

# CLOCK AND CATEGORY: IS QUANTUM GRAVITY ALGEBRAIC?

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**ABSTRACT:** We investigate the possibility that the quantum theory of gravity could be constructed discretely using algebraic methods. The algebraic tools are similar to ones used in constructing Topological Quantum Field theories. The algebraic structures are related to ideas about the reinterpretation of quantum mechanics in a general relativistic context.

## I. INTRODUCTION

The histories of mathematics and theoretical physics are so intimately interwoven that neither can really be understood in isolation from the other. There is no example of a fundamental advance in theoretical physics which did not involve a change in the mathematical structure in which the physical theory is formulated. On the other hand, a new construction in mathematics very often represents a distillation of a physical concept. Differential four dimensional manifolds with lorentzian metrics, for example, can be thought of as structures within which Einstein's falling elevators can coexist. Turning points in the development of theoretical physics are moments when mathematical thought and fundamental conceptual physical thought achieve a sort of fusion.

If we examine the current state of fundamental theoretical physics from a mathematical point of view, we find an extremely odd situation. The subject is dominated by the application of a formal pseudomathematical device, the Feynman path integral, which has no rigorous mathematical formulation, and which clearly cannot be given one in any straightforward way, since most of the quantum theories constructed from them do not actually exist. The disharmony between the foundations of mathematics and physics is so extreme and so jarring that most workers in the field tend to blot it out of their minds.

Topological quantum field theories (TQFTs), were originally defined formally in terms of path integrals [1,2]. Subsequently, they were rigorously constructed in low dimensions by various authors, using algebraic tools [3,4,5]. Among the algebraic tools which appear in this process, the techniques of category theory play a special role.

The purpose of this paper is to propose that the same mathematical tools which put TQFT on a rigorous footing can be applied to the problem of constructing the quantum theory of gravity. The physical thought at the heart of this proposal is the idea that path integral technology is inappropriate for quantum gravity (or quantum gravity coupled to matter) because the geometry which needs to be quantized has a cutoff at the Planck scale. Thus, averaging over continuum geometries via a path integral introduces unphysical infinities, which destroy the mathematical structure of the theory.

One family of algebraic constructions of TQFTs uses topological state sums. These can be thought of as discretized path integrals which avoid the usual pitfalls of lattice theories by a sort of algebraic magic.

What will be argued here is that the categorical tools which have been applied to TQFT can be interpreted as producing quantum gravity as well.

Perhaps it is not very useful to try to argue the plausibility of such a program in advance. It can really be substantiated only by empirical success, which still lies in the future. Nevertheless, let me make a few general remarks. It is hardly a controversial observation that the quantization of gravity is a deep problem. Hence it is reasonable to suggest that the formulation of the quantum theory of gravity requires a new mathematical structure to contain it. The insistence that fundamental physics should be formulated in well defined, non-divergent terms seems almost eccentric in the context of contemporary theoretical physics, where the dramatic success of Feynmanology has created a certain mind set over the last half century. I cannot help feeling though that anyone who pauses to consider the matter would find a rigorous mathematical foundation for a truly fundamental physical theory highly desirable.

Let me also remark that categorical algebra and relativity theory have similar philosophical roots. Both have in their foundations the idea that things should be defined independently of a coordinate system/observer. The reinterpretation of quantum mechanics which will be explored below is that the probability interpretation must be relativized, and that the Hilbert spaces related to different observers have natural linear maps between them. Thus, the observers form a category, which we can construct, and which has a very special structure. The invariant language of categories provides a form for discussing the relativity of the observation process and probability interpretation with respect to the observer.

The purpose of this paper is twofold: first to describe a new branch of mathematics, namely the algebraic approach to topological quantum field theory (TQFT) in three and four dimensions, and the transition between the two by the picture called the dimensional ladder. Secondly, to propose a line of reasoning by which this mathematical structure could be used to produce a quantum theory of gravity. The two parts are not at an equal level of development. The algebraic constructions are understood at least in outline, while the physical interpretation revolves around a reconsideration of the role of the observer in quantum mechanics, and is still unclear. Nevertheless, the resonances between the mathematical structure presented here and the development of the Ashtekar/loop variables approach to quantum general relativity [6] are so strong that the suggestion that they are related deserves to be pursued.

In a certain sense, I am proposing that a particular three dimensional TQFT, namely the CSW theory, already contains the quantum theory of gravity, but that in order to introduce clocks into the theory and connect it with experiment, it is necessary to relate it to a four dimensional theory.

Accordingly, the structure of this paper is as follows. We first review the

structure of CSW theory in 3D, and explain its relationship to quantum spin networks. Next, we explain how this can be thought of as a quantum theory of gravity, assuming a suitable reformulation of the principles of quantum mechanics. Then we shall discuss how to relate 3D and 4D TQFTs, and last, discuss how this may be relevant to the problem of introducing clocks into the theory.

