ON THE DISTRIBUTION AND ASYMPTOTIC RESULTS FOR EXPONENTIAL FUNCTIONALS OF LÉVY PROCESSES

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ABSTRACT. The aim of this note is to study the distribution and the asymptotic behavior of the exponential functional $A_t := \int_0^t e^{\xi_s}\, ds$, where $(\xi_s, s \geq 0)$ denotes a Lévy process. When $A_\infty < \infty$, we show that in most cases, the law of A_∞ is a solution of an integro-differential equation ; moreover, this law is characterized by its integral moments. When the process ξ is asymptotically α-stable, we prove that $t^{-1/\alpha} \log A_t$ converges in law, as $t \to \infty$, to the supremum of an α-stable Lévy process ; in particular, if $\mathbb{E}[\xi_1] > 0$, then $\alpha = 1$ and $(1/t) \log A_t$ converges almost surely to $\mathbb{E}[\xi_1]$. Eventually, we use Girsanov's transform to give the explicit behavior of $\mathbb{E}\left[(a + A_t(\xi))^{-1}\right]$ as $t \to \infty$, where a is a constant, and deduce from this the rate of decay of the tail of the distribution of the maximum of a diffusion process in a random Lévy environment.

1. INTRODUCTION

We first describe three different sources of interest for exponential functionals of Brownian motion with drift.
a) In [19], Kawazu and Tanaka investigate the asymptotic behavior of the tail of the distribution of the maximum of a diffusion process in a drifted Brownian environment, precisely : given a Brownian motion $(W(x), x \in \mathbb{R})$ with $W(0) = 0$,and a constant $c > 0$, consider $V(x) = W(x) + cx$ as a random potential ; then, in [19], the authors associate to V a process $(X(t, V) ; t \geq 0)$ which, conditionally on V, is a diffusion process starting from zero, with generator

$$\frac{1}{2} e^{V(x)} \frac{d}{dx}\left(e^{-V(x)} \frac{d}{dx}\right).$$

X may be constructed from a Brownian motion through suitable Feller changes of scale and time. The problem, treated in [19], is the following : how fast does $\mathbb{P}(\max_{t \geq 0} X(t) > x)$ decay as $x \to \infty$? Since

$$\mathbb{P}(\max_{t \geq 0} X(t) > x) = \mathbb{E}\left[\frac{A}{A + B_x}\right],$$

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where

$$A = \int_{-\infty}^{0} e^{V(t)}\,dt \quad \text{and} \quad B_x = \int_0^x e^{V(t)}\,dt$$

are independent, the problem may be reduced to :

1. find the law of $A_\infty(\xi) = \int_0^\infty e^{\xi_s}\,ds$, when ξ is a Brownian motion with negative drift;
2. find the asymptotic behavior of $A_t(\xi) = \int_0^t e^{\xi_s}\,ds$, when ξ is a Brownian motion with positive drift.

b) In Mathematical finance, particularly for the computation of the price of Asian options, one is interested in the distribution of $A_t(\xi) = \int_0^t e^{\xi_s}\,ds$ when ξ is a Brownian motion with drift, and t may be finite or infinite. This has been studied by a number of authors, including Bouchaud-Sornette [6], Dufresne [11], Geman-Yor [15] and Yor ([31], [30]).
c) In her PhD Thesis [23], Cécile Monthus shows how the exponential functional naturally arises in the study of a one-dimensional diffusion in a frozen random potential V. Let us give examples of physical quantities considered in [23] and in Comtet-Monthus-Yor [9] :

- The *thermal*-average time spent in the interval $[x, x+dx]$ is $T(x)\,dx$, where
$$T(x) = 2\int_x^{+\infty} e^{V(y)}\,dy\,.$$
- The stationary flow J_N of particles going through a disordered sample of size N with prescribed concentrations P_0 at 0 and $P_N = 0$ at N, is given by :
$$J_N = \frac{1}{2}P_0\left(\int_0^N e^{V(y)}\,dy\right)^{-1}.$$
- The free energy of the disordered system :
$$F_L = -\frac{1}{\beta}\log\left(\int_0^L e^{V(y)}\,dy\right).$$

It is now natural to replace, in the preceding examples, Brownian motion with drift by a more general Lévy process which we will continue to denote by ξ. This paper provides three ways to determine the distribution of $A_\infty(\xi)$ and to obtain asymptotic results for $A_t(\xi)$:
(a) Under some mild assumptions, one of which is $\mathbb{E}[\xi_1] < 0$ (this ensures that $A_\infty < \infty$ a.s.), we prove in Proposition 2.1 that the density k of A_∞ is a solution of an integro-differential equation :

(1.1)
$$-\frac{\sigma^2}{2}\frac{d}{dx}(x^2k(x)) + ((\frac{\sigma^2}{2} - c)x + 1)k(x)$$
$$= \int_x^\infty \bar{\nu}_+(\log(u/x))\, k(u)\, du - \int_0^x \bar{\nu}_-(\log(x/u))\, k(u)\, du\,,$$
by identifying the law of A_∞ with the unique invariant probability measure of a generalized Ornstein-Uhlenbeck process. Examples show that for "nice" Lévy processes ξ, we are able to solve this equation (Example A is Brownian motion with negative drift).
(b) Let T be an exponential random variable of parameter $\theta > 0$, independent from ξ. In Proposition 3.1, we establish the moments formulae :

$$\mathbb{E}\left[A_T^\lambda\right] = \frac{\lambda}{\theta + \phi(\lambda)}\,\mathbb{E}\left[A_T^{\lambda-1}\right] \qquad (\lambda > 0) \tag{1.2}$$

$$\mathbb{E}\left[A_\infty^\lambda\right] = \frac{\lambda}{\phi(\lambda)}\,\mathbb{E}\left[A_\infty^{\lambda-1}\right] \qquad (\lambda > 0) \tag{1.3}$$

where ϕ denotes the Laplace exponent of ξ : $\mathbb{E}\left[e^{\lambda\xi_t}\right] = e^{-t\phi(\lambda)}$.
(c) Eventually, when $\mathbb{E}[\xi_1] > 0$, we deduce in Proposition 4.1 the almost sure convergence :

$$\frac{1}{t}\log A_t \underset{t\to\infty}{\longmapsto} \mathbb{E}[\xi_1]\,,$$

and other asymptotic results which we apply to the study of the maximum of a diffusion in a random Lévy environment.
Since the proof of Proposition 2.1 relies heavily on the properties of generalized Ornstein-Uhlenbeck processes (see Hadjiev[18], Novikov [24] and Samorodnitsky and Taqqu [27], section 3.6), we give in Appendix 1 a summary of the properties of these processes. Eventually, we devote Appendix 2 to the study of a Poisson process with drift : there, we apply the results of Sections 2, 3, 4, and make explicit the connection with the Azéma-Emery martingales (see Emery [13]).

Contents

2. An integro-differential equation satisfied by the density of the exponential functional

Consider the Lévy process

$$\xi_t = -\left(ct + \sigma B_t + \tau_t^+ - \tau_t^-\right),$$

where $(B_t\,; t \geq 0)$ is a standard Brownian motion, c is a real constant, $(\tau_t^{\pm}\,; t \geq 0)$ are subordinators without drift, and the three processes B, τ^+ and τ^- are mutually independent. Therefore, the Lévy-Khinchine exponent of ξ, defined by $\mathbb{E}\left[e^{iu\xi_t}\right] = e^{t\psi(u)}$, is

$$\psi(u) = -ciu - \frac{\sigma^2}{2}u^2 + \int_0^\infty \nu_+(dx)(e^{-iux} - 1) + \int_0^\infty \nu_-(dx)(e^{iux} - 1).$$

The hypotheses $\mathbb{E}[|\xi_1|] < +\infty$, and $\mathbb{E}[\xi_1] < 0$, which imply $\frac{1}{t}\xi_t \to \mathbb{E}[\xi_1]$ a.s. as $t \to \infty$, ensure that $A_\infty < \infty$ a.s. These hypotheses can be translated, in terms of $\nu_\pm$, as :

$$\int_0^\infty \bar{\nu}_\pm(x)\,dx < +\infty\,, \quad c + \int_0^\infty \bar{\nu}_+(x)\,dx - \int_0^\infty \bar{\nu}_-(x)\,dx > 0\,,$$

where $\bar{\nu}_\pm(x) = \nu_\pm(x, +\infty)$. We make the additional (technical) hypothesis : $\mathbb{E}\left[e^{\tau_t^-}\right] < \infty$, that is $\int_0^\infty e^x\,\bar{\nu}_-(x)dx < \infty$.

Proposition 2.1. *Under the preceding assumptions, the random variable*

$$A_\infty = \int_0^\infty e^{\xi_s}\,ds$$

admits a density $k(x)$ which is infinitely differentiable on $(0,\infty)$ (minus $\left\{\frac{1}{c}\right\}$ if $\sigma = 0$ and $c > 0$) and is a solution of the equation (1.1). Conversely, if a probability density on $\mathbb{R}_+$ is a solution of equation (1.1), then it is the density of A_∞.

Before we prove Proposition 2.1, we illustrate it with the following examples :

Example A. Brownian motion with negative drift : $\xi_t = -(ct + \sigma B_t)$. The function $k(x)$, which is a solution of

$$-\frac{\sigma^2}{2}\frac{d}{dx}(x^2k(x)) + ((\frac{\sigma^2}{2} - c)x + 1)k(x) = 0\,,$$

is given by

$$k(x) = K\,x^{-(1+\frac{2c}{\sigma^2})}\,\exp(-\frac{2}{x\sigma^2})\,,$$

for a positive constant K. Consequently,

$$A_\infty \stackrel{d}{=} \frac{2}{\sigma^2\,Z_{\frac{2c}{\sigma^2}}}\,,$$

where Z_a denotes a gamma variable of shape parameter $a > 0$, i.e. of density :

$$\mathbb{P}(Z_a \in dx) = \frac{x^{a-1}}{?\,(a)}e^{-x}\,dx\,1_{(x>0)}\,.$$

Example B. The opposite of a subordinator with drift : $\xi_t = -(ct + \tau_t^+)$ with : $c + \int_0^\infty \bar{\nu}_+(x)\,dx > 0$. There, $k(x)$ is a solution of

$$(1-cx)k(x) = \int_x^\infty \bar{\nu}_+(\log(u/x))\,k(u)du\,.$$

We consider the case $\bar{\nu}_+(x) = ae^{-bx}$ where $a > 0,\ b > 0$ and $\mathbb{E}[-\xi_1] = c + (a/b) > 0$. Then, there exists a normalizing constant K_c such that :

$$k(x) = \begin{cases} K_c\ x^b(1-cx)^{(a/c)-1}\ 1_{(x<\frac{1}{c})} & \text{if } c > 0; \\ K_0\ x^b e^{-ax} & \text{if } c = 0; \\ K_c\ x^b(1-cx)^{(a/c)-1} & \text{if } c < 0. \end{cases}$$

In other words :

$$A_\infty \stackrel{d}{=} \begin{cases} \frac{1}{c} Z_{b+1,a/c} & \text{if } c > 0 \; ; \\ \frac{1}{a} Z_{b+1} & \text{if } c = 0 \; ; \\ \frac{1}{|c|} \frac{Z_{b+1}}{Z_{(a/|c|)-b}} & \text{if } c < 0, \end{cases}$$

where the two gamma random variables appearing on the right hand side when $c < 0$ are assumed to be independent and $Z_{\alpha,\beta}$ denotes a beta variable with parameters $\alpha > 0, \beta > 0$:

$$\mathbb{P}(Z_{\alpha,\beta} \in dx) = \frac{dx}{B(\alpha,\beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1}\, 1_{(0<x<1)}\,.$$

Remark 2.2. If $c > 0$ and $\xi_t = -(ct + \tau_t^+)$, then $\xi_t \leq -ct$ so that $A_\infty \leq \int_0^\infty e^{-ct}\, dt = \frac{1}{c}$. On the other hand, if $c > 0$ and $\xi_t = -(ct - \tau_t^-)$, then $\xi_t \geq -ct$ and $A_\infty \geq \frac{1}{c}$.

