Expressive ABox Reasoning with Number Restrictions, Role Hierarchies, and Transitively Closed Roles

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Abstract

We present a new tableaux calculus deciding the ABox consistency problem for the expressive description logic $\mathcal{ALCNH}_{R^+}$. Prominent language features of $\mathcal{ALCNH}_{R^+}$ are number restrictions, role hierarchies, transitively closed roles, and generalized concept inclusions. The ABox description logic system RACE is based on the calculus for $\mathcal{ALCNH}_{R^+}$.

1 Introduction

Experiences with concept languages indicate that at least description logics (DLs) with negation and disjunction are required to solve practical modeling problems without resorting to ad hoc extensions. The requirements derived from practical applications of DLs ask for even more expressive languages. For instance, in [14] the need for transitive roles is demonstrated for representing part-whole relations, family relations or partial orders in general. It is argued that the trade-off between expressivity and complexity favors the integration of transitively closed roles instead of a transitive closure operator for roles. Other examples are given in [8], where the area of medical terminology is discussed. Design studies for the Galen project identified the need for modeling of transitive part-whole, causal and compositional relations, and to organize these relations into a hierarchy. Moreover, generalized concept inclusions were also required as a modeling tool, e.g. for expressing sufficient conditions of concepts.

2 The Description Logic $\mathcal{ALCNH}_{R^+}$

Motivated by the above-mentioned requirements we introduce in this paper an ABox tableaux calculus for the description logic $\mathcal{ALCNH}_{R^+}$. It augments the basic logic $\mathcal{ALC}$ [15] with number restrictions, role hierarchies, and transitively closed roles. Note that these language features imply the presence of generalized concept inclusions and cyclic concepts. The use of number restrictions in combination with transitive roles and role hierarchies is syntactically restricted: no number restrictions are possible for (i) transitive roles and (ii) for any role which has a transitive subrole. Furthermore, we assume that the unique name assumption holds for ABox individuals.

$\mathcal{ALCNH}_{R^+}$ is an extension of $\mathcal{ALCNH}$ that itself can be polynomially reduced to $\mathcal{ALCNR}$ [1] and vice versa. It is possible to rephrase every hierarchy of role names with a set of role conjunctions and vice versa [1]. Thus, our work on $\mathcal{ALCNH}_{R^+}$ extends the work on $\mathcal{ALCNR}$ by additionally providing transitively closed roles. $\mathcal{ALCNH}_{R^+}$ also extends other related description logics such as $\mathcal{ALC}_{R^+}$ [14] and $\mathcal{ALCH}f_{R^+}$ [8]. Recently, the work on these logics has been extended and a tableaux calculus for deciding concept consistency for the language $\mathcal{ALCQHI}_{R^+}$ has been presented in [11]. Another approach is presented in [2] where the logic $\mathcal{CIQ}$ for reasoning with TBoxes and ABoxes is introduced. In comparison to $\mathcal{ALCNH}_{R^+}$ and the other approaches mentioned above $\mathcal{CIQ}$ offers more operators (e.g. the transitive closure) but does not support role hierarchies and allows number restrictions only for primitive roles.

ABox reasoning truly extends the usefulness of description logics in practical applications. The increase of expressiveness is also reflected in an increase of the complexity of the tableaux rules (see Section 4.1 for more details). An alternative might be the so-called "precompletion approach" originally developed for the language $\mathcal{ALCQ}$ [7] and recently adapted to $\mathcal{ALCH}_{R^+}$ [16]. The idea behind the precompletion approach is to transform given ABoxes in a way such that ABox satisfiability is reduced to concept satisfiability. However, there currently exist no calculi for computing the precompletion of ABoxes for languages such as $\mathcal{ALCNH}_{R^+}$ or even $\mathcal{ALCQHI}_{R^+}$.

2.1 The Concept Language

We present the syntax and semantics of the language for specifying concept and role inclusions.

Definition 1 (Role Inclusions, Role Hierarchy) Let P and T be disjoint sets of non-transitive and transitive *role* names, respectively, and let R be defined as $R = P \cup T$. Let R and S be role names, then $\mathsf{R} \sqsubseteq \mathsf{S}$ *(role inclusion axiom)* is a terminological axiom. Given a set of role inclusion axioms, we define a *role hierarchy* where $\sqsubseteq^*$ is the reflexive transitive closure of $\sqsubseteq$ over R.

Additionally we define the set of ancestors and descendants of a role.

Definition 2 (Role Descendants/Ancestors) Given a role hierarchy the set $\mathsf{R}^{\uparrow} := \{\mathsf{S} \in R \,|\, \mathsf{R} \sqsubseteq^* \mathsf{S}\}$ defines the *ancestors* and $\mathsf{R}^{\downarrow} := \{\mathsf{S} \in R \,|\, \mathsf{S} \sqsubseteq^* \mathsf{R}\}$ the *descendants* of a role R. We also define the set $S := \{\mathsf{R} \in P \,|\, \mathsf{R}^{\downarrow} \cap T = \emptyset\}$ of *simple* roles that are neither transitive nor have a transitive role as descendant.

Definition 3 (Concept Terms) Let C be a set of concept names which is disjoint from R. Any element of C is a *concept term*. If C and D are concept terms, $\mathsf{R} \in R$ is an arbitrary role, $\mathsf{S} \in S$ is a simple role, $n > 1$, and $m > 0$, then the following expressions are also concept terms:

- $\top$ *(top concept)*,
- $\bot$ *(bottom concept)*,
- $\mathsf{C} \sqcap \mathsf{D}$ *(conjunction)*,
- $\mathsf{C} \sqcup \mathsf{D}$ *(disjunction)*,
- $\neg \mathsf{C}$ *(negation)*,
- $\forall \mathsf{R} \,.\, \mathsf{C}$ *(concept value restriction)*,
- $\exists \mathsf{R} \,.\, \mathsf{C}$ *(concept exists restriction)*,
- $\exists_{\leq m} \mathsf{S}$ *(at most number restriction)*,
- $\exists_{\geq n} \mathsf{S}$ *(at least number restriction)*.

For an arbitrary role R, the term $\exists_{\geq 1} \mathsf{R}$ can be rewritten as $\exists \mathsf{R} \,.\, \top$, $\exists_{\geq 0} \mathsf{R}$ as $\top$, and $\exists_{\leq 0} \mathsf{R}$ as $\forall \mathsf{R} \,.\, \bot$. Thus, we do not consider these terms as number restrictions in our language.

The concept language is syntactically restricting the combination of number restrictions and transitive roles. Number restrictions are only allowed for *simple* roles. This restriction is motivated by doubtful semantics for an unrestricted combinability and a simplified tableaux decision procedure. Moreover, this decision is supported by a recent undecidability result for the logic $\mathcal{ALCHNI}_{R^+}$ in case of an unrestricted combinability [11].

Definition 4 (Generalized Concept Inclusions) If C and D are concept terms, then $\mathsf{C} \sqsubseteq \mathsf{D}$ *(generalized concept inclusion* or *GCI)* is a terminological axiom as well.

A finite set of terminological axioms $\mathcal{T}$ is called a *terminology* or *TBox*. GCIs can be used to represent terminological cycles. There exist at least two ways to deal with GCIs in a tableaux calculus. The 'internalization' approach (e.g. see in [9]) makes use of the fact that the expressiveness of GCIs is already implied by the combination of role hierarchies and transitive roles. However, with the presence of arbitrary ABoxes one has also to consider unrelated individuals. Therefore, we pursue a different and more direct approach that extends an ABox tableaux calculus by new constructs and rules directly dealing with GCIs (see Definition 7).

The next definition gives a set-theoretic semantics to the language introduced above.

Definition 5 (Semantics) An *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a set $\Delta^{\mathcal{I}}$ (the domain) and an interpretation function $\cdot^{\mathcal{I}}$. The interpretation function maps each concept name C to a subset $\mathsf{C}^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$, each role name R to a subset $\mathsf{R}^{\mathcal{I}}$ of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. Let the symbols C, D be concept expressions, and R, S be role names. Then the interpretation function can be extended to arbitrary concept and role terms as follows ($\|\cdot\|$ denotes the cardinality of a set):

$$
\begin{aligned}
(\mathsf{C} \sqcap \mathsf{D})^{\mathcal{I}} &:= \mathsf{C}^{\mathcal{I}} \cap \mathsf{D}^{\mathcal{I}} \\
(\mathsf{C} \sqcup \mathsf{D})^{\mathcal{I}} &:= \mathsf{C}^{\mathcal{I}} \cup \mathsf{D}^{\mathcal{I}} \\
(\neg \mathsf{C})^{\mathcal{I}} &:= \Delta^{\mathcal{I}} \setminus \mathsf{C}^{\mathcal{I}} \\
(\exists \mathsf{R} \,.\, \mathsf{C})^{\mathcal{I}} &:= \{a \in \Delta^{\mathcal{I}} \,|\, \exists b \in \Delta^{\mathcal{I}} : (a,b) \in \mathsf{R}^{\mathcal{I}}, b \in \mathsf{C}^{\mathcal{I}}\} \\
(\forall \mathsf{R} \,.\, \mathsf{C})^{\mathcal{I}} &:= \{a \in \Delta^{\mathcal{I}} \,|\, \forall b \in \Delta^{\mathcal{I}} : (a,b) \in \mathsf{R}^{\mathcal{I}} \Rightarrow b \in \mathsf{C}^{\mathcal{I}}\} \\
(\exists_{\geq n} \mathsf{R})^{\mathcal{I}} &:= \{a \in \Delta^{\mathcal{I}} \,|\, \|\{b \,|\, (a,b) \in \mathsf{R}^{\mathcal{I}}\}\| \geq n\} \\
(\exists_{\leq n} \mathsf{R})^{\mathcal{I}} &:= \{a \in \Delta^{\mathcal{I}} \,|\, \|\{b \,|\, (a,b) \in \mathsf{R}^{\mathcal{I}}\}\| \leq n\}
\end{aligned}
$$

An interpretation $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$ iff it satisfies (1) $\mathsf{C}^{\mathcal{I}} \subseteq \mathsf{D}^{\mathcal{I}}$ for all terminological axioms (GCIs) $\mathsf{C} \sqsubseteq \mathsf{D}$ in $\mathcal{T}$ and $\mathsf{R}^{\mathcal{I}} \subseteq \mathsf{S}^{\mathcal{I}}$ for all terminological axioms $\mathsf{R} \sqsubseteq \mathsf{S}$ (role inclusions) in $\mathcal{T}$, and (2) iff for every

$R \in T : R^{\mathcal{I}} = (R^{\mathcal{I}})^+$. A concept term C *subsumes* a concept term D w.r.t. a TBox $\mathcal{T}$ (written $D \preceq_{\mathcal{T}} C$), iff $D^{\mathcal{I}} \subseteq C^{\mathcal{I}}$ for all models $\mathcal{I}$ of $\mathcal{T}$. A concept term C is *satisfiable* w.r.t. a TBox $\mathcal{T}$ iff there exists a model $\mathcal{I}$ of $\mathcal{T}$ such that $C^{\mathcal{I}} \neq \emptyset$.

