerical simulation. Section 4 derives a mathematic presentation of the CPPI strategy and compares its terminal utility with portfolios from inter-temporal optimization and a minimum terminal wealth insurance (MTWI) strategy by numerical simulation. Section 5 summarizes the results in this paper and concludes.

2. Market structure, inter-temporal optimization and numerical procedure

This section presents a market model with three types of asset, riskless assets, bonds and equities (stocks), review key results for optimal asset allocation problem without borrowing or short-sale constraints, and describe the numerical procedure for simulation and comparison between different strategies.

2.1. Market structure and wealth growth model

The uncertainty in the financial market is generated by two standard and independent Brownian motions $Z_r(t)$ and $Z_S(t)$ with $t \in [0,T]$, defined on a complete probability space (Ω, $\mathcal{F}$, P) where P is the real world probability (Boulier et al 2001; Deelstra et al 2003; Battocchio and Menoncin 2004). The filtration $\mathcal{F}=\mathcal{F}(\mathrm{t})\ \ \forall t \in [0,T]$ can be interpreted as the information set available to the investor at time t.

The instantaneous risk-free rate of interest $r(t)$ follows an Ornstein-Uhlenbeck process

$$dr(t) = \alpha(\beta - r(t))dt + \sigma_r dZ_r(t)\,,$$

$$r(0) = r_0\,. \qquad (1)$$

In the above equation, α and β are strictly positive constants, and σ_r is the volatility of interest rate (Vasicek, 1977).

There are three types of asset in the financial market: cash, bonds and equities. The riskless asset, which can be considered as a cash fund, has a price process governed by

$$dR(t) = R(t)r(t)dt\,,$$

$$R(0) = R_0\,. \qquad (2)$$

There are zero-coupon bonds for any date of maturity, and a bond rolling over zero coupon bonds with constant maturity K (Boulier et al, 2001). The price of the zero coupon bond with constant maturity K is denoted by $B_K(t, r)$ with

$$\frac{dB_K(t,r)}{B_K(t,r)} = [r(t) + b_K \sigma_r \xi]dt - b_K \sigma_r dZ_r(t) \quad , \tag{3}$$

where

$$b_K = \frac{1-e^{-\alpha K}}{\alpha} \quad .$$

One equity asset, a stock, which can be considered to represent the index of a stock market, has a total return (the value of a single premium investment in the stock with reinvestment of dividend income) follows the stochastic differential equation (SDE)

$$dS(t) = S(t)\left[(r(t) + m_S)dt + v_{rS}\sigma_r dZ_r(t) + \sigma_S dZ_S(t)\right],$$

$$S(0) = S_0, \tag{4}$$

where v_{rS} represents a volatility scale factor measuring how the interest rate volatility affects the stock volatility and m_S is the risk premium on the stock.

The market as assumed above has a diffusion matrix given by

$$\Sigma \equiv \begin{bmatrix} -b_K\sigma_r & 0 \\ v_{rS}\sigma_r & \sigma_S \end{bmatrix}, \tag{5}$$

and σ_r and σ_S are assumed to be different from zero and the diffusion matrix is invertible.

The plan member's wage, $Y(t)$, evolves according to the SDE (Battochio and Menoncin 2004; Cairns et al 2006)

$$dY(t) = Y(t)\left[(\mu_Y + r(t))dt + v_{rY}\sigma_r dZ_r(t) + v_{SY}\sigma_S dZ_S(t) + \sigma_Y dZ_Y(t)\right],$$

$$Y(0) = Y_0, \tag{6}$$

where $\mu_Y(t)$ and σ_Y are assumed to be constant for simplicity. Here $Z_Y(t)$ is a standard Brownian motion independent of $Z_r(t)$ and $Z_S(t)$; v_{rY} and v_{SY} are volatility scaling factors measuring how interest rate volatility and stock volatility affect wage volatility, respectively. When $\sigma_Y = 0$, the market is complete and the wage income is fully hedgeable. Otherwise the market is incomplete.

If the pension fund invests in cash and stock, the SDE governing the wealth process is

$$
\begin{aligned}
dW(t) &= W(t)\left(\theta_R \frac{dR}{R} + \theta_S \frac{dS}{S}\right) + \pi Y(t)dt \\
&= [W(t)\theta_R r + W(t)\theta_S (r + m_S) + \pi Y(t)]dt + W(t)\theta_S v_{rS}\sigma_r dZ_r + W(t)\theta_S \sigma_S dZ_S \ .
\end{aligned}
\tag{7}
$$

If the pension fund invests in bonds and stock, the SDE governing the wealth process is

$$
\begin{aligned}
dW(t) &= W(t)\left(\theta_B \frac{dR}{R} + \theta_S \frac{dS}{S}\right) + \pi Y(t)dt \\
&= [W(t)\theta_B (r + b_K \sigma_r \xi) + W(t)\theta_S (r + m_S) + \pi Y(t)]dt \\
&\quad + W(t)(-\theta_B b_K + \theta_S v_{rS})\sigma_r dZ_r + W(t)\theta_S \sigma_S dZ_S \ .
\end{aligned}
\tag{8}
$$

2.2. Optimal asset allocation for power terminal utility

With fully hedgeable wage income, the market value at time t of future contributions payable between t and T is then

$$
\begin{aligned}
& E_Q\left[\int_t^T \exp\left\{-\int_t^\tau r(s)ds\right\}\pi Y(\tau)d\tau \mid F_t\right] \\
&= \pi E_Q\left[\int_t^T Y(t)\exp\left\{\int_t^\tau \mu_Y(s)ds - (\xi_r v_{rY}\sigma_r + \xi_S v_{SY}\sigma_S + \frac{1}{2}v_{rY}{}^2\sigma_r{}^2 + \frac{1}{2}v_{SY}{}^2\sigma_S{}^2)(\tau - t)\right.\right. \\
&\left.\left. + v_{rY}\sigma_r[\tilde{Z}_r(\tau) - \tilde{Z}_r(t)] + v_{SY}\sigma_S[\tilde{Z}_S(\tau) - \tilde{Z}_S(t)]\right\}d\tau \mid F_t\right] \\
&= \pi Y(t)\int_t^T \exp\left\{\int_t^\tau \mu_Y(s)ds - (\xi_r v_{rY}\sigma_r + \xi_S v_{SY}\sigma_S)(\tau - t)\right\}d\tau \\
&= \pi Y(t) f(t) .
\end{aligned}
\tag{9}
$$

where Q is the risk-neutral pricing measure (Cairns et al 2006), ξ_r is a measure of how interest/bond volatility will affect wage, and ξ_S is a scale factor measuring how stock price volatility affects wages. The pension plan can have an additional wealth of $\pi Y(t) f(t)$ by short-selling a replicating portfolio of value $-\pi Y(t) f(t)$, which will be paid off exactly by future contributions from wage incomes. The total pension wealth enhanced with the present market value of future contributions is the augmented wealth $\widetilde{W}(t) = W(t) + \pi Y(t) f(t)$.

If there is a minimum wealth position W_m over the lifetime of pension plan like the CPPI strategy and the terminal utility can be considered as a function of excess wealth-to-wage ratio, the minimum wealth position W_m need to be subtracted from the augmented wealth as the argument of utility function.