Example C. A Poisson process with drift.
Here $\xi_t = -ct + aN_t$, where $(N_t\, ; t \geq 0)$ is a Poisson process of parameter $\theta > 0$, a, c are real constants such that $0 < \mathbb{E}[-\xi_1] = c - a\theta$ (this example is developed in Appendix 2). We shall see that the distribution function of A_∞, $F(x) := \int_0^x k(u)\, du$, is the solution of a difference-differential equation.
Suppose first that $a > 0$, that is $\bar{\nu}_+(x) = 0$ and $\bar{\nu}_-(x) = \theta 1_{(x<a)}$. We have $c > a\theta > 0$ and :

$$(1-cx)k(x) = -\theta \int_{xe^{-a}}^x k(u)\, du.$$

Thanks to the Remark 2.2, $F(x)$ is a solution of :

$$\begin{aligned} (cx-1)\, F'(x) &= \theta\,(F(x) - F(xe^{-a})) && (x > \frac{1}{c}) \\ F(x) &= 0 && (x \leq \frac{1}{c})\,. \end{aligned}$$

Suppose now that $a < 0$, that is $\bar{\nu}_-(x) = 0$ and $\bar{\nu}_+(x) = \theta\, 1_{(x<-a)}$. We have :

$$(1-cx)\, k(x) = \theta \int_x^{xe^{-a}} k(u)\, du\,.$$

If $c > 0$, then F is a solution of :

$$\begin{aligned} (1-cx)\, F'(x) &= \theta\,(F(xe^{-a}) - F(x)) && (0 < x < \frac{1}{c}) \\ F(x) &= 1 && (x \geq \frac{1}{c})\,. \end{aligned}$$

If $c \leq 0$, then F is a solution of :

$$(1 - cx)\, F'(x) = \theta\,(F(xe^{-a}) - F(x))\,.$$

Example D. The "drift" is a compound Poisson process.

$$\xi_t = -(ct + \sigma B_t + \tau_t^+)\,.$$

We suppose that $\bar{\nu}_+(x) = ae^{-bx}$, with $a > 0$, $b > 0$ and $\mathbb{E}[-\xi_1] = c + (a/b) > 0$. We get :

$$\frac{\sigma^2}{2}x^2k''(x)+\left((\frac{\sigma^2}{2}(3-b)+c)x-1\right)k'(x)+\left((1-b)(\frac{\sigma^2}{2}+c)-a+\frac{b}{x}\right)k(x)=0\,.$$

Thanks to the scaling property of Brownian motion, we may take $\frac{\sigma^2}{2} = 1$. If we suppose that $c = -(1+b)$ then, letting $y(x) = e^x\,(2x)^{b-\frac{1}{2}}\,k(1/2x)$, we find that $y(x)$ is a solution of Bessel's equation :

$$y''(x) + \frac{1}{x}y'(x) - (1 + \frac{\mu^2}{x^2})y(x) = 0\,,$$

with $\mu = \sqrt{a + (1/4)}$, so that

$$k(x) = x^{b-\frac{1}{2}}e^{-\frac{1}{2x}}(A\,I_\mu(1/2x) + B\,K_\mu(1/2x))\,,$$

with A, B two constants. The integrability condition $\int_0^\infty k(u)du = 1$ implies that $B > 0$, $A = 0$, and

$$k(x) = B\,x^{b-\frac{1}{2}}e^{-\frac{1}{2x}}K_\mu(1/2x)\,.$$

2.1. **Proof of Proposition 2.1.** We shall need the following

Lemma 2.3. *Let $(\xi_t\,; t \geq 0\,)$ be a Lévy process and $A_t(\xi) = \int_0^t e^{\xi_s}\,ds$. Then, for each $t > 0$, the following identity in law holds :*

$$(e^{\xi_t}A_t(-\xi), \xi_t) \stackrel{d}{=} (A_t(\xi), \xi_t)\,.$$

Proof. Observe that the process $(\xi_t - \xi_{(t-s)-}, s \leq t)$ has the same law as $(\xi_s, s \leq t)$, and that :

$$e^{\xi_t}A_t(-\xi) = \int_0^t e^{\xi_t - \xi_s}\,ds = \int_0^t e^{\xi_t - \xi_{t-s}}\,ds$$

□

Consider the generalized Ornstein-Uhlenbeck process associated with ξ (see Appendix 1):

$$U_t = e^{\xi_t}\left(x + \int_0^t e^{-\xi_s}\,ds\right) \qquad (x \in \mathbb{R}, t \geq 0)\,.$$

It is easy to prove (see Appendix 1) that, since $\mathbb{E}[\xi_1] < 0$ and $\mathbb{E}\left[e^{\tau_t^-}\right] < +\infty$, the process $(U_t\,; t \geq 0\,)$ is an homogeneous Markov process, the generator of which satisfies

$$L^U f(x) = f'(x) + L^\xi (f \circ \exp)(\log x)\,, \qquad (x > 0)$$

for each function f with a continuous and bounded derivative, and such that the function $f \circ \exp(x) = f(e^x)$ belongs to the domain $D(L^\xi)$ of the generator L^ξ of the process $(\xi_t\,; t \geq 0\,)$.
From Lemma 2.3, we deduce that, for any fixed $t \geq 0$:

$$U_t \stackrel{d}{=} x e^{\xi_t} + \int_0^t e^{\xi_s}\, ds\ ,$$

Since $\xi_t \underset{t\to\infty}{\longmapsto} -\infty$ a.s., for each $x \in \mathbb{R}$

$$U_t \underset{t\to\infty}{\overset{d}{\longmapsto}} A_\infty\,.$$

Consequently the law $m(dx)$ of A_∞ is the unique invariant probability measure of the Markov process $(U_t\,; t \geq 0\,)$, and as such it satisfies :

$$\int_0^\infty L^U f(x)\, m(dx) = 0 \qquad (f \in D(L^U))\,. \tag{2.1}$$

In our framework, for each infinitely differentiable function g with compact support in $\mathbb{R}$, $g \in D(L^\xi)$ and we have :

$$\begin{aligned} L^\xi g(x) &= -cg'(x) + \frac{\sigma^2}{2} g''(x) + \int_0^\infty (g(x-y) - g(x))\, \nu_+(dy) + \int_0^\infty (g(x+y) - g(x))\, \nu_-(dy) \\ &= -cg'(x) + \frac{\sigma^2}{2} g''(x) - \int_0^\infty g'(x-y)\, \bar\nu_+(y)\, dy + \int_0^\infty g'(x+y)\, \bar\nu_-(y)\, dy \end{aligned}$$

Hence, if $f(x)$ is an infinitely differentiable function with compact support in $(0, +\infty)$,

$$\begin{aligned} L^U f(x) = &-\frac{\sigma^2}{2} x^2 f''(x) + ((\frac{\sigma^2}{2} - c)x + 1) f'(x) \\ &- \int_0^x f'(u) \bar\nu_+(\log(x/u))\, du + \int_x^\infty f'(u) \bar\nu_-(\log(u/x))\, du\,. \end{aligned}$$

We remark the following general equalities, the first one being understood in terms of Schwartz distributions :

$$\int_0^\infty m(dx)\, x^2 f''(x) = \langle m, x^2 f''(x) \rangle = \left\langle -\frac{d}{dx}(x^2 m), f'(x) \right\rangle.$$

$$\int_0^\infty m(dx) \int_0^x f'(u)\, \bar\nu_+(\log(x/u))\, du = \int_0^\infty f'(u)\, du \int_u^\infty m(dx)\, \bar\nu_+(\log(x/u))$$
$$\int_0^\infty m(dx) \int_x^\infty f'(u)\, \bar\nu_-(\log(u/x))\, du = \int_0^\infty f'(u)\, du \int_0^u m(dx)\, \bar\nu_-(\log(u/x))$$

Now, since $f(x)$ is an arbitrary infinitely differentiable function with compact support in $(0,+\infty)$, the relation (2.1) implies that m is a solution (in the Schwartz distributions sense) of

(2.2)
$$-\frac{\sigma^2}{2}\frac{d}{dx}(x^2 m) + ((\frac{\sigma^2}{2} - c)x + 1)\, m$$
$$= \int_x^\infty m(du)\, \bar\nu_+(\log(u/x)) - \int_0^x m(du)\, \bar\nu_-(\log(x/u)) \,+ C\,,$$

with C a real constant. The right hand side being a distribution associated to a locally bounded function, we conclude that m is absolutely continuous, $m(dx) = k(x)\, dx$, with $k(x)$ a solution of equation

(2.3)
$$-\frac{\sigma^2}{2}\frac{d}{dx}(x^2 k(x)) + ((\frac{\sigma^2}{2} - c)x + 1)k(x)$$
$$= \int_x^\infty \bar\nu_+(\log(u/x))\, k(u)\, du - \int_0^x \bar\nu_-(\log(x/u))\, k(u)\, du + C$$

To finish our proof of Proposition 2.1, we only need to show that $C = 0$. Indeed, if k is a probability density on $(0,\infty)$ that satisfies equation (1.1), it satisfies (2.3) and, going backwards, we prove that for $m(dx) = k(x)\, dx$ and every infinitely differentiable function f with compact support in $(0,\infty)$, we have :

$$\int L^U f(x)\, m(dx) = 0\,.$$

But, for such an f and $t > 0$:

$$P_t^U f - f = \int_0^t L^U P_s^U f\, ds\,,$$

where $(P_s^U, s \geq 0)$ is the semi-group of the process U. Integrating this identity with respect to $m(dx)$ yields

$$\int P_t f(x)\, m(dx) = \int f(x)\, m(dx)\,.$$

The two positive measures m and mP_t are equal because they coincide on infinitely differentiable functions f with compact support in $(0,\infty)$.

Hence m is the unique invariant probability measure associated to the process U.

We now prove that if k is a probability density on $(0,\infty)$ that satisfies (2.3), then $C=0$. Assume that $C\neq 0$, $\sigma\neq 0$ and consider the function

$$y(x)=x^{-(1+\frac{2c}{\sigma^2})}\exp(-\frac{2}{\sigma^2 x})$$

which verifies

$$-\frac{\sigma^2}{2}\frac{d}{dx}(x^2y(x))+((\frac{\sigma^2}{2}-c)x+1)y(x)=0\,.$$

Let $k(x)=y(x)z(x)$; then, z is a solution of :

$$-\frac{\sigma^2}{2}x^2y(x)z'(x)=\int_x^\infty \bar\nu_+(\log(u/x))\,k(u)\,du-\int_0^x \bar\nu_-(\log(x/u))\,k(u)\,du+C\,.$$

Since k is positive, $\int_0^\infty k(x)\,dx=1$, and the functions $\bar\nu_\pm$ are finite positive decreasing, the first two terms of the preceding equality converge to 0 as x converges to $+\infty$. Hence, we have the asymptotics, as x goes to infinity :

$$z'(x)\sim -\frac{2}{\sigma^2}\,C\,x^{\frac{2}{\sigma^2}c-1}\,.$$

This entails that the function z is ultimately monotone and satisfies :

$$z(x)\sim -\frac{C}{c}\,x^{\frac{2}{\sigma^2}c}\,.$$

We obtain the following contradiction :

$$k(x)\sim -\frac{C}{cx}\quad\text{and}\quad \int_0^\infty k(x)\,dx=1\,.$$

We obtain a similar contradiction if we suppose $\sigma=0$ and $C\neq 0$.

Remark 2.4. The key of the proof relies on the fact that although the process $(A_t=\int_0^t e^{\xi_s}\,ds\,;t\geq 0)$ is not (generally) a Markov process, for each fixed $t>0$ we have : $A_t\stackrel{d}{=}U_t$, where U is a Markov process starting from zero.

3. The moments formulae

3.1. **A functional equation.** Let $(\xi_t\,;t\geq 0)$ be a Lévy process with Lévy exponent $\phi(\lambda)\in[-\infty,+\infty)$ determined by

$$\mathbb{E}\left[e^{\lambda\xi_t}\right]=e^{-t\phi(\lambda)}.$$

and let T be an independent exponential random variable of parameter $\theta>0$.