One of the basic reasoning services for a description logic formalism is computing the subsumption relationship for atomic concepts. This inference is needed in the TBox to build a hierarchy of concept names w.r.t. specificity. Satisfiability and subsumption can be mutually reduced to each other since $C \preceq_{\mathcal{T}} D$ iff $C \sqcap \neg D$ is not satisfiable w.r.t. $\mathcal{T}$ and C is unsatisfiable w.r.t. $\mathcal{T}$ iff $C \preceq_{\mathcal{T}} \bot$.

2.2 The Assertional Language

In the following, the language for representing knowledge about individual worlds is introduced. An *ABox* $\mathcal{A}$ is a finite set of assertional axioms which are defined as follows:

Definition 6 (ABox Assertions)
Let $O = O_O \cup O_N$ be a set of individual names, where the set O_O of "old" names is disjoint with the set O_N of "new" names. If C is a concept term, R a role name, and $a, b \in O$ are individual names, then the following expressions are *assertional axioms*:

- $a : C$ *(concept assertion)*,
- $(a, b) : R$ *(role assertion)*.

The interpretation function $\cdot^{\mathcal{I}}$ of the interpretation $\mathcal{I}$ for the concept language can be extended to the assertional language by additionally mapping every individual name from O to a single element of $\Delta^{\mathcal{I}}$ in a way such that for $a, b \in O_O$, $a^{\mathcal{I}} \neq b^{\mathcal{I}}$ if $a \neq b$ *(unique name assumption)*. This ensures that different individuals in O_O are interpreted as different objects. The unique name assumption does not hold for elements of O_N, i.e. for $a, b \in O_N$, $a^{\mathcal{I}} = b^{\mathcal{I}}$ may hold even if $a \neq b$. An interpretation satisfies an assertional axiom $a : C$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$ and $(a, b) : R$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$.

An interpretation is a *model* of an ABox $\mathcal{A}$ w.r.t. a TBox $\mathcal{T}$ iff it is a model of $\mathcal{T}$ and furthermore satisfies all assertional axioms in $\mathcal{A}$. An ABox is *consistent* w.r.t. a TBox $\mathcal{T}$ iff it has a model w.r.t. $\mathcal{T}$. An individual b is called a *direct successor* of an individual a in an ABox $\mathcal{A}$ iff $\mathcal{A}$ contains the assertional axiom $(a, b) : R$. An individual b is called a *successor* of a if it is either a direct successor of a or there exists in $\mathcal{A}$ a chain of assertions $(a, b_1) : R_1, (b_1, b_2) : R_2, \ldots, (b_n, b) : R_{n+1}$. In case that $R_i = R_j$ or $R_i \in R^{\downarrow}$ for all $i, j \in 1..n+1$ we call b the (direct) R-*successor* of a. A (direct) *predecessor* is defined analogously. An individual a is called an *instance* of a concept term C in an interpretation $\mathcal{I}$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$. The *direct types* of an individual are the most specific atomic concepts which the individual is an instance of.

The ABox consistency problem is to decide whether a given ABox $\mathcal{A}$ is consistent w.r.t. a TBox $\mathcal{T}$. Satisfiability of concept terms can be reduced to ABox consistency as follows: A concept term C is satisfiable iff the ABox $\{a : C\}$ is consistent. *Instance checking* tests whether an individual a is an instance of a concept term C w.r.t. an ABox $\mathcal{A}$ and a TBox $\mathcal{T}$, i.e. whether $\mathcal{A}$ entails $a : C$ w.r.t. $\mathcal{T}$. This problem is reduced to the problem of deciding if the ABox $\mathcal{A} \cup \{a : \neg C\}$ is inconsistent.

3 An ABox Example

Before we continue with the calculus for $\mathcal{ALCNH}_{R^+}$, we illustrate in the following the expressiveness of $\mathcal{ALCNH}_{R^+}$ with a TBox and ABox example about family relationships. This example uses prominent features of $\mathcal{ALCNH}_{R^+}$ such as transitive roles, role hierarchies, number restrictions and generalized concept inclusions.

In the TBox *family* we assume a role has_descendant which is declared to be *transitive*, has_gender which is declared as a feature (e.g. this can be achieved by adding the axiom $\top \sqsubseteq \exists_{\leq 1}$ has_gender), and a role has_sibling. The TBox *family* contains the following role axioms.

$\text{has\_child} \sqsubseteq \text{has\_descendant}$

$\text{has\_sister} \sqsubseteq \text{has\_sibling}$

$\text{has\_brother} \sqsubseteq \text{has\_sibling}$

The TBox *family* contains concept axioms specifying the domain and/or range of the roles introduced above (the domain A of a role R can be expressed by the axiom $\exists_{\geq 1} R \sqsubseteq A$ and the range B by $\top \sqsubseteq \forall R . B$).

$\exists_{\geq 1} \text{has\_descendant} \sqsubseteq \text{human}$

$\top \sqsubseteq \forall \text{has\_descendant} . \text{human}$

$\exists_{\geq 1} \text{has\_child} \sqsubseteq \text{parent}$

$\exists_{\geq 1} \text{has\_sibling} \sqsubseteq \text{sibling}$

$\top \sqsubseteq \forall \text{has\_sibling} . \text{sibling}$

$\top \sqsubseteq \forall \text{has\_sister} . \text{sister}$

$\top \sqsubseteq \forall \text{has\_brother} . \text{brother}$

$\top \sqsubseteq \forall \text{has\_gender} . (\text{female} \sqcup \text{male})$

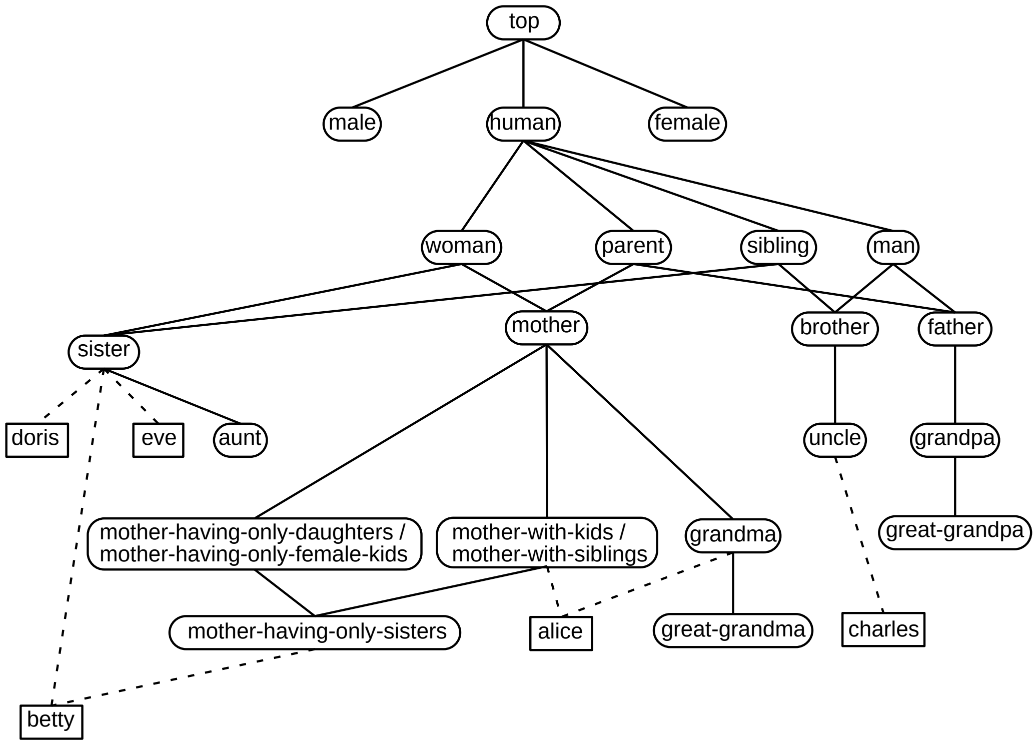

Figure 1: Concept hierarchy of the TBox *family* augmented with the individuals from the ABox *smith_family*. Ovals represent atomic concepts, rectangles denote ABox individuals, solid lines show the direct subsumption relationship, and dashed lines the instance membership of the individuals for their direct types.

The next axioms guarantee the disjointness between the concepts female, male, and human.

$$\mathsf{female} \sqsubseteq \neg(\mathsf{human} \sqcup \mathsf{male})$$
$$\mathsf{male} \sqsubseteq \neg(\mathsf{human} \sqcup \mathsf{female})$$
$$\mathsf{human} \sqsubseteq \neg(\mathsf{female} \sqcup \mathsf{male})$$

After these preliminaries we start with axioms expressing basic knowledge about family members. We use $\mathsf{C} \doteq \mathsf{D}$ as an abbreviation for $\mathsf{C} \sqsubseteq \mathsf{D}$ and $\mathsf{D} \sqsubseteq \mathsf{C}$.

$$\mathsf{human} \sqsubseteq \exists_{\geq 1}\ \mathsf{has\_gender}$$
$$\mathsf{woman} \doteq \mathsf{human} \sqcap \forall\, \mathsf{has\_gender} \,.\, \mathsf{female}$$
$$\mathsf{man} \doteq \mathsf{human} \sqcap \forall\, \mathsf{has\_gender} \,.\, \mathsf{male}$$
$$\mathsf{parent} \doteq \exists_{\geq 1}\ \mathsf{has\_child}$$
$$\mathsf{mother} \doteq \mathsf{woman} \sqcap \mathsf{parent}$$
$$\mathsf{father} \doteq \mathsf{man} \sqcap \mathsf{parent}$$

The next axioms describe some aspects of relatives of a family. Note the inferred equivalences between the concept pairs "$\mathsf{mother\_with\_}\ldots$" and "$\mathsf{mother\_having\_}\ldots$" as shown in Figure 1.

