Common Assumption of Rationality
Preliminary Version

H. Jerome Keisler

January 15, 2009

Abstract

We solve a problem which was posed in the paper [2], "Admissibility in Games". Given a finite game in strategic form with two players, we show that there is a complete lexicographic structure with the following property. A pair of strategies is iteratively admissible if and only if it belongs to a state with rationality and common assumption of rationality.

1 Introduction

Given a finite game in strategic form with two players, say Ann and Bob, we show that the epistemic framework in [2] allows the following.

(a) Each player considers all possibilities.

(b) Each player is rational in the sense of [2] (avoids weakly dominated strategies but rules nothing out).

(c) Ann assumes that Bob is rational, that Bob assumes Ann is rational, that Bob assumes Ann assumes Bob is rational, and so on.

(d) Same as (c) with Ann and Bob reversed.

Our result will show that a pair of strategies belongs to the IA set (i.e. survives iterated deletion of weakly dominated strategies) if and only if it can be played under the conditions (a)–(d). This provides some support for iterated admissibility as a solution concept.

Property (a) corresponds to the notion of completeness in [2]. Property (b) is rationality, and properties (c) and (d) together correspond to common assumption of rationality in [2].

In this introduction we will state our result formally and explain how it fits in with related results in [2].

In the next section we will give a brief review of the framework developed in [2], in order to explain how our theorem delivers the desired conditions (a)–(d).

The paper [2] introduces lexicographic type structures, iterated admissible sets of strategies S^a_m, S^b_m, and iterated rationality sets of strategy-type pairs R^a_m, R^b_m. Here is the formal statement of our result in these terms.

Theorem 1.1 *There is a complete lexicographic type structure*

$$\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$$

such that

$$proj_{S^a}\left(\bigcap_{m=1}^{\infty} R^a_m\right) \times proj_{S^b}\left(\bigcap_{m=1}^{\infty} R^b_m\right) = \bigcap_{m=1}^{\infty} S^a_m \times \bigcap_{m=1}^{\infty} S^b_m. \tag{1}$$

In Theorem 8.1 of [2] states that there is a lexicographic type structure which satisfies condition (1). This shows that conditions (b)–(d) are possible, but does not give condition (a). Our theorem here improves this result by adding the completeness property. As we will see in the discussion below, this delivers all of the conditions (a)–(d).

Theorem 1.1 is also a natural complement to two other results in [2]. Theorem 9.1 of [2] showed that in any complete lexicographic type structure, we have

$$proj_{S^a} R^a_m \times proj_{S^b} R^b_m = S^a_m \times S^b_m$$

for each m. This gives conditions (a)–(d) for any finite number of iterations of assumptions about assumptions, but does not show that the players can reason "all the way down".

Theorem 10.1 of [2] is a negative result showing that in any complete lexicographic type structure which is continuous, both sets $\bigcap_{m=1}^{\infty} R^a_m$ and $\bigcap_{m=1}^{\infty} R^b_m$ are empty. This result shows that there must be cases where (a)–(d) hold for any finite number of iterations, but the players cannot iterate all the way. Our result here shows that for each game there are lexicographic type structures that have all the desired properties (a)–(d), iterating all the way.

Theorem 1.1 raises the question of whether there are complete lexicographic type structures which satisfies condition (1) and in addition has other completeness properties, such as containing all hierarchies of beliefs in the sense of Friedenberg [4].

The author thanks Adam Brandenburger and Amanda Friedenberg for helpful discussions related to this paper.

2 The Underlying Framework

In this section we give a brief review of the concepts we will need from the paper [2].

Iteratively admissible strategies: Consider a finite game

$$G = \langle S^a, S^b, \pi^a, \pi^b \rangle$$

with strategy sets S^a, S^b and payoff functions π^a, π^b. Let $G(X, Y)$ be the sub-game formed by restricting Ann to strategies in X and Bob to strategies in Y. A strategy $s^a \in S^a$ is **admissible** for $G(X, Y)$ if $s^a \in X$ and s^a is not weakly dominated in $G(X, Y)$. Put $S^a_0 = S^a$ and $S^b_0 = S^b$. Inductively define S^a_{m+1} to be the set of all s^a which are admissible for the sub-game $G(S^a_m, S^b_m)$, and define S^b_{m+1} similarly. Note that $S^a_{m+1} \subseteq S^a_m$. The set $\bigcap_{m=1}^{\infty} S^a_m \times \bigcap_{m=1}^{\infty} S^b_m$ is called the **iteratively admissible set** (or IA set). This is the set of strategy pairs which survive iterated deletion of weakly dominated strategies. Since the sets S^a, S^b are finite, we have $\bigcap_{m=1}^{\infty} S^a_m \times \bigcap_{m=1}^{\infty} S^b_m = S^a_M \times S^b_M$ for some M, and the IA set is nonempty.

Lexicographic probability systems: Given a Polish space Ω, a **lexicographic probability system** (LPS) on Ω is a finite sequence of Borel probability measures $\sigma = (\mu_0, \ldots, \mu_{n-1})$ on Ω such that σ is mutually singular, that is, there are Borel sets $U_i, i < n$ such that $\mu_i(U_i) = 1$ and $\mu_i(U_j) = 0$ for $i \neq j$. The set of all LPS's on Ω is denoted by $\mathcal{L}(\Omega)$. As explained in [2], $\mathcal{L}(\Omega)$ has a natural metric, its completion $\overline{\mathcal{L}}(\Omega)$ is a Polish space, and $\mathcal{L}(\Omega)$ is a Borel set. An LPS σ has **full support** if the union of the supports of the measures $\mu_i, i < n$ is the whole space Ω. The set of all full-support LPS's on Ω is denoted by $\mathcal{L}^+(\Omega)$.

Assumption: An LPS on Ω represents a system of beliefs about possible events in Ω, and a full-support LPS represents a system of beliefs which rules nothing out. Fix a full-support LPS $\sigma = (\mu_0, \ldots, \mu_{n-1})$ on Ω. Intuitively, σ **assumes** an event E if every part of E is infinitely more likely than the complement of E under σ. In [2], the notion that σ assumes E is formally defined for any Borel event E in Ω. This definition is simplest when E is open, and that is the only case we will need here. We say that σ assumes an open set E at level j if $\mu_i(E) = 1$ for each $i \leq j$, and $\mu_i(E) = 0$ for each $i > j$ [1]. The events assumed at level 0 can be thought of as primary hypotheses, whose complements are considered to be infinitely unlikely. The events assumed at level 1 are first alternative hypotheses, whose complements are considered to be even more unlikely, and so on.

Lexicographic type structures: A **lexicographic type structure** is a structure

$$\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$$

[1]This will be justified Lemma 4.4 below

where T^a, T^b are Polish spaces and $\lambda^a : T^a \to \overline{\mathcal{L}}(S^b \times T^b)$, $\lambda^b : T^b \to \overline{\mathcal{L}}(S^a \times T^a)$ are Borel mappings. Each type $t^a \in T^a$ for Ann corresponds to a system of beliefs $\lambda^a(t^a)$ about Bob, and vice versa. In [2], a lexicographic type structure is called **complete** if the range of λ^a properly contains $\mathcal{L}^+(S^b \times T^b)$, and similarly with a, b reversed. Completeness is the mathematical counterpart of the heuristic condition (a) at the beginning of this introduction, that each player considers all possibilities. It says that each player considers the possibility that the other player's beliefs correspond to any full-support LPS, and also considers the possibility that it does not correspond to a full-support LPS.

Rationality: A strategy $s^a \in S^a$ is **optimal** under an LPS σ on $S^b \times T^b$ if there is no strategy $r^a \in S^a$ whose associated sequence of expected payoffs under σ is lexicographically greater than that of s^a. A strategy-type pair (s^a, t^a) is **rational** if $\lambda^a(t^a)$ is a full-support LPS and s^a is optimal under $\lambda^a(t^a)$. The set of all rational pairs for Ann is denoted by R_1^a. The set R_1^b is defined similarly with a, b reversed.