The process governing the augmented wealth-to-wage ratio $\widetilde{X}(t) = \widetilde{W}(t)/Y(t)$ can be written as

$$d\widetilde{X}(t) = (\theta M + u)\widetilde{X}dt + (\theta\Gamma' + \Lambda')\widetilde{X}dZ, \tag{10}$$

$$\widetilde{X}(T) = X(T).$$

Using θ as the proportion of wealth invested in stock, for pension funds investing in cash and stock,

$$M \equiv m_S - v_{rS}v_{rY}\sigma_r^{\ 2} - v_{SY}\sigma_S^{\ 2},$$

$$u \equiv -\mu_Y + v_{rY}^{\ 2}\sigma_r^{\ 2} + v_{SY}^{\ 2}\sigma_S^{\ 2},$$

$$\Gamma \equiv \left[v_{rS}\sigma_r \quad \sigma_S\right]' ,$$

$$\Lambda \equiv \left[-v_{rY}\sigma_r \quad -v_{sY}\sigma_S\right]',$$

$$Z \equiv \left[Z_r \quad Z_S\right]'. \tag{11}$$

For pension funds investing in bond and stock, M, u and Γ are different

$$M \equiv m_S - b_K\sigma_r\xi - (v_{rY}b_K + v_{rS}v_{rY})\sigma_r^{\ 2} - v_{SY}\sigma_S^{\ 2},$$

$$u \equiv b_K\sigma_r\xi - \mu_Y + v_{rY}(v_{rY} + b_K)\sigma_r^{\ 2} + v_{SY}(v_{SY} - 1)\sigma_S^{\ 2},$$

$$\Gamma \equiv \left[(b_K + v_{rS})\sigma_r \quad \sigma_S\right]' . \tag{12}$$

The optimal allocation problem for terminal utility that is a function of terminal wealth-to-wage ratio is:

$$\max_{\theta} E\left[U(X(T),T)\right]$$

subject to

$$d\begin{bmatrix} w \\ \tilde{X} \end{bmatrix} = \begin{bmatrix} \mu_w \\ \theta M\tilde{X} \end{bmatrix} dt + \begin{bmatrix} \Omega' \\ \theta\Gamma'\tilde{X} \end{bmatrix} dZ ,$$

$$w(0) = w_0, \tilde{X}(0) = \tilde{X}_0, \forall 0 \le t \le T , \qquad (13)$$

where,

$$\underset{2\times1}{w} \equiv \begin{bmatrix} r & Y \end{bmatrix}',$$

$$\underset{2\times1}{\mu_w} \equiv \begin{bmatrix} \alpha(\beta - r) & Y(\mu_Y + r) \end{bmatrix}' ,$$

$$\underset{2\times2}{\Omega'} \equiv \begin{bmatrix} \sigma_r & 0 \\ Y v_{rY}\sigma_r & Y v_{SY}\sigma_S \end{bmatrix} . \qquad (14)$$

The corresponding Hamilton-Jacobi-Bellman equation is

$$H(J) = J_t + \mu'_w \frac{\partial J}{\partial w} + (\theta M + u)\tilde{X}\frac{\partial J}{\partial \tilde{X}} + \frac{1}{2} tr\left(\Omega'\Omega \frac{\partial^2 J}{\partial w^2}\right) + (\theta\Gamma' + \Lambda')\Omega\tilde{X}\frac{\partial^2 J}{\partial w \partial \tilde{X}}$$
$$+ \frac{1}{2}(\theta\Gamma'\Gamma\theta + 2\theta\Gamma'\Lambda + \Lambda'\Lambda)\tilde{X}^2 \frac{\partial^2 J}{\partial \tilde{X}^2}$$

(15)

The system of the first order conditions on H with respect to θ is

$$\frac{\partial H}{\partial \theta} = M\tilde{X}\frac{\partial J}{\partial \tilde{X}} + \Gamma'\Omega\frac{\partial^2 J}{\partial w \partial \tilde{X}}\tilde{X} + (\Gamma'\Gamma\theta + \Gamma'\Lambda)\tilde{X}^2\frac{\partial^2 J}{\partial \tilde{X}^2} = 0 \qquad (16)$$

The optimal portfolio composition is

$$\theta^* = -(\Gamma'\Gamma)^{-1}\Gamma'\Lambda - (\Gamma'\Gamma)^{-1}M\frac{J_{\tilde{X}}}{\tilde{X}J_{\tilde{X}\tilde{X}}} - (\Gamma'\Gamma)^{-1}\Gamma'\Omega\frac{J_{w\tilde{X}}}{\tilde{X}J_{\tilde{X}\tilde{X}}} \qquad (17)$$

Equation (20) shows that the optimal allocation in the stock contains three components, which is consistent with earlier studies (Battocchio and Menoncin 2004; Cairns et al 2006).

Assuming that the maximized expected terminal utility of plan members has the functional form

$$J(t,x,w) = \frac{1}{1-\gamma} g(t,w)^{\gamma} x^{1-\gamma}, \tag{18}$$

the optimal asset allocation problem can be solved analytically. The optimal proportion invested in stocks for pension funds investing in cash and stock is

$$\begin{aligned} \theta^* &= -(\Gamma'\Gamma)^{-1}\Gamma'\Lambda + (\Gamma'\Gamma)^{-1} M \frac{1}{\gamma} \\ &= \frac{v_{rS} v_{rY} \sigma_r{}^2 + v_{SY} \sigma_S{}^2}{v_{rS}{}^2 \sigma_r{}^2 + \sigma_S{}^2} + \frac{m_S - v_{rS} v_{rY} \sigma_r{}^2 - v_{SY} \sigma_S{}^2}{(v_{rS}{}^2 \sigma_r{}^2 + \sigma_S{}^2)\gamma}. \end{aligned} \tag{19}$$

The optimal proportion of pension wealth invested in stocks for pension funds investing in bond and stock is

$$\theta^* = \frac{(b_K + v_{rY})(b_K + v_{rS})\sigma_r{}^2 + v_{SY}\sigma_S{}^2}{(b_K + v_{rS})^2 \sigma_r{}^2 + \sigma_S{}^2} + \frac{m_S - b_K \sigma_r \xi - (b_K + v_{rS}) v_{rY} \sigma_r{}^2 - v_{SY} \sigma_S{}^2}{\gamma[(b_K + v_{rS})^2 \sigma_r{}^2 + \sigma_S{}^2]} \tag{20}$$

The optimal proportion of pension wealth invested in cash or bonds is $1-\theta^*$. The optimal portfolio composition is horizon independent. It is easy to see that the optimal portfolio composition for either DC plans investing in cash and equities or DC plans investing in bonds and equities is horizon independent. The pension plan financial wealth (wealth of pension portfolio + short-sold wage replicating portfolio) is horizon-dependent because the short-sold wage replicating portfolio is paid off gradually by future wage contributions. For details of the above derivation see Ma (2007).

2.3. Parameters for numerical simulation

The values of parameters used for numerical simulation are listed in Table 1, which are commonly used in other pension studies (Boulier et al 2001; Deelstra et al 2003; Cairns et al 2006; Battocchio and Menoncin 2004) and chosen to facilitate comparison. The commonly used value of relative risk aversion in pension studies is 2 to 3 (Boulier et al 2001; Deelstra et al 2003). Table 2 shows the optimal proportions invested in different assets for different allocation strategies with parameters in Table 1.

Table 1 Parameters used in numerical simulation

Interest rate	Value
Mean reversion, α,	0.2
Mean rat, β	0.05
Volatility, σ_r	0.02
Initial rate, r_0	0.05
Fixed maturity bond	
Maturity, K	20 years
Market price of risk, ξ	0.15
Stock	
Risk Premium, m_S	0.06
Stock own volatility, σ_S	0.19
Interest volatility scale factor, v_{rS}	1
Wage	
Wage premium, μ_Y	0.01
Non-hedgeable volatility, σ_Y	0.01
Interest volatility scale factor, v_{rY}	0.7
Stock volatility scale factor, v_{SY}	0.9
Initial wage, Y_0	10k
Contribution rate, π	10%
Risk aversion	
Relative risk aversion, γ	2
Length of pension plan, T	45

The optimal proportions shown in Table 2 are analytical solution for fully hedgeable wage income ($\pi \neq 0$ and $\sigma_Y = 0$), but they are also used for testing numerically the scenario where $\pi \neq 0$ and $\sigma_Y \neq 0$. These tests show that the presence of non-hedgeable risk has little effect on the performance of different portfolios and the optimal asset allocation, suggesting that if the optimal portfolio composition for $\pi \neq 0$ and $\sigma_Y \neq 0$ scenario is solved numerically, it would be similar to the optimal composition for $\pi \neq 0$ and $\sigma_Y = 0$ scenario. Minus sign (-) indicates short-sale; -0.271 in cash means short-selling cash asset valued as 27.1% of the net pension wealth. With short-sale of cash asset, the proportion invested in stock is 127.1% of the net pension wealth. For the power utility, the optimal proportions are dependent on the relative risk aversion.