Proposition 3.1. *(i) If $\lambda > 0$ and $\theta + \phi(\lambda) > 0$, then*

$$\mathbb{E}\left[A_T^\lambda\right] = \frac{\lambda}{\theta + \phi(\lambda)} \mathbb{E}\left[A_T^{\lambda-1}\right] \tag{1.2}$$

(ii) If $\lambda \geq 1$ and $\phi(\lambda) > 0$, then

$$\mathbb{E}\left[A_\infty^\lambda\right] = \frac{\lambda}{\phi(\lambda)} \mathbb{E}\left[A_\infty^{\lambda-1}\right] \tag{1.3}$$

(iii) If $0 < \lambda < 1$, $\phi(\lambda) > 0$ and $\mathbb{E}\left[A_\infty^\lambda\right] < \infty$ then (1.3) is satisfied.
(iv) Assume that the right derivative $\phi'_+(0) = \lim_{\lambda \downarrow 0} \frac{\phi(\lambda)}{\lambda}$ exists and satisfies $\phi'_+(0) > 0$. Then :

$$\mathbb{E}\left[A_\infty^{-1}\right] = \phi'_+(0) = -\mathbb{E}\left[\xi_1\right].$$

Proof. (i) The integration by parts formula

$$(A_t - A_u)^\lambda = A_t^\lambda - \lambda \int_0^u (A_t - A_v)^{\lambda-1} e^{\xi_v} \, dv \quad (u \leq t)$$

yields, for $u = t$,

$$A_t^\lambda = \lambda \int_0^t (A_t - A_v)^{\lambda-1} e^{\xi_v} \, dv. \tag{3.1}$$

Moreover, observe that

$$A_t - A_v = e^{\xi_v} \int_0^{t-v} e^{\xi_{s+v} - \xi_v} \, ds,$$

and, since $(\xi_{s+v} - \xi_v, s \geq 0)$ is a Lévy process which is independent of $(\xi_u, u \leq v)$, and which has the same law as $(\xi_s, s \geq 0)$, we get, by taking expectations in (3.1) :

$$\mathbb{E}\left[A_t^\lambda\right] = \lambda \int_0^t dv \, e^{-v\phi(\lambda)} \mathbb{E}\left[A_{t-v}^{\lambda-1}\right].$$

Applying Fubini's Theorem yields

$$\begin{aligned}
\mathbb{E}\left[A_T^\lambda\right] &= \theta \int_0^\infty dt \, e^{-\theta t} \mathbb{E}\left[A_t^\lambda\right] \\
&= \lambda\theta \int_0^\infty dv \, e^{-v\phi(\lambda)} \int_v^{+\infty} dt \, e^{-\theta t} \mathbb{E}\left[A_{t-v}^{\lambda-1}\right] \\
&= \lambda\theta \int_0^\infty dv \, e^{-v(\theta + \phi(\lambda))} \int_0^\infty du \, e^{-\theta u} \mathbb{E}\left[A_u^{\lambda-1}\right] \\
&= \frac{\lambda}{\theta + \phi(\lambda)} \mathbb{E}\left[A_T^{\lambda-1}\right].
\end{aligned}$$

(ii) For $\mu \geq 0$,

$$\mathbb{E}\left[A_T^\mu\right] = \theta \int_0^\infty e^{-\theta t} \mathbb{E}\left[A_t^\mu\right] dt = \int_0^\infty e^{-s} \mathbb{E}\left[A_{s/\theta}^\mu\right] ds\,. \tag{3.2}$$

Hence, as θ decreases to 0, this quantity increases to $\mathbb{E}[A_\infty^\mu]$. Formula (1.3) is obtained from (1.2) by applying (3.2) to $\mu = \lambda$ and $\mu = \lambda - 1$, and then letting θ decrease to 0.
(iii) Since $\mathbb{E}\left[A_\infty^\lambda\right] < +\infty$, we have $\mathbb{E}\left[A_T^\lambda\right] < +\infty$ and, by (1.2), $\mathbb{E}\left[A_T^{\lambda-1}\right] < +\infty$. Hence, for each $u > 0$, $\mathbb{E}\left[A_u^{\lambda-1}\right] < +\infty$. Observe that

$$\mathbb{E}\left[A_T^{\lambda-1}\right] = \int_0^\infty e^{-s} \mathbb{E}\left[A_{s/\theta}^{\lambda-1}\right] ds. \tag{3.3}$$

We may apply the dominated convergence theorem to show that for each $u > 0$, $\mathbb{E}\left[A_{u/\theta}^{\lambda-1}\right]$ decreases to $\mathbb{E}\left[A_\infty^{\lambda-1}\right]$. Hence, both sides of (3.3) being finite, we get, by monotone convergence :
$\lim_{\theta\downarrow 0} \mathbb{E}\left[A_T^{\lambda-1}\right] = \mathbb{E}\left[A_\infty^{\lambda-1}\right]$. We conclude as in (ii).
(iv) Assume that there exists $\delta > 0$ such that : $\mathbb{E}\left[A_\infty^\delta\right] < \infty$. Then, there is a $\eta \in (0, \delta)$ such that for any $\lambda \in (0, \eta)$: $\phi(\lambda) > 0$ and $\mathbb{E}\left[A_\infty^\lambda\right] < \infty$. The desired result is obtained by letting λ decrease to zero in formula (1.3).
Observe first that if $k > 0$ is small enough, then the Lévy process $(\xi'_t = \xi_t + kt\,; t \geq 0\,)$ has the same properties as ξ and

$$A_\infty(\xi) = \int_0^\infty e^{\xi'_s} e^{-ks}\, ds \leq e^{M^*(\xi')} \int_0^\infty e^{-ks}\, ds = \frac{1}{k} e^{M^*(\xi')}\,,$$

with $M^*(\xi') = \sup_{s\geq 0} \xi'_s$. Thus,

$$\mathbb{E}\left[A_\infty^\delta(\xi)\right] \leq \frac{1}{k^\delta} \mathbb{E}\left[(M^*(\xi'))^\delta\right].$$

Hence, all we have to show is that for a Lévy process ξ satisfying the hypothesis of (iv) and $\delta > 0$ small enough, we have :
$\mathbb{E}\left[(M^*(\xi))^\delta\right] < +\infty$.
It is clear that ξ may be decomposed as the sum of two independent Lévy processes satisfying the hypothesis of (iv), $\xi = \xi_1 + \xi_2$, such that ξ_1 (resp. ξ_2) has no positive (resp. negative) jumps. Since $M^*(\xi) \leq M^*(\xi_1) + M^*(\xi_2)$ we may suppose that the jumps ξ are of a fixed sign, and apply Theorem 8, Chapter IV of Gihman-Skorohod [16], to conclude. □

Remark 3.2. For every positive measurable function f, we have :

$$\mathbb{E}\left[f(A_T)\right] = \theta \int_0^\infty e^{-\theta t}\, \mathbb{E}\left[f(A_t)\right] dt\,.$$

Hence, the moments formula (1.2) characterizes the law of A_t at fixed time t. However, even in simple cases, we are not able to invert the

Laplace transform. For example, consider Example B, with $c = 0$ and $\bar{\nu}_+(x) = ae^{-bx}$ with $a > 0, b > 0$. The Lévy exponent of ξ is

$$\phi(\lambda) = \lambda \int_0^\infty \bar{\nu}_+(x) e^{-\lambda x} dx = \frac{a\lambda}{b+\lambda} \quad (\lambda > 0)$$

so that

$$\mathbb{E}\left[A_T^\lambda\right] = \frac{\lambda}{\theta + \frac{a\lambda}{b+\lambda}} \mathbb{E}\left[A_T^{\lambda-1}\right] \quad (\lambda > 0)$$

which implies :

$$A_T \stackrel{d}{=} \frac{1}{\theta + a} Z_{b+1} \, Z_{1, \frac{b\theta}{\theta+a}} \, ,$$

where on the right hand side, the gamma and beta variables are assumed to be independent.

Suppose now that $\xi_t = -\tau_t^+$, where τ^+ is a subordinator without drift. Even when $\mathbb{E}[\xi_1] < 0$ we are not able to solve, in the general case, equation (1.1), which states that $k(x)$ is a solution of

$$k(x) = \int_x^{+\infty} \bar{\nu}_+(\log(u/x)) \, k(u) \, du \, .$$

However,the moments formulae enable us to determine completely the law of A_∞ which has exponential moments. Indeed, if $M_t = \mathbb{E}[A_\infty / \mathcal{F}_t]$, we have :

$$M_t = A_t + e^{-\tau_t^+} \mathbb{E}[A_\infty] = A_t + \frac{1}{\phi(1)} \, e^{-\tau_t^+} .$$

So, we deduce:

$$M_\infty - M_{t-} = A_\infty - A_t - \frac{1}{\phi(1)} \, e^{-\tau_{t-}^+},$$

then, for every $(\mathcal{F}_t)$-stopping time T:

$$\mathbb{E}[|M_\infty - M_{T-}| / \mathcal{F}_T] \leq \frac{2}{\phi(1)} \, e^{-\tau_{T-}^+} \leq \frac{2}{\phi(1)}.$$

Hence, the martingale $(M_t; t \geq 0)$ is in BMO, and $\|M\|_{BMO_1} \leq \frac{2}{\phi(1)}$. Thanks to John-Nirenberg's inequality (see [10], Chapter VI, Theorem 109), if $0 \leq \alpha < \frac{1}{2\phi(1)}$, we have:

$$\mathbb{E}\left[e^{\alpha A_\infty}\right] < +\infty \, .$$

In fact, we obtain better results :

Proposition 3.3. *The law of A_∞ is determined by its integral moments*

$$m_n := \mathbb{E}[A_\infty^n] = \frac{n!}{\prod_{j=1}^n \phi(j)} \qquad (n \in \mathbb{N}) \, . \tag{3.4}$$

More precisely, if $0 \le \alpha < \phi(\infty)$*, then :*

$$\mathbb{E}\left[e^{\alpha A_\infty}\right] < \infty\,.$$

Proof. Formula (3.4) is an easy consequence of (1.3). Since $(\tau_t^+\,; t \ge 0\,)$ is an increasing process, $\phi(\lambda)$ is a positive increasing function on $\mathbb{R}_+$. Let $\alpha < \phi(\infty)$; there exists $k \in \mathbb{N}$ such that $\phi(k) > \alpha$. For $n \ge k$, we have :

$$m_n = \frac{n!}{\prod_{j=1}^n \phi(j)} \le \frac{n!}{\left(\prod_{j=1}^k \phi(j)\right)\phi(k)^{n-k}}\,.$$

Hence

$$\mathbb{E}\left[e^{\alpha A_\infty}\right] = \sum_{n\ge 0} \frac{\alpha^n}{n!}\, m_n < \infty\,.$$

□

Example E. The opposite of a stable subordinator of index α.
Let $(\tau_t^{(\alpha)}\,; t \ge 0\,)$ be a stable subordinator of index $\alpha \in (0,1)$ and let

$$H_\alpha := \int_0^\infty e^{-\tau_t^{(\alpha)}}\,dt\,.$$

Without any loss of generality we may suppose that $\phi(\lambda) = \lambda^\alpha$, since, thanks to the scaling property of $\tau^{(\alpha)}$, we have for all $\lambda > 0$:

$$H_\alpha \stackrel{d}{=} \int_0^\infty \exp(-\lambda^{-\frac{1}{\alpha}}\tau_{\lambda s}^{(\alpha)})ds = \frac{1}{\lambda}\int_0^\infty \exp(-\lambda^{-\frac{1}{\alpha}}\tau_t^{(\alpha)})dt\,.$$

Let $\alpha \in (0,1)$, $\beta \in (0,1)$ such that $\alpha+\beta-1 \in (0,1)$. If H_α and H_β are independent, then Proposition 3.3 implies that :

$$H_\alpha\, H_\beta \stackrel{d}{=} H_{\alpha+\beta-1}\,.$$

Let $\bar\alpha = 1-\alpha$, and $L_{\bar\alpha} = \log H_\alpha$. The preceding identity in law may be written :

$$L_{\bar\alpha} + L_{\bar\beta} \stackrel{d}{=} L_{\bar\alpha+\bar\beta}\,.$$

Consequently, $L_{\bar\alpha}$ is infinitely divisible, and, below, we shall study the corresponding Lévy process. We also deduce from (3.4) that :

$$\mathbb{E}\left[\exp(nL_{\bar\alpha})\right] = (n!)^{\bar\alpha} = \mathbb{E}\left[\exp(n\log Z_1)\right]^{\bar\alpha}\,. \qquad (n \in \mathbb{N})$$

where Z_1 denotes a standard exponential.