The RCAR set: For $m > 0$, we inductively define R_{m+1}^a as the set of all pairs $(s^a, t^a) \in R_m^a$ such that $\lambda^a(t^a)$ assumes R_m^b. As usual, the set R_{m+1}^b is defined analogously. We let $R_\infty^a = \bigcap_{m=1} \infty R_m^a$, $R_\infty^b = \bigcap_{m=1}^\infty R_m^b$. If

$$(s^a, t^a, s^b, t^b) \in R_\infty^a \times R_\infty^b,$$

we say that there is **rationality and common assumption of rationality** (RCAR) at the state (s^a, t^a, s^b, t^b), and we call $R_\infty^a \times R_\infty^b$ the RCAR set.

By Property 6.3 in [2], if each set R_m^a is assumed under an LPS σ, then the intersection R_∞^a is also assumed under σ (and similarly for b). Therefore, for any state (s^a, t^a, s^b, t^b) in the RCAR set, $\lambda^a(t^a)$ assumes R_∞^b and $\lambda^b(t^b)$ assumes R_∞^a.

One can now see that Theorem 1.1 does show that conditions (a)–(d) at the beginning of this introduction are possible. We have already discussed condition (a). We have noted that the IA set $S_M^a \times S_M^b$ is nonempty, so by Theorem 1.1 the RCAR set is also nonempty, and thus contains a state (s^a, t^a, s^b, t^b). By definition, s^a belongs to R_1^a and s^b belongs to R_1^b, so condition (b) holds. Finally, for each m, (s^a, t^a) belongs to R_{m+1}^a and hence $\lambda^a(t^a)$ assumes R_m^b, and similarly with a, b reversed. This gives conditions (c) and (d).

As a corollary to Theorem 1.1, we get the characterization the IA set stated at the beginning of this introduction: the IA set is the set of all strategy pairs that can be played under the conditions (a)–(d).

Corollary 2.1 *A pair of strategies (s^a, s^b) belongs to the IA set if and only if in some complete lexicographic structure there is a state (s^a, t^a, s^b, t^b) which belongs to the RCAR set.*

Proof. Suppose first that there is a complete lexicographic structure with a state (s^a, t^a, s^b, t^b) belonging to the RCAR set. Then $(s^a, t^a, s^b, t^b) \in R^a_M \times R^b_M$. By Theorem 9.1 of [2], $(s^a, s^b) \in S^a_M \times S^b_M$, which is the IA set.

Suppose now that (s^a, s^b) belongs to the IA set. By Theorem 1.1 there is a complete lexicographic structure $\langle S^a, T^a, S^b, T^b, \lambda^a, \lambda^b \rangle$ which satisfies (1). Then there is a state (s^a, t^a, s^b, t^b) which belongs to the RCAR set, as required. ■

In the course of the proof of Theorem 1.1, we will show that the type structure can be taken to be mathematically well-behaved in several ways. We state this formally in the next two propositions.

Proposition 2.2 *In Theorem 1.1, the type structure* $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ *can be taken so that:*

(i) The sets T^a, T^b *are compact.*

(ii) λ^a *maps* T^a *onto* $\mathcal{L}(S^b \times T^b)$*, and* λ^b *maps* T^b *onto* $\mathcal{L}(S^a \times T^a)$*.*

(iii) The iterated rationality sets R^a_m, R^b_m, R^a_∞*, and* R^b_∞ *are open.*

(iv) The mappings λ^a, λ^b *are continuum-to-one, that is, the pre-image of each point in the range has cardinality* $2^{\aleph_0}$*.*

Part (ii) of Proposition 2.2 strengthens the assertion that the players consider all possibilities. It says that each player considers the possibility that the other player uses any LPS, full-support or not. Part (iii) shows that our result is robust with respect to assumption, that is, it depends only on the definition of assumption for open events. As we have mentioned above, the notion of assumption for open events is simpler than the general notion of assumption.

Proposition 2.3 *The type structure can also be taken so that (i)–(iii) of Proposition 2.2 hold and*

(iv') The mappings λ^a, λ^b *are one-to-one.*

The rest of this paper is devoted to the proof of Theorem 1.1 and Propositions 2.2 and 2.3. Section 3 develops some refinements of the admissibility concept that we will need. The main proof is in Section 4. The Appendix contains some results we will use about Borel sets and mappings. To streamline our exposition, we will adopt the following convention:

Convention: Whenever we state a definition or result involving the players a *and* b*, it will be understood that we also make the analogous statement with* a *and* b *reversed.*

3 Admissibility

We suppose throughout that $G = \langle S^a, S^b, \pi^a, \pi^b \rangle$ is a finite game in strategic form. In order to prove Theorem 1.1, it will be helpful to refine the usual notion of iterated admissibility to apply to sets of strategies rather than only to single strategies. Recall that a strategy $s^a \in S^a$ is **admissible** if there is a probability measure ν with support S^b such that s^a is optimal under ν, and S_1^a is the set of all admissible strategies. We define $\mathbb{S}_1^a$ to be the set of all $X^a \subseteq S^a$ such that for some probability measure ν with support S^b, X^a is the set of all $s^a \in S^a$ which are optimal under ν. Thus X^a is the set of all strategies $s^a \in S^a$ which are admissible because of ν. In the literature, the notion of an admissible strategy is iterated to form the sets S_m^a of m-admissible strategies. In an analogous manner, we will iterate the collection $\mathbb{S}_1^a$ to form a collection $\mathbb{S}_m^a$ of subsets of S_m^a. Heuristically, $X^a \in \mathbb{S}_m^a$ means that there is a sequence of probability measures ν on S^b such that X^a is exactly the set of strategies s^a which are m-admissible because of ν.

We first introduce some notation. For $r^a, s^a \in S^a$ and a sequence $\nu = (\nu_0, \dots, \nu_m)$ of probability measures on S^b, we say that s^a is **preferred** to r^a under ν, in symbols $s^a \succ_\nu r^a$, if the payoff for s^a under ν is lexicographically greater than that for r^a. (Thus $s^a \succ_\nu r^a$ if and only if for some $i \le m$, $\pi^a(s^a, \nu_j) = \pi^a(r^a, \nu_j)$ for all $j < i$, but $\pi^a(s^a, \nu_i) > \pi^a(r^a, \nu_i)$.)

Given two finite sequences μ, ν of probability measures on S^b, we write $\mu \sim \nu$ if for all $r^a, s^a \in S^a$, $s^a \succ_\mu r^a$ if and only if $s^a \succ_\nu r^a$. It is easy to see that $\sim$ is an equivalence relation. Moreover, if $\mu \sim \nu$ and $\mu' \sim \nu'$, then $\mu\nu \sim \mu'\nu'$, where $\mu\nu$ is the concatenation of μ and ν.

For each finite sequence ν of probability measures on S^b, $\mathbb{O}(\nu)$ denotes the set of all $s^a \in S^a$ such that s^a is optimal under $\succ_\nu$. Note that if $\mu \sim \nu$, then $\mathbb{O}(\mu) = \mathbb{O}(\nu)$. Let P_m^b be the set of all sequences of probability measures $\nu = (\nu_0, \dots, \nu_m)$ on S^b such that for each $i \le m$, ν_i has support S_{m-i}^b. Finally, we define $\mathbb{S}_0^a$ be the set of all subsets of S^a, and for each $m \in \mathbb{N}$, we let

$$\mathbb{S}_{m+1}^a = \{X^a \in \mathbb{S}_0^a : X^a = \mathbb{O}(\nu) \text{ for some } \nu \in P_m^b\}.$$

Note that P_0^b is the set of all length one sequences of probability measures with support S^b, and $\mathbb{S}_1^a$ is the set we described at the beginning of this section.

The reason we need the sets of strategies $\mathbb{S}_m^a$ will become evident in the next section. The idea is that the family of sets $\mathbb{O}^{-1}(X^a)$, $X^a \in \mathbb{S}_m^a$, form a partition of P_m^b into finitely many pairwise disjoint classes, and elements of the same class behave in a similar way with respect to strategies.

We will need the following result, which is Proposition 1 in [1], .