Table 2 Optimal proportions invested in different assets

Strategies	Utility	Cash	Bond	Stock
Cash-stock	power	-0.271		1.271
Bond-stock	power		-0.0252	1.0252

2.4. Numerical simulation method

The Euler-Maruyama method is used for numerical simulation of stochastic differential equation (Higham 2001). The SDE

$$dX(t) = f(X(t))dt + g(X(t))dZ(t) \tag{21}$$

is simulated over [0,T] by using

$$X_j = X_{j-1} + f(X_{j-1})\Delta t + g(X_{j-1})(Z(\tau_j) - Z(\tau_{j-1}) \quad \text{j=1,2,…,N.} \tag{22}$$

In the above difference equation, $\Delta t = T/N = Rh$, $\tau_j = j\Delta t$ and

$$Z(\tau_j) - Z(\tau_{j-1}) = Z(jRh) - Z((j-1)Rh) = \sum_{k=jR-R+1}^{jR} dZ_k \, . \tag{23}$$

The terminal utility is calculated with the result of each simulation and 1000 simulations performed for each allocation strategy. The cumulative terminal utility distribution density as well as the mean and the standard deviation (SD) are then calculated for each allocation strategy. The program was written in Visual Basic for Applications.

3. Threshold or funded status strategy for DC pension plans

The optimization problem for threshold or funded status strategy can be stated as follows:

Find optimal lower threshold pension ratio T_L and upper threshold pension ratio T_U. A proportion

$$\min\left(1, \max\left(0, \frac{T_U - \phi(t)}{T_U - T_L}\right)\right) \tag{24}$$

of pension fund wealth will be invested in high risk equities, and a proportion

$$\min\left(1, \max\left(0, \frac{\phi(t) - T_L}{T_U - T_L}\right)\right) \tag{25}$$

of pension fund wealth will be invested in low risk cash fund or bonds. Following Blake et al (2001), the lower and upper thresholds are in terms of current pension ratio, and the definition of current pension ratio in Blake et al (2001) is used here:

$$\phi(t) = \frac{W(t)/a(t,r(t))}{\frac{2}{3}Y(t)} = \frac{X(t)}{\frac{2}{3}a(t,r(t))}. \tag{26}$$

Pure equity instead of pension-fund-average (PFA) will be used for high risk assets and cash or bond instead of 50/50 (T-bills/bonds) portfolio for low risk assets. Other definition of current pension ratio can also be used. For example, if the targeted pension income is equal to the final salary, the current pension ratio will be defined to be

$$\phi(t) = \frac{W(t)/a(t,r(t))}{Y(t)} = \frac{X(t)}{a(t,r(t))}.$$

Using 2/3 of final salary is to compare with a corresponding DB plan, which is often used as benchmark for assessing DC plan performance.

3.1. Replicating threshold strategies with static allocations

Because of the cumulative nature of pension wealth which starts from zero initial wealth, if borrowing and short-selling to get an augmented initial wealth are not allowed, the threshold strategy will appear a typical lifestyle strategy with a complete switch-over date before the end of pension plan (the lifestyle strategy usually switch over entirely to riskless assets only at the end of pension plan). Before the pension wealth reaches the lower threshold T_L, all wealth is invested in risky assets; between the lower threshold T_L and the upper threshold T_U, the proportion invested in the risky assets linearly increasing; after reaching the upper threshold T_U, all wealth is invested in riskless assets. This lifestyle feature has been caught in the optimal asset allocation solution for DC pension plans whose objective is minimization of time discounted lifetime disutility, the squared difference between the actual fund and the targeted fund (Vigna and Haberman 2001; Haberman and Vigna 2002). The threshold strategy suits the minimization of such a disutility better than the maximization of power utility.

The similarity between lifestyle and threshold strategies can be seen from the following analysis. For simplicity, the instantaneous interest rate r is assumed to be constant (implying constant annuity price a); the initial wealth is W_0 and there is no further contribution; the investment decision is based on the current wealth-to-targeted wealth ratio, $\varphi(t) = W(t)/W_{\mathrm{Target}}$ instead of pension ratio. Given that the design of threshold strategies restricts the maximum proportion invested in high risk assets (stocks) to 100% of pension wealth, a constraint on short-selling low risk assets (cash assets or bonds) is in place. It can be reasoned that the maximum expected time t wealth is that derived from investing all wealth in equities,

$$
\begin{aligned}
&E_0\left[W_{\max}(t)\right] = W(0)E_{t_0}\left\{\exp\int_0^t\left[(r + m_S - \frac{1}{2}\sigma_S{}^2)ds + \sigma_S dZ_S(s)\right]\right\} \\
&= W(0)E_0\left\{\exp\left[t(r + m_S - \frac{1}{2}\sigma_S{}^2) + \sigma_S Z_S(t)\right]\right\} \qquad (27) \\
&= W(0)\exp\left[t(r + m_S)\right].
\end{aligned}
$$

The minimum expected time t wealth is that derived from investing all wealth in the riskless asset,

$$E_0\left[W_{\min}(t)\right] = W(0)E_0\left\{\exp\int_0^t rds\right\} = W(0)\exp(rt)\,. \tag{28}$$

The threshold strategy is in fact a three-stage operation: 1) 100% in stocks from beginning (t_0) to the time (t_{TL}) when the current wealth-to-targeted wealth ratio φ(t) reaches T_L; 2) $\frac{T_U - \varphi(t)}{T_U - T_L}$ in stock from t_{TL} to the time (t_{TU}) when φ(t) reaches T_U; 3) 100% in risk free assets from t_{TU}.

From the above analysis, the threshold strategy is very similar to the lifestyle strategy, except that the switching points in the threshold strategy marked with the size of pension wealth relative to annuity price rather than fixed dates. It is conceivable to add a period with 100% riskless assets to a lifestyle strategy. It has been shown that the corresponding static allocation has the same expected return and a smaller variance than the lifestyle strategy (Ma 2007). In an extreme scenario, if T_L and T_U are set at same values, the threshold strategy will be one-off switching between 100% risky assets and 100% riskless assets, resembling the "all stocks half the time" approach (Kritzman 2000). The allocation rule between T_L and T_U provides a transition zone resembling the gradual switching of the lifestyle strategy. Since in reality pension plans usually start with zero or very small initial wealth and there is a flow of contribution from wage incomes, the period that the current wealth-to-targeted wealth ratio stays between T_L and T_U (stage 2) will be much shorter than the case of only one initial lump sum contribution assumed here. Comparatively, stage 1 (100% stocks) and stage 3 (100% riskless) will be longer.

The comparison of variance between threshold strategy and its static replicating allocation is not straightforward, because of the negative feedback mechanism in the threshold strategy. The threshold strategy leads to a lower expected return with a lower variance than a corresponding static allocation. The threshold strategy can be compared with a hybrid replicating allocation to see their variance implications. The hybrid replicating allocation contains two stages: 1) a static allocation that has roughly the same expected return before the current wealth-to-targeted wealth ratio reaches T_U; 2) the threshold strategy once the current wealth-to-targeted wealth ratio reaches T_U. In the second stage, the hybrid strategy and the pure threshold strategy is the same. Their expected returns and variances only need to be compared in the first stage of the hybrid strategy.

The composition of the static allocation can be derived by estimating the expected duration of the second stage, between the time when all wealth is invested in the riskless assets after the pension wealth reaches T_U and the termination of pension plan, which is

$$t_2 = \frac{\log(1/T_U)}{r}.$$

The proportion invested in the risky assets before the portfolio value reaches T_U is then determined by

$$\theta = \frac{\log\left(T_U W_{\mathrm{Target}} / W_0\right)/(T - t_2) - r}{m_S},$$

From the analysis on the lifestyle strategy (Ma 2007), when the current wealth-to-targeted wealth ratio of the hybrid strategy reaches the target level T_U for the first time, the static allocation in the hybrid strategy has a smaller variance than the pure threshold strategy. Therefore, the hybrid replicating allocation strategy has the same expected terminal wealth as the pure threshold strategy, but with a smaller variance. If the static allocation does not switch to the threshold strategy after reaching T_U, its expected return and the expected terminal pension ratio will be higher than the pure threshold strategy. Although the variance will be higher, the "Sharpe ratio" (expected return per unit standard deviation (the square root of variance)) is higher than the pure threshold strategy.

3.2. Threshold strategy for power terminal utility

The threshold strategy provides an asset allocation strategy θ, the proportion invested in stock,

$$\theta = \min\left(1, \max\left(0, \frac{T_U - \phi(t)}{T_U - T_L}\right)\right). \tag{29}$$

Obviously there are no such T_U and T_L that $\min\left(1,\max\left(0,\frac{T_U-\phi(t)}{T_U-T_L}\right)\right)$ leads to θ* in equations (19) or (20). Therefore, there are no such optimal upper or lower thresholds in the threshold strategy that can optimize asset allocation for power terminal utility function.