3.2. **The log-gamma process.** We first recall that a gamma process $(Z_a, a \geq 0)$ is a Lévy process such that for each $a > 0$, Z_a is a gamma variable of shape parameter a. Then, we consider $(Z^{(j)}, j \in \mathbb{N})$ a family of independent gamma processes $(Z_a^{(j)}, a \geq 0)$.Following Gordon [17], we fix $b > 0$ and define

$$\eta_a^{(b)} := -a\gamma + \sum_{j=0}^{\infty} \frac{a}{j+1} - \frac{Z_a^{(j)}}{j+b},$$

where $\gamma = -?'(1)$ denotes the Euler constant. This definition makes sense, as we may write alternatively :

$$\eta_a^{(b)} = -a\gamma + a\sum_{j=0}^{\infty} \frac{b-1}{(j+1)(j+b)} + \sum_{j=0}^{\infty} \frac{a - Z_a^{(j)}}{j+b},$$

where the third term on the right hand side is understood as the L^2 (resp. a.s.) limit of an L^2-bounded martingale.

Proposition 3.4. *(i) The log-gamma process $(\eta_a^{(b)}, a \geq 0)$ is a Lévy process.*
(ii) $\eta_1^{(b)}$ is the logarithm of a gamma variable with shape parameter b, i.e. : $\eta_1^{(b)} \stackrel{d}{=} \log Z_b$.
(iii) The Lévy-Khintchine exponent of $\eta^{(b)}$, defined by

$$\mathbb{E}\left[e^{iu\eta_a^{(b)}}\right] = e^{a\phi^{(b)}(u)} \qquad (u \in \mathbb{R},\, a \geq 0)$$

admits the representation :

$$\phi^{(b)}(u) = \log\left(\frac{?\,(b+iu)}{?\,(b)}\right) = iu\,\psi(b) + \int_{-\infty}^{0} \frac{e^{bs}}{|s|(1-e^s)}\,(e^{ius} - ius - 1)\,ds\,,$$

where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the logarithmic derivative of the Gamma function.
(iv) For each $\alpha \in (0,1)$, we have :

$$L_{\bar{\alpha}} = \log H_{1-\bar{\alpha}} \stackrel{d}{=} \eta_{\bar{\alpha}}^{(1)}$$

Remark 3.5. The fact that $\log Z_a$ is infinitely divisible is well-known: see e.g. Shanbhag [28], who shows that $\log Z_a$ is self-decomposable, hence infinitely divisible.

Proof. (i) Let $a_0 > 0$. Since, for each $j \in \mathbb{N}$, $(Z_{a+a_0}^{(j)} - Z_{a_0}^{(j)}, a \geq 0)$ is a gamma process independent of $(\, Z_a^{(j)}\,, a \leq a_0\,)$, we see that $(\eta_{a+a_0}^{(b)} - \eta_{a_0}^{(b)}, a \geq 0)$ is a process independent of $(\eta_a^{(b)}, a \leq a_0)$, which has the same law as $(\eta_a^{(b)}, a \geq 0)$.
(ii) is Theorem 2 of Gordon [17].
(iii) for each $u \in \mathbb{R}$, (ii) implies that

$$\mathbb{E}\left[e^{iu\eta_1^{(b)}}\right] = \mathbb{E}\left[e^{iu\log Z_b}\right] = \frac{?\,(b+iu)}{?\,(b)}\,,$$

so that :

$$\phi^{(b)}(u) = \log\left(\frac{?\,(b+iu)}{?\,(b)}\right).$$

Then, the representation of $\phi^{(b)}$ derives from the formulae, valid for $\mathcal{R}e(z) > 0$, (see Formula (1.3.14) of Lebedev [22]) :

$$\log ?\,(z) = \int_1^z \psi(x)\,dx\,, \qquad \psi(z) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-tz}}{1 - e^{-t}}\,dt\,.$$

(iv) Hölder's inequality implies that the function $g(\lambda) = \frac{1}{1-\alpha}\log\mathbb{E}\left[H_\alpha^\lambda\right]$ is convex on $\mathbb{R}_+$.
Since $g(0) = 0$ and $g(\lambda+1) = g(\lambda) + \log(\lambda+1)$, we conclude from Theorem 2.1 of Artin [2] (see also Artin [1] or Bourbaki [7], Chapter 7 Proposition 1 and exercises 1 and 2), that for $\lambda > 0$, $g(\lambda) = \log ?\,(\lambda+1)$.
Hence, by analytic continuation :

$$\mathbb{E}\left[H_\alpha^{iu}\right] = ?\,(1+iu)^{1-\alpha} \quad (u \in \mathbb{R})\,,$$

which implies the desired identity in law. □

4. Asymptotic results

The behavior of $\log A_t$ is particularly simple when ξ admits a positive first moment : $\mathbb{E}[\xi_1] > 0$.

Proposition 4.1. *Let $(\xi_t\,; t \geq 0)$ be a Lévy process such that $\mathbb{E}[|\xi_1|] < +\infty$ and $\mathbb{E}[\xi_1] > 0$. Then, almost surely,*

$$\frac{1}{t}\log\left(\int_0^t e^{\xi_s}\,ds\right) \underset{t\to\infty}{\longmapsto} \mathbb{E}[\xi_1]\,.$$

Proof. This is an immediate consequence of the strong law of large numbers : almost surely $\dfrac{\xi_t}{t} \underset{t\to\infty}{\longmapsto} \mathbb{E}[\xi_1]$. □

When ξ admits a null first moment : $\mathbb{E}[\xi_1] = 0$, or when ξ has no first moment, we shall rely upon a self-similarity assumption. Recall that an α-stable Lévy process ξ is a Lévy process such that for each $t > 0$, the variable ξ_t has a strictly stable distribution of order $\alpha \in (0, 2]$.

Proposition 4.2. *Let $(\xi_t\,; t \geq 0)$ be an α-stable Lévy process. Then :*

$$\frac{1}{t^{1/\alpha}}\log A_t(\xi) \overset{d}{\underset{t\to\infty}{\longmapsto}} \sup_{s\leq 1}\,\xi_s\,.$$

Proof. Our hypothesis implies :

$$\frac{1}{t^{1/\alpha}} \log A_t \stackrel{d}{=} \frac{1}{t^{1/\alpha}} \log t + \frac{1}{t^{1/\alpha}} \log \left(\int_0^1 e^{t^{1/\alpha} \xi_u} \, du \right).$$

Laplace's method shows that the second term on the right hand side converges almost surely to $\sup_{s \le 1} \xi_s$. □

Remark 4.3. When $\alpha = 2$, i.e. ξ is a Brownian motion, the preceding result is well-known (see e.g. Durrett [12], Pitman-Yor [25], Revuz-Yor [26], Chapter I, Exercise 1.18, or Bertoin-Werner [3] and [4]), and is a main step in the proof of Spitzer's theorem on the winding number of planar Brownian motion.

It is natural to think that the assumptions of the preceding Proposition were too restrictive, and that we may merely assume that the $(1/\alpha)$-scaled process $(\xi_t^{(\lambda)} = \frac{1}{\lambda^{1/\alpha}} \xi_{\lambda t} \, ; t \ge 0)$ converges in distribution to an α-stable Lévy process. We conjecture that it is really the case, and we prove it in the case $\alpha = 2$, i.e. the target process $\xi^{(\infty)}$ is a Brownian motion.

Proposition 4.4. *Let* $(\xi_t \, ; t \ge 0)$ *be a Lévy process such that the* $(1/\alpha)$*-scaled process* $(\xi_t^{(\lambda)} = \frac{1}{\lambda^{1/\alpha}} \xi_{\lambda t} \, ; t \ge 0)$ *converges in law to an* α*-stable Lévy process* $\xi^{(\infty)}$*. If* $\alpha = 2$ *then,*

$$\frac{1}{\sqrt{t}} \log A_t \xrightarrow[t \to \infty]{d} \sup_{s \le 1} B_s \stackrel{d}{=} |B_1| \, ,$$

where B *denotes a standard Brownian motion.*

Remark 4.5. A necessary and sufficient condition for the preceding convergence is that the Lévy-Khintchine exponent of ξ satisfies : $\lambda \, \psi(u/\lambda^{1/\alpha}) \to \psi^{\infty}(u)$ as $\lambda \to \infty$, that is ξ_1 belongs to the domain of attraction of a strictly α-stable law.

Proof. Let $M_t^{(\lambda)} = M_t(\xi^{(\lambda)}) = \sup_{s \le t} \xi_s^{(\lambda)}$. We have :

$$\frac{1}{t^{1/\alpha}} \log A_t = \frac{1}{t^{1/\alpha}} \log \left(t \int_0^1 e^{t^{1/\alpha} \xi_u^{(t)}} \, du \right). \tag{4.1}$$

Hence :

$$\frac{1}{t^{1/\alpha}} \log A_t \le \frac{1}{t^{1/\alpha}} \log t + M_1^{(t)} . \tag{4.2}$$

Let D_E be the space of càdlàg functions on $\mathbb{R}_+$ endowed with Skorokhod's topology. Since the function $f : D_E \to D_E$, such that $f(x)(t) = \sup_{s \le t} x(s)$, is continuous (see Ethier-Kurtz [14], Chapter 3, exercise 26, page 153), we have :

$$(\xi^{(\lambda)}, M^{(\lambda)}) \xrightarrow[\lambda \to \infty]{d} (\xi^{(\infty)}, M^{(\infty)}).$$

Let $\mu \in (0,1)$ and let $?(\xi^{(\lambda)}) = \int_0^1 1_{(\xi_s^{(\lambda)} > \mu M_1^{(\lambda)})} \, ds$. From (4.1), we deduce the inequality :

$$\frac{1}{t^{1/\alpha}} \log A_t \geq \frac{1}{t^{1/\alpha}} \log t + \mu M_1^{(t)} + \frac{1}{t^{1/\alpha}} \log ?(\xi^{(t)}) \tag{4.3}$$

Assume the convergence in law :

$$(?(\xi^{(\lambda)}), M^{(\lambda)}) \underset{\lambda \to \infty}{\overset{d}{\longmapsto}} (?(\xi^{(\infty)}), M^{(\infty)}). \tag{4.4}$$

Then, we conclude from relations (4.2) and (4.3), that the family $(\frac{1}{t^{1/\alpha}} \log A_t \,;\, t \geq 0)$ is relatively compact, and that for every convergent (in law) subsequence, we have :

$$\mu M_1^{(\infty)} \prec \lim \frac{1}{t_n^{1/\alpha}} \log A_{t_n} \prec M_1^{(\infty)} ,$$

where $\prec$ denotes stochastic comparison. Then, letting μ increase to 1 gives the desired result.
We now prove (4.4). Unfortunately, the function $\gamma : D_E \to \mathbb{R}$,

$$\gamma(x) = \int_0^1 1_{(x(s) > \mu \sup_{u \leq 1} x(u))} \, ds ,$$

is not continuous. However, the set C_γ of continuity points of γ contains the set C' of $x \in D_E$ such that for each $T > 0, \epsilon > 0$ there exists $\delta > 0$ such that

$$\int_0^T 1_{(|x(s) - \mu \sup_{u \leq 1} x(u)| < \delta)} \, ds < \epsilon .$$

Since $\xi^{(\infty)}$ is a Brownian motion, we have $\mathbb{P}(\xi^{(\infty)} \in C') = 1$ and we have proven (4.4). □

Corollary 4.6. *Let $(\xi_t \,;\, t \geq 0)$ be a Lévy process such that $(\lambda^{-\frac{1}{2}} \xi_{\lambda t} \,;\, t \geq 0)$ converges in law to a Brownian motion B. Suppose furthermore that either ξ or $-\xi$ has no positive jumps. Then,*

$$\frac{1}{\sqrt{t}} \mathbb{E}[\log A_t(\xi)] \underset{t \to \infty}{\longmapsto} \mathbb{E}[|B_1|] = \sqrt{\frac{2}{\pi}} . \tag{4.5}$$

Consequently,

$$\mathbb{E}\left[\frac{1}{A_t(\xi)}\right] \sim \sqrt{\frac{1}{2\pi t}} . \tag{4.6}$$

Proof. Since

$$\frac{1}{\sqrt{t}} \log A_t \underset{t \to \infty}{\overset{d}{\longmapsto}} |B_1| ,$$

we only need to show that the family $(\frac{1}{\sqrt{t}}\log A_t\,;t\geq 0)$ is uniformly integrable. We consider the process : $M_t(\xi)=\sup_{s\leq t}(\xi_s)$. Observe first that

$$\frac{1}{\sqrt{t}}\log t-\frac{1}{\sqrt{t}}M_t(-\xi)\leq\frac{1}{\sqrt{t}}\log A_t(\xi)\leq\frac{1}{\sqrt{t}}\log t+\frac{1}{\sqrt{t}}M_t(\xi)\,,$$