Fact 3.1 *Let* $(\nu_0, \ldots, \nu_n)$ *be a sequence of probability measures on* S^b. *Then there is a probability measure* μ *on* S^b *such that*

(i) The support of μ *is the union of the supports of* $\nu_i, i \leq n$.

(ii) $(\mu) \sim (\nu_0, \ldots, \nu_n)$.

Lemma 3.2 *For every sequence* $(\nu_0, \ldots, \nu_m) \in P^b_m$, *there is a sequence* $(\mu_0, \ldots, \mu_m) \in P^b_m$ *such that whenever* $0 \leq i \leq j \leq m$, $(\mu_i, \ldots, \mu_j) \sim (\nu_0, \ldots, \nu_j)$.

Proof. By Fact 3.1, for each $i \leq m$ there is a probability measure μ_i on S^b with support S^B_{m-i} such that $(\mu_i) \sim (\nu_0, \ldots, \nu_i)$. We claim that for each $i < m$, $(\mu_i \mu_{i+1}) \sim (\mu_{i+1})$. To see this, first note that $(\mu_{i+1}) \sim (\nu_0, \ldots, \nu_i, \nu_{i+1}) \sim (\mu_i \nu_{i+1})$. Consider any $r^a, s^a \in S^a$. If $s^a \succ_{(\mu_i)} r^a$, then both $s^a \succ_{(\mu_i \nu_{i+1})} r^a$ and $s^a \succ_{(\mu_i \mu_{i+1})} r^a$. Similarly if $r^a \succ_{(\mu_i)} s^a$. If neither of these happens, then $s^a \succ_{(\mu_i \mu_{i+1})} r^a$ if and only if $s^a \succ_{(\mu_{i+1})} r^a$. This proves the claim. We therefore have $(\mu_i, \mu_{i+1}) \sim (\nu_0, \ldots, \nu_{i+1})$. The Lemma now follows by an easy induction on $j - i$. ■

Lemma 3.3 *For each* $m \in \mathbb{N}$, $\mathbb{S}^a_{m+1} \subseteq \mathbb{S}^a_m$.

Proof. The result is trivial when $m = 0$. Let $m > 0$ and suppose $X^a \in \mathbb{S}^a_{m+1}$. By Lemma 3.2, there is a sequence $\mu = (\mu_0, \ldots, \mu_m) \in P^b_m$ such that $\mathbb{O}(\mu) = X^a$ and for all $i \leq m$, $(\mu_i, \ldots, \mu_m) \sim \mu$. Then $(\mu_1, \ldots, \mu_m) \in P^b_{m-1}$ and $\mathbb{O}((\mu_1, \ldots, \mu_{m-1})) = X^a$. Therefore $X^a \in \mathbb{S}^a_m$. ■

By Lemma 3.3 and the fact that S^a, S^b are finite, there must be a least $M \in \mathbb{N}$ such that

$$\mathbb{S}^a_M = \bigcap_{m=1}^{\infty} \mathbb{S}^a_m, \qquad \mathbb{S}^b_M = \bigcap_{m=1}^{\infty} \mathbb{S}^b_m,$$

and

$$S^a_M = \bigcap_{m=1}^{\infty} S^a_m, \qquad S^b_M = \bigcap_{m=1}^{\infty} S^b_m.$$

We note that $S^a_M \times S^b_M$ is the IA set.

Lemma 3.4 *For each* $m \in \mathbb{N}$, $\bigcup\{X^a : X^a \in \mathbb{S}^a_m\} = S^a_m$. *Thus each* $s^a \in S^a_m$ *belongs to some* $X^a \in \mathbb{S}^a_m$, *and* $\mathbb{S}^a_m$ *is a set of subsets of* S^a_m.

Proof. We argue by induction on m. The result is trivial for $m = 0$. We assume the result for m and prove it for $m + 1$. First suppose $s^a \in \bigcup \mathbb{S}^a_{m+1}$. Then $s^a \in X^a$ for some $X^a \in \mathbb{S}^a_{m+1}$. We have $X^a = \mathbb{O}(\nu)$ for some $\nu = (\nu_0, \ldots, \nu_m) \in P^b_m$. Then s^a is optimal for ν_0, and ν_0 has support S^b_m. By Lemma 3.3, $X^a \in \mathbb{S}^a_m$, and by inductive hypothesis, $\bigcup \mathbb{S}^a_m = S^a_m$, so $s^a \in X^a \subseteq S^a_m$. Therefore $s^a \in S^A_{m+1}$.

Now suppose $s^a \in S^a_{m+1}$. By Lemma E.1 in [2], there is a sequence $\nu \in P^b_m$ such that s^a is optimal for ν. Then $s^a \in X^a = \mathbb{O}(\nu) \in \mathbb{S}^a_{m+1}$, and $s^a \in \bigcup \mathbb{S}^a_{m+1}$. ■

Since S^a_m is nonempty for each m, it follows from Lemma 3.4 that S^a_M and $\mathbb{S}^a_M$ are nonempty.

4 Complete Type Structures with RCAR

In this section we give the proof of our main result, Theorem 1.1, and the refinements stated in Propositions 2.2 and 2.3.

Let both T^a and T^b be the Cantor space $\{0,1\}^{\mathbb{N}}$, which is the set of all infinite sequences of 0's and 1's with the product topology. This space is compact. We will first construct sets $Q^a_m \subseteq S^a \times T^a, Q^b_m \subseteq S^b \times T^b$ with certain special properties involving the admissibility families $\mathbb{S}^a_m$. These sets will satisfy the condition

$$proj_{S^a}\left(\bigcap_{m=1}^{\infty} Q^a_m\right) \times proj_{S^b}\left(\bigcap_{m=1}^{\infty} Q^b_m\right) = S^a_M \times S^b_M.$$

We will then use these special properties to show that there exist Borel mappings λ^a, λ^b such that the corresponding rationality sets at level m are $R^a_m = Q^a_m$ and $R^b_m = Q^b_m$.

Let υ be the natural Borel probability measure on the Cantor space formed by taking the product of $\mathbb{N}$ copies of the uniform measure on $\{0,1\}$. υ is the measure such that for each $k \in \mathbb{N}$, the marginal of υ on $\{0,1\}^k$ is the measure such that each point has weight 2^{-k}. We start by partitioning the Cantor space T^a into $|\mathbb{S}^a_0| = 2^{|S^a|}$ clopen (both closed and open) sets $C^a_0(X^a), X^a \in \mathbb{S}^a_0$, each of measure $\upsilon(C^a_0(X^a)) = 1/|\mathbb{S}^a_0|$. To do this, take $C^a_0(X^a)$ to be the set of all types $t^a \in T^a$ such that for each $n < |S^a|$, the n-th element of S^a belongs to X^a if and only if $t^a(n) = 1$.

Let $X^a \in \mathbb{S}^a_0$. By Corollary A.8 in the Appendix, there is a decreasing chain $A_0 \supseteq A_1 \supseteq \cdots$ of open sets (which depend on X^a) such that $A_0 = C^a_0(X^a)$, $A_m \setminus A_{m+1}$ is uncountable for each m, and $\bigcap_m A_m$ is open and has υ-measure $1/|\mathbb{S}^a_0|$. Define $C^a_m(X^a) = A_m$ if $X^a \in \mathbb{S}^a_m$, and $C^a_m(X^a) = \varnothing$ if $X^a \notin \mathbb{S}^a_m$. Let $C^a_\infty(X^a) = \bigcap_{m=1}^{\infty} C^a_m(X^a)$.

For each $m > 0$ define

$$Q^a_m = \bigcup \left\{X^a \times C^a_m(X^a) : X^a \in \mathbb{S}^a_m\right\}.$$

We also let $Q^a_0 = S^a \times T^a$ and $Q^a_\infty = \bigcap_{m=1}^{\infty} Q^a_m$.

Our plan is to have $C^a_1(X^a)$ map to the set of all full-support LPS's σ such that X^a is the set of all optimal strategies for σ, and to have $C^a_{m+1}(X^a)$ map to the set of all full-support LPS's σ such that X^a is the set of optimal strategies for σ, and σ assumes $Q_1^b, \dots, Q_m^b$.