$$\mathsf{mother\_having\_only\_female\_kids} \doteq \mathsf{mother} \sqcap \forall\, \mathsf{has\_child} \,.\, \forall\, \mathsf{has\_gender} \,.\, \mathsf{female}$$
$$\mathsf{mother\_having\_only\_daughters} \doteq \mathsf{mother} \sqcap \exists_{\geq 1}\ \mathsf{has\_child} \sqcap \forall\, \mathsf{has\_child} \,.\, \mathsf{woman}$$
$$\mathsf{mother\_with\_kids} \doteq \mathsf{mother} \sqcap \exists_{\geq 2}\ \mathsf{has\_child}$$
$$\mathsf{grandpa} \doteq \mathsf{man} \sqcap \exists\, \mathsf{has\_child} \,.\, \mathsf{parent}$$
$$\mathsf{great\_grandpa} \doteq \mathsf{man} \sqcap \exists\, \mathsf{has\_child} \,.\, (\exists\, \mathsf{has\_child} \,.\, \mathsf{parent})$$
$$\mathsf{grandma} \doteq \mathsf{woman} \sqcap \exists\, \mathsf{has\_child} \,.\, \mathsf{parent}$$
$$\mathsf{great\_grandma} \doteq \mathsf{woman} \sqcap \exists\, \mathsf{has\_child} \,.\, (\exists\, \mathsf{has\_child} \,.\, \mathsf{parent})$$
$$\mathsf{aunt} \doteq \mathsf{woman} \sqcap \exists\, \mathsf{has\_sibling} \,.\, \mathsf{parent}$$
$$\mathsf{uncle} \doteq \mathsf{man} \sqcap \exists\, \mathsf{has\_sibling} \,.\, \mathsf{parent}$$
$$\mathsf{sibling} \doteq \mathsf{sister} \sqcup \mathsf{brother}$$
$$\mathsf{sister} \doteq \mathsf{woman} \sqcap \exists_{\geq 1}\ \mathsf{has\_sibling}$$
$$\mathsf{brother} \doteq \mathsf{man} \sqcap \exists_{\geq 1}\ \mathsf{has\_sibling}$$
$$\mathsf{mother\_with\_siblings} \doteq \mathsf{mother} \sqcap \forall\, \mathsf{has\_child} \,.\, \mathsf{sibling}$$

There still exists no formal relationship between the notions "having kids" and "having siblings." This is expressed by the next two axioms. The last axiom defines a concept $\mathsf{mother\_having\_only\_sisters}$ which has the other specific "$\mathsf{mother\_\ldots}$" concepts as parents (see Figure 1).

$$\exists_{\geq 2}\,\mathsf{has\_child} \sqsubseteq \forall\,\mathsf{has\_child}\,.\,\mathsf{sibling}$$
$$\exists\,\mathsf{has\_child}\,.\,\mathsf{sibling} \sqsubseteq \exists_{\geq 2}\,\mathsf{has\_child}$$
$$\mathsf{mother\_having\_only\_sisters} \doteq \mathsf{mother} \sqcap \forall\,\mathsf{has\_child}\,.\,(\mathsf{sister} \sqcap \forall\,\mathsf{has\_sibling}\,.\,\mathsf{sister})$$

Using the TBox *family*, the ABox *smith_family* is specified. It consists of several assertions about the individuals alice, betty, charles, doris, and eve. The individual alice is the mother of her two children betty and charles.

$$\begin{aligned}
\mathsf{alice} &: \mathsf{woman} \sqcap \exists_{\leq 2}\,\mathsf{has_child}\\
(\mathsf{alice}, \mathsf{betty}) &: \mathsf{has_child}\\
(\mathsf{alice}, \mathsf{charles}) &: \mathsf{has_child}
\end{aligned}$$

The individual betty is the sibling of charles and the mother of doris and eve, who are the only siblings of each other. The individual charles is the only brother of betty.

$$\begin{aligned}
\mathsf{betty} &: \mathsf{woman} \sqcap \exists_{\leq 2}\,\mathsf{has_child} \sqcap \exists_{\leq 1}\,\mathsf{has_sibling}\\
(\mathsf{betty}, \mathsf{doris}) &: \mathsf{has_child}\\
(\mathsf{betty}, \mathsf{eve}) &: \mathsf{has_child}\\
(\mathsf{betty}, \mathsf{charles}) &: \mathsf{has_sibling}\\
\mathsf{charles} &: \mathsf{brother} \sqcap \exists_{\leq 1}\,\mathsf{has_sibling}\\
(\mathsf{charles}, \mathsf{betty}) &: \mathsf{has_sibling}\\
\mathsf{doris} &: \exists_{\leq 1}\,\mathsf{has_sibling}\\
\mathsf{eve} &: \exists_{\leq 1}\,\mathsf{has_sibling}\\
(\mathsf{doris}, \mathsf{eve}) &: \mathsf{has_sister}\\
(\mathsf{eve}, \mathsf{doris}) &: \mathsf{has_sister}
\end{aligned}$$

Figure 1 also shows the inferred *direct types* of the individuals in ABox *smith_family*. alice has as direct types $\{\mathsf{mother\_with\_siblings}, \mathsf{grandma}\}$, betty has $\{\mathsf{mother\_having\_only\_sisters}, \mathsf{sister}\}$, charles has $\{\mathsf{uncle}\}$, and doris and eve have $\{\mathsf{sister}\}$. These inferences demonstrate the expressiveness of $\mathcal{ALCNH}_{R^+}$. The ABox *smith_family* contains only minimal knowledge about the individuals and their relationships.

4 A Tableaux Calculus for $\mathcal{ALCNH}_{R^+}$

In the following we devise a *tableaux* algorithm to decide the consistency of $\mathcal{ALCNH}_{R^+}$ ABoxes. The algorithm is characterized by a set of tableaux or *completion* rules and by a particular *completion strategy* ensuring a specific order for applying the completion rules to assertional axioms of an ABox. The strategy is essential to guarantee the completeness of the ABox consistency algorithm. First, we have to introduce new assertional axioms needed to define the augmentation of an ABox.

Definition 7 (Additional ABox Assertions) Let C be a concept term, the individual names $\mathsf{a}, \mathsf{b} \in O$, and $x \notin O$, then the following expressions are also assertional axioms:

- $\forall\, x\,.\, x\colon \mathsf{C}$ *(universal concept assertion)*,[1]
- $\mathsf{a} \neq \mathsf{b}$ *(inequality assertion)*.

An interpretation $\mathcal{I}$ satisfies an assertional axiom $\forall\, x\,.\, x\colon \mathsf{C}$ iff $\mathsf{C}^{\mathcal{I}} = \Delta^{\mathcal{I}}$ and $\mathsf{a} \neq \mathsf{b}$ iff $\mathsf{a}^{\mathcal{I}} \neq \mathsf{b}^{\mathcal{I}}$.

Given the new ABox assertions we define for any concept term its negation normal form that is needed to introduce the notion of an augmented ABox.

Definition 8 (Negation Normal Form) The same naming conventions as in Definition 3 are assumed. The negation normal form is defined by applying the following transformations in such a way that a negation sign may occur only in front of concept names. This transformation is possible in linear time:

- $\neg\top \equiv \bot$,
- $\neg\bot \equiv \top$,
- $\neg(\mathsf{C} \sqcap \mathsf{D}) \equiv \neg\mathsf{C} \sqcup \neg\mathsf{D}$,
- $\neg(\mathsf{C} \sqcup \mathsf{D}) \equiv \neg\mathsf{C} \sqcap \neg\mathsf{D}$,
- $\neg\forall\,\mathsf{R}\,.\,\mathsf{C} \equiv \exists\,\mathsf{R}\,.\,\neg\mathsf{C}$,
- $\neg\exists\,\mathsf{R}\,.\,\mathsf{C} \equiv \forall\,\mathsf{R}\,.\,\neg\mathsf{C}$,
- $\neg\exists_{\leq m}\,\mathsf{S} \equiv \exists_{\geq m+1}\,\mathsf{S}$,
- $\neg\exists_{\geq m}\,\mathsf{S} \equiv \exists_{\leq m-1}\,\mathsf{S}$.

Definition 9 (Augmented ABox) For an initial ABox $\mathcal{A}$ w.r.t a TBox $\mathcal{T}$ we define its *augmented* ABox $\mathcal{A}'$ by applying the following rules to $\mathcal{A}$. For every GCI $\mathsf{C} \sqsubseteq \mathsf{D}$ in $\mathcal{T}$ the assertion $\forall\, x\,.\, x\colon(\neg\mathsf{C} \sqcup \mathsf{D})$ is added to $\mathcal{A}'$. Every concept term occurring in $\mathcal{A}$ is transformed into its negation normal form. Let $O_O := \{\mathsf{a}_1, \ldots, \mathsf{a}_n\}$ be the set of old individual names mentioned in $\mathcal{A}$, then the set of inequality assertions $\{\mathsf{a}_i \neq \mathsf{a}_j \mid \mathsf{a}_i, \mathsf{a}_j \in O_O, \mathsf{i}, \mathsf{j} \in 1..\mathsf{n}, \mathsf{i} \neq \mathsf{j}\}$ is added to $\mathcal{A}$. From this point on, if we refer to an initial ABox $\mathcal{A}$ we always mean its augmented ABox.

The tableaux rules also require the notion of *blocking* their applicability. This is based on so-called concept

[1] $\forall\, x\,.\, x\colon \mathsf{C}$ should be read as $\forall\, x\,.\,(x\colon \mathsf{C})$.

sets, an ordering for new individuals, and blocking individuals.

Definition 10 (Concept Set, $\mathcal{A}$-blocked) Given an ABox $\mathcal{A}$ and an individual a occurring in $\mathcal{A}$, we define the *concept set* of a as $\sigma(\mathcal{A}, \mathsf{a}) := \{\top\} \cup \{\mathsf{C} \mid \mathsf{a}:\mathsf{C} \in \mathcal{A}\}$. We define two individuals as $\mathcal{A}$-*equivalent*, written $\mathsf{a} \equiv_{\mathcal{A}} \mathsf{b}$, if their concept sets are equal, i.e. $\sigma(\mathcal{A}, \mathsf{a}) = \sigma(\mathcal{A}, \mathsf{b})$. We say that an individual b is $\mathcal{A}$-*blocked*[2] by a, written $\mathsf{a} \succcurlyeq_{\mathcal{A}} \mathsf{b}$, if $\sigma(\mathcal{A}, \mathsf{a}) \supseteq \sigma(\mathcal{A}, \mathsf{b})$.

Definition 11 (Individual Ordering) We define an *individual ordering* '$\prec$' for new individuals (elements of O_N) occurring in an ABox $\mathcal{A}$. If $\mathsf{b} \in O_N$ is introduced in $\mathcal{A}$, then $\mathsf{a} \prec \mathsf{b}$ for all new individuals a already present in $\mathcal{A}$.