We deduce from Lemma 2.3 that :

$$\frac{1}{\sqrt{t}}\log A_t(\xi)\stackrel{d}{=}\frac{1}{\sqrt{t}}\xi_t+\frac{1}{\sqrt{t}}\log A_t(-\xi)\,.$$

Hence, if $\prec$ denotes stochastic comparison, then :

$$\frac{1}{\sqrt{t}}\log t+\frac{1}{\sqrt{t}}\xi_t-\frac{1}{\sqrt{t}}M_t(\xi)\prec\frac{1}{\sqrt{t}}\log A_t(\xi)\prec\frac{1}{\sqrt{t}}\log t+\frac{1}{\sqrt{t}}\xi_t+\frac{1}{\sqrt{t}}M_t(-\xi)\,.$$

Consequently, it is enough to prove that the families $(\frac{1}{\sqrt{t}}M_t(\xi)\,;t\geq 0)$ and $(\frac{1}{\sqrt{t}}\xi_t\,;t\geq 0)$ are bounded in L^2, when ξ has no positive jumps. Since ξ has no positive jumps, we may use Zolotarev's formula :

$$\mathbb{P}(M_t>x)=-x\,\frac{\partial}{\partial x}\left(\int_0^t\mathbb{P}(\xi_u>x)\,\frac{du}{u}\right),$$

to obtain :

$$\mathbb{E}\left[M_t^2\right]=2\int_0^t\frac{du}{u}\;\mathbb{E}\left[\xi_u^2\;1_{(\xi_u>0)}\right].$$

From the convergence

$$(\lambda^{-\frac{1}{2}}\;\xi_{\lambda t}\,;t\geq 0)\underset{\lambda\to\infty}{\overset{d}{\longmapsto}}(B_t\,;t\geq 0)$$

we deduce that $\mathbb{E}[\xi_1]=0$ and $\mathbb{E}[\xi_1^2]=1$. Hence, for all $u\geq 0$, $\mathbb{E}[\xi_u^2]=u$,

$$\mathbb{E}\left[M_t^2\right]\leq 2\int_0^t\frac{du}{u}\mathbb{E}\left[\xi_u^2\right]\leq 2t\,.$$

and : $\sup_{t\geq 0}\frac{1}{t}\mathbb{E}[M_t^2]\leq 2$.
Now, remark that the equivalent (4.6) is an immediate consequence of the convergence (4.5) applied to the process $-\xi$, the Monotone Density Theorem (see e.g. [5], Theorem 1.7.2), and the relation :

$$\frac{d}{dt}\mathbb{E}[\log A_t(-\xi)]=\mathbb{E}\left[\frac{e^{-\xi_t}}{A_t(-\xi)}\right]=\mathbb{E}\left[\frac{1}{A_t(\xi)}\right],$$

(the last equality coming from Lemma 2.3). $\square$

We now show how to use Girsanov transform to estimate the asymptotics of quantities of the type $\mathbb{E}[(a+A_t(\xi))^{-p}]$. Recall that the Laplace exponent $\phi:\mathbb{R}\to[-\infty,+\infty[$ of ξ is defined by :

$$\mathbb{E}\left[e^{m\xi_t}\right] = e^{-t\phi(m)} \qquad (t \geq 0,\ m \in \mathbb{R}).$$

Assume that $\phi(m) > -\infty$. Then, the process $(e^{m\xi_t + t\phi(m)}\,; t \geq 0)$ is a martingale with respect to the natural filtration $(\mathcal{F}_t\,; t \geq 0)$ of ξ. The *Girsanov or Esscher transform* $\mathbb{P}^{(m)}$ of the probability $\mathbb{P}$ is defined by

$$\mathbb{P}^{(m)} = e^{m\xi_t + t\phi(m)}\,.\,\mathbb{P} \qquad \text{on } \mathcal{F}_t$$

Under $\mathbb{P}^{(m)}$, the process ξ is a Lévy process with Laplace exponent :

$$\phi^{(m)}(p) = \phi(p+m) - \phi(m)\,.$$

${}^{(m)}\xi$ will denote a process which, under $\mathbb{P}$, has the same law as ξ under $\mathbb{P}^{(m)}$.

Lemma 4.7. *Assume that for m in a neighbourhood of $-p$, $p > 0$, we have $\phi(m) > -\infty$, and suppose that :*

$$\mathbb{E}\left[(A_\infty(-{}^{(p)}\xi))^{-p}\right] < +\infty\,.$$

Then, for all $a \geq 0$ we have :

$$\begin{aligned}
&\mathbb{E}\left[(a + A_t(\xi))^{-p}\right] \sim e^{-t\phi(-p)}\,\mathbb{E}\left[(A_\infty(-{}^{(p)}\xi))^{-p}\right] && \text{if } \phi'(-p) < 0\\
&\mathbb{E}\left[(a + A_t(\xi))^{-p}\right] = o\left(e^{-t\phi(-p)}\right) && \text{if } \phi'(-p) \geq 0
\end{aligned}$$

Proof. Let $a \geq 0$. Using first Lemma 2.3, and then the Girsanov Transform, we obtain :

$$\begin{aligned}
\mathbb{E}\left[(a + A_t(\xi))^{-p}\right] &= \mathbb{E}\left[(a + e^{\xi_t}A_t(-\xi))^{-p}\right] = \mathbb{E}\left[e^{-p\xi_t}\left(ae^{-\xi_t} + A_t(-\xi)\right)^{-p}\right]\\
&= e^{-t\phi(-p)}\mathbb{E}^{(-p)}\left[(ae^{-\xi_t} + A_t(-\xi))^{-p}\right]
\end{aligned}$$

Observe that :

$$\mathbb{E}^{(-p)}\left[\xi_1\right] = -\phi'(-p)$$

The strong law of large numbers says that $\dfrac{{}^{(-p)}\xi_t}{t}$ converges almost surely to $-\phi'(-p) > 0$. It implies that :

- if $\phi'(-p) < 0$, then $ae^{-{}^{(-p)}\xi_t} \to 0$ and $A_t(-{}^{(-p)}\xi) \to A_\infty(-{}^{(-p)}\xi)$;
- if $\phi'(-p) > 0$, then $A_\infty(-{}^{(-p)}\xi) = +\infty$.

We conclude by noting that since ϕ is defined in a neighborhood of $-p$, the r.v. ${}^{(-p)}\xi_1$ has moments of all orders. Hence, if $\phi'(-p) = 0$, ${}^{(-p)}\xi_1$ is in the domain of attraction of a Gaussian r.v., and, by Corollary 4.6, $A_\infty(-{}^{(-p)}\xi) = +\infty$. □

We shall also need the following upper bound :

Lemma 4.8. *For all $s, t \geq 0$ and $p \geq 1$, we have :*

$$\mathbb{E}\left[e^{\xi_{t+s}}A_{t+s}(\xi)^{-p}\right] \leq \mathbb{E}\left[A_s(\xi)^{-(p-1)}\right]\mathbb{E}\left[A_t(-\xi)^{-1}\right].$$

Proof. Since the process $(\ \xi_{u+s} - \xi_s\ , u \geq 0\ )$ is independent from $(\xi_v, 0 \leq v \leq s)$, and has the same law as ξ, we have :

$$
\begin{aligned}
\mathbb{E}\left[e^{\xi_{t+s}} A_{t+s}(\xi)^{-p}\right] &= \mathbb{E}\left[\frac{e^{\xi_{t+s}-\xi_s}}{A_s(\xi)e^{-\xi_s} + \int_0^t \exp(\xi_{s+u} - \xi_s)du}\ (A_s(\xi) + (A_{t+s}(\xi) - A_s(\xi)))^{1-p}\right] \\
&\leq \mathbb{E}\left[\frac{e^{\xi_{t+s}-\xi_s}}{\int_0^t \exp(\xi_{s+u} - \xi_s)du}\ (A_s(\xi))^{1-p}\right] \\
&= \mathbb{E}\left[e^{\xi_t} A_t(\xi)^{-1}\right]\ \mathbb{E}\left[A_s(\xi)^{1-p}\right] \\
&= \mathbb{E}\left[A_t(-\xi)^{-1}\right]\ \mathbb{E}\left[A_s(\xi)^{1-p}\right]
\end{aligned}
$$

(the last equality coming from Lemma 2.3). □

We shall now see how these different techniques may be applied to the study of

4.1. **The maximum of a diffusion process in a random Lévy environment.** Following Kawazu and Tanaka [19], we consider $(\xi_t\,; t \geq 0\,)$ and $(\eta_t\,; t \geq 0\,)$ two independent Lévy processes starting from zero and admitting first moments : $\mathbb{E}[\xi_1] > 0$ and $\mathbb{E}[\eta_1] < 0$.
Let $(X(t,V)\,; t \geq 0\,)$ be a diffusion process starting from zero with infinitesimal generator

$$\frac{1}{2} e^{V(x)} \frac{d}{dx}\left(e^{-V(x)}\, \frac{d}{dx}\right),$$

where

$$V(x) = \begin{cases} \xi_x & \text{if } x \geq 0, \\ \eta_{-x} & \text{if } x \leq 0. \end{cases}$$

Our goal is to determine the rate of decay, as $t \to \infty$, of

$$\mathbb{P}(\max_{s \geq 0} X(s) > t) = \mathbb{E}\left[\frac{A_\infty(\eta)}{A_\infty(\eta) + A_t(\xi)}\right].$$

Proposition 4.9. *Assume that :*

- *the Laplace exponent of ξ, $\phi(m)$, is defined in a neighbourhood of -1;*
- *the Laplace exponent of η, $\psi(m)$, is defined in a neighbourhood of 1, and $\psi(1) > 0$.*

(1) If $\phi'(-1) < 0$ then, as $t \to \infty$,

$$\mathbb{P}(\max_{s \geq 0} X(s) \geq t) \sim e^{-t\phi(-1)} \left|\frac{\phi'(-1)}{\psi(1)}\right|.$$

(2) If $\phi'(-1) = 0$ and $\phi''(-1) < 0$ then :

$$\mathbb{P}(\max_{s\geq 0} X(s) \geq t) \sim e^{-t\phi(-1)}\sqrt{\left|\frac{\phi''(-1)}{2\pi t}\right|}\,\frac{1}{\psi(1)}\,.$$

(3) If $\phi'(-1) > 0$, then :

$$\mathbb{P}(\max_{s\geq 0} X(s) \geq t) = o\left(e^{-t\phi(-1)}\right).$$

Remark 4.10. (1) Since

$$\mathbb{E}\,[A_t(\eta)] = \int_0^t \mathbb{E}\,[e^{\eta_s}]\,ds = \int_0^t e^{-s\psi(1)}\,ds$$

assuming $\mathbb{E}\,[A_\infty(\eta)] < \infty$ is the same as assuming $\psi(1) > 0$; in that case, $\mathbb{E}\,[A_\infty(\eta)] = \dfrac{1}{\psi(1)}$.

(2) The function ϕ is concave so $\phi''(-1) \leq 0$. If $\phi''(-1) = 0$, we have $\mathbb{E}^{(-1)}[\xi_1] = -\phi'(-1) = 0$ and $\mathbb{E}^{(-1)}[\xi_1^2] = \phi'(-1)^2 - \phi''(-1) = 0$. Hence ξ is the null process and we may discard this case.

Example 4.11. It is natural to consider a Lévy process over $\mathbb{R}$, that is

$$(\eta_t\,;t\geq 0\,) \stackrel{d}{=} (-\xi_t\,;t\geq 0\,)\,.$$

In this case, $\psi(m) = \phi(-m)$; since it is a concave function, if $\phi'(-1) < 0$, then $\phi(-1) > 0$.

Brownian motion with drift : $\xi_t = B_t + ct$ with $c > 0$. This is the example considered by Kawazu and Tanaka.