Lemma 4.1 *(i) For each m, $C_m^a(X^a) \supseteq C_{m+1}^a(X^a)$, and $Q_m^a \supseteq Q_{m+1}^a$.*

(ii) For each m the sets $C_m^a(X^a), X^a \in \mathbb{S}_m^a$, are open and pairwise disjoint, and the set Q_m^a is open. Moreover, the sets $C_\infty^a(X^a)$ and Q_∞^a are open.

(iii) For each $m > 0$ and $X^a \in \mathbb{S}_m^a$, $\upsilon(C_m^a(X^a)) = 1/|\mathbb{S}_0^a|$.

(iv) For each $X^a \in \mathbb{S}_M^a$, $\upsilon(C_\infty^a(X^a)) = 1/|\mathbb{S}_0^a|$.

Proof. (i) follows from Lemma 3.3. Parts (ii)–(iv) are straightforward. ■

Lemma 4.2 *For each $m > 0$ we have*

$$proj_{S^a} Q_m^a \times proj_{S^b} Q_m^b = S_m^a \times S_m^b.$$

Proof. Let $s^a \in proj_{S^a} Q_m^a$, so that $(s^a, t^a) \in Q_m^a$ for some t^a. Then $(s^a, t^a) \in X^a \times C_m^a(X^a)$ for some $X^a \in \mathbb{S}_m^a$. By Lemma 3.4, $X^a \subseteq S_m^a$, so $s^a \in S_m^a$.

For the other direction, let $s^a \in S_m^a$. By Lemma 3.4, $s^a \in X^a$ for some $X^a \in \mathbb{S}_m^a$. We also have $X^a \times C_m^a(X^a) \subseteq Q_m^a$. By Lemma 4.1 $C_m^a(X^a)$ has positive measure under υ and thus is nonempty. Hence there exists $t^a \in C_m^a(X^a)$. Therefore $(s^a, t^a) \in Q_m^a$. ■

Lemma 4.3

$$proj_{S^a} Q_\infty^a \times proj_{S^b} Q_\infty^b = S_M^a \times S_M^b.$$

Proof. We have $S_M^a = \bigcap_{m=1}^\infty S_m^a$ and $Q_\infty^a \subseteq Q_M^a$. By Lemma 4.2,

$$proj_{S^a} Q_\infty^a \subseteq proj_{S^a} Q_M^a = S_M^a.$$

For the other inclusion, let $s^a \in S_M^a$. By Lemma 3.4, s^a belongs to some $X^a \in \mathbb{S}_M^a$. By Lemma 4.1, $C_\infty^a(X^a)$ has positive measure and hence is nonempty, so $s^a \in proj_{S^a} Q_\infty^a$. ■

Let us verify the simple condition for assumption given in the introduction when an event is open.

Lemma 4.4 *Let $\sigma = (\mu_0, \dots, \mu_n) \in \mathcal{L}^+(S^b \times T^b)$ and let U be an open subset of $S^b \times T^b$. Then σ assumes U at level j if and only if*

$$\begin{array}{lrl} (i) & \mu_0(U) & = \cdots = \mu_j(U) = 1, \text{ and} \\ (ii) & \mu_{j+1}(U) & = \cdots = \mu_n(U) = 0. \end{array}$$

Proof. Proposition 5.1 in [2] shows that for an arbitrary Borel set U, σ assumes U at level j if and only if we have (i), (ii), and the following additional condition:

(iii) For every open set V which meets U, $\mu_i(U \cap V) > 0$ for some $i \leq n$.

Now suppose that U is open. Then $U \cap V$ is open. σ has full support, so by Lemma C.1 in [2], there is an $i \leq n$ with $\mu_i(U \cap V) > 0$. Therefore condition (iii) holds for U. ∎

We now define sets $D^a_m(X^a)$ of LPS's on $S^b \times T^b$ which will be matched to the sets $C^a_m(X^a)$.

For each full-support LPS $\sigma \in \mathcal{L}^+(S^b \times T^b)$, let $\mathbb{O}(\sigma)$ be the set of all $s^a \in S^a$ such that s^a is optimal for σ. If $\sigma = (\mu_0, \ldots, \mu_m)$, the marginal of σ is defined as the sequence of measures $marg_{S^b}\sigma = (marg_{S^b}\mu_0, \ldots, marg_{S^b}\mu_m)$ on S^b. Note that $\mathbb{O}(\sigma) = \mathbb{O}(marg_{S^b}\sigma)$, and that $\mathbb{O}(\sigma)$ is always a nonempty subset of S^a. For each $X^a \subseteq S^a$, let $D^a_1(X^a)$ be the set of all full-support LPS's σ on $S^b \times T^b$ such that $\mathbb{O}(\sigma) = X^a$. Continuing inductively, for each $m > 0$ and $X^a \subseteq S^a$ we define

$$D^a_{m+1}(X^a) = \{\sigma \in D^a_m(X^a) : \sigma \text{ assumes } Q^b_m\}.$$

Note that for each $X^a \subseteq S^a$, $D^a_1(X^a) \supseteq D^a_2(X^a) \supseteq \cdots$. The sets $D^a_1(X^a), X^a \subseteq S^a$ are obviously pairwise disjoint. Since $D^a_m(X^a) \subseteq D^a_1(X^a)$, for each m the sets $D^a_m(X^a), X^a \subseteq S^a$, are also pairwise disjoint.

Lemma 4.5 *$D^a_1(X^a)$ is nonempty if and only if $X^a \in \mathbb{S}^a_1$. Also,*

$$\mathcal{L}^+(S^b \times T^b) = \bigcup \{D^a_1(X^a) : X^a \in \mathbb{S}^a_1\}.$$

Proof. Suppose first that $X^a \in \mathbb{S}^a_1$. Then $X^a = \mathbb{O}(\nu)$ for some $\nu \in P^b_1$. We have $\nu = (\nu_0)$ where ν_0 is a probability measure on S^b with full support. Let $\sigma = (\mu_0)$ where μ_0 is the product measure $\mu_0 = \nu_0 \otimes \upsilon$. Then σ is a full-support LPS on $S^b \times T^b$, $marg_{S^b}\sigma = \nu$, and $X^a = \mathbb{O}(\sigma)$. Therefore $\sigma \in D^a_1(X^a)$, so $D^a_1(X^a)$ is nonempty.

Suppose now that $D^a_1(X^a)$ is nonempty, and let $\sigma \in D^a_1(X^a)$ and $\nu = marg_{S^b}\sigma$. Then $X^a = \mathbb{O}(\sigma) = \mathbb{O}(\nu)$, and ν has full support on S^b. By Fact 3.1, there is a probability measure τ_0 with support S^b such that $\tau \sim \nu$, where $\tau = (\tau_0)$. Then $\tau \in P^b_1$ and $X^a = \mathbb{O}(\tau)$, so $X^a \in \mathbb{S}^a_1$.

By definition, $D^a_1(X^a) \subseteq \mathcal{L}^+(S^b \times T^b)$ for each X^a. For the other direction, if $\sigma \in \mathcal{L}^+(S^b \times T^b)$ and $X^a = \mathbb{O}(\sigma)$, then $\sigma \in D^a_1(X^a)$, so $D^a_1(X^a)$ is nonempty and $X^a \in \mathbb{S}^a_1$. ∎

In what follows, will use λ^a to denote an arbitrary Borel mapping from T^a onto $\mathcal{L}(S^b \times T^b)$. Since the m-th order rationality sets depend on the mappings λ^a, λ^b, we will write $R^a_m(\lambda^a, \lambda^b)$ and $R^b_m(\lambda^a, \lambda^b)$ instead of R^a_m, R^b_m. We now give conditions under which $R^a_m(\lambda^a, \lambda^b) = Q^a_m$.