Definition 12 (Blocking Individual) Let $\mathcal{A}$ be an ABox and $\mathsf{a}, \mathsf{b} \in O_N$ be individuals in $\mathcal{A}$. We call a the blocking individual of b if the following conditions hold:

1. $\mathsf{a} \succcurlyeq_{\mathcal{A}} \mathsf{b}$
2. $\mathsf{a} \prec \mathsf{b}$
3. $\neg\exists\, \mathsf{c}$ in $\mathcal{A} : \mathsf{c} \in O_N, \mathsf{c} \prec \mathsf{a}, \mathsf{c} \succcurlyeq_{\mathcal{A}} \mathsf{b}$.

4.1 Completion Rules

We are now ready to define the *completion rules* that are intended to generate a so-called completion of an ABox (see also below).

Definition 13 (Completion Rules)

R$\sqcap$ The conjunction rule.
if 1. $\mathsf{a}:\mathsf{C} \sqcap \mathsf{D} \in \mathcal{A}$, and
2. $\{\mathsf{a}:\mathsf{C},\ \mathsf{a}:\mathsf{D}\} \not\subseteq \mathcal{A}$
then $\mathcal{A}' = \mathcal{A} \cup \{\mathsf{a}:\mathsf{C},\ \mathsf{a}:\mathsf{D}\}$

R$\sqcup$ The disjunction rule.
if 1. $\mathsf{a}:\mathsf{C} \sqcup \mathsf{D} \in \mathcal{A}$, and
2. $\{\mathsf{a}:\mathsf{C},\ \mathsf{a}:\mathsf{D}\} \cap \mathcal{A} = \emptyset$
then $\mathcal{A}' = \mathcal{A} \cup \{\mathsf{a}:\mathsf{C}\}$ **or** $\mathcal{A}' = \mathcal{A} \cup \{\mathsf{a}:\mathsf{D}\}$

R$\forall$C The role value restriction rule.
if 1. $\mathsf{a}:\forall\, \mathsf{R}\,.\,\mathsf{C} \in \mathcal{A}$, and
2. $\exists\, \mathsf{b} \in O, \mathsf{S} \in \mathsf{R}^{\downarrow} : (\mathsf{a}, \mathsf{b}):\mathsf{S} \in \mathcal{A}$, and
3. $\mathsf{b}:\mathsf{C} \notin \mathcal{A}$
then $\mathcal{A}' = \mathcal{A} \cup \{\mathsf{b}:\mathsf{C}\}$

[2] We may omit the reference to $\mathcal{A}$ by speaking of *blocked* if the context is obvious.

R$\forall_+$C The transitive role value restriction rule.
if 1. $\mathsf{a}:\forall\, \mathsf{R}\,.\,\mathsf{C} \in \mathcal{A}$, and
2. $\exists\, \mathsf{b} \in O, \mathsf{T} \in \mathsf{R}^{\downarrow}, \mathsf{T} \in T, \mathsf{S} \in \mathsf{T}^{\downarrow}$: $(\mathsf{a}, \mathsf{b}):\mathsf{S} \in \mathcal{A}$, and
3. $\mathsf{b}:\forall\, \mathsf{T}\,.\,\mathsf{C} \notin \mathcal{A}$
then $\mathcal{A}' = \mathcal{A} \cup \{\mathsf{b}:\forall\, \mathsf{T}\,.\,\mathsf{C}\}$

R$\forall_x$ The universal concept restriction rule.
if 1. $\forall\, x\,.\,x:\mathsf{C} \in \mathcal{A}$, and
2. $\exists\, \mathsf{a} \in O$: a mentioned in $\mathcal{A}$, and
3. $\mathsf{a}:\mathsf{C} \notin \mathcal{A}$
then $\mathcal{A}' = \mathcal{A} \cup \{\mathsf{a}:\mathsf{C}\}$

R$\exists$C The role exists restriction rule.
if 1. $\mathsf{a}:\exists\, \mathsf{R}\,.\,\mathsf{C} \in \mathcal{A}$, and
2. $\mathsf{a} \in O_N \Rightarrow (\neg\exists\, \mathsf{c}$ in $\mathcal{A}$: $\mathsf{c} \in O_N$, c is a blocking individual for a), and
3. $\neg\exists\, \mathsf{b} \in O, \mathsf{S} \in \mathsf{R}^{\downarrow} : \{(\mathsf{a}, \mathsf{b}):\mathsf{S},\ \mathsf{b}:\mathsf{C}\} \subseteq \mathcal{A}$
then $\mathcal{A}' = \mathcal{A} \cup \{(\mathsf{a}, \mathsf{b}):\mathsf{R},\ \mathsf{b}:\mathsf{C}\}$ where $\mathsf{b} \in O_N$ is not used in $\mathcal{A}$

R$\exists_{\geq n}$ The number restriction exists rule.
if 1. $\mathsf{a}:\exists_{\geq n}\, \mathsf{R} \in \mathcal{A}$, and
2. $\mathsf{a} \in O_N \Rightarrow (\neg\exists\, \mathsf{c}$ in $\mathcal{A}$: $\mathsf{c} \in O_N$, c is a blocking individual for a), and
3. $\neg\exists\, \mathsf{b}_1, \ldots, \mathsf{b}_\mathsf{n} \in O, \mathsf{S}_1, \ldots, \mathsf{S}_\mathsf{n} \in \mathsf{R}^{\downarrow}$: $\{(\mathsf{a}, \mathsf{b}_\mathsf{k}):\mathsf{S}_\mathsf{k} \mid \mathsf{k} \in 1..\mathsf{n}\} \cup \{\mathsf{b}_\mathsf{i} \neq \mathsf{b}_\mathsf{j} \mid \mathsf{i}, \mathsf{j} \in 1..\mathsf{n}, \mathsf{i} \neq \mathsf{j}\} \subseteq \mathcal{A}$
then $\mathcal{A}' = \mathcal{A} \cup \{(\mathsf{a}, \mathsf{b}_\mathsf{k}):\mathsf{R} \mid \mathsf{k} \in 1..\mathsf{n}\} \cup \{\mathsf{b}_\mathsf{i} \neq \mathsf{b}_\mathsf{j} \mid \mathsf{i}, \mathsf{j} \in 1..\mathsf{n}, \mathsf{i} \neq \mathsf{j}\}$ where $\mathsf{b}_1, \ldots, \mathsf{b}_\mathsf{n} \in O_N$ are not used in $\mathcal{A}$

R$\exists_{\leq n}$ The number restriction merge rule.
if 1. $\mathsf{a}:\exists_{\leq n}\, \mathsf{R} \in \mathcal{A}$, and
2. $\exists\, \mathsf{b}_1, \ldots, \mathsf{b}_\mathsf{m} \in O, \mathsf{S}_1, \ldots, \mathsf{S}_\mathsf{m} \in \mathsf{R}^{\downarrow}$: $\{(\mathsf{a}, \mathsf{b}_1):\mathsf{S}_1, \ldots, (\mathsf{a}, \mathsf{b}_\mathsf{m}):\mathsf{S}_\mathsf{m}\} \subseteq \mathcal{A}$ with $\mathsf{m} > \mathsf{n}$, and
3. $\exists\, \mathsf{b}_\mathsf{i}, \mathsf{b}_\mathsf{j} \in \{\mathsf{b}_1, \ldots, \mathsf{b}_\mathsf{m}\} : \mathsf{i} \neq \mathsf{j},\ \mathsf{b}_\mathsf{i} \neq \mathsf{b}_\mathsf{j} \notin \mathcal{A}$
then $\mathcal{A}' = \mathcal{A}[\mathsf{b}_\mathsf{i}/\mathsf{b}_\mathsf{j}]$, i.e. replace every occurrence of b_i in $\mathcal{A}$ by b_j

We call the rules R$\sqcup$ and R$\exists_{\leq n}$ *nondeterministic* rules since they can be applied in different ways to the same ABox. The remaining rules are called *deterministic* rules. Moreover, we call the rules R$\exists$C and R$\exists_{\geq n}$ *generating* rules since they are the only rules that introduce new individuals in an ABox.

The increase of expressiveness in $\mathcal{ALCNH}_{R^+}$ gained by supporting ABox reasoning is reflected in tableaux rules that are more complex than in comparable approaches for concept consistency. The universal concept restriction rule takes care of GCIs and usually causes additional complexity by adding disjunctions to an ABox. The generating rules have a more complex premise since they may test only for a blocking situation if they are applied to new individuals, i.e. a blocking situation can never occur for old individuals. The necessity of this additional precondition is illustrated by the following example. We define a concept D where R is a transitive superrole of S.

$$\mathsf{D} \doteq \mathsf{C} \sqcap \exists \mathsf{S} . \mathsf{C} \sqcap \exists_{\leq 1} \mathsf{S} \sqcap \forall \mathsf{R} . \exists \mathsf{S} . \mathsf{C}$$

$$\mathcal{A} := \{(\mathsf{i},\mathsf{j}) : \mathsf{S},\ (\mathsf{j},\mathsf{k}) : \mathsf{S},\ \mathsf{i} : \mathsf{D},\ \mathsf{j} : \mathsf{D},\ \mathsf{k} : \neg\mathsf{C}\}$$

Then, we define an ABox $\mathcal{A}$ which is obviously unsatisfiable due to a clash for the individual k with $\mathsf{C} \sqcap \neg\mathsf{C}$. However, if blocking were allowed for old individuals, the role exists restriction rule would not create a S-successor with qualification C for the individual j. As a consequence, the number restriction merge rule would never merge this successor with the individual k which results in the unsatisfiability of $\mathcal{A}$.

Proposition 14 (Invariance) Let $\mathcal{A}$ and $\mathcal{A}'$ be ABoxes. Then:

1. If $\mathcal{A}'$ is derived from $\mathcal{A}$ by applying a deterministic rule, then $\mathcal{A}$ is satisfiable iff $\mathcal{A}'$ is satisfiable.
2. If $\mathcal{A}'$ is derived from $\mathcal{A}$ by applying a nondeterministic rule, then $\mathcal{A}$ is satisfiable if $\mathcal{A}'$ is satisfiable. Conversely, if $\mathcal{A}$ is satisfiable and a nondeterministic rule is applicable to $\mathcal{A}$, then it can be applied in such a way that it yields a satisfiable ABox $\mathcal{A}'$.