- If $c > 1$, then

$$\mathbb{P}(\max_{s\geq 0} X(s) \geq t) \sim \frac{c-1}{c-\frac{1}{2}} e^{-t(c-\frac{1}{2})}\,.$$

- If $c = 1$ then,

$$\mathbb{P}(\max_{s\geq 0} X(s) \geq t) \sim \sqrt{\frac{2}{\pi t}}\, e^{-t/2}\,.$$

Poisson process with drift : $\xi_t = a\ N_t\ +\ c\ t$, where a, c are constants, N is a Poisson process of parameter $\theta > 0$, and we assume $\mathbb{E}\,[\xi_1] = a\theta + c > 0$. Then, $\phi(m) = -cm + \theta(1 - e^{ma})$. Hence :

- If $c > -\theta a e^{-a}$, then

$$\mathbb{P}(\max_{s\geq 0} X(s) \geq t) \sim \frac{c+\theta a e^{-a}}{c+\theta(1-e^{-a})}\, e^{-t(c+\theta(1-e^{-a}))}\,.$$

- If $c = -\theta a e^{-a}$ then :

$$\mathbb{P}(\max_{s\geq 0} X(s) \geq t) \sim \sqrt{\frac{\theta}{2\pi t}}\frac{a e^{-a/2}}{\theta(1-e^{-a}-ae^{-a})}\ \exp(-t\theta(1-e^{-a}-ae^{-a}))\,.$$

Proof of Proposition 4.9. Since ξ and η are independent, we have :

$$\mathbb{P}(\max_{s\geq 0} X(s) \geq t) = \mathbb{E}[A_\infty(\eta) f(A_\infty(\eta), t)] \tag{4.7}$$

with $f(a,t) = \mathbb{E}[(a + A_t(\xi))^{-1}]$.
(1) Combining Lemma 4.7 with Proposition 3.1 gives :

$$e^{t\phi(-1)} f(a,t) \underset{t\to\infty}{\longmapsto} \mathbb{E}\left[A_\infty(-{}^{(-1)}\xi)^{-1}\right] = \mathbb{E}\left[{}^{(-1)}\xi_1\right] = |\phi'(-1)|.$$

Since $\sup_{a\geq 0} af(a,t) \leq 1$ we conclude by applying the dominated convergence theorem.
(2) Note that $\phi^{(-1)}$ is defined in a neighbourhood of 0. Hence ${}^{(-1)}\xi_1$ admits moments of all orders, $\mathbb{E}\left[\ {}^{(-1)}\xi_1\ \right] = -\phi'(-1) = 0$,
$Var({}^{(-1)}\xi_1) = -\phi''(-1) > 0$. Let $\sigma = \sqrt{|\phi''(-1)|}$. We have the convergence to a standard normal random variable :

$$\frac{1}{\sigma\sqrt{t}}\,{}^{(-1)}\xi_t \underset{t\to\infty}{\overset{d}{\longmapsto}} N.$$

A straightforward modification of Proposition 4.4 and Corollary 4.6 gives that :

$$\mathbb{E}\left[\frac{1}{A_t({}^{(-1)}\xi)}\right] \sim \sigma\sqrt{\frac{1}{2\pi t}}. \tag{4.8}$$

The combination of Lemma 2.3, Girsanov transform and relation (4.8) gives :

$$\mathbb{E}\left[\frac{1}{A_t(\xi)}\right] = \mathbb{E}\left[\frac{1}{e^{\xi_t} A_t(\xi)}\right] \sim e^{-t\phi(-1)}\sigma\sqrt{\frac{1}{2\pi t}}. \tag{4.9}$$

Since $\psi(1) > 0$, we may choose $p \in (1,2)$ close enough to 1, so that $\psi(p) > 0$. Proposition 3.1 yields :

$$\mathbb{E}[A_\infty(\eta)^p] = \frac{p}{\psi(p)}\mathbb{E}\left[A_\infty(\eta)^{p-1}\right] \leq \frac{p}{\psi(p)}\mathbb{E}[A_\infty(\eta)]^{p-1} = \frac{p\,\psi(1)^{1-p}}{\psi(p)} < \infty.$$

Let $C_p = \sup_{u>0} \dfrac{u^{2-p}}{1+u}$. For all positive constants α, β we have :

$$0 \leq \frac{\alpha}{\beta} - \frac{\alpha}{\alpha+\beta} \leq C_p\,\alpha^p\,\beta^{-p}.$$

Hence,

(4.10)
$$0 \leq \mathbb{E}\left[\frac{A_\infty(\eta)}{A_t(\xi)}\right] - \mathbb{P}(\max_{s\geq 0} X(s) \geq t) \leq C_p\,\mathbb{E}[A_\infty(\eta)^p]\,\mathbb{E}\left[A_t(\xi)^{-p}\right].$$

Consequently, it is enough to show that as $t \to \infty$,

$$\mathbb{E}\left[A_t(\xi)^{-p}\right] = o\left(\mathbb{E}\left[\frac{1}{A_t(\xi)}\right]\right).$$

Using first Girsanov transform, then Lemma 4.8 and Hölder's inequality, we obtain the upper bound :

$$\begin{aligned}
\mathbb{E}\left[A_t(\xi)^{-p}\right] &= e^{-t\phi(-1)}\,\mathbb{E}^{(-1)}\left[e^{\xi_t}A_t(\xi)^{-p}\right] \\
&\leq e^{-t\phi(-1)}\,\mathbb{E}^{(-1)}\left[A_{t/2}(\xi)^{-(p-1)}\right]\mathbb{E}^{(-1)}\left[A_{t/2}(-\xi)^{-1}\right] \\
&\leq e^{-t\phi(-1)}\,\mathbb{E}^{(-1)}\left[A_{t/2}(\xi)^{-1}\right]^{(p-1)}\mathbb{E}^{(-1)}\left[A_{t/2}(-\xi)^{-1}\right]
\end{aligned}$$

Corollary 4.6 gives an equivalent of $\mathbb{E}^{(-1)}\left[A_{t/2}(\xi)^{-1}\right]$ and $\mathbb{E}^{(-1)}\left[A_{t/2}(-\xi)^{-1}\right]$. Thus there exists a positive constant K such that :

$$\mathbb{E}\left[A_t(\xi)^{-p}\right] \leq e^{-t\phi(-1)}\,\frac{K}{t^{p/2}}.$$

(3) is an immediate consequence of relation (4.7) and Lemma 4.7 :

$$f(a,t) = o\left(e^{-t\phi(-1)}\right).$$

□

5. Appendix 1 : Generalized Ornstein-Uhlenbeck processes

The Ornstein-Uhlenbeck process of parameter λ, starting from x, may be constructed from a standard Brownian motion $(B_t\,; t \geq 0)$ as

$$U_t = x + B_t + \lambda\int_0^t U_s\,ds.$$

This stochastic differential equation has a unique solution :

$$U_t = e^{\lambda t}\left(x + \int_0^t e^{-\lambda s}\,dB_s\right). \tag{5.1}$$

If we represent the Wiener integral $(\int_0^t e^{-\lambda s}\,dB_s\,; t \geq 0)$ as the time change of a standard Brownian motion $(\hat{B}_u, u \geq 0)$, we obtain :

$$U_t = e^{\lambda t}\left(x + \hat{B}\left(\int_0^t e^{-2\lambda s}\,ds\right)\right). \tag{5.2}$$

The aim of this section is to define generalizations of representations (5.1) and (5.2), our goal being to define a Markov process $(U_t\,; t \geq 0)$ from a two-dimensional process that will play the rôle of either $(e^{\lambda t}, B_t\,; t \geq 0)$ or $(e^{\lambda u}, \hat{B}_u\,;\, u \geq 0)$.

We shall prove that these representations coincide for a large class of processes. Eventually, we investigate the properties of the generalization based on (5.2), one of the most important being the existence, under mild assumptions, of a unique invariant probability measure.

5.1. **The Ornstein-Uhlenbeck process associated to a two-dimensional Lévy process.** Our aim is to unify the generalizations of (5.1) proposed by Hadjiev [18], Novikov [24], and Samorodnitsky-Taqqu [27]. Let (ξ, η) be a two-dimensional Lévy process starting from $(0,0)$, and let

$$V_t = e^{\xi_t}\left(x + \int_0^t e^{-\xi_{s-}}\, d\eta_s\right) \qquad (t \geq 0) \tag{5.3}$$

We shall need the following

Lemma 5.1. *Let $(Z_t, \mathcal{F}_t, t \geq 0\,; \mathbb{P}_z, z \in E)$ be a Markov process, let $(A_t\,; t \geq 0)$ be an additive functional of Z that is also a $(\mathcal{F}_t)$-semimartingale, and let $(M_t\,; t \geq 0)$ be a strictly positive multiplicative functional of Z, adapted and càdlàg. Then the pair*

$$(Y_t := \frac{1}{M_t}(y + \int_0^t M_{u-}\, dA_u)\,;\, Z_t\,; t \geq 0\,)$$

is an homogeneous Markov process of semi-group

$$Q_t F(y,z) = \mathbb{E}_z\left[F(\frac{1}{M_t}(y + \int_0^t M_{u-}\, dA_u), Z_t)\right].$$

Proof. Let $s, t \geq 0$. Our assumptions imply that :

$$M_{t+s} = M_s(M_t \circ \theta_s)\,, \qquad A_{t+s} = A_s + A_t \circ \theta_s\,.$$

Let $C_t = \int_0^t M_{u-}\, dA_u$. For every bounded measurable function $F(y,z)$, we have :

$$\begin{aligned}
\mathbb{E}[F(Y_{t+s}, Z_{t+s}) \mid \mathcal{F}_s] &= \mathbb{E}\left[F(\frac{1}{M_s(M_t \circ \theta_s)}(y + C_s + M_s(C_t \circ \theta_s))\,;\, Z_{t+s}) \mid \mathcal{F}_s\right] \\
&= \mathbb{E}\left[F(\frac{1}{M_t \circ \theta_s}(Y_s + C_t \circ \theta_s)\,;\, Z_t \circ \theta_s) \mid \mathcal{F}_s\right] \\
&= Q_t F(Y_s, Z_s)
\end{aligned}$$

□

Corollary 5.2. *The process V given by (5.3) is an homogeneous Markov process.*

Proof. Let us apply the preceding Lemma to $Z_t = (\xi_t, \eta_t)$, $M_t = e^{-(\xi_t - \xi_0)}$ and $A_t = (\eta_t - \eta_0)$. What remains to be shown is that if $F(y,z) = f(y)$ does not depend on z, then $Q_t F(y,z)$ does not depend on z. Letting $z = (a,b)$, we have :

$$\begin{aligned} Q_t F(y,z) &= \mathbb{E}_{(a,b)}\left[f(e^{\xi_t - \xi_0}(y + \int_0^t e^{-(\xi_{s-} - \xi_0)}\, d\eta_s)) \right] \\ &= \mathbb{E}_{(0,0)}\left[f(e^{\xi_t}(y + \int_0^t e^{-(\xi_{s-})}\, d\eta_s)) \right] \end{aligned}$$

□

Assume now that ξ and η are independent Lévy processes starting from zero. Let $\psi(u)$ be the Lévy-Khintchine exponent of η, i.e.

$$\mathbb{E}\left[e^{iu\eta_t}\right] = e^{t\psi(u)} \qquad (t \geq 0, u \in \mathbb{R})$$

The law of the process V is determined by the

Proposition 5.3. *Under the preceding assumptions, the semi-group of the process V given by* (5.3) *has characteristic function:*

$$\mathbb{E}_x\left[e^{iuV_t}\right] = \mathbb{E}\left[\exp\left(iuxe^{\xi_t} + \int_0^t \psi(ue^{\xi_s})\, ds\right)\right].$$

Proof. Since ξ and η are independent, all we have to do is to condition with respect to the whole process $(\eta_t\,; t \geq 0\,)$ and then use the well-known

Lemma 5.4. *Let $(\eta_t\,; t \geq 0\,)$ be a Lévy process of Lévy-Khintchine exponent $\psi(u)$. Then, for every measurable locally bounded function f and for every $t \geq 0$, we have :*

$$\mathbb{E}\left[\exp\left(i\int_0^t f(s)\, d\eta_s\right)\right] = \exp\left(\int_0^t \psi(f(s))\, ds\right).$$

□

5.2. **The Ornstein-Uhlenbeck process associated to a self-similar Markov process.** Let $(X_t\,; t \geq 0\,)$ be a ρ-self-similar Markov process $(\rho > 0)$, that is a Markov process $(X_t, \mathcal{F}_t, t \geq 0\,; \mathbb{P}_x, x \in E)$, with state space $E = \mathbb{R}_+, \mathbb{R}$ or $\mathbb{R}^n$, such that :

$$(\frac{1}{a^\rho} X_{at}\,; t \geq 0\ ;\ \mathbb{P}_{a^\rho x}) \stackrel{d}{=} (X_t\,; t \geq 0\ ;\ \mathbb{P}_x) \qquad (a > 0, x \in E).$$

Such Markov processes have been considered by Lamperti [21] in relation with exponentials of Lévy processes. Let $(\xi_t\,; t \geq 0\,)$ be a Lévy process, starting from zero, independent from X, and let $A_t = A_t(-\xi) = \int_0^t e^{-\xi_s}\, ds$.