Lemma 4.6 *Suppose that for each $n > 0$ and $X^a \in \mathbb{S}_1^a$, we have $(\lambda^a)^{-1}(D_n^a(X^a)) = C_n^a(X^a)$, and similarly for λ^b. Then for each $m > 0$ we have*

$$R_m^a(\lambda^a, \lambda^b) = Q_m^a.$$

Proof. We first prove the result for $m = 1$. Suppose $(s^a, t^a) \in Q_1^a$. Then for some $X^a \in \mathbb{S}_1^a$, $(s^a, t^a) \in X^a \times C_1^a(X^a)$, so $\lambda^a(t^a) \in D_1^a(X^a)$. Hence $s^a \in X^a$, $\lambda^a(t^a)$ is a full-support LPS, and $\mathbb{O}(\lambda^a(t^a)) = X^a$. This means that s^a is optimal for $\lambda^a(t^a)$, and thus $(s^a, t^a) \in R_1^a(\lambda^a, \lambda^b)$. Now suppose $(s^a, t^a) \in R_1^a(\lambda^a, \lambda^b)$. Then $\lambda^a(t^a)$ is a full-support LPS and s^a is optimal for $\lambda^a(t^a)$. Let $X^a = \mathbb{O}(\lambda^a(t^a))$. Then $s^a \in X^a$ and $\lambda^a(t^a) \in D_1^a(X^a)$. By Lemma 4.5, $X^a \in \mathbb{S}_1^a$, so $(s^a, t^a) \in X^a \times C_1^a(X^a) \subseteq Q_1^a$.

We suppose the result holds for m and prove it for $m+1$. By inductive hypothesis, $R_m^a(\lambda^a, \lambda^b) = Q_m^a$ and similarly for b. Consider any $(s^a, t^a) \in S^a \times T^a$, and let $\sigma = \lambda^a(t^a)$.

Suppose first that $(s^a, t^a) \in R_{m+1}^a(\lambda^a, \lambda^b)$. Then $(s^a, t^a) \in R_m^a(\lambda^a, \lambda^b) = Q_m^a$, so there is an $X^a \in \mathbb{S}_m^a$ such that $s^a \in X^a$ and $t^a \in C_m^a(X^a)$. Since $(\lambda^a)^{-1}(D_m^a(X^a)) = C_m^a(X^a)$, $\sigma \in D_m^a(X^a)$. Moreover, σ assumes Q_m^b. Therefore $\sigma \in D_{m+1}^a(X^a)$, and hence $t^a \in C_{m+1}^a(X^a)$ and $(s^a, t^a) \in Q_{m+1}^a$.

Now suppose $(s^a, t^a) \in Q_{m+1}^a$. Then $(s^a, t^a) \in Q_m^a = R_m^a(\lambda^a, \lambda^b)$. Moreover, for some $X^a \in \mathbb{S}_{m+1}^a$, $s^a \in X^a$ and $t^a \in C_{m+1}^a(X^a)$. Since $(\lambda^a)^{-1}(D_{m+1}^a(X^a)) = C_{m+1}^a(X^a)$, $\sigma \in D_{m+1}^a(X^a)$. Therefore $\sigma \in D_m^a(X^a)$ and σ assumes Q_m^b. Thus $(s^a, t^a) \in R_{m+1}^a(\lambda^a, \lambda^b)$. ■

Let

$$C^a = T^a \setminus \bigcup \{C_1^a(X^a) : X^a \in \mathbb{S}_1^a\},$$

and let $\mathcal{C}^a$ be the family of sets

$$\mathcal{C}^a = \{C^a\} \cup \{C_m^a(X^a) \setminus C_{m+1}^a(X^a) : X^a \in \mathbb{S}_1^a, m > 0\} \cup \{C_\infty^a(X^a) : X^a \in \mathbb{S}_1^a\}.$$

Let

$$D^a = \mathcal{L}(S^b \times T^b) \setminus \bigcup \{D_1^a(X^a) : X^a \in \mathbb{S}_1^a\}$$

and let $\mathcal{D}^a$ be the family

$$\mathcal{D}^a = \{D^a\} \cup \{D_m^a(X^a) \setminus D_{m+1}^a(X^a) : X^a \in \mathbb{S}_1^a, m > 0\} \cup \{D_\infty^a(X^a) : X^a \in \mathbb{S}_1^a\}.$$

To complete the proof of Theorem 1.1, it will suffice to show that:

(2) Each set in $\mathcal{C}^a$ is Borel and each set in $\mathcal{D}^a$ is Borel (and similarly for b).

(3) For each matching pair of sets in $\mathcal{C}^a$ and $\mathcal{D}^a$, either both sets are empty or both sets are nonempty (and similarly for b).

Once (2) and (3) are established, the proof of Theorem 1.1 and Proposition 2.2 are completed as follows. (We deal with Proposition 2.3 later). The set C^a has positive υ-measure, and hence is uncountable. By Lemma 4.1, for each $X^a \in \mathbb{S}^a_M$, the set $C^a_\infty(X^a)$ has positive υ-measure and is uncountable. By definition, each of the remaining sets in $\mathcal{C}^a$ is either empty or uncountable. We note that $\mathcal{C}^a$ is a countable family of pairwise disjoint sets whose union is T^a. Similarly, $\mathcal{D}^a$ is a countable family of pairwise disjoint sets whose union is $\mathcal{L}(S^b \times T^b)$. Then by Corollary A.5, there is a continuum-to-one Borel mapping λ^a from T^a onto $\mathcal{L}(S^b \times T^b)$ which sends each set in $\mathcal{C}^a$ onto its counterpart in $\mathcal{D}^a$. It follows that $(\lambda^a)^{-1}(D^a_m(X^a)) = C^a_m(X^a)$ for each $X^a \in \mathbb{S}^a_1$ and $m > 0$. Then by Lemma 4.6 we have $R^a_m(\lambda^a, \lambda^b) = Q^a_m$ for each $m > 0$. Since T^b has more than one element, there are $\sigma \in \mathcal{L}(S^b \times T^b)$ which do not have full support, so $\mathcal{L}(S^b \times T^b)$ properly contains $\mathcal{L}^+(S^b \times T^b)$. Theorem 1.1 and Proposition 2.2 now follow by Lemma 4.3.

We now prove (2). By definition, $C^a_m(X^a)$ is open for each $X^a \subseteq S^a$ and $m > 0$. It follows that each member of $\mathcal{C}^a$ is Borel. The following lemma shows that each member of $\mathcal{D}^a$ is Borel, establishing (2).

Lemma 4.7 *For each $X^a \subseteq S^a$, $D^a_m(X^a)$ is Borel.*

Proof. We argue by induction on m. It follows from [2] (Lemma C.2 and the proof of Lemma C.4) that each set $D^a_1(X^a)$ is Borel. Now suppose that $D^a_m(X^a)$ is Borel and let $X^a \subseteq S^a$. Since Q^b_m is open, Lemma C.3 in [2] shows that $D^a_{m+1}(X^a)$ is Borel, and completes the induction. ■

We now prove a series of lemmas that will establish (3).

For a measure μ on $S^b \times T^b$, let us write $\mu \ll \upsilon$ if $\mu(S^b \times U) = 0$ whenever $\upsilon(U) = 0$. We now prove the key lemma.

Lemma 4.8 *Suppose $\nu \in P^b_m$. Then there is a full-support LPS $\sigma = (\mu_0, \ldots, \mu_m)$ on $S^b \times T^b$ such that:*

(i) $marg_{S^b}\sigma = \nu$,

(ii) For each $j \leq m$, σ assumes Q^b_{m-j} at level j,

(iii) $\mu_0 \ll \upsilon$.

Proof. For each $U \subseteq S^b \times T^b$ and $s^b \in S^b$, let $U[s^b]$ denote the section

$$U[s^b] = \{t^b \in T^b : (s^b, t^b) \in U\}.$$

Given sets $X, Y \subseteq T^b$ with $\upsilon(Y) > 0$, $\upsilon(X|Y)$ denotes the conditional probability $\upsilon(X \cap Y)/\upsilon(Y)$.

