

the collection $E = \{U_k^* U_{k+1}^* \chi_{I_k} W_k e_{k,j} : j, k \in \mathbf{Z}\}$ is an orthonormal basis for $L^2(\mathbf{R})$ consisting of functions of compact support. If in addition all the functions $e_{k,j}$ and r_k , $k, j \in \mathbf{Z}$, belong to $C^d(\mathbf{R})$ for some $0 \leq d \leq \infty$, then the functions in E belong to $C_0^d(\mathbf{R})$.

Proof. Adjacent intervals I_k, I_{k+1} are compatible, so $L^2(\mathbf{R}) = \bigoplus_{k \in \mathbf{Z}} P_{I_k} L^2(\mathbf{R})$. Each of the spaces $P_I L^2(\mathbf{R})$ has an orthonormal basis $E_k = \{U_k^* U_{k+1}^* \chi_{I_k} W_k e_{k,j} : j \in \mathbf{Z}\}$. Putting these bases together into $E = \bigcup_{k \in \mathbf{Z}} E_k$ yields the result. \square

The second main result is just the adjoint of Theorem 2. There are better-tested computer programs for expanding a periodic function in a periodic basis, so the adjoint result is more useful in practice:

Theorem 4. *Suppose that the triplets $(r_0, \alpha_0, \epsilon_0), (r_1, \alpha_1, \epsilon_1)$ and $(r, \alpha_0, \epsilon), (r, \alpha_1, \epsilon)$ satisfy the consistency conditions. Write $I = (\alpha_0, \alpha_1)$, $U_0 = U(r_0, \alpha_0, \epsilon_0)$, $U_1 = U(r_1, \alpha_1, \epsilon_1)$, and $W_I^* = W^*(r, I, \epsilon)$. If f belongs to $C^d(\mathbf{R})$, then $T_I f \stackrel{\text{def}}{=} W_I^* \chi_I U_0 U_1 f$ has an I -periodic extension which belongs to $C^d(\mathbf{R})$. Also, T_I is a unitary isomorphism from $P_I L^2(\mathbf{R})$ to $L^2(I)$.*

Proof. Since $\chi_I U_0 U_1 f$ satisfies Eq.(5), we have by Eq.(12) that $[W_I^* \chi_I U_0 U_1 f]^{(n)}(\alpha_0+) = [W_I^* \chi_I U_0 U_1 f]^{(n)}(\alpha_1-)$ for all $0 \leq n \leq d$. Hence $W_I^* \chi_I U_0 U_1 f$ has a unique continuous periodic extension in $C^d(\mathbf{R})$.

For the second part, we note that $U_0 U_1$ is a unitary isomorphism from $U_0^* U_1^* \chi_I L^2(\mathbf{R}) \cong P_I L^2(\mathbf{R})$ to $\chi_I L^2(\mathbf{R})$, and $W_I^* \chi_I$ is a unitary automorphism on $\chi_I L^2(\mathbf{R}) \cong L^2(I)$. \square

A natural choice for the periodic basis is $e_{k,j}(t) = \exp(2\pi i j t / |I_k|)$. Another is periodized compactly supported wavelets, which avoid the big support problem of smoother wavelets. Likewise periodic wavelet packets can be used to construct local “best-bases” with no artifacts from cutting or windowing. Discrete versions exist for all of these transforms.

REFERENCES

- [AWW] Pascal Auscher, Guido Weiss, and M. V. Wickerhauser, *Local sine and cosine bases of Coifman and Meyer and the construction of smooth wavelets*, Wavelets—A Tutorial in Theory and Applications, C. K. Chui (ed.) ISBN 0-12-174590-2, Academic Press, Boston, 1992, pp. 237–256.
- [CM] R. R. Coifman and Y. Meyer, *Remarques sur l'analyse de Fourier à fenêtre*, série I, C. R. Acad. Sci. Paris **312** (1991), 259–261.
- [M] H. Malvar, *Lapped transforms for efficient transform/subband coding*, IEEE Trans. Acoustics, Speech, and Signal Processing **38** (1990), 969–978.