I conjecture that the best upper and lower thresholds for a threshold strategy are such that the average proportion over the accumulation phase invested in equities (i.e. the equilibrium ratio) equals to the θ* in equations (19) or (20). However, such "second best" allocation strategies do not exist for people with lower relative risk aversion. The value of relative risk aversion (RRA) for at least 100% stock with an equity-cash strategy is

$$\gamma \le \frac{m_S - v_{rS}v_{rY}\sigma_r{}^2 - v_{SY}\sigma_S{}^2}{(v_{rS}{}^2\sigma_r{}^2+\sigma_S{}^2) - v_{rS}v_{rY}\sigma_r{}^2 - v_{SY}\sigma_S{}^2}. \tag{30}$$

The assumptions on the market used by the present study and most pension studies give a RRA of 7.3. Lower than 7.3, the pension fund should invest all its wealth in stocks, and short-sell cash assets to finance further stock holding if there is no short-sale constraint.

3.3. Comparison with inter-temporal optimization

Three sets of threshold are used in the numerical simulation for comparison with inter-temporal optimal allocation, 100% cash or 100% bond, and 100% stock strategies. For simplicity, $a(t,r(t))$ is assumed to be of the form $\exp[d_0 - d_1 r(t)]$ with d_0=3 and d_1=3.5, as in Cairns et al (2006). This assumption implies that $a(t,r(t))$ behaves like a zero-coupon bond with 8.318 years to maturity (Cairns et al 2006). In the threshold strategies, the wage contribution is added to the pension wealth as it comes in and no short-sale involved. In the inter-temporal optimal allocation strategy "optimal power, augmented wealth" ("power solution borrow" in Fig.1), the present value of future wage contributions is used as pension wealth by short-selling a wage replicating portfolio, which will be paid off by future wage contributions. In the inter-temporal optimal allocation strategy "optimal power, non-augmented" ("power solution no borrow" in Fig.1), the wage contribution is added to the pension wealth as it comes in and no short-sale of the wage replicating portfolio is involved. In the 100% cash, 100% bond, and 100% equity (stock) strategies, the wage contribution is added to the pension wealth as it comes in and no short-sale of the wage replicating portfolio is involved. Terminal financial wealth (wealth of pension port portfolio + remaining value of the short-sold wage replicating portfolio) is used to calculate the terminal utility. In the $\sigma_Y = 0$ scenario, terminal financial wealth always equals the terminal wealth of pension portfolio, because the short-sold wage replicating portfolio will be fully paid by the contributions from future wage incomes. In the $\sigma_Y \neq 0$ scenario, terminal financial wealth may be smaller or larger than the terminal wealth of pension portfolio because of the nonhedgeable wage risk.

Results from numerical simulations are shown in Fig.1 and Table 3. The allocation derived from optimization for power utility ("Optimal Power, augmented wealth") dominates those of threshold strategies. Even when the optimal allocation derived from borrowing against future wages (short-sale of wage replicating portfolio) is used for allocating contribution from wage income and cumulated wealth only (without transforming the future wage income contributions into an initial augmented pension wealth by short-selling a replicating portfolio) ("Optimal power, non-augmented"), the inter-temporal optimization still has a higher expected terminal utility than the threshold strategies. In the "Optimal power, non-augmented" case, short-sale in implementing the optimal asset allocation derived for $\sigma_Y = 0$ scenario is still allowed; "non-augmented" means no short-selling the wage replicating portfolio in order to transform the future wage income contributions into the initial augmented pension wealth. When borrowing and short-sale are not allowed, with market parameters commonly used in pension studies the optimal asset allocation becomes the 100% stock strategy. The 100% stock strategy or the optimal asset allocation without borrowing and short-sale also has a higher expected terminal utility than the lifestyle strategies. In contrast, the 100% cash or 100% bonds strategy is least efficient (Table 3).

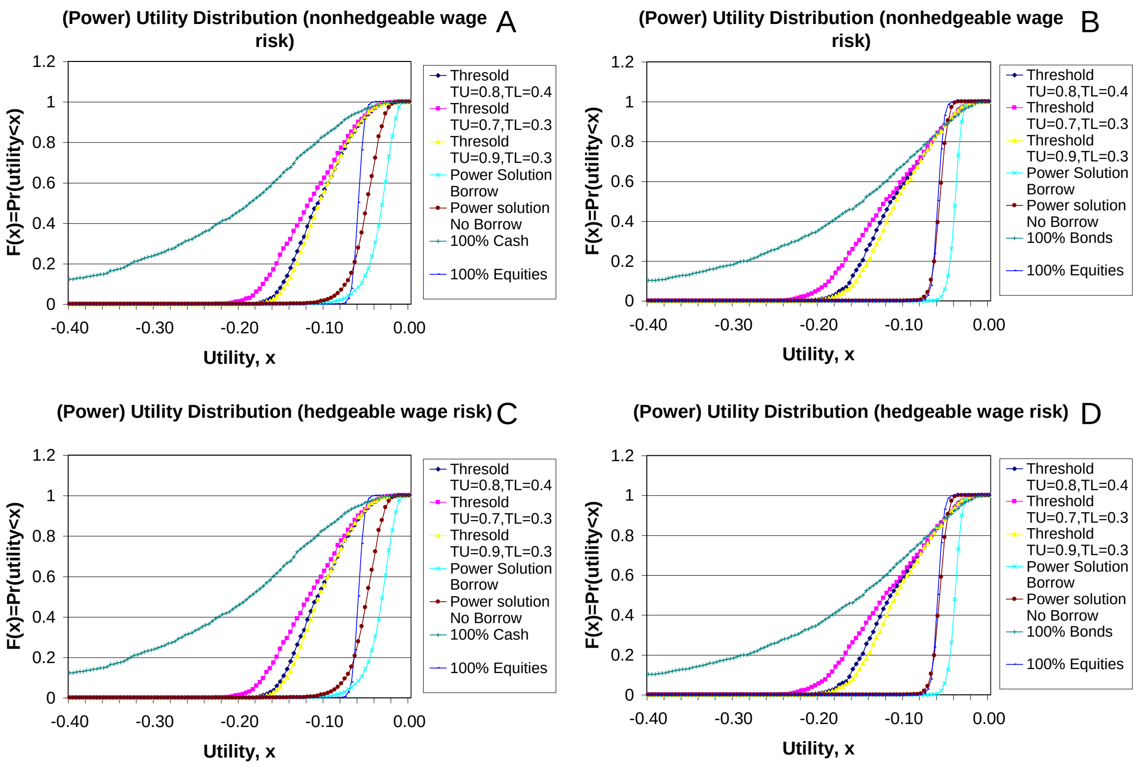


Fig.1 Comparison between threshold strategies and inter-temporal optimization, evaluated for power utility γ=2. Results are from 1000 simulations. A. Equity-cash scenario, $\sigma_Y \neq 0$. B. Equity-bond scenario, $\sigma_Y \neq 0$. C. Equity-cash scenario, $\sigma_Y = 0$. D. Equity-bond scenario, $\sigma_Y = 0$.

When the optimal asset allocation derived for $\sigma_Y = 0$ scenario is applied for the $\pi \neq 0$ and $\sigma_Y \neq 0$ scenario, the "optimal power, augmented wealth", "optimal power, non-augmented" and 100% stock cases still dominate threshold strategies. There is little difference in the performance of the optimal asset allocation between the $\sigma_Y = 0$ scenario and the $\pi \neq 0$ and $\sigma_Y \neq 0$ scenario (Table 3). With $\sigma_Y \neq 0$, the wage replicating portfolio only hedges for the interest and stock market risks; the wage specific risk is not hedged.

Table 3 Expected utility of threshold strategies and optimal asset allocation

Strategy	$\sigma_Y = 0$		$\sigma_Y \neq 0$	
	Equity-cash, γ=2	Equity-bond, γ=2	Equity-cash, γ=2	Equity-bond, γ=2
T_U=0.8, T_L=0.4	-0.10539	-0.10985	-0.10521	-0.10967
T_U=0.7, T_L=0.3	-0.11660	-0.11925	-0.11643	-0.11909
T_U=0.9, T_L=0.3	-0.10356	-0.10552	-0.10342	-0.10537
Optimal power, augmented wealth	-0.03516	-0.04023	-0.03534	-0.04031
Optimal power, non-augmented	-0.05176	-0.05809	-0.05179	-0.05809
100% cash	-0.22877		-0.22905	
100% bond		-0.19374		-0.19393
100% equity	-0.06009	-0.06009	-0.06014	-0.06014

The order of expected terminal utility for different allocation strategies using bonds and stock is the same as that using cash and stock. The 100% bonds strategy has a larger expected terminal utility than 100% cash strategy (Table 3).