We argue by induction on m. We first prove the result for $m = 0$. Let $\nu = (\nu_0) \in P^b_0$. The product measure $\mu_0 = \nu \otimes \upsilon$ is a full-support probability measure on $S^b \times T^b$ such that $marg_{S^b}\mu_0 = \nu_0$ and $\mu_0 \ll \upsilon$. It is clear that $\sigma = (\mu_0)$ is a full-support LPS which assumes $Q_0^a = S^b \times T^b$, so σ satisfies (i)–(iii) for $m = 0$.

We now suppose $m > 0$ and the result holds for $m-1$, and prove it for m. Let $\nu = (\nu_0, \dots, \nu_m) \in P_m^b$. Then $\nu' = (\nu_1, \dots, \nu_m) \in P_{m-1}^b$. By inductive hypothesis there is a full-support LPS $\sigma' = (\mu_0', \dots, \mu_{m-1}')$ such that (i)–(iii) hold for $m-1$. That is, $marg_{S^b}\sigma' = \nu'$, σ' assumes Q_{m-1-j}^b at level j for each $j < m$, and $\mu_0' \ll \upsilon$. We will construct a new full-support LPS $\sigma = (\mu_0, \dots, \mu_m)$ which satisfies (i)–(iii) for m.

Since $\nu \in P_m^b$, ν_0 has support S_m^b. By Lemma 3.4, for each $s^b \in S_m^b$ we have $s^b \in X^b$ for some $X^b \in \mathbb{S}_m^b$, so by Lemma 4.1 we have $\upsilon(Q_m^b[s^b]) > 0$. So we may define a measure μ_0 on $S^b \times T^b$ by

$$\mu_0(U) = \sum_{s^b \in S_m^b} \nu(s^b) \cdot \upsilon(U[s^b] | Q_m^b[s^b]).$$

It is clear that $marg_{S^b}\mu_0 = \nu_0$. If $m > 1$, we also define $\mu_2 = \mu_1', \dots, \mu_m = \mu_{m-1}'$.

Finally, we define the measure μ_1 as follows. We note that $\mu_0'(Q_{m-1}^b) = 1$ and $Q_{m-1}^b = \bigcup\{X^b \times C_{m-1}^b(X^b) : X^b \in \mathbb{S}_{m-1}^b\}$. For each $X^b \in \mathbb{S}_m^b$, $C_{m-1}^b(X^b) \setminus C_m^b(X^b)$ is nonempty, so there is a measure μ_1 on T^b with the following properties:

(4) For $X^b \in \mathbb{S}_m^b$, there is a point $y \in C_{m-1}^b(X^b) \setminus C_m^b(X^b)$ such that $\mu_1(\{y\}) = \mu_0'(C_{m-1}^b(X^b))$, and $\mu_1(C_{m-1}^b(X^b) \setminus \{y\}) = 0$.

(5) For $X^b \in \mathbb{S}_{m-1}^b \setminus \mathbb{S}_m^b$, μ_1 agrees with μ_0' on $X^b \times C_{m-1}^b(X^b)$ that is, $\mu_1(U) = \mu_0'(U)$ for each $U \subseteq X^b \times C_{m-1}^b(X^b)$.

It follows that $marg_{S^b}\mu_1 = marg_{S^b}\mu_0' = \nu_1$. Therefore (i) holds for m. It is easily seen that $\mu_0' \ll \upsilon$, so (iii) holds for m. It is clear that whenever $i \le m$, $\mu_j(Q_i^b) = 1$ for $j \le m-i$ and $\mu_j(Q_i^b) = 0$ for $m-i < j \le m$. Therefore the measures $\mu_0, \dots, \mu_m$ are mutually singular, and σ is an LPS. Condition (ii) for m now follows from the fact that each set Q_n^b is open.

It remains to show that σ has full-support. By inductive hypothesis, σ' is a full-support LPS. We claim that for each nonempty open set $U \subseteq S^b \times T^b$, $\mu_j(U) > 0$ for some $j \le m$. Suppose first that $U \cap Q_m^b \neq \varnothing$. Then for some $X^b \in \mathbb{S}_m^b$, $V = U \cap (X^b \times C_m^b(X^b)) \neq \varnothing$. Then $V[s^b] \neq \varnothing$ and $V[s^b] \subseteq C_{m+1}^b(X^b)$ for some $s^b \in X^b$. Since $C_m^b(X^b)$ is open and U is open, $V[s^b]$ is open. Therefore $\upsilon(V[s^b]) > 0$, and hence $\mu_0(U) \ge \mu_0(V) > 0$. Now suppose that $U \cap Q_m^b = \varnothing$. Then $U \subseteq Q_0^b \setminus Q_m^b$. Since σ' has full support, we have $\mu_i'(U) > 0$ for some $i \le m$. But μ_1 agrees with μ_0' on the complement of Q_m^b, and for $i > 0$, $\mu_{i+1} = \mu_i'$. Therefore $\mu_{i+1}(U) > 0$. This proves the claim. By Lemma C.1 in [2], it follows that σ has full support. ■

Lemma 4.9 *For each $m > 0$ and $X^a \subseteq S^a$, $X^a \in \mathbb{S}_m^a$ if and only if $D_m^a(X^a) \neq \varnothing$.*

Proof. The case $m = 1$ is proved in Lemma 4.5. We now suppose the result holds for m and prove it for $m+1$. Suppose $X^a \in \mathbb{S}^a_{m+1}$. Then $X^a = \mathbb{O}(\nu)$ for some $\nu \in P^b_m$, and hence there is a full-support LPS σ which satisfies conditions (i)–(iii) of Lemma 4.8. Conditions (i) and (ii) show that $\sigma \in D^a_{m+1}(X^a)$.

Now suppose $D^a_{m+1}(X^a) \neq \varnothing$. By Lemma 3.3, $D^a_m(X^a) \neq \varnothing$, and by inductive hypothesis we have $X^a \in \mathbb{S}^a_m$. This means that $X^a = \mathbb{O}(\mu)$ for some $\mu \in P^b_{m-1}$. We will find a probability measure ν_0 on S^b such that the sequence $\nu = (\nu_0, \mu)$ formed by adding ν_0 to the beginning of μ satisfies $\nu \in P^b_m$ and $\mathbb{O}(\nu) = X^a$. This will establish that $X^a \in \mathbb{S}^a_{m+1}$.

Since $D^a_{m+1}(X^a) \neq \varnothing$, there exists $\tau = (\tau_0, \dots, \tau_k) \in D^a_{m+1}(X^a)$. τ assumes Q^b_m at some level $j \leq k$, so $\tau_i(Q^b_m) = 1$ for each $i \leq j$, and $\tau_i(Q^b_m) = 0$ for each $i > j$. By Lemma 4.2, $proj_{S^b} Q^b_m = S^b_m$, and hence $Q^b_m \subseteq S^b_m \times T^b$, and $\tau_i(S^b \times T^b) = 1$ for each $i \leq j$. Let $\theta = (\theta_0, \dots, \theta_k) = marg_{S^b}\tau$. Then $\theta_i(S^b_m) = 1$ for each $i \leq j$. Furthermore, for each $s^b \in S^b_m$, the set $U = (\{s^b\} \times T^b) \cap Q^b_m$ is open and nonempty, and because τ has full support, there is an $n \leq k$ with $\tau_n(U) > 0$. But $\tau_i(U) \leq \tau_i(Q^b_m) = 0$ for each $i > j$, so we must have $n \leq j$. Then $\tau_n(\{s^b\} \times T^b) = \theta_n(\{s^b\}) > 0$. Therefore S^b_m is equal to the union of the supports of θ_i for $i \leq j$. Since $\tau \in D^a_{m+1}(X^a) \subseteq D^a_1(X^a)$, $X^a = \mathbb{O}(\tau)$. Therefore every $s^a \in X^a$ is optimal for τ, and hence is lexicographically optimal for $(\theta_0, \dots, \theta_j)$. By Fact 3.1, there is a probability measure ν_0 on S^b with support S^b_m such that $(\nu_0) \sim (\theta_0, \dots, \theta_j)$, and hence each $s^a \in X^a$ is optimal for ν_0. Let $\nu = (\nu_0, \mu)$. Since $\mu \in P^b_{m-1}$ and ν_0 has support S^b_m, ν belongs to P^b_m. Moreover, $X^a = \mathbb{O}(\mu)$ and $X^a \subseteq \mathbb{O}(\nu_0)$, so $X^a = \mathbb{O}(\nu)$, as required. ■

Lemma 4.10 *For each $m > 0$ and $X^a \in \mathbb{S}^a_1$, the sets $U = C^a_m(X^a) \setminus C^a_{m+1}(X^a)$ and $V = D^a_m(X^a) \setminus D^a_{m+1}(X^a)$ are either both empty or both nonempty.*

Proof. If $X^a \in \mathbb{S}^a_1 \setminus \mathbb{S}^a_m$, then $C^a_m(X^a)$ is empty, and by Lemma 4.9 the set $D^a_m(X^a)$ is empty. Therefore U and V are both empty.