The proposal discussed here creates a rather difficult communication problem, since vanishingly few theoretical physicists study category theory. An ambitious reader might want to study the subject from Maclane [7]. It may also be possible to wing it and just follow my development as it stands. Although it was clearly impossible to make it self contained, I have interspersed my discussion with thumbnail definitions which may suffice. Perhaps a physically minded reader would prefer to skip to chapter III on a first reading, where the physical ideas are discussed.

## II. THE BASIC STRUCTURE OF 3D TQFT

### A. TQFT

The definition of a TQFT can be explained in many ways. A major theme of this paper is that fundamental physical ideas related to quantum gravity can best be understood in the language of categorical algebra. Since this area is rather strange to physicists (and by no means universally popular among mathematicians), let us begin by showing how TQFTs, which, after all, are a type of physical theory, have a natural description in categorical language.

Remember that a category has objects and morphisms between the objects, which can be composed when the range of the first is the domain of the second. The classical example is sets and functions.

Accordingly, let us begin by recalling the structure of a cobordism category. The category of oriented  $n$ -dimensional cobordisms has oriented compact  $n-1$  dimensional manifolds as objects and cobordisms as morphisms. (A cobordism from  $M$  to  $N$  is an oriented  $n$ -dimensional manifold  $P$  with boundary, together with an oriented diffeomorphism between the boundary of  $P$  and  $M^* \cup N$ .) Composition of morphisms comes from gluing of manifolds along shared boundary components.

The category of oriented  $n$ -cobordisms has the natural structure of a tensor category with duality. (A tensor category is one where two objects can be combined into a third by an operation we call tensor product, and where morphisms can be similarly combined. The prototype example is the category of vector spaces.) The tensor product is disjoint union and the duality is reversal of orientation.

The most elegant definition of an  $n$ -dimensional TQFT is that it is a monoidal functor from the category of oriented  $n$ -cobordisms with disjoint union as tensor product to the category **VECT** of finite dimensional vector spaces with the

usual tensor product i.e. a functor which preserves tensor product up to canonical coherent isomorphism. (It is a point often missed that this suffices—that a manifold with opposite orientation is sent to the dual space of the image with the given orientation is an easy theorem, not a necessary part of the definition). Let me remind the reader that a functor is a mapping from one category to another, taking objects to objects and morphisms to morphisms, and taking composition to composition.

We often modify the definition of TQFTs by modifying the cobordism category. For instance, we can specify a framing of the tangent bundle of the cobordisms and of a formal neighborhood of the closed manifolds. Another possibility is to include insertions of submanifolds in the manifolds and matching insertions in the cobordisms. We also refer to tensor and duality preserving functors from such modified cobordism categories to **VECT** as TQFTs.

Let us spell out this definition for the less categorically inclined. An  $n$ -dimensional TQFT assigns a vector space to each oriented  $n$ -dimensional manifold, and a linear map to each oriented cobordism in such a way that the composition of cobordisms corresponds to the composition of linear maps, the disjoint union of manifolds gets the tensor product of vector spaces, and the manifold with opposite orientation gets assigned the dual space. As John Baez has pointed out [8], there is a natural inner product on the vector spaces whenever they are associated to a bounding 3-manifold, generated by gluing pairs of links along the boundary. Thus, we can think of them as Hilbert spaces.

Thus, for  $n=3$ , a TQFT assigns a vector space to a surface, and a linear map to a 3-dimensional cobordism. We also think of an assignment of a vector space to a surface with labelled punctures and a linear map to a 3D cobordism which contains a link with components which can end on the punctures, with labels on the components and framings of the cobordism and link as a 3D-TQFT. CSW theory in fact provides us with a 3D-TQFT in this extended sense. The similarity between this and the loop variable picture for quantum gravity [6] should be evident to the initiated.

## B THE CSW TQFT

The 3D TQFT which seems to be related to the quantum theory of gravity was first constructed in physicist's language by Witten from a topologically invariant lagrangian by means of a path integral [1]. Subsequently, it was reconstructed by algebraic methods by a number of authors. We will be interested in two approaches to the algebraic construction of 3D-TQFTs: the construction from a modular tensor category in [5], and the topological state sum, also from a modular tensor category in [4], but since generalized to any tensor category with only a weak condition on its duals in [9]. (See [10] for the definition of a modular tensor category.) The construction in [3] is also elegant and historically important, but does not play as big of a role in what follows.

The point which is most important for us here is the form of the construction of the finite dimensional Hilbert spaces on the surfaces, possibly with labelled punctures, from the modular tensor category.

As originally explained in [10], the vector space attached to a surface with labelled punctures is constructed by translating the surface into a word in the tensor category. This is accomplished by cutting the surface up into three holed spheres, or trinions, and attaching a tensor product to each trinion. The axioms of a modular tensor category are just what is needed so that if the same surface is decomposed in two different ways, an isomorphic object results. In order to assign a vector space to the punctured surface, we extract the invariant part of the object (this is the set of homomorphisms of the identity object of the category into the object; hom sets in these categories form vector spaces.)