Proposition 5.5. *The generalized Ornstein-Uhlenbeck process*

$$U_t = e^{\rho\xi_t} X\left(\int_0^t e^{-\xi_s}\, ds\right) \qquad (t \geq 0)$$

is an homogeneous Markov process with semi-group

$$Q_t f(x) = \mathbb{E}_x \left[f(e^{\rho\xi_t} X_{A_t}) \right].$$

Proof. We shall prove that $(U_t\,; t \geq 0)$ is Markovian with respect to the filtration

$$\mathcal{U}_t := \sigma(X_{A_s}, \xi_s, s \leq t).$$

Observe that, given $s, t \geq 0$, the process $(\bar{\xi}_u = \xi_{u+s} - \xi_s, u \geq 0)$ is independent from $\mathcal{U}_s$ and has the same law as ξ. Furthermore,

$$A_{t+s} = A_s + e^{-\xi_s} A_t(-\bar{\xi}) \qquad e^{\rho\xi_{t+s}} = e^{\rho\bar{\xi}_t} e^{\rho\xi_s}.$$

Hence, for any measurable bounded function f, applying the Markov property for X at time A_s yields :

$$\mathbb{E}[f(U_{t+s}) \mid \mathcal{U}_s] = \phi(X_{A_s}, \xi_s),$$

where

$$\phi(x, n) = \mathbb{E}_x \left[f(e^{\rho(n+\xi_t)} X_{e^{-n} A_t}) \right].$$

Since X is ρ-self-similar, we have :

$$\phi(x, n) = \mathbb{E}_{xe^{\rho n}} \left[f(e^{\rho\xi_t} X_{A_t}) \right] = Q_t f(xe^{\rho n}).$$

Hence,

$$\mathbb{E}[f(U_{t+s}) \mid \mathcal{U}_s] = Q_t f(U_s)$$

□

5.3. **Two representations of a generalized Ornstein-Uhlenbeck process.** We extend to other pairs than $(e^{\lambda t}, B_t\,; t \geq 0)$ the representations (5.1) and (5.2).

Proposition 5.6. *Let $(\eta_t\,; t \geq 0)$ be an α-stable Lévy process, with $\alpha \in (0, 2]$ and let $(\xi_t\,; t \geq 0)$ be a Lévy process independent from η. We assume that $\xi_0 = \eta_0 = 0$. Then, the two homogeneous Markov processes*

$$U_t = e^{\frac{1}{\alpha}\xi_t}(x + \eta(\int_0^t e^{-\xi_s}\, ds))$$

and

$$V_t = e^{\frac{1}{\alpha}\xi_t}(x + \int_0^t e^{-\frac{1}{\alpha}\xi_{s-}}\, d\eta_s)$$

have the same law.

Proof. Since η is a Lévy process and a $\frac{1}{\alpha}$-self-similar Markov process, U and V are homogeneous Markov processes. Hence, it is enough to show that for given x, t we have : $U_t \stackrel{d}{=} V_t$.
We deduce from Lemma 5.4 that for every bounded positive measurable function f, we have :

$$\eta\left(\int_0^t f(s)\,ds\right) \stackrel{d}{=} \int_0^t f(s)^{\frac{1}{\alpha}}\,d\eta_s \tag{5.4}$$

We conclude by conditioning with respect to the process ξ, using (5.4) and Lemma 2.3:

$$\begin{aligned}
U_t &= xe^{\frac{1}{\alpha}\xi_t} + e^{\frac{1}{\alpha}\xi_t}\eta(A_t(-\xi)) \\
&\stackrel{d}{=} xe^{\frac{1}{\alpha}\xi_t} + \eta(e^{\xi_t}A_t(-\xi)) \\
&\stackrel{d}{=} xe^{\frac{1}{\alpha}\xi_t} + \eta(A_t(\xi)) \\
&\stackrel{d}{=} xe^{\frac{1}{\alpha}\xi_t} + \int_0^t e^{\frac{1}{\alpha}\xi_{s-}}\,d\eta_s \\
&\stackrel{d}{=} xe^{\frac{1}{\alpha}\xi_t} + e^{\frac{1}{\alpha}\xi_t}\int_0^t e^{-\frac{1}{\alpha}(\xi_t-\xi_{(t-s)-})}\,d\eta_s \\
&\stackrel{d}{=} V_t
\end{aligned}$$

The last equality in law derives from the equality in law

$$(\xi_t - \xi_{(t-s)-}, s \le t) \stackrel{d}{=} (\xi_s, s \le t)$$

5.4. **The invariant measure.** Let $(X_t\,; t \ge 0)$ be a ρ-self-similar Markov process, let $(\xi_t\,; t \ge 0)$ be a Lévy process, starting from zero, independent from X, and let $A_t = A_t(-\xi) = \int_0^t e^{-\xi_s}\,ds$.

Proposition 5.7. *Assume that ξ admits a first moment : $\mathbb{E}[\xi_1] < 0$. Then, if U denotes the generalized Ornstein-Uhlenbeck process defined in Proposition 5.5, for every continuous bounded function f, and every x, we have :*

$$\mathbb{E}_x\left[f(U_t)\right] \underset{t\to\infty}{\longmapsto} \mathbb{E}\left[f(X_1 A_\infty^\rho)\right].$$

Hence, $\mu(dx) = \mathbb{P}_0(X_1 A_\infty^\rho \in dx)$ is the unique invariant probability measure of the semi-group $(Q_t\,; t \ge 0)$ of U.

Proof. Since X is ρ-self-similar, we have for all $a > 0, t \ge 0$:

$$P_t H_{a^\rho} = H_{a^\rho} P_{at} \tag{5.5}$$

where $(P_t\,; t \ge 0)$ denotes the semi-group of X, and H_a is the dilatation $H_a f(x) = f(ax)$. Hence,

$$\begin{aligned}
\mathbb{E}_x\left[f(U_t)\right] &= \mathbb{E}_x\left[H_{e^{\rho\xi_t}} f(X_{A_t(-\xi)})\right] \\
&= \mathbb{E}\left[P_{A_t(-\xi)} H_{e^{\rho\xi_t}} f(x)\right] \\
&= \mathbb{E}\left[H_{e^{\rho\xi_t}} P_{e^{\xi_t} A_t(-\xi)} f(x)\right] \\
&= \mathbb{E}\left[H_{e^{\rho\xi_t}} P_{A_t(\xi)} f(x)\right] \qquad \text{(from Lemma 2.3)}
\end{aligned}$$

Observe that $\mathbb{E}[\xi_1] < 0$ implies that almost surely $\frac{1}{t}\xi_t \underset{t\to\infty}{\longmapsto} \mathbb{E}[\xi_1]$ and $A_t(\xi) \underset{t\to\infty}{\longmapsto} A_\infty(\xi) < \infty$. Thus,

$$\mathbb{E}_x\left[f(U_t)\right] \underset{t\to\infty}{\longmapsto} \mathbb{E}\left[H_0 P_{A_\infty} f(x)\right] = \mathbb{E}_0\left[f(X_{A_\infty})\right] = \mathbb{E}_0\left[f(X_1 A_\infty^\rho)\right].$$

In other terms, $Q_t f(x) \underset{t\to\infty}{\longmapsto} \mu f$, so that for each $s > 0$, $Q_{t+s} f(x) \underset{t\to\infty}{\longmapsto} \mu f$. Since $Q_{t+s} f(x) = Q_t(Q_s f)(x)$, we get that

$$\mu Q_s f = \mu f \qquad\qquad (s > 0, f \text{ bounded continuous}),$$

i.e. μ is an invariant measure.
Suppose ν is another invariant probability measure. Then for every continuous bounded f, we have $Q_t f(x) \underset{t\to\infty}{\longmapsto} \mu f$ and $\sup_{t\geq 0} \|Q_t f\|_\infty \leq \|f\|_\infty$. Hence, by dominated convergence :

$$\nu f = \nu Q_t f \underset{t\to\infty}{\longmapsto} \mu f$$

5.5. **The infinitesimal generator.** Let L^X (resp. L^ξ) be the infinitesimal generator, with domain $D(L^X)$ (resp. $D(L^\xi)$), of the process X (resp. ξ). Assume that X has a Feller semi-group $(P_t\,; t \geq 0)$ and that there exists $\epsilon > 0$ such that

$$\mathbb{E}\left[\sup_{t\leq\epsilon}\left(\frac{1}{t} e^{\xi_t} A_t(-\xi)\right)\right] < \infty \tag{5.6}$$

Proposition 5.8. *Under the preceding assumptions, if $f \in D(L^X)$ is such that for all x $\tilde{f}(y) = f(xe^{\rho y})$ is in $D(L^\xi)$, then $f \in D(L^U)$ and*

$$L^U f(x) = L^X f(x) + L^\xi \tilde{f}(0).$$

Proof. We have :

$$\frac{1}{t}\mathbb{E}_x\left[f(U_t) - f(x)\right] = \frac{1}{t}\mathbb{E}_x\left[f(e^{\rho\xi_t} X_{A_t}) - f(e^{\rho\xi_t} x)\right] + \frac{1}{t}\mathbb{E}\left[f(e^{\rho\xi_t} x) - f(x)\right].$$

The second term on the right hand side is $\frac{1}{t}(P_t^\xi \tilde{f} - \tilde{f})(0)$ and converges to $L^\xi \tilde{f}(0)$. The first term on the right hand side may be written, thanks to relation (5.5):

$$\begin{aligned} F(t) &= \frac{1}{t}\mathbb{E}\left[P_{A_t} H_{e^{\rho\xi_t}} f(x) - H_{e^{\rho\xi_t}} f(x)\right] \\ &= \mathbb{E}\left[\frac{1}{t} e^{\xi_t} A_t H_{e^{\rho\xi_t}} \left(\frac{P_{e^{\xi_t} A_t} f - f}{e^{\xi_t} A_t}\right)(x)\right] \end{aligned}$$

Since $(P_t\,; t \geq 0\,)$ is a Feller semi-group, we may use

$$\frac{1}{u}(P_u f - f) = \int_0^1 P_{uv}(L^X f)\,dv \qquad\qquad (f \in D(L^X), u > 0)$$

to get

$$F(t) = \mathbb{E}\left[\frac{1}{t} e^{\xi_t} A_t H_{e^{\rho\xi_t}} \left(\int_0^1 P_{e^{\xi_t} A_t\, v}(L^X f)\,dv\right)(x)\right].$$

Observe that as t decreases to 0, $\frac{1}{t} e^{\xi_t} A_t$ converges almost surely to 1. It is now clear that (5.6) allows us to use the dominated convergence theorem, in order to get the desired result. □

We shall prove next that the assumption

$$\mathbb{E}\left[\sup_{0 \leq s \leq t \leq \epsilon} e^{\xi_t - \xi_s}\right] < \infty \tag{5.7}$$

is stronger than (5.6) and easier to check.

Lemma 5.9. *(i) The relation* (5.7) *implies* (5.6)*.*
(ii) If ξ^1 and ξ^2 are two independent Lévy processes satisfying (5.7)*, then $\xi = \xi^1 + \xi^2$ satisfies* (5.7)*.*

Remark 5.10. (ii) is useful because (5.7) is trivially satisfied by a pure drift $\xi_t = ct$; it is also easily shown that (5.7) is satisfied by a Brownian motion. It is trivially satisfied if $(-\xi_t\,; t \geq 0\,)$ is a subordinator, since

$$\sup_{0 \leq s \leq t \leq \epsilon} e^{\xi_t - \xi_s} \leq 1$$

Eventually, when ξ is a subordinator, then

$$\sup_{0 \leq s \leq t \leq \epsilon} e^{\xi_t - \xi_s} = e^{\xi_\epsilon}$$

so that (5.7) is equivalent to $\mathbb{E}\left[e^{\xi_\epsilon}\right] < \infty$.