The results in Fig.1 and Table 3 are from simulations for RRA=2, which is at the lower end of usual RRA estimates. Table 4 summarizes the results for more risk averse individuals, with RRA=6. The general pattern of expected terminal utility for RRA=6 is very similar to that for RRA=2. The inter-temporal optimal allocation has a smaller proportion of pension wealth invested in the (more) risky asset. The "optimal power, augmented wealth", "optimal power, non-augmented" and 100% stock strategies with RRA=6 still outperform the threshold strategies.

Since the objective of the threshold strategy is to keep the terminal wealth around a target value of pension wealth, whereas the objective of the inter-temporal optimization is maximization of expected terminal utility, the comparison between threshold strategy and inter-temporal optimization does not take into account the objectives of threshold strategies. In the following subsection, I will examine the performance of the threshold strategies when the objective of pension funds is to minimize disutility (Vigna and Haberman 2001; Haberman and Vigna 2002).

Table 4 Expected utility of threshold strategies and optimal asset allocation ($\times 10^{-4}$)

Strategy	$\sigma_Y = 0$		$\sigma_Y \neq 0$	
	Equity-cash, γ=6	Equity-bond, γ=6	Equity-cash, γ=6	Equity-bond, γ=6
T_U=0.8, T_L=0.4	-0.05611	-0.08609	-0.05588	-0.08585
T_U=0.7, T_L=0.3	-0.10709	-0.1609	-0.10675	-0.16051
T_U=0.9, T_L=0.3	-0.04886	-0.07019	-0.04868	-0.06996
Optimal power, augmented wealth	-0.00003	-0.00024	-0.00027	-0.00025
Optimal power, non-augmented	-0.00164	-0.00155	-0.00168	-0.00158
100% cash	-25.965		-26.3845	
100% bond		-30.8723		-31.9074
100% equity	-0.00171	-0.00171	-0.00176	-0.00176

3.4. Threshold strategy for minimizing expected terminal disutility

The threshold strategy tries to minimize the difference between the actual wealth and a target wealth. Once the actual wealth approaches the target wealth (reaching $T_U < 1$), all wealth is invested in the riskless assets to slow down the wealth growth. Since such a strategy by design cannot maximize a utility that is monotonically increasing in wealth, it might miss the point to compare threshold strategies with inter-temporal optimization that maximizes expected terminal utility monotonically increasing in wealth. The threshold strategy is likely to be more appropriate for minimizing the disutility defined as a function of deviation from a targeted level (Vigna and Haberman 2001; Haberman and Vigna 2002). The upper threshold T_U can be viewed as an intermediate target aimed to prevent the pension plan from outperforming the final target (a final pension ratio of 2/3) by a big difference as well as losses from a sudden drop in the stock market.

Even if the objective of a pension plan is really the minimization of disutility, which is the squared difference between actual fund and targeted fund (Vigna and Haberman 2001; Haberman and Vigna 2002), the threshold strategy is still not optimal. Such an objective requires the pension plan has an expected terminal wealth with minimum variance, whereas the threshold strategy will invest 100% in stocks at the beginning and 100% in riskless assets in the final period of pension plan, which leads to larger variances than the static allocations (Kritzman 2000; Ma 2007). Using the parameters in Table 1 and assuming constant interest rate *r* and only an initial lump sum without further contributions for simplicity, the threshold strategy are compared with the hybrid strategy discussed in subsection 3.1. The proportion invested in the risky assets before the portfolio value reaches T_U is determined by

$$\theta = \frac{\log\left(T_U W_{\mathrm{Target}} / W_0\right)/(T - t_2) - r}{m_S},$$

where W_{Target} is the targeted terminal wealth. The initial wealth W_0 is 10 and the targeted wealth W_{Target} is 500 when equity risk premium is 0.06 in the numerical simulation. And the targeted wealth W_{Target} is 250 for an equity risk premium of 0.04, and 160 for an equity risk premium of 0.02, to reflect the achievability with the assumed market parameters. The terminal disutility is

$$Disutility = [W_{\text{Target}} - W_T]^2 .$$

As shown in Table 5, the hybrid strategy (a static replicating allocation before reaching T_U and 100% riskless asset after reaching T_U) outperforms the threshold strategy in all three sets of threshold with three different values of equity risk premium. The equity risk premium of 0.06 is an estimate based on historical stock return data in the United States. Obviously from Table 5, the hybrid strategy outperforms the threshold strategy with an equity risk premium of 0.06. It has been argued that average stock returns are likely to be lower in the future than they have been in the past (Blanchard 1993; Campbell and Shiller 2001; Fama and French 2002; Jagannathan, McGrattan and Scherbina 2001). Here the two strategies are also compared with an equity risk premium of 0.04, which is fairly common choice in recent literature (Fama and French 2002; Campbell and Viceira 2002; Gomes and Michaelides 2005), and the hybrid strategy leads to better outcomes. Even with a much lower equity risk premium of 0.02, the hybrid strategy still outperforms the threshold strategy.

Table 5 Expected disutility of threshold strategies and hybrid strategies

Strategy	Threshold			Hybrid		
	m_S=0.06	m_S=0.04	m_S=0.02	m_S=0.06	m_S=0.04	m_S=0.02
T_U=0.8, T_L=0.4	96317.99	35680.99	17469.24	61786.76	15392.22	9006.576
T_U=0.7, T_L=0.3	73724.66	27460.75	14141.75	58094.15	15163.52	9772.816
T_U=0.9, T_L=0.3	99155.70	35821.83	17138.41	64252.65	15629.82	8873.757

Since the disutility from the hybrid strategies is noticeably smaller than that of the threshold strategies, an optimal asset allocation from inter-temporal optimization, which outperforms the threshold strategy, must exist even if the objective of a pension plan is the minimization of disutility defined as squared difference between actual fund and targeted fund. The optimal asset allocation is either the hybrid allocation with expected terminal wealth equal to the targeted wealth, as demonstrated here, or a static or dynamic allocation strategy that outperforms this type of hybrid allocation strategy.

4. CPPI strategy for DC pension plans

In this paper two assets, cash and stocks or bonds and stocks, are used in the CPPI strategy; the weight in stocks =C_M(1-C_F(Floor/Fund), and the remaining proportion of the fund is invested in cash or bonds. As in Blake et al (2001), the floor is set at the level of the liabilities in a comparable DB plan. The analysis is similar if the floor is set at a different level. The control variables are C_M and C_F.

It can be seen from the expression for weight in stocks that there is an inherent shortcoming for CPPI strategy to be applied in pension fund management. Because of the cumulative nature of pension wealth which starts from zero initial wealth, if borrowing and short-selling to get an augmented initial wealth are not allowed, the initial value of Floor/Fund will be very large. All contributions and cumulated pension wealth will be invested in riskless or low risk assets before C_F*Floor/Fund<1. This may lead to very slow growth of pension wealth in the early stage of the DC pension plan. After C_F*Floor/Fund<1, the proportion invested in stocks will generally change along with the stock market.