Proof. **1.** "$\Leftarrow$" Due to the structure of the deterministic rules one can immediately verify that $\mathcal{A}$ is a subset of $\mathcal{A}'$. Therefore, $\mathcal{A}$ is satisfiable if $\mathcal{A}'$ is satisfiable.

"$\Rightarrow$" In order to show that $\mathcal{A}'$ is satisfiable after applying a deterministic rule to the satisfiable ABox $\mathcal{A}$, we examine each applicable rule separately. We assume that $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ satisfies $\mathcal{A}$.

If the conjunction rule is applied to $\mathsf{a} : \mathsf{C} \sqcap \mathsf{D} \in \mathcal{A}$, then we get a new Abox $\mathcal{A}' = \mathcal{A} \cup \{\mathsf{a} : \mathsf{C},\ \mathsf{a} : \mathsf{D}\}$. Since $\mathcal{I}$ satisfies $\mathsf{a} : \mathsf{C} \sqcap \mathsf{D}$, $\mathcal{I}$ satisfies $\mathsf{a} : \mathsf{C}$ and $\mathsf{a} : \mathsf{D}$ and therefore $\mathcal{A}'$.

If the role value restriction rule is applied to $\mathsf{a} : \forall \mathsf{R} . \mathsf{C} \in \mathcal{A}$, then there must be a role assertion $(\mathsf{a}, \mathsf{b}) : \mathsf{S} \in \mathcal{A}$ with $\mathsf{S} \in \mathsf{R}^{\downarrow}$ such that $\mathcal{A}' = \mathcal{A} \cup \{\mathsf{b} : \mathsf{C}\}$. Since $\mathcal{I}$ satisfies $\mathcal{A}$, it holds that $(\mathsf{a}^{\mathcal{I}}, \mathsf{b}^{\mathcal{I}}) \in \mathsf{S}^{\mathcal{I}} \subseteq \mathsf{R}^{\mathcal{I}}$. Since $\mathcal{I}$ satisfies $\mathsf{a} : \forall \mathsf{R} . \mathsf{C}$, it holds that $\mathsf{b}^{\mathcal{I}} \in \mathsf{C}^{\mathcal{I}}$. Thus, $\mathcal{I}$ satisfies $\mathsf{b} : \mathsf{C}$ and therefore $\mathcal{A}'$.

If the transitive role value restriction rule is applied to $\mathsf{a} : \forall \mathsf{R} . \mathsf{C} \in \mathcal{A}$, there must be an assertion $(\mathsf{a}, \mathsf{b}) : \mathsf{S} \in \mathcal{A}$ with $\mathsf{S} \in \mathsf{T}^{\downarrow} \subseteq \mathsf{R}^{\downarrow}$, $\mathsf{T} \in T$ such that we get $\mathcal{A}' = \mathcal{A} \cup \{\mathsf{b} : \forall \mathsf{T} . \mathsf{C}\}$. Since $\mathcal{I}$ satisfies $\mathcal{A}$, we have $\mathsf{a}^{\mathcal{I}} \in (\forall \mathsf{R} . \mathsf{C})^{\mathcal{I}}$ and $(\mathsf{a}^{\mathcal{I}}, \mathsf{b}^{\mathcal{I}}) \in \mathsf{S}^{\mathcal{I}} \subseteq \mathsf{T}^{\mathcal{I}} \subseteq \mathsf{R}^{\mathcal{I}}$. Since $\mathcal{I}$ satisfies $\mathsf{a} : \forall \mathsf{T} . \mathsf{C}$ and $\mathsf{T} \in T, \mathsf{T} \in \mathsf{R}^{\downarrow}$, it holds that $\mathsf{b}^{\mathcal{I}} \in (\forall \mathsf{T} . \mathsf{C})^{\mathcal{I}}$ unless there exists a successor c of b such that $(\mathsf{b}, \mathsf{c}) : \mathsf{S}' \in \mathcal{A}$, $(\mathsf{b}^{\mathcal{I}}, \mathsf{c}^{\mathcal{I}}) \in \mathsf{S}'^{\mathcal{I}} \subseteq \mathsf{T}^{\mathcal{I}}$ and $\mathsf{c}^{\mathcal{I}} \notin \mathsf{C}^{\mathcal{I}}$. It follows from $(\mathsf{a}^{\mathcal{I}}, \mathsf{b}^{\mathcal{I}}) \in \mathsf{T}^{\mathcal{I}}$, $(\mathsf{b}^{\mathcal{I}}, \mathsf{c}^{\mathcal{I}}) \in \mathsf{T}^{\mathcal{I}}$, and $\mathsf{T} \in T$ that $(\mathsf{a}^{\mathcal{I}}, \mathsf{c}^{\mathcal{I}}) \in \mathsf{T}^{\mathcal{I}} \subseteq \mathsf{R}^{\mathcal{I}}$ and $\mathsf{a}^{\mathcal{I}} \notin (\forall \mathsf{R} . \mathsf{C})^{\mathcal{I}}$ in contradiction to the assumption. Thus, $\mathcal{I}$ satisfies $\mathsf{b} : \forall \mathsf{T} . \mathsf{C}$ and therefore $\mathcal{A}'$.

If the universal concept restriction rule is applied to an individual a in $\mathcal{A}$ because of $\forall x . x : \mathsf{C} \in \mathcal{A}$, then $\mathcal{A}' = \mathcal{A} \cup \{\mathsf{a} : \mathsf{C}\}$. Since $\mathcal{I}$ satisfies $\mathcal{A}$, it holds that $\mathsf{C}^{\mathcal{I}} = \Delta^{\mathcal{I}}$. Thus, it holds that $\mathsf{a}^{\mathcal{I}} \in \mathsf{C}^{\mathcal{I}}$ and $\mathcal{I}$ satisfies $\mathcal{A}'$.

If the role exists restriction rule is applied to $\mathsf{a} : \exists \mathsf{R} . \mathsf{C} \in \mathcal{A}$, then we get the ABox $\mathcal{A}' = \mathcal{A} \cup \{(\mathsf{a}, \mathsf{b}) : \mathsf{R},\ \mathsf{b} : \mathsf{C}\}$. Since $\mathcal{I}$ satisfies $\mathcal{A}$, there exists a $y \in \Delta^{\mathcal{I}}$ such that $(\mathsf{a}^{\mathcal{I}}, y) \in \mathsf{R}^{\mathcal{I}}$ and $y \in \mathsf{C}^{\mathcal{I}}$. We define the interpretation function $\cdot^{\mathcal{I}'}$ such that $\mathsf{b}^{\mathcal{I}'} := y$ and $x^{\mathcal{I}'} := x^{\mathcal{I}}$ for $x \neq \mathsf{b}$. It is easy to show that $\mathcal{I}' = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}'})$ satisfies $\mathcal{A}'$.

If the number restriction exists rule is applied to $\mathsf{a} : \exists_{\geq n} \mathsf{R} \in \mathcal{A}$, then we get $\mathcal{A}' = \mathcal{A} \cup \{(\mathsf{a}, \mathsf{b_k}) : \mathsf{R} \mid \mathsf{k} \in 1..\mathsf{n}\} \cup \{\mathsf{b_i} \neq \mathsf{b_j} \mid \mathsf{i}, \mathsf{j} \in 1..\mathsf{n}, \mathsf{i} \neq \mathsf{j}\}$. Since $\mathcal{I}$ satisfies $\mathcal{A}$, there must exist n distinct individuals $y_i \in \Delta^{\mathcal{I}}$, $i \in 1..n$ such that $(\mathsf{a}^{\mathcal{I}}, y_i) \in \mathsf{R}^{\mathcal{I}}$. We define the interpretation function $\cdot^{\mathcal{I}'}$ such that $\mathsf{b_i}^{\mathcal{I}'} := y_i$ and $x^{\mathcal{I}'} := x^{\mathcal{I}}$ for $x \notin \{\mathsf{b}_1, \ldots, \mathsf{b_n}\}$. It is easy to show that $\mathcal{I}' = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}'})$ satisfies $\mathcal{A}'$.

2. "$\Leftarrow$" Assume that $\mathcal{A}'$ is satisfied by $\mathcal{I}' = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}'})$. We show that $\mathcal{A}$ is also satisfiable by examining the nondeterministic rules.

If $\mathcal{A}'$ is obtained from $\mathcal{A}$ by applying the disjunction rule, then $\mathcal{A}$ is a subset of $\mathcal{A}'$ and therefore satisfied by $\mathcal{I}'$.

If $\mathcal{A}'$ is obtained from $\mathcal{A}$ by applying the number restriction merge rule to $\mathsf{a} : \exists_{\leq n} \mathsf{R} \in \mathcal{A}$, then there exist $\mathsf{b_i}, \mathsf{b_j}$ in $\mathcal{A}$ such that $\mathcal{A}' = \mathcal{A}[\mathsf{b_i}/\mathsf{b_j}]$. We define the interpretation function $\cdot^{\mathcal{I}}$ such that $\mathsf{b_i}^{\mathcal{I}} := \mathsf{b_j}^{\mathcal{I}'}$ and $\mathsf{x}^{\mathcal{I}} := \mathsf{x}^{\mathcal{I}'}$ for every $\mathsf{x} \neq \mathsf{b_i}$. Obviously $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ satisfies $\mathcal{A}$.

"$\Rightarrow$" We suppose that $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ satisfies $\mathcal{A}$ and a nondeterministic rule is applicable to an individual a in $\mathcal{A}$.

If the disjunction rule is applicable to $\mathsf{a} : \mathsf{C} \sqcup \mathsf{D} \in \mathcal{A}$ and $\mathcal{A}$ is satisfiable, it holds $\mathsf{a}^{\mathcal{I}} \in (\mathsf{C} \sqcup \mathsf{D})^{\mathcal{I}}$. It follows that either $\mathsf{a}^{\mathcal{I}} \in \mathsf{C}^{\mathcal{I}}$ or $\mathsf{a}^{\mathcal{I}} \in \mathsf{D}^{\mathcal{I}}$ (or both). Hence, the disjunction rule can be applied in a way that $\mathcal{I}$ also satisfies the ABox $\mathcal{A}'$.