In the case of the modular tensor category associated to  $SU(2)$  at any level, we can write a basis for the vector space attached to the surface by writing all ways to label the cuts with irreducible objects in the category, consistent with the quantum version of the Clebsch-Gordon relations. (A modular tensor category possesses only a finite set of irreducibles.) The formula for a general modular tensor category uses combinations of the spaces of tensor operators on irreducibles. We will not need it here, see [10]. Two different decompositions of the surface give two different bases, and the transition coefficients between them are determined by the structure of the category.

The example which seems relevant to quantizing gravity is the  $SU(2)$  case, ie the category either of representations of the loop group on  $SU(2)$  at a central extension, or of the quantum group at a root of unity with dimension zero modules quotiented out. The simple form of the vector spaces in that case seems to be closely related to a geometric interpretation. The thing which needs observing at this stage is that the irreducible objects of the category are labelled by spins, much like the irreducible representations of  $SU(2)$ . It may turn out that natural models for gravity coupled to matter emerge from more complex modular tensor categories.

In summary, the vector spaces attached to a punctured surface have bases described by cutting the surfaces into trinions (with the punctures, labelled with spins, at open ends of trinions) and giving labels to the internal cuts from the set of allowable spins, so that the triple of spins on each trinion satisfies the Clebsch-Gordon conditions for the truncated category of representations of a quantum group at a root of unity.

We now need to know a little about how to extract numbers and linear maps from 3D manifolds in this theory. There are two simple observations which solve this problem. One is that only the trivial representation can be extended across a closed disk. This means that the vector in the vector space associated to a compact surface which is associated to a solid handlebody which bounds the surface is the vector in the basis corresponding to any trinion decomposition which extends to the solid handlebody in which all the cuts are labelled by the trivial representation.

The second observation is that the change of basis from one trinion decomposition to another allows us to obtain a linear representation of the group of large diffeomorphisms (diffeomorphisms modulo isotopy) of the surface. Thus, a vector in the basis labelled by some spins on the cuts of a decomposition is mapped into some linear combination of similar vectors by a map of the surface to itself, where the coefficients are determined by the structure of the category.

When these two ideas are combined, we can derive formulas for the invariants on closed manifolds, and the linear maps on manifolds with boundary. The long and short of it is that everything can be written as linear combinations of the numbers we get by taking the dual pairing of a state on a surface embedded in  $R^3$  given by labelling the cuts of the trinion decomposition by spins with the state obtained by labelling the cuts of any trinion decomposition which extends to the exterior with the trivial representation. (The second state is the "exterior vacuum").

We can shrink the labelled embedded surface until it is very thin, and think of the number we are obtaining as an invariant of a labelled knotted trivalent graph. This is a generalization of the Jones invariant to knotted graphs. (The generalisation to an arbitrary modular tensor category requires us to label the edges of the embedded graph with irreducible objects of the category and the vertices with tensor operators).

### C. CSW THEORY AND DEFORMED SPIN NETWORKS

The invariants of embedded labelled graphs at the heart of the CSW TQFT are closely related to the evaluations of spin networks due to Penrose [11]. Spin networks are not sensitive to embeddings, but that is due precisely to the fact that Penrose did not have quantum groups to work with, and used representations of ordinary  $SU(2)$  instead [12].

Spin networks have been undeservedly forgotten. They were an effort to produce the kind of quantum geometry necessary to quantize gravity, and they succeeded to the point of giving a plausible form of quantized 2+1 gravity. One point of view on the present work is that it is an effort to bring Penrose's program up one dimension by adding better algebra.

The best way to understand how spin networks apply to quantizing geometry is to follow the approach of Regge and Ponzano [13]. What they do is to triangulate a region of space surrounded by the embedded graph, and prove by an inductive argument that the evaluation of the graph is given by a formula involving a sum over labellings of a product of one  $6J$  symbol for each tetrahedron.

In [14], it was observed that this formula is equally valid for the CSW invariants of embedded graphs. Let us note that this formula, using the representations of a quantum group at a root of unity, is identical with the formula used to construct a 3D-TQFT in [5]. It is being applied here in a different way to obtain the CSW invariant instead. Specifically, it is being applied to a triangulation

of one half of the manifold only. The Viro-Turaev formula gives the absolute square of the CSW invariant by summing over the entire manifold. (It would be very attractive to find a new formula of a similar form which bridged this gap, ie gave CSW when computed on the entire manifold. Kuperberg [15] has given a candidate for such a formula, which reproduces CSW for embedded graphs, but then gives zero on most manifolds. It is not yet clear if his work admits a modification to reproduce CSW on the nose.)

This formula is the classical example of what we mean by a topological state sum. It is a summation on a triangulation which does not change if we alter the triangulation. It has the form of a lattice field theory, with a strange lattice-independence which comes from abstract algebra. (The basic topological invariance of our formula comes from the Biedenharn-Eliot relation [16], as the chemists know it, which is the Stasheff pentagon for the associator of the tensor category).