4.1. Parameters in CPPI strategies

The CPPI strategy is designed to have a minimum wealth (the floor) protected, which appears to be more appropriate for utility that is a function of excess wealth over a guaranteed minimum. When $C_F \neq 1$, the effective floor is $C_F * Floor$, rather than the Floor. What the CPPI strategies can achieve depends on the values of C_M and C_F. C_F determines at what level of liability/fund ratio that pension plan will no longer invest in the risky asset. It is not difficult to see the impact of C_F, if C_F is defined in [0,1]. When $C_F = 0$, all wealth will be invested in risky assets; when $C_F = 1$, since liability is usually larger than fund between the beginning and the termination of a pension plan, all wealth will be invested in the riskless assets. This role of C_F can be illustrated by the following condition

$$\frac{Fund}{Liability} = C_F\,. \tag{31}$$

When this condition is met, all wealth will be invested in the riskless asset no matter what value C_M has. If all wealth is invested at the beginning as a lump sum, the upper bound of C_F should be the ratio of initial wealth-to-initial present value of the expected terminal liability to avoid a 100% riskless asset allocation because of a small initial fund wealth. Therefore, the range of meaningful C_F at time t is

$$0 < C_F < \frac{\text{fund (t)}}{E[\text{terminal liability}]\exp\left[-\int_t^T r(s)ds\right]}\,. \tag{32}$$

C_M determines whether deviation from a (stated or unstated) target will be attenuated or enhanced. The impact of C_M is constrained by both C_F and the expected returns from the riskless and risky assets. For each value of C_M>1, there is a positive (i.e. >0) value of the fund wealth-to-liability ratio above which all wealth will be invested in risky assets. A proper C_M value depends on the market parameters

Assuming that the interest rate is constant, the actual value and the expected value of 100% in the riskless asset are the same

$$E_0[W(t)] = W(t) = W(0)\exp(rt)\,.$$

The expected value of 100% in the risky asset is

$$E_0[W(t)] = W(0)\exp[(r + m_S)t].$$

If the terminal fund equals the terminal liability, the ratio of the present value of expected terminal fund to the present value of expected terminal liability by investing all wealth in risky assets is

$$\begin{aligned} E_0\frac{Fund(t)}{Liability(t)} &= \frac{\exp[-(r+m_s)(T-t)]}{\exp[-r(T-t)]} \\ &= \exp[-(m_s(T-t)]. \end{aligned} \tag{33}$$

Therefore, to ensure that the expected terminal wealth equals the expected terminal liability,

$$C_M(1-C_F\exp[m_S(T-t)])\geq 1. \tag{34}$$

The impact of C_M depends on the value of C_F, fund wealth and allocation strategy. If k of the wealth is invested in risky assets and the rest (1-k) invested in riskless assets, the expected value of the portfolio is

$$E_0[W(t)]=W(0)\exp[(r+km_S)t].$$

Then k has to be chosen to ensure $W(t)\exp[km_S(T-t)]\geq liability(t)$; this is the condition for the expected terminal wealth to be equal to, or larger than, the expected terminal liability.

When a proportion k is invested in the risky asset and the condition $W(t)\exp[km_S(T-t)]\geq liability(t)$ satisfied, there is a positive feedback in the CPPI strategy if

$$C_M > \frac{k}{1-C_F\dfrac{liability}{fund}}. \tag{35}$$

The positive feedback means that an increase in pension wealth to liability ratio leads to an increasing proportion invested in the risky asset until all pension wealth being invested in the risky asset. If

$$C_M < \frac{k}{1-C_F\dfrac{liability}{fund}}, \tag{36}$$

a decreasing proportion of wealth will be invested in the risky asset until all pension wealth being invested in the riskless asset.

The closer C_M to $\frac{k}{1-C_F\frac{liability}{fund}}$, the CPPI portfolio behave more like a static portfolio. From the above analysis, it can be seen that unless $C_M=\frac{k}{1-C_F\frac{liabiliy}{fund}}$, the CPPI strategies encourage the deviation from the level determined by the allocation of k in risky assets and 1-k in riskless assets. This is in sharp contrast with the threshold strategy which has a negative feedback from the funded status and therefore keeps the pension wealth around a target. Since the CPPI strategy does not have a stable expected wealth, it cannot be readily mimicked by a static allocation strategy. A pure equity strategy and a pure riskless asset strategy are the upper and lower limits of the expected wealth from a CPPI strategy.

4.2. CPPI strategy for power terminal utility

Since the ratio of the liabilities to the fund value at retirement is equal to the inverse of the pension ratio, here the inverse of current fund to liability ratio $\psi(t)$ is set equal to the present value of expected liability from the targeted terminal pension ratio divided by current wealth

$$\frac{1}{\psi(t)} = \frac{\exp\left(\int_t^T -r(s)ds\right)E_t[W(T)]}{W(t)} = \frac{\exp\left(\int_t^T -r(s)ds\right)E_t\left[\frac{2}{3}Y(T)a(T)\right]}{W(t)}. \tag{37}$$

In the above equation the targeted pension is assumed to be two thirds of the final salary. This definition of current liability to fund value ratio is to compare the fund performance with the usual defined benefit (DB) pension plans. The CPPI strategy provides the proportion invested in equities

$$\theta = \min\left(1, C_M\left(\max\left(0, 1 - \frac{C_F}{\psi(t)}\right)\right)\right). \tag{38}$$

Obviously there are no such C_M and C_F that $\min\left(1, C_M\left(\max\left(0, 1 - \frac{C_F}{\psi(t)}\right)\right)\right)$ leads to the constant optimal proportion θ* for stock in equations (19) and (20). Therefore, there are no such C_M and C_F in a CPPI strategy that can optimize asset allocation for power terminal utility.

4.3. Comparison with inter-temporal optimization

The results from numerical simulations are shown in Table 6 and Fig.2. The Vasicek interest rate model is used for calculating the present value of terminal liability and the same assumption for $a(t, r(t))$ in calculating the expected terminal liability as for threshold strategies. The allocations derived from optimization for power utility with short-sale of wage replicating portfolio ("Optimal Power, augmented wealth", "power solution borrow" in Fig.2), the "Optimal power, non-augmented" strategy ("power solution no borrow" in Fig.2) and 100% stock strategy dominate those of CPPI strategies for DC plans investing in cash and stocks or investing in bonds and stocks. With market parameters commonly used in pension studies, the 100% stock strategy is the optimal asset allocation when borrowing and short-sale are not allowed (Table 6).

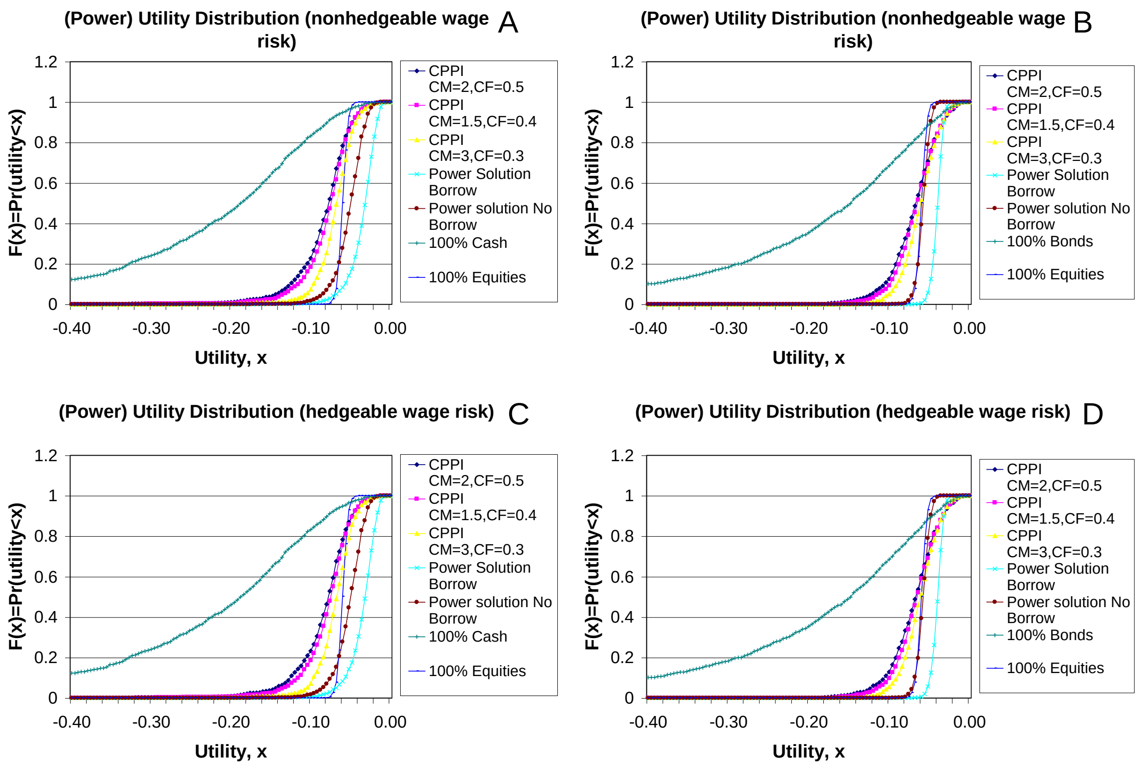
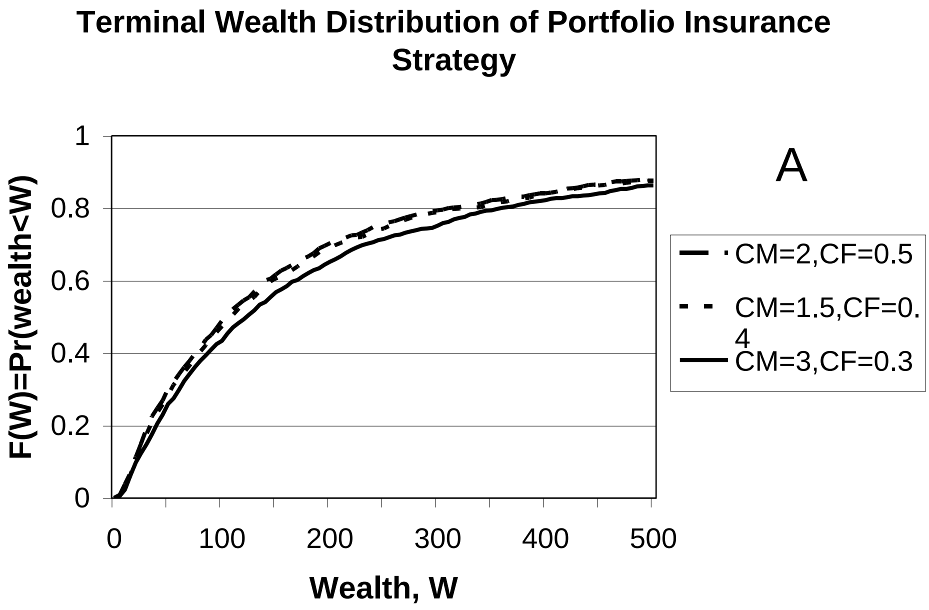