If the number restriction merge rule is applicable to $\mathsf{a} : \exists_{\leq n} \mathsf{R} \in \mathcal{A}$ and $\mathcal{A}$ is satisfiable, it holds

$a^{\mathcal{I}} \in (\exists_{\leq n} R)^{\mathcal{I}}$ and $\|\{b \mid (a,b) \in R^{\mathcal{I}}\}\| \leq n$. However, it also holds $\|\{b \mid (a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}\}\| > m$ with $m \geq n$.[3] Thus, we can conclude by the Pigeonhole Principle (e.g. see [13, page 26]) that there exist at least two R-successors b_i, b_j of a such that $b_i^{\mathcal{I}} = b_j^{\mathcal{I}}$. Since $\mathcal{I}$ satisfies $\mathcal{A}$, we have $b_i \neq b_j \notin \mathcal{A}$ and at least one of the two individuals must be a new individual. Let us assume that $b_i \in O_N$ and $b_i = b_j$, then $\mathcal{I}$ obviously satisfies $\mathcal{A}[b_i/b_j]$. □

Given an initial ABox $\mathcal{A}$, more than one rule might be applicable to $\mathcal{A}$. This is controlled by a completion strategy in accordance to the ordering for new individuals (see Definition 11).

Definition 15 (Completion Strategy) We define a *completion strategy* that must observe the following restrictions.

- Meta rules:
  - Apply a rule to an individual $b \in O_N$ only if no rule is applicable to an individual $a \in O_O$.
  - Apply a rule to an individual $b \in O_N$ only if no rule is applicable to another individual $a \in O_N$ such that $a \prec b$.
- The completion rules are always applied in the following order. A step is skipped in case the corresponding set of applicable rules is empty.
  1. Apply all nongenerating rules (R⊓, R⊔, R∀C, R$\forall_+$C, R$\forall_x$, R$\exists_{\leq n}$) as long as possible.
  2. Apply a generating rule (R∃C, R$\exists_{\geq n}$) and restart with step 1 as long as possible.

In the following we always assume that rules are applied in accordance to this strategy. It ensures that the rules are applied to new individuals w.r.t. the ordering '$\prec$'.

Definition 16 (Clash Triggers) We assume the same naming conventions as used above. An ABox $\mathcal{A}$ is called *contradictory* if one of the following *clash triggers* is applicable. If none of the clash triggers is applicable to $\mathcal{A}$, then $\mathcal{A}$ is called *clash-free*.

- *Primitive clash*:
  $a : \bot \in \mathcal{A}$ or $\{a : C, a : \neg C\} \subseteq \mathcal{A}$, where C is a concept name.
- *Number restriction merging clash*:
  $\exists S_1, \ldots, S_m \in R^{\downarrow} : \{a : \exists_{\leq n} R\} \cup$
  $\{(a, b_i) : S_i \mid i \in 1..m\} \cup$
  $\{b_i \neq b_j \mid i, j \in 1..m, i \neq j\} \subseteq \mathcal{A}$ with $m > n$

[3] Without loss of generality we only need to consider the case that $m = n + 1$.

A clash-free ABox $\mathcal{A}$ is called *complete* if no completion rule is applicable to $\mathcal{A}$. A complete ABox $\mathcal{A}'$ derived from an ABox $\mathcal{A}$ is also called a *completion* of $\mathcal{A}$. Any ABox containing a clash is obviously unsatisfiable. The purpose of the calculus is to generate a completion for an initial ABox $\mathcal{A}$ that proves the satisfiability of $\mathcal{A}$ or its unsatisfiability if no completion can be found. In the following we have to show that a model can be constructed for any complete ABox.

4.2 Decidability of the ABox Consistency Problem

The following lemma proves that whenever a generating rule has been applied to an individual a, the concept set $\sigma(\cdot, a)$ of a does not change in succeeding ABoxes.

Lemma 17 (Stability) Let $\mathcal{A}$ be an ABox and $a \in O_N$ be in $\mathcal{A}$. Let a generating rule be applicable to a according to the completion strategy. Let $\mathcal{A}'$ be any ABox derivable from $\mathcal{A}$ by any (possibly empty) sequence of rule applications. Then:

1. No rule is applicable in $\mathcal{A}'$ to an individual $b \in O_N$ with $b \prec a$
2. $\sigma(\mathcal{A}, a) = \sigma(\mathcal{A}', a)$, i.e. the concept set of a remains unchanged in $\mathcal{A}'$.
3. If $b \in O_N$ is in $\mathcal{A}$ with $b \prec a$ then b is an individual in $\mathcal{A}'$, i.e. the individual b is not substituted by another individual.

Proof. **1.** By contradiction: Suppose $\mathcal{A} = \mathcal{A}_0 \rightarrow_* \cdots \rightarrow_* \mathcal{A}_n = \mathcal{A}'$, where $*$ is element of the completion rules and a rule is applicable to an individual b with $b \prec a$ in $\mathcal{A}'$. Then there has to exist a minimal i with $i \in 1..n$ such that this rule is also applicable in $\mathcal{A}_i$. If a rule is applicable to a in $\mathcal{A}$ then no rule is applicable to b in $\mathcal{A}$ due to our strategy. So no rule is applicable to any individual c such that $c \prec a$ in $\mathcal{A}_0, \ldots, \mathcal{A}_{i-1}$. It follows that from $\mathcal{A}_{i-1}$ to $\mathcal{A}_i$ a rule is applied to a or to a d such that $a \prec d$. Using an exhaustive case analysis of all rules we can show that no new assertion of the form $b : C$ or $(b, e) : R$ can be added to $\mathcal{A}_{i-1}$. Therefore, no rule is applicable to b in $\mathcal{A}_i$. This is a contradiction to our assumption.

2. By contradiction: Suppose $\sigma(\mathcal{A}, a) \neq \sigma(\mathcal{A}', a)$. Let b be the direct predecessor of a with $b \prec a$. A rule must have been applied to a and not to b because of point 1. Due to our strategy only generating rules are applicable to a that cannot add new elements to $\sigma(\cdot, a)$. This is an obvious contradiction.

3. This follows from point 1 and the completion strategy. □

The next lemma guarantees the uniqueness of a blocking individual for a blocked individual. This is a precondition for defining a particular interpretation from $\mathcal{A}$.

Lemma 18 Let $\mathcal{A}'$ be an ABox and a be a new individual in $\mathcal{A}'$. If a is blocked then

1. a has no direct successor and
2. a has exactly one blocking individual.

Proof. **1.** By contradiction: Suppose that a is blocked in $\mathcal{A}'$ and $(\mathsf{a},\mathsf{b}):\mathsf{R} \in \mathcal{A}'$. There must exist an ancestor ABox $\mathcal{A}$ where a generating rule has been applied to a in $\mathcal{A}$. It follows from the definition of the generating rules that for every new individual c with $\mathsf{c} \prec \mathsf{a}$ in $\mathcal{A}$ we had $\sigma(\mathcal{A},\mathsf{c}) \not\supseteq \sigma(\mathcal{A},\mathsf{a})$. Since $\mathcal{A}'$ has been derived from $\mathcal{A}$ we can use Lemma 17 and conclude that for every new individual c with $\mathsf{c} \prec \mathsf{a}$ in $\mathcal{A}'$ we also have $\sigma(\mathcal{A}',\mathsf{c}) \not\supseteq \sigma(\mathcal{A}',\mathsf{a})$. Thus there cannot exist a blocking individual c for a in $\mathcal{A}'$. This is a contradiction to our hypothesis.

2. This follows directly from condition 3 in Definition 12. □

Definition 19 Let $\mathcal{A}$ be an ABox. We define the *canonical interpretation* $\mathcal{I}_\mathcal{A} = (\Delta^{\mathcal{I}_\mathcal{A}}, \cdot^{\mathcal{I}_\mathcal{A}})$ as follows:

1. $\Delta^{\mathcal{I}_\mathcal{A}} := \{\mathsf{a} \mid \mathsf{a} \text{ is an individual in } \mathcal{A}\}$
2. $\mathsf{a}^{\mathcal{I}_\mathcal{A}} := \mathsf{a}$ iff a is mentioned in $\mathcal{A}$
3. $\mathsf{a} \in \mathsf{A}^{\mathcal{I}_\mathcal{A}}$ iff $\mathsf{a}:\mathsf{A} \in \mathcal{A}$
4. $(\mathsf{a},\mathsf{b}) \in \mathsf{R}^{\mathcal{I}_\mathcal{A}}$ iff
   (a) $(\mathsf{a},\mathsf{b}):\mathsf{S} \in \mathcal{A}$ for a role $\mathsf{S} \in \mathsf{R}^{\downarrow}$ *or*
   (b) $\exists\, \mathsf{c}_1,\ldots,\mathsf{c}_{n-1}$ in $\mathcal{A}$: $(\mathsf{a},\mathsf{c}_1):\mathsf{S}_1$, $(\mathsf{c}_1,\mathsf{c}_2):\mathsf{S}_2$, $\ldots$, $(\mathsf{c}_{n-1},\mathsf{b}):\mathsf{S}_n \in \mathcal{A}$, $\mathsf{n} > 1$, $\mathsf{S}_i \in \mathsf{R}^{\downarrow}$ for $\mathsf{i} \in 1..\mathsf{n}$ and $\mathsf{R} \in T$, *or*
   (c) $\exists\, \mathsf{c}$ in $\mathcal{A}$, $\mathsf{c} \in O_N$, c is a blocking individual for a, and $(\mathsf{c},\mathsf{b}):\mathsf{S} \in \mathcal{A}$, for a role $\mathsf{S} \in \mathsf{R}^{\downarrow}$, *or*
   (d) $\exists\, \mathsf{c}$ in $\mathcal{A}$, $\mathsf{c} \in O_N$, c is a blocking individual for a, and $(\mathsf{c},\mathsf{b}_1):\mathsf{S}_1 \in \mathcal{A}$, and $\exists\, \mathsf{b}_2,\ldots,\mathsf{b}_{n-1}$ in $\mathcal{A}$: $(\mathsf{b}_1,\mathsf{b}_2):\mathsf{S}_2$, $\ldots$, $(\mathsf{b}_{n-1},\mathsf{b}):\mathsf{S}_n \in \mathcal{A}$, $\mathsf{n} > 1$, $\mathsf{S}_i \in \mathsf{R}^{\downarrow}$ for $\mathsf{i} \in 1..\mathsf{n}$ and $\mathsf{R} \in T$.

Theorem 20 (Soundness) Let $\mathcal{A}$ be a complete ABox, then $\mathcal{A}$ is satisfiable.

Proof. Let $\mathcal{I}_\mathcal{A} = (\Delta^{\mathcal{I}_\mathcal{A}}, \cdot^{\mathcal{I}_\mathcal{A}})$ be the canonical interpretation for the ABox $\mathcal{A}$. In the following we prove that $\mathcal{I}_\mathcal{A}$ satisfies every assertion in $\mathcal{A}$.