### III. CSW THEORY AND QUANTUM GRAVITY

The above formula, in either context, has a natural interpretation as a quantum geometry. We can reinterpret the spins as lengths. We are then summing over all assignments of lengths to edges in a triangulation, and we can interpret this as a discretized summation over metrics. If we make this interpretation, and ask which terms in our sum have stationary phase, we find that they are discrete analogs of flat metrics. It was this observation which made Regge and Ponzano interpret the formula, in the case of representations of ordinary  $SU(2)$ , as a discretized path integral for 2+1 dimensional gravity. (The equations of motion for 2+1 general relativity are exactly flatness).

Thus, if I were to argue that CSW theory gives us 2+1 gravity, the argument would be quite straightforward. Indeed, it would also not be terribly original. A number of authors have made just such a connection in a number of ways [17]. Nobody expects 2+1 dimensional gravity to behave very differently from a TQFT.

Why then is the suggestion in this paper so novel? The physical objection to the idea that 3+1 gravity has a similar formulation to the 2+1 theory is based on the observation that the **linearized** theory of 3+1 dimensional gravity contains excitations called gravitons, which are similar to photons in that they come in all frequencies. These can be combined into wave packets of arbitrarily small volume, leading to the twin conclusions that quantum gravity A) contains an infinite dimensional Hilbert space, and B) has local excitations, both of which are properties alien to TQFT.

I want to argue that both of these arguments are misleading. Linearized general relativity is an extremely poor model for the full theory. It possesses a background metric, which solves the probability interpretation problems, at the cost of destroying the fully invariant nature of the theory. In particular, both properties A and B above are artifacts of the linearized theory, which do not

survive in the full theory. As regards property A, it is extremely implausible that gravitons of wavelength below the Planck scale survive the interactions of the full theory. They are more likely to disappear into a cloud of Planck scale black holes under their self gravitation. This puts an ultraviolet cutoff on the local excitations, and makes the Hilbert space finite dimensional. Recent works of several authors have pointed out that the energy of states in any bounded region is cut off at the Schwarzschild mass, so the dimension of the space of states is finite, and related to the surface area.

Far worse is the idea that excitations in general relativity are "local" in the same sense as in the linearized theory. General relativity is diffeomorphism invariant. This means that a state where an excitation occupies a given point (or, more plausibly, small volume) on the manifold is gauge equivalent to a state in which it is someplace else. This is well known to relativists, who say that "the points disappear".

General relativity has two other properties of a similar type. One is that the Hamiltonian vanishes, ie, that the time evolution operator is the identity. (Let us observe that the identical fact holds for a TQFT). Another is that the inner product on Hilbert space which provides a measure for the probability interpretation is missing, or at least well hidden. Various relativists have attempted to find the inner product by elaborate measures, but without clear success. It is, in any case, hard to imagine what a probability interpretation for whole universes would mean, since there can be no external observer.

These properties are so maddeningly counterintuitive that most physicists contrive to forget them. I propose to approach the problem of quantum gravity by recasting the form of the theory in such a way as to make these properties central. My proposal was originally motivated more by the mathematical tools described above than by direct reflection on the subtleties of quantum gravity. Nevertheless, I would like to state it in physical terms before pointing out how the techniques of TQFT could be used to construct the theory.

What I am proposing is that quantum gravity is not a quantum field theory at all. Instead of a Hilbert space and operator fields there is a much subtler structure, which I call a model for quantum gravity. Here is an outline of the principles and corresponding structure of such a model:

## **PRINCIPLES FOR A MODEL FOR THE QUANTUM THEORY OF GRAVITY**

1. *No observation is possible without an observer.* Hence there is no Hilbert space associated with a closed universe. Any observer is part of a universe, hence occupies a 3-manifold with boundary, and makes observations on another such with a shared boundary.

2. *There is no observation at a distance.* Thus the Hilbert spaces in the theory reflect the interface between observer and system. This means they are associated to surfaces.



3. *Observers observe each other.* Hence there are linear maps between the Hilbert spaces attached to surfaces in the same 3-manifold. These maps satisfy a natural consistency condition.

4. *States for quantum gravity can be described by embedded graphs or knots.* Hence the assignments of Hilbert spaces and maps above can be extended to surfaces with punctures and 3-manifolds with graphs which end on the punctures of their boundaries.

5. *General relativity is diffeomorphism invariant.* Hence all the above assignments depend only on topological type.

6. *General relativity is a theory of geometry.* Hence a state in a special basis for the Hilbert space on a surface must assign values to lengths on the surface, and probability amplitudes to combinations of lengths elsewhere.

7. *The Hamiltonian for general relativity is 0.* Hence there is no time evolution for states in themselves. The core theory is three dimensional. The experience of time evolution in the world must be described by treating variables in the theory as local clocks. There are no global clocks.