Fig.2 Comparison between CPPI strategies and intertemporal optimization, evaluated for power utility γ=2. Results are from 1000 simulations. A. Equity-cash scenario, $\sigma_Y \neq 0$. B. Equity-bond scenario, $\sigma_Y \neq 0$. C. Equity-cash scenario, $\sigma_Y = 0$. D. Equity-bond scenario, $\sigma_Y = 0$.

When the optimal asset allocation derived for $\sigma_Y = 0$ scenario is applied for the $\pi \neq 0$ and $\sigma_Y \neq 0$ scenario, the "optimal power, augmented wealth", "optimal power, non-augmented" and 100% stock strategies still dominate CPPI strategies. There is little difference in the performance of the optimal asset allocation between the $\sigma_Y = 0$ scenario and the $\pi \neq 0$ and $\sigma_Y \neq 0$ scenario (Table 5). Like the threshold strategies, the CPPI strategies perform better in the $\pi \neq 0$ and $\sigma_Y \neq 0$ scenario than in the $\sigma_Y = 0$ scenario, indicating that the CPPI strategies may take advantage of the variability in wage growth.

Table 6 Expected utility of CPPI strategies and optimal asset allocation

Strategy	$\sigma_Y = 0$		$\sigma_Y \neq 0$	
	Equity-cash, γ=2	Equity-bond, γ=2	Equity-cash, γ=2	Equity-bond, γ=2
C_M=2,C_F=0.5	-0.08375	-0.06954	-0.08360	-0.06933
C_M=1.5,C_F=0.4	-0.07947	-0.06691	-0.07930	-0.06673
C_M=3,C_F=0.3	-0.06933	-0.06227	-0.06926	-0.06210
Optimal power, augmented wealth	-0.03516	-0.04023	-0.03534	-0.04031
Optimal power, non-augmented	-0.05176	-0.05809	-0.05179	-0.05809
100% cash	-0.22877		-0.22905	
100% bond		-0.19374		-0.19393
100% equity	-0.06009	-0.06009	-0.06014	-0.06014



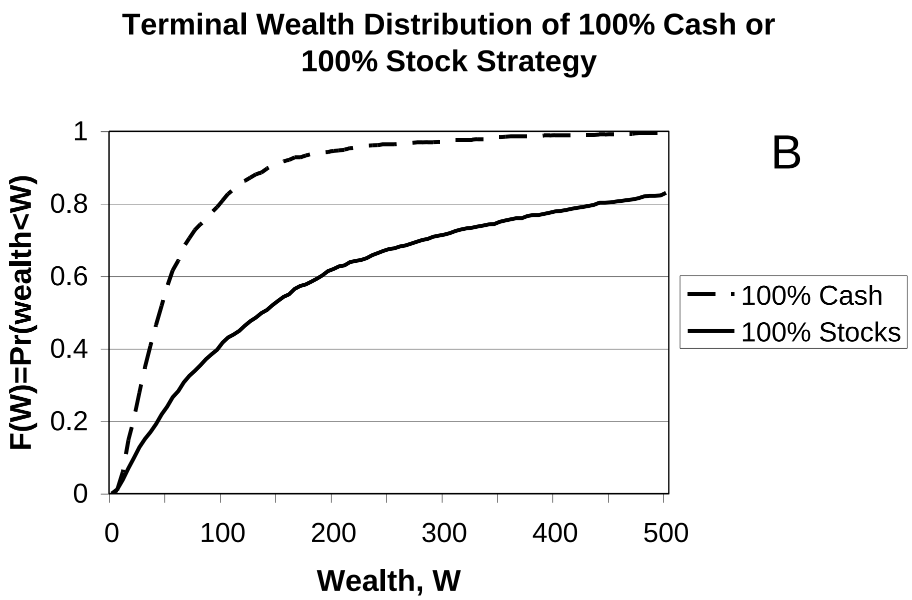


Fig.3 Distribution of terminal pension wealth (wealth at retirement) of CPPI strategies, 100% cash and 100% stocks. Results are from 1000 simulations of wealth growth paths. A. CPPI strategies. B. 100% cash and 100% stocks.

The analysis in the preceding section indicates that for C_M substantially different from $\dfrac{k}{1-C_F\dfrac{liability}{fund}}$, the portfolio will move to a 100% stock or 100% cash allocation. This inference is not confirmed by the wealth distribution of CPPI strategies in Fig.3A, which shows a smooth curve over a wide range rather than two steps.

The distribution pattern shown in Fig.3A suggests three possibilities: 1) the values of C_M is close to $\dfrac{k}{1-C_F\,\dfrac{liability}{fund}}$; 2) the length of the pension plan is not long enough for the portfolios to diverge far enough; 3) the terminal wealth of 100% stock and 100% cash portfolios per se is distributed over a wide range. The first possibility implies that C_F is close to zero, which can be excluded because the C_F values used here are not close to zero. The second possibility is implausible because 45 years is used and there is no obvious diversion; a much longer pension plan will be irrelevant to reality. Fig.3B supports the third possibility; the terminal wealth of 100% stock and 100% cash portfolios per se is distributed over a wide range. The three CPPI strategy curves in Fig.3A will lie between the 100% cash curve and the 100% stock curve and closer to the 100% stock curve if they are plotted together, suggesting that the move toward 100% stock is more prevalent.

The results in Fig.2 and Table 6 are from simulations for RRA=2, which is at the lower end of usual RRA estimates. Table 7 summarizes the results for more risk averse individuals, with RRA=6. The general pattern of expected terminal utility for RRA=6 is very similar to that for RRA=2. The inter-temporal optimal allocation has a smaller proportion of pension wealth invested in the (more) risky asset. The "optimal power, augmented wealth", "optimal power, non-augmented" and 100% stock strategies with RRA=6 still outperform the three CPPI strategies.

Since the objective of the CPPI strategy is to maximize the utility derived from wealth over a guaranteed minimum, the comparison between the CPPI strategy and the inter-temporal optimal strategy based on utility derived from final pension wealth relative to the final wages does not take into account the objectives of the CPPI strategies. The more appropriate comparison should be based on the utility derived from wealth over a guaranteed minimum. In the following subsection, I will examine the performance of the CPPI strategies when the objective of pension funds is to maximize utility derived from final pension wealth relative to the final wages does not take into account the objectives of the CPPI strategies.