For these to be well defined, we must assume that (r, α_0, ϵ) and (r, α_1, ϵ) satisfy the consistency conditions. Then using Ω , we can write the relationship between W and U . For all $t \in I = (\alpha_0, \alpha_1)$, we have:

$$(11) \quad \begin{aligned} W(r, I, \epsilon) f(t) &= U(r, \alpha_0, \epsilon)U(r, \alpha_1, \epsilon)\Omega_{|I|\chi_I} f(t); \\ W^*(r, I, \epsilon) f(t) &= U^*(r, \alpha_0, \epsilon)U^*(r, \alpha_1, \epsilon)\Omega_{|I|\chi_I} f(t). \end{aligned}$$

When there is no possibility of confusion, we will write W_I for $W(r, I, \epsilon)$ where $I = (\alpha_0, \alpha_1)$, suppressing the r and ϵ . We observe that W_I and W_I^* are unitary isomorphisms of $L^2(\mathbf{R})$ (i.e., $W_I^*W_I = W_IW_I^* = Id$) because $|r(t)|^2 + |r(-t)|^2 = 1$ for all t . Also, if $t < \alpha_0$ or $t > \alpha_1$, then $W_I f(t) = f(t)$ and $W_I^* f(t) = f(t)$. Thus if I and J are disjoint intervals, the operators W_I, W_I^*, W_J , and W_J^* all commute. We also note that if f is smooth in the interval I , then $W_I f$ is also smooth there. Furthermore, $W_I f$ satisfies the same boundary conditions at α_0+ and α_1- as U_0U_1f . If f is periodic and satisfies Eq.(5) at α_0+ and α_1- , then $W_I^* f$ satisfies the equation:

$$(12) \quad \lim_{t \rightarrow \alpha_0+} [W_I^* f]^{(n)}(t) = \lim_{t \rightarrow \alpha_1-} [W_I^* f]^{(n)}(t), \quad \text{for all } 0 \leq n \leq d.$$

Thus $W_I^* f$ has a continuous periodic extension in $C^d(\mathbf{R})$.

The first main point of this paper is that from arbitrary smooth *periodic* orthonormal bases we can construct smooth *compactly supported* orthonormal bases:

Theorem 2. *Suppose that $(r, \alpha_0, \epsilon), (r, \alpha_1, \epsilon)$ and $(r_0, \alpha_0, \epsilon_0), (r_1, \alpha_1, \epsilon_1)$ satisfy the consistency conditions, and write $I = (\alpha_0, \alpha_1)$ and $W_I = W(r, I, \epsilon)$. If $\{e_j\}_{j \in \mathbf{Z}}$ is a collection of $|I|$ -periodic functions which form an orthonormal basis for $L^2(I)$ when restricted to I , then $E_0 = \{U_0^*U_1^*\chi_I W_I e_j\}_{j \in \mathbf{Z}}$ is an orthonormal basis of $P_I L^2(\mathbf{R})$. In addition, if $\{e_j\}_{j \in \mathbf{Z}} \subset C^d(\mathbf{R})$, then $E_0 \subset C_0^d(\mathbf{R})$. \square*

We can get an orthonormal basis for $L^2(\mathbf{R})$ out of this by decomposing \mathbf{R} into adjacent compatible intervals. We write $\mathbf{R} = \bigcup_{k \in \mathbf{Z}} I_k \stackrel{\text{def}}{=} \bigcup_{k \in \mathbf{Z}} [a_k, a_{k+1})$ and define $U_k^* = U^*(r_k, \alpha_k, \epsilon_k)$ and $W_k = W(\tilde{r}_k, I_k, \tilde{\epsilon}_k)$.

Corollary 3. *Suppose that for each $k \in \mathbf{Z}$, the triplets $(r_k, \alpha_k, \epsilon_k), (r_{k+1}, \alpha_{k+1}, \epsilon_{k+1})$ and $(\tilde{r}_k, \alpha_k, \tilde{\epsilon}_k), (\tilde{r}_k, \alpha_{k+1}, \tilde{\epsilon}_k)$ satisfy the consistency conditions, and the family of $|I_k|$ -periodic functions $\{e_{k,j} : j \in \mathbf{Z}\}$ forms an orthonormal basis of $L^2(I_k)$ when restricted to I_k ; Then*

Then conjugation with δ_ϵ and τ_α dilates and translates the “range of influence” of the folding operators to an arbitrary interval $(\alpha-\epsilon, \alpha+\epsilon)$. We obtain a family of folding (respectively unfolding) operators indexed by (r, α, ϵ) :

$$(7) \quad U(r, \alpha, \epsilon) = \tau_\alpha^* \delta_\epsilon^* U(r) \delta_\epsilon \tau_\alpha; \quad U^*(r, \alpha, \epsilon) = \tau_\alpha^* \delta_\epsilon^* U^*(r) \delta_\epsilon \tau_\alpha.$$