For any $(\mathsf{a},\mathsf{b}):\mathsf{R} \in \mathcal{A}$ or $\mathsf{a} \neq \mathsf{b} \in \mathcal{A}$, $\mathcal{I}_\mathcal{A}$ satisfies them by definition. Next we consider assertions of the form $\mathsf{a}:\mathsf{C}$. We show by induction on the structure of C that $\mathsf{a} \in \mathsf{C}^{\mathcal{I}_\mathcal{A}}$.

If C is a concept name, then $\mathsf{a} \in \mathsf{C}^{\mathcal{I}_\mathcal{A}}$ by definition of $\mathcal{I}_\mathcal{A}$. If $\mathsf{C} = \top$, then obviously $\mathsf{a} \in \top^{\mathcal{I}_\mathcal{A}}$. The case $\mathsf{C} = \bot$ cannot occur since $\mathcal{A}$ is clash-free.

If $\mathsf{C} = \neg\mathsf{D}$, then D is a concept name since all concepts are in negation normal form (see Definition 9). $\mathcal{A}$ is clash-free and cannot contain $\mathsf{a}:\mathsf{D}$. Thus, $\mathsf{a} \notin \mathsf{D}^{\mathcal{I}_\mathcal{A}}$, i.e. $\mathsf{a} \in \Delta^{\mathcal{I}_\mathcal{A}} \setminus \mathsf{D}^{\mathcal{I}_\mathcal{A}}$. Hence $\mathsf{a} \in (\neg\mathsf{D})^{\mathcal{I}_\mathcal{A}}$.

If $\mathsf{C} = \mathsf{C}_1 \sqcap \mathsf{C}_2$ then (since $\mathcal{A}$ is complete) $\mathsf{a}:\mathsf{C}_1 \in \mathcal{A}$ and $\mathsf{a}:\mathsf{C}_2 \in \mathcal{A}$. By induction hypothesis, $\mathsf{a} \in \mathsf{C}_1{}^{\mathcal{I}_\mathcal{A}}$ and $\mathsf{a} \in \mathsf{C}_2{}^{\mathcal{I}_\mathcal{A}}$. Hence $\mathsf{a} \in (\mathsf{C}_1 \sqcap \mathsf{C}_2)^{\mathcal{I}_\mathcal{A}}$.

If $\mathsf{C} = \mathsf{C}_1 \sqcup \mathsf{C}_2$ then (since $\mathcal{A}$ is complete) either $\mathsf{a}:\mathsf{C}_1 \in \mathcal{A}$ or $\mathsf{a}:\mathsf{C}_2 \in \mathcal{A}$. By induction hypothesis, $\mathsf{a} \in \mathsf{C}_1{}^{\mathcal{I}_\mathcal{A}}$ or $\mathsf{a} \in \mathsf{C}_2{}^{\mathcal{I}_\mathcal{A}}$. Hence $\mathsf{a} \in (\mathsf{C}_1 \sqcup \mathsf{C}_2)^{\mathcal{I}_\mathcal{A}}$.

If $\mathsf{C} = \forall\,\mathsf{R}\,.\,\mathsf{D}$, then we have to show that for all b with $(\mathsf{a},\mathsf{b}) \in \mathsf{R}^{\mathcal{I}_\mathcal{A}}$ it holds that $\mathsf{b} \in \mathsf{D}^{\mathcal{I}_\mathcal{A}}$. If $(\mathsf{a},\mathsf{b}) \in \mathsf{R}^{\mathcal{I}_\mathcal{A}}$, then according to Definition 19 the following cases can occur: (4a) b is a direct S-successor of a for a role $\mathsf{S} \in \mathsf{R}^{\downarrow}$ with $\mathsf{S}^{\mathcal{I}_\mathcal{A}} \subseteq \mathsf{R}^{\mathcal{I}_\mathcal{A}}$; then we have $\mathsf{b}:\mathsf{D} \in \mathcal{A}$ since $\mathcal{A}$ is complete and by induction hypothesis $\mathsf{b} \in \mathsf{D}^{\mathcal{I}_\mathcal{A}}$. (4b) b is a R-successor of a via a subrole chain of S_i's with $\mathsf{S}_i{}^{\mathcal{I}_\mathcal{A}} \subseteq \mathsf{R}^{\mathcal{I}_\mathcal{A}}$, $\mathsf{R} \in T$; then we have $\mathsf{c}_{n-1}:\forall\,\mathsf{R}\,.\,\mathsf{D} \in \mathcal{A}$ and $\mathsf{b}:\mathsf{D} \in \mathcal{A}$ since $\mathcal{A}$ is complete and by induction hypothesis we have $\mathsf{b} \in \mathsf{D}^{\mathcal{I}_\mathcal{A}}$. (4c) There has to exist a blocking individual c such that $\mathsf{c}:\forall\,\mathsf{R}\,.\,\mathsf{D} \in \mathcal{A}$ and $(\mathsf{c},\mathsf{b}):\mathsf{S} \in \mathcal{A}$ for a role $\mathsf{S} \in \mathsf{R}^{\downarrow}$ and because $\mathcal{A}$ is complete we have $\mathsf{b}:\mathsf{D} \in \mathcal{A}$ and again by induction hypothesis it holds $\mathsf{b} \in \mathsf{D}^{\mathcal{I}_\mathcal{A}}$. (4d) This case combines the cases (4b-c) because the individual b is reachable from the blocking individual c via a chain of subroles of the transitive role R. It can be proven analogously.

If $\mathsf{C} = \exists\,\mathsf{R}\,.\,\mathsf{D}$, then we have to show that there exists an individual $\mathsf{b} \in \Delta^{\mathcal{I}_\mathcal{A}}$ with $(\mathsf{a},\mathsf{b}) \in \mathsf{R}^{\mathcal{I}_\mathcal{A}}$ and $\mathsf{b} \in \mathsf{D}^{\mathcal{I}_\mathcal{A}}$. Since ABox $\mathcal{A}$ is complete, we have either $(\mathsf{a},\mathsf{b}):\mathsf{R} \in \mathcal{A}$ and $\mathsf{b}:\mathsf{D} \in \mathcal{A}$ or a is blocked by an individual c and $(\mathsf{c},\mathsf{b}):\mathsf{R} \in \mathcal{A}$. In the first case we have $(\mathsf{a},\mathsf{b}) \in \mathsf{R}^{\mathcal{I}_\mathcal{A}}$ and $\mathsf{b} \in \mathsf{D}^{\mathcal{I}_\mathcal{A}}$ by induction hypothesis and the definition of $\mathcal{I}_\mathcal{A}$. In the second case there exists the blocking individual c with $\mathsf{c}:\exists\,\mathsf{R}\,.\,\mathsf{D} \in \mathcal{A}$. By definition c cannot be blocked and by hypothesis $\mathcal{A}$ is complete. So we have an individual b with $(\mathsf{c},\mathsf{b}):\mathsf{R} \in \mathcal{A}$ and $\mathsf{b}:\mathsf{D} \in \mathcal{A}$. By induction hypothesis we have $\mathsf{b} \in \mathsf{D}^{\mathcal{I}_\mathcal{A}}$ and by the definition of $\mathcal{I}_\mathcal{A}$ (case 4c) we have $(\mathsf{a},\mathsf{b}) \in \mathsf{R}^{\mathcal{I}_\mathcal{A}}$.

If $\mathsf{C} = \exists_{\geq n}\,\mathsf{R}$, we prove the hypothesis by contradiction. We assume that $\mathsf{a} \notin (\exists_{\geq n}\,\mathsf{R})^{\mathcal{I}_\mathcal{A}}$. Then there exist at most m ($0 \leq m < n$) distinct R-successors of a. Two cases can occur: (1) the individual a is not blocked in $\mathcal{I}_\mathcal{A}$. Then we have less than n R-successors of a in $\mathcal{A}$ and the R$\exists_{\geq n}$-rule is applicable to a. This contradicts the assumption that $\mathcal{A}$ is complete. (2) a is blocked by an individual c but the same argument as in case (1) holds and leads to the same contradiction.

For $\mathsf{C} = \exists_{\leq n}\,\mathsf{R}$ we show the goal by contradiction. Suppose that $\mathsf{a} \notin (\exists_{\leq n}\,\mathsf{R})^{\mathcal{I}_\mathcal{A}}$. Then there exist at least $n+1$ distinct individuals $\mathsf{b}_1, \ldots, \mathsf{b}_{n+1}$ such that $(\mathsf{a}, \mathsf{b}_i) \in \mathsf{R}^{\mathcal{I}_\mathcal{A}}$, $i \in 1..n+1$. According to Definition 19 the following two cases can occur. (1) We have $n+1$ $(\mathsf{a}, \mathsf{b}_i) : \mathsf{S}_i \in \mathcal{A}$ with $\mathsf{S}_i \in \mathsf{R}^{\downarrow}$ and $\mathsf{S}_i \notin T$, $i \in 1..n+1$. The R$\exists_{\leq n}$ rule cannot be applicable since $\mathcal{A}$ is complete and the b_i are distinct, i.e. $\mathsf{b}_i \neq \mathsf{b}_j \in \mathcal{A}$, $i, j \in 1..n+1$, $i \neq j$. This contradicts the assumption that $\mathcal{A}$ is clash-free. (2) There exists a blocking individual c with $(\mathsf{c}, \mathsf{b}_i) : \mathsf{S}_i \in \mathcal{A}$, $\mathsf{S}_i \in \mathsf{R}^{\downarrow}$, and $\mathsf{S}_i \notin T$, $i \in 1..n+1$. This leads to an analogous contradiction.

If $\forall\, x\, .\, x : \mathsf{D} \in \mathcal{A}$, then –due to the completeness of $\mathcal{A}$– for each individual a in $\mathcal{A}$ we have $\mathsf{a} : \mathsf{D} \in \mathcal{A}$ and, by the previous cases, $\mathsf{a} \in \mathsf{D}^{\mathcal{I}_\mathcal{A}}$. Thus, $\mathcal{I}_\mathcal{A}$ satisfies $\forall\, x\, .\, x : \mathsf{D}$. Finally, since $\mathcal{I}_\mathcal{A}$ satisfies all assertions in $\mathcal{A}$, $\mathcal{I}_\mathcal{A}$ satisfies $\mathcal{A}$. □

Theorem 21 (Completeness) Let $\mathcal{A}$ be a satisfiable ABox, then there exists at least one completion of $\mathcal{A}$ computed by applying the completion rules.