8. *A global Hilbert space can be recovered in a semiclassical limit.* If we associate a family of observers in some natural way to a triangulation of the manifold, we can use the maps between the vector spaces to take an inverse limit. In the limit as the triangulation is refined this reproduces a global Hilbert space, which can be thought of as corresponding to a universe full of classical observers whose quantum correlations can be neglected.

We invite the reader to contemplate the intrinsic plausibility of this proposal before considering the following observation:

THE CSW TQFT PROVIDES EXACTLY WHAT WE HAVE SPECIFIED ABOVE AS A MODEL FOR QUANTUM GRAVITY, except for the local clocks.

Let us spell this out: the vector spaces on embedded surfaces correspond to the spaces which an observer on one side can observe when interacting with the other. The linear maps between these are the maps corresponding to the cobordisms between them given by the regions of space they jointly bound. The formula given above for CSW invariants as a summation is interpreted as a probability amplitude for lengths in the volume they enclose. Spins are reinterpreted as lengths, as explained above.

In some sense, then, we are suggesting that we already know the quantum theory of gravity. The deformed spin network description of states in CSW on a surface allows us to interpret them as quantum 3-geometries on regions thought of as the interiors of observed systems.

In a theory with vanishing Hamiltonian, the state of the whole universe cannot change. A model for the quantum theory of gravity is the same as a state for the universe. Thus, this proposal is consistent with the observation

that the Chern Simons lagrangian gives a state for quantum general relativity in the Ashtekar variables, with cosmological constant [18]. This proposal is also related to the suggestion of Witten that the Chern Simons theory is the topological phase of gravity [19], together with the suggestion that we never left the topological phase.

Of course, this is not what anybody really wants from a theory. In order to relate to experiment, it is necessary to describe a time in which things change.

The rest of my proposal is a line of development which attempts to solve this problem by using an important fact about TQFTs, namely that the algebraic constructions of TQFTs in adjacent dimensions are closely related. The idea is that a 4D-TQFT which is closely related to CSW theory would contain the secret of time.

#### IV. 4D TQFT AND THE DIMENSIONAL LADDER

The goal of the mathematical program which is outlined here is to produce a 4D-TQFT which has three properties: it is factorizable, related by categorification to the 3D CSW theory, and admits a topological state sum formulation.

We shall explain here what each of these three condition means, and suggest how they can be obtained. In the last part of this paper, we shall try to show what each of them has to do with quantizing gravity, (and incidently why categories are related to clocks).

##### A. Factorizability

A TQFT can be thought of as an invariant of closed manifolds which can be "factorized" when the manifold is cut along a hypersurface so that the two halves (manifolds with boundary) are assigned vectors in a vector space, and the invariant is recovered as a dual pairing. A 3D-TQFT with factorizability has an analogous structure one layer farther down in dimension so that we can cut surfaces along sets of circles and write the vector space on the surface as the hom-space between objects associated to the pieces in a suitable category. (the categorical analogue of a dual pairing!). The definition of a TQFT with factorizability is similar in higher dimension, except that there are successive categorical layers at higher codimension. See [20] for the general definition. We shall give the 3D definition in its full glory in a minute.

The importance of factorizability is that it allows us to obtain the objects on manifolds with boundary by cutting them up into manifolds with corners. In particular, a 3D theory can be extended to include three manifolds with corners. If we have a 3-manifold with boundary with an inserted link with ends on the boundary, we can describe this as a 3 manifold with corners by hollowing out a neighborhood of the link. The circles around the punctures at the ends of the links then are the corners. This means that the extension of CSW theory to include punctures and links or embedded graphs is a direct consequence of

factorizability. As we shall see, factorizability will be essential in trying to make a physical interpretation of our theory.

For completeness, we reproduce here the definition from [20] in both a categorical and not-so categorical form.

To set up the formal definition embodying this notion, we must remind the reader that a finitely generated semisimple linear category is one in which each object is isomorphic to a direct sum of irreducible (simple) objects chosen from a finite set of such objects, hom-sets (the set of morphisms between two objects) are complex vector spaces, and composition is bilinear. As categories, they are equivalent to  $\mathbf{VECT}^n$  for some  $n$ . For the theory of such categories, also called  $\mathbf{VECT}$  modules, see [21]. As shown in [21], these categories form a monoidal bicategory: objects are  $\mathbf{VECT}$  modules, 1-arrows are exact  $\mathbf{C}$ -bilinear functors, 2-arrows are natural transformations, and the tensor product is given up to canonical equivalence by using pairs of the generating simple objects in the tensorands as a set of generating simple objects.

Similarly observe that there is a monoidal bicategory of 3-dimensional cobordisms with corners,  $3\text{-cobord}_2$ : its objects are 1-manifolds, its 1-arrows are (2-dimensional) cobordisms of 1-manifolds, and its 2-arrows are cobordisms with corners between pairs of 2-dimensional cobordisms with the same source and target. To be precise, a 3-dimensional cobordism with corners is a 3-manifold with corners, whose boundary is a union along a family of circles joining corresponding boundary components of the two surfaces. The 1-dimensional composition of 1-arrows and 2-dimensional composition of 2-arrows are just given by glueing target to source. The 1-dimensional composition of 2-arrows consists of glueing along the corners and glueing on a “collar” . It is trivial to verify that disjoint union gives this bicategory the structure of a monoidal bicategory.