Where convenient we will write U_0 for $U(r_0, \alpha_0, \epsilon_0)$, and so on. We note that if r_0 and r_1 satisfy Eq.(1), and the intervals $(\alpha_0-\epsilon_0, \alpha_0+\epsilon_0)$ and $(\alpha_1-\epsilon_1, \alpha_1+\epsilon_1)$ are disjoint, then the operators U_0, U_1, U_0^* and U_1^* all commute. In this case we will say that $(r_0, \alpha_0, \epsilon_0)$ and $(r_1, \alpha_1, \epsilon_1)$ satisfy the *consistency conditions*. For such triplets, the projection $P_{(\alpha_0, \alpha_1)}$ of [AWW] factors as follows:

$$(8) \quad P_{(\alpha_0, \alpha_1)} = U_0^* U_1^* \chi_{(\alpha_0, \alpha_1)} U_1 U_0,$$

where $\chi_I f(t) = f(t)$ if $t \in I$, and $\chi_I f(t) = 0$ otherwise. The map P_I replaces the traditional windowing of periodic bases. If we use a single ϵ and a single r , then the interval $I = (\alpha_0, \alpha_1)$ and its adjacent translate $J = (\alpha_0+|I|, \alpha_1+|I|)$ are compatible, i.e., $P_I + P_J = P_{I \cup J}$ and $P_I P_J = 0$. The main consequence of this observation is that P_I composed with $|I|$ -periodization is a unitary isomorphism from $P_I L^2(\mathbf{R})$ to $L^2(I)$. We make this more precise by defining the λ -periodization $\Omega_\lambda f$ of a function $f = f(t)$ by the formula $\Omega_\lambda f(t) \stackrel{\text{def}}{=} \sum_{k \in \mathbf{Z}} f(t + k\lambda) = \sum_{k \in \mathbf{Z}} \tau_{k\lambda} f(t)$. If f belongs to $L^2(\mathbf{R})$ and is compactly supported, then $\Omega_\lambda f$ belongs to $L^2_{loc}(\mathbf{R})$ and is periodic of period λ . If in addition f belongs to $C^d(\mathbf{R})$, then $\Omega_\lambda f$ also belongs to $C^d(\mathbf{R})$.

We can now define a “periodized” version of the folding and unfolding operators. The periodization is hidden in the definition of the following operators:

$$(9) \quad W(r, (\alpha_0, \alpha_1), \epsilon) f(t) = \begin{cases} r \left(\frac{t-\alpha_0}{\epsilon} \right) f(t) + r \left(\frac{\alpha_0-t}{\epsilon} \right) f(\alpha_0 + \alpha_1 - t), & \text{if } \alpha_0 < t \leq \alpha_0 + \epsilon, \\ r \left(\frac{\alpha_1-t}{\epsilon} \right) f(t) - r \left(\frac{t-\alpha_1}{\epsilon} \right) f(\alpha_0 + \alpha_1 - t), & \text{if } \alpha_1 - \epsilon \leq t < \alpha_1, \\ f(t), & \text{otherwise;} \end{cases}$$

$$(10) \quad W^*(r, (\alpha_0, \alpha_1), \epsilon) f(t) = \begin{cases} r \left(\frac{t-\alpha_0}{\epsilon} \right) f(t) - r \left(\frac{\alpha_0-t}{\epsilon} \right) f(\alpha_0 + \alpha_1 - t), & \text{if } \alpha_0 < t \leq \alpha_0 + \epsilon, \\ r \left(\frac{\alpha_1-t}{\epsilon} \right) f(t) + r \left(\frac{t-\alpha_1}{\epsilon} \right) f(\alpha_0 + \alpha_1 - t), & \text{if } \alpha_1 - \epsilon \leq t < \alpha_1, \\ f(t), & \text{otherwise.} \end{cases}$$

A general construction for such functions is given in [AWW]. We now define the *folding* operator $U = U(r)$ and its adjoint *unfolding* operator $U^* = U^*(r)$:

$$(2) \quad Uf(t) = \begin{cases} r(t)f(t) + r(-t)f(-t), & \text{if } t > 0, \\ \overline{r(-t)}f(t) - \overline{r(t)}f(-t), & \text{if } t < 0; \end{cases}$$

$$(3) \quad U^*f(t) = \begin{cases} \overline{r(t)}f(t) - r(-t)f(-t), & \text{if } t > 0, \\ r(-t)f(t) + \overline{r(t)}f(-t), & \text{if } t < 0. \end{cases}$$