Proof. Obviously, an Abox containing a clash is unsatisfiable. If every completion of $\mathcal{A}$ is unsatisfiable, then it follows from Proposition 14 that ABox $\mathcal{A}$ is unsatisfiable. □

Definition 22 For any augmentation of an initial ABox $\mathcal{A}$, we define the *concept size* $n_\mathcal{A}$ as the number of concepts or subconcepts occurring in $\mathcal{A}$.[4] Note that $n_\mathcal{A}$ is bound by the length of the string expressing $\mathcal{A}$. The *size* of an ABox $\mathcal{A}$ is defined as $n_\mathcal{A} \times \|T\| + \|O_O\|$.

Lemma 23 Let $\mathcal{A}$ be an ABox and let $\mathcal{A}'$ be a completion of $\mathcal{A}$. In any set X consisting of individuals occurring in $\mathcal{A}'$ with a cardinality greater than $2^{n_\mathcal{A}}$ there exist at least two individuals $\mathsf{a}, \mathsf{b} \in X$ whose concept sets are equal ($\mathsf{a} \equiv_{\mathcal{A}'} \mathsf{b}$).

Proof. Each assertion $\mathsf{a} : \mathsf{C_i} \in \mathcal{A}'$ may contain at most $n_\mathcal{A}$ different concepts $\mathsf{C_i}$. So there cannot exist more than $2^{n_\mathcal{A}}$ different concept sets for the individuals in $\mathcal{A}'$. □

Lemma 24 Let $\mathcal{A}$ be an ABox and let $\mathcal{A}'$ be a completion of $\mathcal{A}$. Then there occur at most $2^{n_\mathcal{A}}$ non-blocked new individuals in $\mathcal{A}'$.

Proof. Suppose we have $2^{n_\mathcal{A}} + 1$ non-blocked new individuals in $\mathcal{A}'$. From Lemma 23 we know that there exist at least two individuals a, b in $\mathcal{A}'$ such that $\mathsf{a} \equiv_{\mathcal{A}'} \mathsf{b}$. By Definition 11 we have either $\mathsf{a} \prec \mathsf{b}$ or $\mathsf{b} \prec \mathsf{a}$. Assume without loss of generality that $\mathsf{a} \prec \mathsf{b}$ holds and $\mathsf{a} \equiv_{\mathcal{A}'} \mathsf{b}$ implies $\sigma(\mathcal{A}', \mathsf{a}) \supseteq \sigma(\mathcal{A}', \mathsf{b})$. Then we have either $\mathsf{a} \not\succcurlyeq_{\mathcal{A}'} \mathsf{b}$ or there exists an individual c with $\mathsf{c} \not\succcurlyeq_{\mathcal{A}'} \mathsf{b}$ and $\mathsf{c} \prec \mathsf{a}$. Both cases contradict the hypothesis. □

[4]We have to increase $n_\mathcal{A}$ by 1 if $\top$ does not occur in $\mathcal{A}$.

Theorem 25 (Termination) Let $\mathcal{A}_\mathcal{T}$ be the augmented ABox w.r.t a TBox $\mathcal{T}$ and let n be the size of $\mathcal{A}_\mathcal{T}$. Every completion of $\mathcal{A}_\mathcal{T}$ is finite and its size is $O(2^{4n})$.

Proof. Let $\mathcal{A}'$ be a completion of $\mathcal{A}_\mathcal{T}$. From Lemma 24 we know that $\mathcal{A}'$ has at most 2^n non-blocked new individuals. Therefore, a total of at most $m \times 2^n$ new individuals may exists in $\mathcal{A}'$, where m is the maximum number of direct successors for any individual in $\mathcal{A}'$.

Note that m is bound by the number of $\exists\,\mathsf{R}\,.\,\mathsf{C}$ concepts ($\leq n$) plus the total sum of numbers occurring in $\exists_{\geq n}\,\mathsf{R}$. Since numbers are expressed in binary, their sum is bound by 2^n. Hence, we have $m \leq 2^n + n$. Since the number of individuals in the initial ABox is also bound by n, the total number of individuals in $\mathcal{A}'$ is at most $m \times (2^n + n) \leq (2^n + n) \times (2^n + n)$, i.e. $O(2^{2n})$.

The number of different assertions of the form $\mathsf{a} : \mathsf{C}$ or $\forall\, x\, .\, x : \mathsf{C}$ in which each individual in $\mathcal{A}'$ can be involved, is bound by n and each assertion has a size linear in n. Hence, the total size of these assertions is bound $n \times n \times 2^{2n}$, i.e. $O(2^{3n})$.

The number of different assertions of the form $(\mathsf{a}, \mathsf{b}) : \mathsf{R}$ or $\mathsf{a} \neq \mathsf{b}$ is bound by $(2^{2n})^2$, i.e. $O(2^{4n})$. In conclusion, we have a size of $O(2^{4n})$ for $\mathcal{A}'$. □

Theorem 26 (Decidability) Let $\mathcal{A}_\mathcal{T}$ be an ABox w.r.t. a TBox $\mathcal{T}$. Checking whether $\mathcal{A}_\mathcal{T}$ is satisfiable is a decidable problem.

Proof. This follows immediately from the Theorems 20, 21, and 25. □

5 Practical Reasoning with RACE

The tableaux calculus introduced in the previous sections is of theoretical interest for proving the decidability of the ABox consistency problem. For practical purposes such calculi are highly inefficient. Therefore, the development of optimization techniques is a very important research topic. In order to support practical ABox reasoning with $\mathcal{ALCNH}_{R^+}$ and to empirically

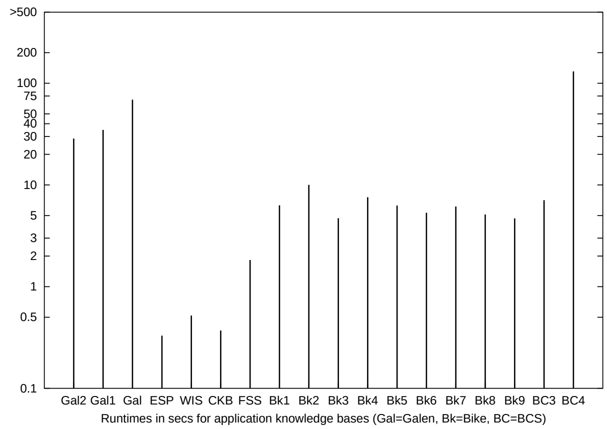

Figure 2: Runtimes for application KBs.

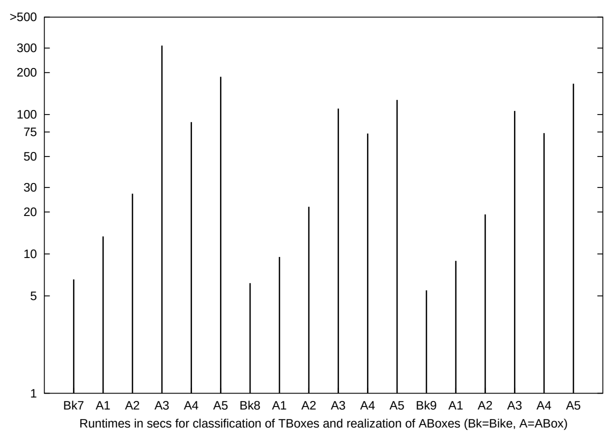

Figure 3: Runtimes for Bike TBoxes and ABoxes.

evaluate optimization techniques for this tableaux calculus, the DL system RACE[5] has been developed [6]. RACE implements an $\mathcal{ALCNH}_{R^+}$ reasoner for answering queries concerning ABoxes and TBoxes. It is a successor of HAM-ALC [3]. The RACE architecture incorporates established and novel optimization techniques for TBox and ABox reasoning [6, 5].

The combined effectiveness of these and other techniques are demonstrated with knowledge bases (KBs) derived from actual applications (Figure 2) and a set of ABox benchmark problems (Figures 3, 4). The 'Galen' application KBs are described in [8]. Their employed DLs range from $\mathcal{ALE}$ to $\mathcal{ALCH}f_{R^+}$. The KBs 'ESPR', 'WISBER', 'CKB', and 'FSS' (using $\mathcal{ALCNH}$ with GCIs) are taken from previous DL benchmarks but role hierarchies and domain and/or range restrictions for primitive roles (using GCIs) are restored [4]. The

[5]RACE is available from the URL http://kogs-www.informatik.uni-hamburg.de/~moeller/race.html

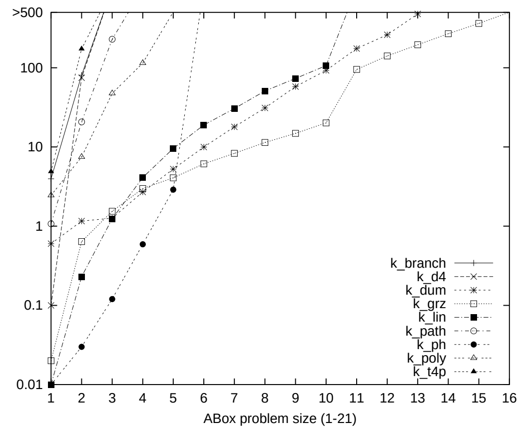

Figure 4: Runtimes for synthetic ABoxes.

'Bike' KBs (using $\mathcal{ALCNH}$ with GCIs) contain configuration knowledge about various types of bicycles. Their corresponding ABoxes describe example configurations of bikes. The KBs 'BCS3' and 'BCS4' (using $\mathcal{ALC}$ with GCIs) are derived from a telecommunication application. Their characteristics is the heavy use of terminological cycles and GCIs.

A set of five ABoxes is iteratively realized using the KBs 'bike7-9' (see Figure 3). The ABoxes describe bike example configurations and exemplify typical classification tasks. The TBoxes 'bike7-9' are almost identical except that they vary in the degree of specifying disjointness between atomic concepts. Figure 4 reports on the runtimes for realizing synthetic ABoxes with an increasing level of difficulty (1-21, see [6] for further explanations).

6 Conclusion

We presented the first treatment for a tableaux calculus deciding the ABox consistency problem for the description logic $\mathcal{ALCNH}_{R^+}$. A highly optimized variant of this calculus is already implemented in the ABox description logic system RACE demonstrating the practical usefulness of $\mathcal{ALCNH}_{R^+}$. Although TBox reasoners for logics such as $\mathcal{ALCQHI}_{R^+}$ are available, the development of $\mathcal{ALCNH}_{R^+}$ and its optimized implementation in RACE is a novel approach. Practical reasoning is only possible with the design and implementation of appropriate optimization techniques. This is supported by recent empirical findings suggesting that RACE dramatically outperforms other known DL reasoners for logics at least as expressive as $\mathcal{ALCNH}_{R^+}$. To the best of our knowledge there currently exists no other ABox DL system with a performance comparable to RACE.

Acknowledgements

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