Suppose $X^a \in \mathbb{S}^a_m \setminus \mathbb{S}^a_{m+1}$. Then $U = C^a_m(X^a)$, and by Lemma 4.1, $\upsilon(U) > 0$ and hence U is nonempty. By Lemma 4.9, $D^a_m(X^a)$ is nonempty and $D^a_{m+1}(X^a)$ is empty, so V is nonempty.

Suppose $X^a \in \mathbb{S}^a_{m+1}$. By definition, the set U is uncountable and hence nonempty in this case. By Lemma 3.2, there is a sequence $\nu = (\nu_0, \dots, \nu_m) \in P^b_m$ such that $\mathbb{O}(\nu) = X^a$, $(\nu_1, \dots, \nu_m) \in P^b_{m-1}$, and $\nu' = (\nu_1, \dots, \nu_m) \sim \nu$. Then $\mathbb{O}(\nu') = X^a$. By Lemma 4.8, there is a full-support LPS $\sigma' = (\mu'_0, \dots, \mu'_{m-1})$ on $S^b \times T^b$ such that $marg_{S^b}\sigma' = \nu'$ and σ assumes Q^b_{m-1-j} at level j for each $j < m-1$. Since $\nu \in P^b_m$, ν_0 has support S^b_m. By Lemma 3.4, for each $s^b \in S^b_m$ there is an $X^b \in \mathbb{S}^b_m$ such that $s^b \in X^b$. If $m > 1$, take X^b such that $s^b \in X^b \in \mathbb{S}^b_m$. If $m = 1$, take $X^b \in \mathbb{S}^b_0 \setminus \mathbb{S}^b_1$. In both cases, the set $C^b_{m-1}(X^b)$ is nonempty, and $C^b_{m-1}(X^b) \setminus C^b_m(X^b)$ is uncountable.

We may therefore choose a point $x_{s^b} = (s^b, t^b)$ such that $t^b \in C^b_{m-1}(X^b) \setminus C^b_m(X^b)$, and $\mu'_i(\{x_{s^b}\}) = 0$ for each $i < m$. Then $x_{s^b} \in Q^b_{m-1} \setminus Q^b_m$. In addition, we may choose these points so that $x_{s^b} \neq x_{r^b}$ whenever $s^b \neq r^b$. Let μ_0 be the probability measure on $S^b \times T^b$ such that $\mu_0(\{x_{s^b}\}) = \nu_0(\{s^b\})$ for each $s^b \in s^b_m$. Let σ be the LPS formed by putting μ_0 in front of σ', that is, $\sigma = (\mu_0, \ldots, \mu_m)$ where $\mu_{i+1} = \mu'_i$ for each $i < m$. It is clear that $marg_{S^b}\sigma = \nu$, that the measures in σ are mutually singular, and that σ has full support, so $\sigma \in \mathcal{L}^+(S^b \times T^b)$. Therefore $X^a = \mathbb{O}(\sigma)$. It is also clear that $\mu_0(Q^b_i) = 1$ whenever $i < m$, so σ assumes Q^b_i whenever $i < m$. Thus $\sigma \in D^b_m(X^a)$. However, $x_{s^b} \notin Q^b_m$ for each $s^b \in S^b_m$, so $\mu_0(Q^b_m) = 0$. Therefore σ does not assume Q^b_m, and hence $\sigma \in V$ and thus V is nonempty. ■

Lemma 4.11 *The sets $C^a_\infty(X^a)$ and $D^a_\infty(X^a)$ are both nonempty if $X^a \in \mathbb{S}^a_M$, and both empty if $X^a \notin \mathbb{S}^a_M$.*

Proof. If $X^a \notin \mathbb{S}^a_M$, then $C^a_M(X^a)$ is empty by definition, and $D^a_M(X^a)$ is empty by Lemma 4.9. Therefore $C^a_\infty(X^a)$ and $D^a_\infty(X^a)$ are both empty.

Let $X^a \in \mathbb{S}^a_M$. We have $X^a \in \mathbb{S}^a_m$ for all m, so by Lemma 4.1, $\upsilon(C^a_\infty(X^a)) > 0$, and C^a_∞ is nonempty. We have $X^a \in \mathbb{S}^a_{M+1} = \mathbb{S}^a_M$, so $X^a = \mathbb{O}(\nu)$ for some $\nu \in P^b_M$. By Lemma 4.8, there is a full-support LPS σ which satisfies conditions (i)–(iii) of Lemma 4.8 for M. Consider any $m \geq M$. We show that σ assumes Q^b_m at level 0. For each $X^b \subseteq S^b$, $\upsilon(C^b_m(X^b)) = \upsilon(C^b_M(X^b))$. Since $\mu_0 \ll \upsilon$, we have $\mu_0(X^b \times C^b_m(X^b)) = \mu_0(X^b \times C^b_M(X^b))$, and therefore $\mu_0(Q^b_m) = \mu_0(Q^b_M) = 1$. We also have $Q^b_m \subseteq Q^b_M$, so $\mu_i(Q^b_m) = \mu_i(Q^b_M) = 0$ when $0 < i \leq M$. Then σ assumes Q^b_m at level 0. This proves that $\sigma \in D^a_\infty(X^a)$, so $D^a_\infty(X^a) \neq \varnothing$. ■

Condition (3) follows from Lemmas 4.10 and 4.11. Theorem 1.1 and Proposition 2.2 now follow by the discussion immediately after Lemma 4.6.

It remains to prove Proposition 2.3. The next lemma will be useful in showing that a set is uncountable. Given $\sigma = (\sigma_0, \ldots, \sigma_n), \tau = (\tau_0, \ldots, \tau_n)$ in $\mathcal{L}(S^b \times T^b)$, we write $\sigma \approx \tau$ if for each $i \leq n$, σ_i and τ_i have the same marginals on S^b and the same null sets.

Lemma 4.12 *Let $\sigma \in \mathcal{L}^+(S^b \times T^b)$.*

(i) If $\sigma \approx \tau$, then $\tau \in \mathcal{L}^+(S^b \times T^b)$.

(ii) There are uncountably many τ with $\sigma \approx \tau$.

(iii) If U is open in $S^b \times T^b$, U is assumed under σ, and $\tau \approx \sigma$, then U is assumed under τ.

Proof. The proof is an easy modification of the proof of Lemma E.2 in [2]. ■

Lemma 4.13 *For each $X^a \subseteq S^a$, $D^a_m(X^a)$ is closed under $\approx$.*

Proof. We argue by induction on m. Let $\sigma \in D_1^a(X^a)$. By Lemma 4.12 (i), every $\tau \approx \sigma$ belongs to $\mathcal{L}^+(S^b \times T^b)$. Since $\mathbb{O}(\sigma)$ depends only $marg_{S^b}\sigma$, $D_1^a(X^a)$ is closed under $\approx$. This proves the result for $m = 1$.