In what follows, we shall refer to a cobordism (resp. cobordism with corners) as “trivial” if its underlying space is a product of one of its boundaries with the interval (resp. a product of one of its boundary strata with the interval modulo collapsing the product of the bounding corner with the interval back onto the corner). Note that a trivial cobordism or cobordism with corners need not be the identity cobordism—the attaching maps at the ends could be different. However, a trivial cobordism is manifestly invertible.

In the definitions below, the non-categorically minded reader is advised on first reading to read only the bold-faced portions of the definitions. These give the essential flavor of the definition, without going into excessive categorical detail.

A **3D TQFT with factorization** is a monoidal bifunctor from  $3\text{-cobord}_2$  to  $\mathbf{VECT} - mod$ .

Less briefly, but more intelligibly to the non-categorically minded, this **entails an assignment of**

1. **A finitely generated semisimple  $\mathbf{C}$ -linear category to each compact 1-manifold.**

2. An exact  $\mathbf{C}$ -linear functor to each 2-dimensional cobordism. In particular, since an exact  $\mathbf{C}$ -linear functor from  $\mathbf{VECT}$  to a semisimple  $\mathbf{C}$ -linear category is completely determined by the image of  $\mathbf{C}$ , we have **a choice of an object in the category associated to the boundary of each oriented surface with boundary, and more particularly, we have an assignment of a vector space to every closed oriented surface.**
3. A natural transformation to each 3-dimensional cobordism with corners. In particular **for a 3-manifold with boundary and corners consisting of two surfaces with boundary sharing their common boundary as a corner, we have a map in the category associated to the boundary of the surfaces between the objects associated to the surfaces.** Likewise, since the empty surface is assigned  $\mathbf{C}$ , **a 3-manifold with boundary is assigned a vector in the vector space associated to its boundary, and finally a 3-manifold without boundary is assigned a number.**

Moreover, these assignments will satisfy:

1. **The disjoint union of two 1-manifolds gets the tensor product in the sense of [21] of the semisimple categories attached to the parts. The empty 1-manifold will be assigned  $\mathbf{VECT}$ .**
2. **The 1-manifold with opposite orientation is assigned the dual category. (cf. Yetter [22])**
3. **The disjoint union of surfaces is assigned the tensor product of the vector spaces on the surfaces.**
4. **The surface with opposite orientation is assigned the dual vector space.**
5. **If we cut an oriented surface along a 1-manifold (union of circles), the vector space on the closed surface is naturally isomorphic to the hom set of the two objects in the category corresponding to the cuts which correspond to the two surfaces with boundary. A similar result holds for the case when we cut a surface with boundary and take hom with respect to the “tensor indices” corresponding to the cuts only.**
6. **If we cut a 3-manifold along a surface with boundary, the number invariant of the manifold is the dual pairing of the vectors associated to the two manifolds with boundary.**
7. **If we join two cobordisms with corners to form a cobordism, the linear map associated to the cobordism is the hom of the two linear maps. If we join two cobordisms with corners along a surface**

**with boundary to form a new cobordism with the same corner, the map corresponding to the new cobordism is the composite of the old maps.**

The reader will no doubt have noticed that these assignments and conditions fall into two analogous tiers, with semisimple linear categories closely paralleling vector spaces. The situation for  $D=4$  will be closely analogous again, with a third categorical tier.

It follows from these axioms that the category on a circle has an associative tensor product, corresponding to the three holed sphere, or trinion, with associativity constraints given by trivial cobordisms with corners. Moreover, the category must be braided, again with structure maps given by trivial cobordisms with corners.

## B. Categorification

As we stated above, factorizable TQFTs have tiers of analogous structures at higher categorical levels. The structure on surfaces in a 3D-TQFT, for example, is a categorical analog of the structure of a 2D-TQFT.

To make this notion precise, we have defined the idea of a categorification [23]. The categorification of any type of algebraic structure is a type of structure with analogous operations one categorical level higher. A categorification of a particular algebraic structure is an example of the categorified type of the structure which when "traced down" gives us back the original structure.

Let us elucidate these operations by means of a concrete example. It is common knowledge that the axioms of a category with direct sum and tensor product (such as the category of vector spaces of finite dimension) satisfies a list of axioms like those of a ring except for additive inverses), only with circles around the operations. The equations for the ring become isomorphisms for the category, which then satisfy certain natural axioms called coherence identities [21].

This is what we mean by an analogous structure one categorical level up. We say that the structure of a semisimple ring category ( a category with direct sum and tensor product and the usual axioms, where every object is a sum of irreducibles and hom sets are vector spaces) is a categorification of the structure of an algebra.