We observe that $Uf(t) = f(t)$ and $U^*f(t) = f(t)$ if $t \geq 1$ or $t \leq -1$. Also, $U^*Uf(t) = UU^*f(t) = (|r(t)|^2 + |r(-t)|^2)f(t) = f(t)$ for all $t \neq 0$, so that U and U^* are unitary isomorphisms of $L^2(\mathbf{R})$. It does not matter how we define $Uf(0)$ or $U^*f(0)$ for functions $f \in L^2$; for smooth f we may just as well define $Uf(0) \stackrel{\text{def}}{=} f(0)$, and for f satisfying certain smoothness and boundary limit conditions we will show that there is a unique smooth extension of U^*f across $t = 0$.

Lemma 1. *Suppose $r \in C^d(\mathbf{R})$ for $0 \leq d \leq \infty$. If $f \in C^d(\mathbf{R})$, then $Uf \in C^d(\mathbf{R} \setminus \{0\})$, the limits $[Uf]^{(n)}(0+)$ and $[Uf]^{(n)}(0-)$ exist for all $0 \leq n \leq d$, and we have:*

$$(4) \quad [Uf]^{(n)}(0+) = 0 \quad \text{if } n \text{ is odd}, \quad [Uf]^{(n)}(0-) = 0 \quad \text{if } n \text{ is even}.$$

Conversely, if $f \in C^d(\mathbf{R} \setminus \{0\})$ has limits $f^{(n)}(0+)$ and $f^{(n)}(0-)$ for all $0 \leq n \leq d$ which satisfy the equations

$$(5) \quad f^{(n)}(0+) = 0 \quad \text{if } n \text{ is odd}, \quad f^{(n)}(0-) = 0 \quad \text{if } n \text{ is even},$$

*then U^*f has a unique continuous extension (across $t = 0$) which belongs to $C^d(\mathbf{R})$. \square*

This lemma shows that just a trivial boundary condition is needed to obtain smoothness. In particular, the 0 function satisfies the condition, and we shall use this fact to clarify the construction of the smooth orthogonal projections in [AWW]. For real numbers α and $\epsilon > 0$, define the translation and rescaling operators:

$$(6) \quad \begin{aligned} \tau_\alpha f(t) &= f(t+\alpha); & \tau_\alpha^* f(t) &= f(t-\alpha); \\ \delta_\epsilon f(t) &= \sqrt{\epsilon}f(\epsilon t); & \delta_\epsilon^* f(t) &= \frac{1}{\sqrt{\epsilon}}f(t/\epsilon). \end{aligned}$$

SMOOTH LOCALIZED ORTHONORMAL BASES

MLADEN VICTOR WICKERHAUSER

Department of Mathematics, Washington University
Campus Box 1146, St. Louis, Missouri 63130

July 30, 1992

ABSTRACT. We describe an orthogonal decomposition of $L^2(\mathbf{R})$ which maps smooth functions to smooth periodic functions. It generalizes previous constructions by Malvar, Coifman and Meyer. The adjoint of the decomposition can be used to construct smooth orthonormal windowed exponential, wavelet and wavelet packet bases.

Orthogonal projections which map smooth functions to smooth compactly supported functions appeared in the work of Malvar [M] and Coifman and Meyer [CM]. In those papers the projections were used to build a smooth overlapping orthogonal basis on the line, composed of windowed sine (or cosine) functions. Many of these bases' properties were only briefly described in the short papers of Malvar, Coifman, and Meyer, but are developed in detail in [AWW]. We do not wish to overlook the original sources, but we will take advantage of some of the later paper's structure and notation. In this short note we sketch a construction of smoothness-preserving unitary maps onto periodic functions. We thereby show that arbitrary smooth periodic bases can be used as smooth "windowed" orthonormal bases on the line. We evade the Balian-Low obstruction by a modification of the definition of "window." The main ingredient in the recipe is a pair of orthogonal projections which can be factored into simple pieces. Let $r = r(t)$ be a function in the class $C^d(\mathbf{R})$ for some $0 \leq d \leq \infty$, satisfying the following conditions:

$$(1) \quad |r(t)|^2 + |r(-t)|^2 = 1 \quad \text{for all } t \in \mathbf{R}; \quad r(t) = \begin{cases} 0, & \text{if } t \leq -1, \\ 1, & \text{if } t \geq 1; \end{cases}$$

Research supported in part by AFOSR Grant F49620-92-J-0106.