Now suppose the result holds for m and let $X^a \subseteq S^a$. The fact that $D_{m+1}^a(X^a)$ is closed under $\approx$ follows from Lemma 4.12 (iii) and the hypothesis that $D_m^a(X^a)$ is closed under $\approx$. ■

Lemma 4.14 *For each matching pair of sets in $\mathcal{C}^a$ and $\mathcal{D}^a$, either both sets are empty or both sets have cardinality $2^{\aleph_0}$.*

Proof. Each set in $\mathcal{C}^a$ is Borel, and either empty or uncountable. By Fact A.1, each nonempty set in $\mathcal{C}^a$ has cardinality $2^{\aleph_0}$. By Lemmas 4.7 and 4.13, each set in $\mathcal{D}^a$ is Borel and closed under $\approx$. By Lemma 4.12, each nonempty set in $\mathcal{D}^a$ is uncountable, and thus by Fact A.1, each nonempty set in $\mathcal{D}^a$ has cardinality $2^{\aleph_0}$. The result now follows from property (3). ■

By Property (2), Lemma 4.14, and Corollary A.3, there is a bijection (one-to-one mapping) λ^a from T^a onto $\mathcal{L}(S^b \times T^b)$ which sends each set in $\mathcal{C}^a$ onto its counterpart in $\mathcal{D}^a$. Now, using the argument immediately after Lemma 4.6, we see that Proposition 2.3 holds.

A Appendix: Borel Sets and Mappings

Fact A.1 *Every uncountable Borel subset of a Polish space has cardinality $2^{\aleph_0}$*

For a proof of Fact A.1, see Theorem 13.6 in [5], .

Fact A.2 *Suppose X and Y are Polish spaces, A is a Borel subset of X, B is a Borel subset of Y, and A and B have the same cardinality. Then there is a Borel bijection from A onto B.*

For a proof of Fact A.2, see Theorem 13.1.1 in [3].

Corollary A.3 *Suppose X and Y are Polish spaces, $A_n, n \in \mathbb{N}$ is a family of pairwise disjoint Borel subsets of X, $B_n, n \in \mathbb{N}$ is a family of pairwise disjoint Borel subsets of Y, and for each n, A_n has the same cardinality as B_n. Then there is a Borel bijection from $\bigcup_n A_n$ onto $\bigcup_n B_n$.*

Proof. By Fact A.2, for each n there is a Borel bijection f_n from A_n onto B_n. Since the sets A_n and B_n are pairwise disjoint, the union $\bigcup_n f_n$ is a bijection from $\bigcup_n A_n$ onto $\bigcup_n B_n$. Countable unions of Borel sets are Borel, so $\bigcup_n f_n$ is Borel. ■

Corollary A.4 *Suppose X and Y are Polish spaces, A is an uncountable Borel subset of X, and B is a nonempty Borel subset of Y. Then there is a Borel mapping f from A onto B such that $f^{-1}(\{b\})$ has cardinality $2^{\aleph_0}$ for each $b \in B$.*

Proof. By Fact A.1, A has cardinality $2^{\aleph_0}$. The set $B \times [0,1]$ is a Borel subset of the Polish space $Y \times [0,1]$ of cardinality $2^{\aleph_0}$. By Fact A.2, there is a Borel bijection g from A onto $B \times [0,1]$. Let h be the projection mapping from $B \times [0,1]$ onto B. Then the mapping $f = g \circ h$ from A to B has the required properties. ■

Corollary A.5 *Suppose X and Y are Polish spaces, $A_n, n \in \mathbb{N}$ is a family of pairwise disjoint Borel subsets of X, $B_n, n \in \mathbb{N}$ is a family of pairwise disjoint Borel subsets of Y, and for each n, A_n is uncountable and B_n is nonempty. Then there is a Borel mapping f from $\bigcup_n A_n$ onto $\bigcup_n B_n$ such that $f^{-1}(\{b\})$ has cardinality $2^{\aleph_0}$ for each $b \in \bigcup_n B_n$.*

Proof. Similar to the proof of Corollary A.3, but using Corollary A.4 instead of Fact A.2. ■

Lemma A.6 *Suppose X is a Polish space and υ is an atomless Borel probability measure on X. Then every Borel set $A \subseteq X$ such that $\upsilon(A) > 0$ has an uncountable compact subset $Z \subseteq A$ such that $\upsilon(Z) = 0$.*

Proof. We use the Cantor binary tree construction. Any Borel set U with $\upsilon(U) > 0$ has a compact subset V with $\upsilon(V) > 0$ (see [5], Theorem 17.11 on page 107). Therefore there are disjoint compact sets $C_0, C_1 \subseteq A$ such that $0 < \upsilon(C_0)$ and $0 < \upsilon(C_1)$. Let $\{0,1\}^n$ be the set of all n-tuples of 0's and 1's. Proceeding inductively, for each n-tuple $s \in \{0,1\}^n$ there are disjoint compact sets $C_{s,0}, C_{s,1} \subseteq C_s$ such that $0 < \upsilon(C_{s,0}) \leq 4^{-n}$ and $0 < \upsilon(C_{s,1}) \leq 4^{-n}$. Let $Z_n = \bigcap\{C_s : s \in \{0,1\}^n\}$ and let $Z = \bigcap_n Z_n$. Z_n is a decreasing chain of compact subsets of A, and $\upsilon(Z_n) \leq 2^n \cdot 4^{1-n} = 2^{1-n}$. Therefore Z is a compact subset of A and $\upsilon(Z) = 0$. For each infinite sequence $s = \{s_0, s_1, s_2, \ldots\}$ of 0's and 1's, the set

$$Z_s = C_{s_1} \cap C_{s_0,s_1} \cap C_{s_0,s_1,s_2} \cap \cdots$$

is a nonempty subset of Z, and Z_s, Z_t are disjoint whenever $s \neq t$. Therefore Z is uncountable. ■

Lemma A.7 *Let Z be a Polish space with no isolated points. Then there is an increasing sequence of closed sets $Z_0 \subseteq Z_1 \subseteq \cdots$ such that $\bigcup_n Z_n = Z$, Z_0 is uncountable, and $Z_{n+1} \setminus Z_n$ is uncountable for each n.*

Proof. Let d be a metric on X whose topology is the topology of X, and let $x \in X$. For each $n > 0$, let

$$B_n = \{y \in X : d(x, y) \geq 1/n\}.$$

Then $B_1 \subseteq B_2 \subseteq \cdots$ and $\bigcup_n B_n = X \setminus \{x\}$. Since X has no isolated points, each neighborhood of x is uncountable (see Theorem 6.4 in [5]), so $X \setminus B_n$ is uncountable for each $n > 0$. It follows that there is an sequence $k(0) < k(1) < \cdots$ such that $B_{k(0)}$ is uncountable, and $B_{k(n+1)} \setminus B_{k(n)}$ is uncountable for each n. Then the sequence of closed sets $Z_n = \{x\} \cup B_{k(n)}$ has the required properties. ■

Corollary A.8 *Suppose X is a Polish space, υ is an atomless Borel probability measure on X, and A is an open subset of X such that $\upsilon(A) > 0$. Then there is a decreasing chain $A = A_0 \supseteq A_1 \supseteq \cdots$ of open sets such that $A_n \setminus A_{n+1}$ is uncountable for each n, $A_\infty = \bigcap_n A_n$ is open, and $\upsilon(A_\infty) = \upsilon(A)$.*

Proof. By Lemma A.6 there is an uncountable compact set $Z \subseteq A$ such that $\upsilon(Z) = 0$. Since Z is closed in X, it is a Polish space. Let $Z_0 \subseteq Z_1 \subseteq \cdots$ be as in Lemma A.7. Then $A_n = A \setminus Z_n$ has the required properties, with $A_\infty = A \setminus Z$. ■

References

[1] L.A. Blume, A. Brandenburger, and E. Dekel. Lexicographic Probabilities and Equilibrium Refinements. Econometrica 59 (1991), 81-98.

[2] A. Brandenburger, A. Friedenberg, and H. J. Keisler. Admissibility in Games. Econometrica 26 (2008), pp, 307-352.

[3] R. M. Dudley. Real Analysis and Probability. Cambridge University Press, 2002.

[4] A. Friedenberg. When do Type Structures Contain All Hierarchies of Beliefs? Preprint 2008.

[5] A. Kechris. Classical Descriptive Set Theory. Springer-Verlag 1995.