We trace out a tensor category by using the operations to induce a ring structure on the space of formal linear combinations of objects in the category. We use real or complex coefficients. Thus, the category of vector spaces is a categorification of the algebra of real or complex numbers. (Every object is a sum of copies of a single generator).

We have discovered that in many situations if a given type of algebraic structure can be used to generate a TQFT in  $D=n$  then a categorification can naturally be used to generate a TQFT in  $D=n+1$ . This principle, which we

have called the dimensional ladder [23], suggests that we can categorically lift particular TQFTs if we can find categorifications of the particular structure used to construct them.

The algebraic operation of going down in categorical level is referred to as tracing out, because the structure spaces of the category are replaced by their dimensions as structure coefficients of the algebra. (Sometimes spaces are assigned exotic or "quantum" dimensions.)

The algebraic operation of tracing out corresponds to the geometric operation of taking the cartesian product with  $S^1$ . We can always turn an  $n+1$  dimensional TQFT into an  $n$  dimensional one by this means, assigning to every object the algebraic structure assigned to its product with the circle.

Categorification is analogous in several respects to quantization. In the first place, it is neither always defined nor unique. In the second, it is the inverse of a well defined procedure (tracing out) which yields a simpler theory from a more complex one. (Like Bohr's correspondence principle). Finally, categorification of a structure requires that it have an integral structure, ie a basis in which the structure coefficients are positive integers. This is reminiscent of the discrete orbitals of quantum theory.

The reason for hoping that a categorification for CSW theory exists is that the quantum groups admit categorifications, as part of the canonical basis program of Lusztig [24]. In [23], we have proposed a construction of a 4D-TQFT based upon this fact.

#### D. Topological State Sums

We have already given an example of what we mean by a topological state sum (TSS) in the formula we cited for  $D=3$ . In general, we say we have a topological state sum whenever we have rules for labelling triangulations at certain places (specifically "combinatorial flags", or places where simplices of different dimensions meet) and rules for extracting numbers from the combinations of labels on simplices, so that the sum over labellings of the product over simplices of the numbers is invariant of the triangulation of the manifold. Such a sum of products has an obvious analogy to a path integral, so it is no surprise that a TQFT can be constructed from it.

In fact, interesting examples of 2D and 3D TSSs were constructed by physicists interested in lattice QFTs in [25,26]. They found that 2D theories were related to semisimple algebras, while 3D theories were related to Hopf algebras. The latter work is equivalent to the work of Kuperberg in [15], although the notation is very different. This is another example of the relationship between TQFT and abstract or categorical algebra.

For the purposes of this work, we are especially interested in the construction of 4D TSSs. The literature on this is quite thin. Aside from the construction related to a finite group in [27], which seems far removed from realistic physics,

(although it does reproduce Dijkgraf-Witten theory), there are only two formulas proposed: the one in [28], which is closely connected to CSW theory, and also uses a modular tensor category, and the one in [23], which is part of the dimensional ladder picture, and utilizes the categorification of a Hopf algebra, or a Hopf category.

The first of these expressions is known to give the euler character and signature of a 4-manifold, while the second has not yet been explored. We do not know at this time if the two formulas are related in any way.

It would probably not be productive for the purposes of this paper to reproduce the explanations of these two formulas. Let me give a simple description of them as background for the discussion of possibilities for their physical interpretation. The first of the two formulas [28] (formula 4A) is a rather direct analog of the 3D formula discussed above. Spins are places on faces in the triangulated 4 manifold, and simple closed networks are associated to the boundaries of the 4-simplices. The formula is then a sum over labellings of a product over simplices, as usual. The second formula [23] (formula 4B), is much more subtle. It uses the structure of a Hopf category associated to the Borel subalgebra of a quantum group. These exist in general because of Lusztig's work, but the  $SU(2)$  case can be done by hand.

## V. TQFT AND QUANTUM GRAVITY

Now let us state a conjecture which summarizes all the aims of the above mathematical program.

**CONJECTURE:** *there exists a 4D-TQFT which is a categorification of CSW theory, is factorizable, and can be written in terms of a topological state sum. The topological state sum can be interpreted as a quantum geometry, and gives Einstein's equations in a suitable classical steepest descent approximation.*

This is a sizable wish list. In a moment I will review how much of this is known or highly likely, and how much is plausible.

First, let us see how a theory of quantum gravity could be constructed along these lines which could be tested by experiment.

We could model an experiment by means of a four manifold with corners. The initial state could be defined by picking a state on the boundary surface of the initial manifold with boundary, and a final state could be similarly defined on the terminal hypersurface with boundary. We could then calculate probability amplitudes by computing the state sum on the 4 manifold with corners. Physically, we would think of this as doing a quantum experiment on the outside of a classical observer, while fixing the state of the observer itself. We would see excitations which propagated, which were localized relative to the observers on the boundaries, but not more finely than the Planck scale. The fact that steepest descent, in a classical limit, for the TSS gave us back Einstein's equation would imply that we had a possible quantization of general relativity. The observers would solve the problem of locality, which is as things should be.