Table 7 Expected utility of CPPI strategies and optimal asset allocation with RRA=6 ($\times 10^{-4}$)

Strategy	$\sigma_Y = 0$		$\sigma_Y \neq 0$	
	Equity-cash, γ=6	Equity-bond, γ=6	Equity-cash, γ=6	Equity-bond, γ=6
C_M=2,C_F=0.5	-0.05396	-0.01342	-0.05643	-0.01295
C_M=1.5,C_F=0.4	-0.03903	-0.00942	-0.04012	-0.00916
C_M=3,C_F=0.3	-0.01362	-0.00527	-0.012	-0.00516
Optimal power, augmented wealth	-0.00003	-0.00024	-0.00027	-0.00025
Optimal power, non-augmented	-0.00164	-0.00155	-0.00168	-0.00158
100% cash	-25.965		-26.3845	
100% bond		-30.8723		-31.9074
100% equity	-0.00171	-0.00171	-0.00176	-0.00176

4.4. CPPI strategy for utility with a guaranteed minimum wealth

Although the CPPI strategy appears to be more appropriate for terminal utility that is a function of wealth over a guaranteed minimum, it is not an optimal allocation rule even for such a utility function. To show this, we can compare the CPPI strategy with a minimum terminal wealth insurance (MTWI) strategy. For simplicity, the constant interest rate and no wage income scenario in subsections 3.1, 3.4 and 4.1 is used here again, and other parameters are those in Table 1. The MTWI strategy is to invest $W_I e^{-r(T-t)}$ in riskless asset where W_I is the guaranteed minimum terminal wealth and T retirement date, and the rest of wealth in the risky asset. The initial wealth W_0 is

10 and the minimum guaranteed wealth or the Floor W_I is 50 in the numerical simulation. The CPPI strategy invests $C_M\left(1-C_F\frac{W_I}{W(t)}\right)$ of wealth in risky assets and the rest in riskless assets. As shown in Table 8, where utility is defined as $U=\frac{1}{1-\gamma}[W(T)-W_I]^{1-\gamma}$ and γ is the relative risk aversion coefficient, the MTWI strategy outperforms the CPPI strategy in all four sets of C_M and C_F with three different values of equity risk premium. Since in the CPPI strategy C_F (<1) implies investing in risky assets before the portfolio value equal to the floor, the MTWI strategy also invest $C_F W_I e^{-r(T-t)}$ in risky assets to reflect the risk tolerance represented by C_F (<1).

Table 8 Expected utility of CPPI strategies and minimum terminal wealth insurance (MTWI) strategy

Strategy	CPPI			MTWI: $C_F W_I e^{-r(T-t)}$		
	m_S=0.06	m_S=0.04	m_S=0.02	m_S=0.06	m_S=0.04	m_S=0.02
C_M=2, C_F=1	-0.02972	-0.12354	-0.37711	-0.00730	-0.0176	-0.04171
C_M=2, C_F=0.5	$-\infty$ (-31.0073)	$-\infty$ (-132.011)	$-\infty$ (-318.073)	$-\infty$ (-8.00804)	$-\infty$ (-50.0286)	$-\infty$ (-180.051)
C_M=1.5, C_F=0.4	$-\infty$ (-25.0114)	$-\infty$ (-116.016)	$-\infty$ (-303.037)	$-\infty$ (-10.0084)	$-\infty$ (-60.036)	$-\infty$ (-202.049)
C_M=3, C_F=0.3	$-\infty$ (-22.0082)	$-\infty$ (-102.082)	$-\infty$ (-267.071)	$-\infty$ (-12.0073)	$-\infty$ (-72.0751)	$-\infty$ (-221.075)

The utility values in Table 8 are calculated with the guaranteed minimum $W_I = 50$. Only when C_F=1, the CPPI and MTWI strategies can truly guarantee the minimum wealth. Since for other C_F values there are some cases with terminal wealth less than the guaranteed minimum $W_I = 50$ in the 1000 simulations, those events lead to negative infinite utility. The numbers in brackets are computed for comparison between otherwise negative infinity utility values by assuming $U[W(T)-W_I] = -1000, \ \forall W(T)-W_I < 0$.

Table 9 Comparison between threshold and CPPI strategies (RRA=2, m_S=0.06)

Strategy		Disutility $D=[W(T)-W_{target}]^2$	Utility $U = \frac{[W(T)-W_I]^{1-\gamma}}{1-\gamma}$
Threshold	T_U=0.8, T_L=0.4	96317.99	$-\infty$ (-19.0085)
	T_U=0.7, T_L=0.3	73724.66	$-\infty$ (-19.0086)
	T_U=0.9, T_L=0.3	99155.70	$-\infty$ (-19.0085)
CPPI	C_M=2,C_F=1	11287094.49	-0.02972
	C_M=2, C_F=0.5	12458181.20	$-\infty$ (-31.0073)
	C_M=1.5, C_F=0.4	12400583.52	$-\infty$ (-25.0114)
	C_M=3, C_F=0.3	12463489.25	$-\infty$ (-22.0082)

It is clear from Table 8 that the CPPI strategy is not the optimal asset allocation strategy even for utility that is a function of excess terminal wealth over a guaranteed minimum wealth. Therefore, an optimal asset allocation strategy must exist, which is either this MTWI strategy, or other static or dynamic asset allocation strategy that outperforms the MTWI strategy.

When the expected terminal disutility defined as squared difference between actual wealth and targeted wealth is examined for the results of CPPI strategies, their expected disutility is much higher than those of the threshold (Table 9) (and static-riskless hybrid strategies, Table 5). When the expected terminal utility defined as a function of excess wealth over a guaranteed minimum is examined for the results of threshold strategies, their expected utility is negative infinity ($-\infty$) in all three cases, indicating that the threshold strategy is inappropriate for guaranteeing a minimum terminal wealth. The MTWI strategy and the real CPPI rule (C_F=1) outperform the threshold strategy. If the downsize utility is cut off at -1000, other CPPI strategies with C_F<1 generally perform less well than the threshold strategy (Table 9).

5. Conclusion

In this paper, the threshold and CPPI strategies are compared with the optimal asset allocations by inter-temporal optimization, a hybrid static-riskless strategy, or a minimum terminal wealth insurance (MTWI) strategy. When terminal utility is a function of terminal wealth-to wage ratio, the optimal asset allocation always outperforms the threshold and CPPI strategy whether borrowing and short-sale are allowed or not. These results are true for both the $\sigma_Y = 0$ and $\sigma_Y \neq 0$ cases no matter whether the pension wealth is augmented by short-selling a replicating portfolio to be paid by future pension contributions.

The threshold strategies can be mimicked by hybrid static-riskless allocation strategies with same expected return and less risk, and therefore they are second order dominated by their corresponding static-riskless hybrid allocation strategies. The threshold strategy is designed to keep the portfolio value around a target, whose objective can be described as minimization of expected terminal disutility defined as squared difference between actual fund and target fund (Vigna and Haberman 2001; Haberman and Vigna 2002). Since the utility of most investors is increasing in wealth, threshold strategies are generally inappropriate for DC pension plans. Since the hybrid static-riskless allocation outperforms the threshold strategy even when the objective of a DC pension plan is to minimize the expected terminal disutility defined as squared difference between actual fund and target fund, a superior asset allocation from inter-temporal optimization exists and outperforms the threshold strategy.

The CPPI strategy is designed to have a minimum protected portfolio wealth as well as take advantage of high returns of risky assets. Because of the cumulative nature of DC pension fund, CPPI strategies tend to lead to slow growth of pension wealth in the early stage of DC pension plans. The optimal allocation strategy with constraints on borrowing and short-sale (100% stock strategy) outperforms the CPPI strategy due to its slow growth at the beginning. When terminal utility is a function of excess wealth over a guaranteed minimum, which seems to describe the objective of CPPI strategy, the MTWI strategy outperforms the CPPI strategy.

To summarize, the optimal asset allocation strategy from intertemporal optimization produces higher expected terminal utility than those of threshold and CPPI strategies no matter whether borrowing and short-sale are allowed and no matter whether the wage incomes are fully hedgeable ($\sigma_Y = 0$) or not ($\sigma_Y \neq 0$). There are better strategies when the objectives of the threshold and CPPI strategies are minimization of expected terminal disutility and maximization of expected terminal utility which is a function of terminal wealth over a guaranteed minimum, respectively. Therefore, threshold and CPPI strategies are suboptimal in managing DC pension plans.

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