Why do we consider it desirable that the 4D TQFT be a categorification of CSW theory? The argument is the following: if the 4D theory is the categorification of CSW, then we must reproduce CSW if we trace out the 4D theory, ie consider it on cartesian products with  $S^1$ .

The operation of tracing out is well known to relativists in a different setting. It is called "quantization with periodic euclidean time." It is generally believed that this procedure produces a thermal state; much of Hawking's work on black hole radiation is based on this idea.

Thus, the idea that CSW is recovered by tracing out is consistent with the suggestion [29] that CSW is a thermal state for quantum gravity. The suggestion in [30] that the CSW lagrangian provides a time parameter for quantum gravity is also closely related.

How likely then, is it that our mathematical conjecture can be demonstrated? Most of it seems to be within reach. Formula 4B is directly a quantization of Kuperberg's formula [15], which comes rather close to reproducing CSW. On the other hand, 4A, although it produces only a classical invariant, is extremely similar to the CSW invariants.

The really difficult piece of our wish list is the geometric interpretation, and the recovery of Einstein's equation from the steepest descent of the formula in the classical limit. (The classical limit is the limit of large spins, ie of classical distances.) It seems likely that if we could produce a state sum which imitated Einstein's equation in 4D, then even the most categorically disinclined relativists would agree that we had something of interest.

The motivation for hoping this from the mathematical point of view is, in the first place, the fact that exactly what we want happens in the 3D case. The evaluation of a spin network reproduces 3D Einstein in the classical limit, see [13]. The CSW invariant is very similar, and only adds a cosmological constant. Now the similarities of the 3D and 4D formulas give us reason to hope. Formula 4A can be thought of as putting spins on surfaces in 4D, which is reminiscent of the recent suggestion [31] that the observables for the loop variables are areas of surfaces, not lengths.

In fact, it is possible, up to a small ambiguity, to define the geometry of a triangulated 4-manifold by assigning areas to faces, instead of lengths to edges. My collaborators and I are currently exploring whether this gives rise to an interesting interpretation of formula 4A.

The suggestion that the quantum theory of gravity can be reconstructed from a topological state sum, and hence indirectly from abstract algebra can be thought of several ways. It is reminiscent of the suggestion of E. Witten that the fundamental unified theory of nature should resemble a "sporadic" algebraic structure. Although Witten's suggestion was motivated by mathematical aspects of particle physics, it has a general philosophical appeal as well. The picture that some very special algebraic structure, whose operations are naturally related to four dimensional geometry creates our world has a strange appealing simplicity.



## VI. ANCESTRAL VOICES

To my great surprise, I recently learned from an essay by John Stachel [32], that the ideas explored here are very similar to the ones that Einstein was exploring near the end of his career. Apparently, these ideas remained obscure because Einstein did not have the mathematical tools to carry them to fruition, and discussed them only orally or in letters.

Let us repeat a few of Einstein's sentences:

*... you have correctly grasped the drawback that the continuum brings. If the molecular view of matter is the correct (appropriate) one, ie., if a part of the universe is to be represented by a finite number of moving points, then the continuum of the present theory contains too great a manifold of possibilities. I also believe that this too great is responsible for the fact that our present means of description miscarry with the quantum theory. The problem seems to me how one can formulate statements about a discontinuum without calling on a continuum as an aid; the latter should be banned from the theory as a supplementary construction not justified by the essence of the problem, which corresponds to nothing "real". But we still lack the mathematical structure unfortunately...*

....Einstein to Walter Dallenbach

So not only was Einstein interested in a discrete theory, but he thought the missing element was mathematical.

*The other possibility leads in my opinion to a renunciation of the space-time continuum, and to a purely algebraic physics.*

...Einstein to Paul Langevin

I am proposing to scrap the metrical continuum, but retain the topological one. Algebra!

*An algebraic theory of physics is affected with just the inverted strengths and weaknesses; aside from the fact that no one has been able to propose a possible logical schema for such a theory. It would be especially difficult to derive something like a spatio-temporal quasi-order from such a schema. I cannot imagine how the axiomatic framework of such a physics would appear, and I don't like it when one talks about it in dark apostrophies. But I hold it entirely possible that the development will lead there...*

.....Einstein to H. S. Joachim.

The line of development in this paper also seems relevant to the objections to quantum mechanics of the **LATE** Einstein. In his later writings, Einstein

was bothered by the fact that quantum mechanics posits a macroworld, and gets its observables from it. Since I am proposing a picture in which the classical world reappears in a suitable limit, I believe that the objection Einstein voiced could be answered by the completion of my program.

I was able to resist the urge to retitle this paper "On some ideas of Professor Einstein," but given the unconventional nature of my proposal, I am happy for the moral support. Einstein, after all, was often ahead of his time.

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