Proof. (i) We have,

$$\begin{aligned}\sup_{t\leq\epsilon}\frac{1}{t}e^{\xi_t}A_t &= \sup_{t\leq\epsilon}\frac{1}{t}\int_0^t e^{\xi_t-\xi_s}\,ds\\ &= \sup_{t\leq\epsilon}\int_0^1 e^{\xi_t-\xi_{tu}}\,du\\ &\leq \sup_{0\leq s\leq t\leq\epsilon} e^{\xi_t-\xi_s}\end{aligned}$$

(ii) Observe that

$$\begin{aligned}\sup_{0\leq s\leq t\leq\epsilon} e^{\xi_t-\xi_s} &= \sup_{0\leq s\leq t\leq\epsilon} e^{\xi^1_t-\xi^1_s}\,e^{\xi^2_t-\xi^2_s}\\ &\leq (\sup_{0\leq s\leq t\leq\epsilon} e^{\xi^1_t-\xi^1_s})(\sup_{0\leq s\leq t\leq\epsilon} e^{\xi^2_t-\xi^2_s})\end{aligned}$$

and take expectations. □

6. Appendix 2 : The exponential functional of a Poisson process with drift

In this section we investigate the asymptotic behavior of $A_t(\xi) = \int_0^t e^{\xi_s}\,ds$, where $(\xi_t = -ct + aN_t\,; t\geq 0\,)$, with a, c two constants, and $(N_t\,; t\geq 0\,)$ a Poisson process of parameter $\theta > 0$. Since ξ admits a first moment $\mathbb{E}[\xi_1] = a\theta - c$, we will use the results of Section 4 to deal with the case $\mathbb{E}[\xi_1]\geq 0$.

Proposition 6.1. *(i) If $a\theta > c$, then*

$$\frac{1}{t}\log A_t(\xi)\overset{d}{\underset{t\to\infty}{\longmapsto}} a\theta - c\,.$$

(ii) If $a\theta = c$, then

$$\frac{1}{\sqrt{t}}\log A_t(\xi)\overset{d}{\underset{t\to\infty}{\longmapsto}}\sqrt{\theta}|aN|\,,$$

where N denotes a standard normal variable.

Proof. (i) is immediate from Proposition 4.1.
(ii) According to Proposition 4.4 and the remark following it, since the Lévy exponent of ξ, defined by : $\mathbb{E}\left[e^{iu\xi_t}\right] = e^{t\psi(u)}$, is given by :

$$\psi(u) = -cui + \theta(e^{iua} - 1)\,,$$

we have $\lambda\psi(u\lambda^{-\frac{1}{2}})\to\psi^{\infty}(u) = -\theta a^2\frac{u^2}{2}$, as $\lambda\to\infty$. Thus,

$$\frac{1}{\sqrt{t}}\log A_t(\xi)\overset{d}{\underset{t\to\infty}{\longmapsto}}\sup_{s\leq 1}\xi^{(\infty)}_s\,,$$

with $(\xi^{(\infty)}_t = \sqrt{\theta}|a|B_t\,; t\geq 0\,)$, and $(B_t\,; t\geq 0\,)$ a standard Brownian motion. □

From now on we suppose that $c - a\theta = -\mathbb{E}[\xi_1] > 0$. In this section we shall :

1. prove that $Y = A_\infty$ (or $Y = 1 - cA_\infty$), is solution of a stochastic difference equation ;
2. give an explicit formula for the density of A_∞, or an explicit recurrence relation for its distribution function F ;
3. give explicit computations of $\mathbb{E}[\{(A_\infty - k)^+\}^n]$ which are key quantities in the study of Asian options pricing.

6.1. **The stochastic (difference) equation.** The stochastic difference equation

$$Y_n = A_n Y_{n-1} + B_n$$

with $\{(A_n, B_n), n \in \mathbb{N}\}$ an i.i.d sequence, has been thoroughly studied by Vervaat [29], Kesten [20], and more recently by Brandt [8]. This equation arises in various disciplines, for example economics, physics, nuclear technology, ... (see Vervaat [29]) If we have the convergence $Y_n \underset{n\to\infty}{\overset{d}{\longmapsto}} Y$, then Y satisfies the stochastic equation : $Y \overset{d}{=} AY + B$.
The study of the exponential functional of a Lévy process provides a natural framework in which to obtain such equations.

Lemma 6.2. *Let T be a stopping time for the Lévy process ξ. If $A_\infty < +\infty$, then A_∞ satisfies the stochastic equation :*

$$A_\infty \overset{d}{=} A_T(\xi) + e^{\xi_T} A'_\infty .$$

Proof. We have :

$$A_\infty(\xi) = A_T(\xi) + e^{\xi_T} \int_0^\infty e^{\xi_{s+T} - \xi_T}\, ds .$$

We conclude with the strong Markov property of the Lévy process ξ, which entails that the process $(\xi_{s+T} - \xi_T, s \geq 0)$ has the same law as ξ, and is independent from $(\xi_s, s \leq T)$. □

Applying this Lemma to T, the first jump time of the Poisson process N, gives, since T is exponentially distributed with parameter $\theta > 0$, the

Proposition 6.3. *The random variable*

$$Y := \begin{cases} A_\infty & \text{if } c = 0, \\ 1 - cA_\infty & \text{otherwise,} \end{cases}$$

satisfies the stochastic equation :

$$Y \overset{d}{=} \begin{cases} \frac{Z}{\theta} + e^{-a} Y & \text{if } c = 0, \\ U^{c/\theta} e^{a}(e^{-a} - 1 + Y) & \text{otherwise,} \end{cases}$$

where Z denotes a standard exponential, U is uniform on $(0,1)$, and (U,Z) is independent from Y.

6.2. **The case $c > 0$, $a < 0$.**

Proposition 6.4. *(i) The random variable A_∞ is determined by its integral moments :*

$$\mathbb{E}[A_\infty^n] = n! \prod_{j=1}^{n} (\theta(1-e^{aj}) + jc)^{-1} \qquad (n \in \mathbb{N}) \tag{6.1}$$

(ii) The distribution function $F(x)$ of A_∞ is the solution of the difference-differential equation

$$\begin{aligned} (1-cx)F'(x) &= \theta(F(xe^{-a}) - F(x)) && (0 < x < 1/c) \\ F(x) &= 1 && (x \geq 1/c) \end{aligned}$$

This equation may be solved step by step on the intervals $[e^{an}/c, e^{a(n+1)}/c]$, for $n \in \mathbb{N}$, the first step being given by :

$$F(x) = 1 - \alpha(1-cx)^{\frac{\theta}{c}} \qquad (\tfrac{e^a}{c} \leq x \leq \tfrac{1}{c}),$$

with

$$\alpha = \mathbb{E}\left[(1 - ce^a A_\infty)^{-\theta/c}\right] = \sum_{p \geq 0} (ce^a)^p \frac{?\,(p+\theta/c)}{?\,(\theta/c)} \prod_{k=1}^{p} (\theta(1-e^{ak}) + kc)^{-1}.$$

(iii) Let $H_n(x) = \mathbb{E}[\{(A_\infty + x - 1/c)^+\}^n]$. Then $H_n(x) = 0$ if $x \leq 0$, and H_n is the unique solution of

$$xH_n'(x) = (n + \frac{\theta}{c})H_n(x) + \frac{\theta}{c}e^{an} H_n(xe^{-a} + \frac{1-e^{-a}}{c}) \qquad (x \geq 0)$$

This equation may be solved step by step, the first step being given by:

$$H_n(x) = \lambda_n \, x^{n+\frac{\theta}{c}} \qquad (0 \leq x \leq \tfrac{1-e^a}{c})$$

with $\lambda_n = \frac{\alpha\theta}{c} c^{\frac{\theta}{c}} B(n+1, \theta/c)$.

Proof. (i) Since $(\xi_t\,; t \geq 0\,)$ is a subordinator with Laplace exponent

$$\phi(\lambda) = c\lambda + \theta(1 - e^{a\lambda}),$$

we deduce from Proposition 3.3 that the law of A_∞ is determined by its entire moments, which are given by formula (6.1).
(ii) The difference-differential equation has already been established in Section 2, Example C. The only problem that remains is to show that $\alpha = \mathbb{E}\left[(1 - ce^a A_\infty)^{-\theta/c}\right]$; the explicit value of α may then be derived from (i).
Let $Y = 1 - cA_\infty$, and let $G(y)$ be its distribution function. We have :

$$G(y) = \alpha y^{\theta/c} \qquad\qquad (0 \le y \le 1 - e^a)$$

Since Y is solution of the stochastic equation

$$Y \overset{d}{=} U^{c/\theta} e^a (e^{-a} - 1 + Y)\,,$$

we get, for $0 \le y \le 1 - e^a$,

$$\begin{aligned} G(y) &= \mathbb{P}\big(U^{c/\theta} e^a (e^{-a} - 1 + Y) \le y\big) \\ &= \mathbb{P}\left(U^{c/\theta} \le \frac{y e^{-a}}{e^{-a} - 1 + Y}\right) \\ &= \mathbb{E}\left[\left(\frac{y e^{-a}}{e^{-a} - 1 + Y}\right)^{\theta/c}\right] \\ &= y^{\theta/c}\,\mathbb{E}\left[(1 - c e^a A_\infty)^{-\theta/c}\right] \end{aligned}$$

(iii) The equation satisfied by H_n may be derived from (ii) or from the stochastic difference equation. The constant λ_n is determined by observing that for $0 < x < \frac{1-e^a}{c}$, we have :

$$H_n(x) = \int_{-x+\frac{1}{c}}^{1/c} (t+x-\frac{1}{c})^n \, dF(t) = \alpha\theta \int_{-x+\frac{1}{c}}^{1/c} (t+x-\frac{1}{c})^n (1-ct)^{(\theta/c)-1} \, dt = \lambda_n x^{n+\frac{\theta}{c}}$$

□

The case $c > a\theta > 0$ is treated in a similar fashion. Seeing that $(-\xi_t\,; t \ge 0\,)$ is not a subordinator, A_∞ is not determined by its entire moments. However, we proved in Section 2, Example C, that

$$\begin{aligned} (1 - cx)F'(x) &= \theta(F(x) - F(xe^{-a})) \qquad && (x > 1/c) \\ F(x) &= 0 && (x \le 1/c) \end{aligned}$$

Thus, the distribution function F may be determined step by step, the first step being given by :

$$F(x) = \gamma(cx - 1)^{\theta/c} \qquad\qquad (1/c \le x \le e^a/c)$$

with γ determined, as was α, by the stochastic equation,

$$\gamma = \mathbb{E}\left[(e^a c A_\infty - 1)^{\theta/c}\right]$$

6.3. **The case** $c = 0$. The results obtained are, as may be expected, simpler and more precise. We just have to consider the case $h(x) = e^{ax}$, $a < 0$, of the

Proposition 6.5. *Let* $h : \mathbb{N} \to \mathbb{R}_+$ *be a measurable function, and let*

$$A_\infty = A_\infty(h) = \int_0^\infty h(N_s)\, ds\,.$$

Then, $A_\infty(h) < \infty$ *almost surely iff* $\sum_p h(p) < \infty$. *Assume this is the case. Then :*
(i) The Laplace transform of A_∞ *is*

$$\mathbb{E}\left[e^{-\lambda A_\infty}\right] = \prod_{p \geq 0} (1 + \frac{\lambda}{\theta} h(p))^{-1}\,.$$

Moreover, when h *is injective, we have:*
(ii) A_∞ *admits the density*

$$k(t) = \sum_{i \geq 0} \frac{\theta}{h(i)} e^{-\frac{t\theta}{h(i)}} \prod_{p \neq i} (1 - \frac{h(p)}{h(i)})^{-1}\,.$$

(iii) Eventually, for every $\lambda \geq 0$ *and* $r \geq 0$, *we have:*

$$\mathbb{E}\left[\left\{(A_\infty - r)^+\right\}^\lambda\right] = \frac{?\,(\lambda+1)}{\theta^\lambda} \sum_{i \geq 0} h(i)^\lambda e^{-\frac{r\theta}{h(i)}} \prod_{p \neq i} (1 - \frac{h(p)}{h(i)})^{-1}\,.$$

Proof. Let T be the first jump time of the Poisson process. We have :

$$A_\infty(h) = h(0)T + \int_0^\infty h(1 + (N_{T+t} - N_T))\, dt\,.$$

Since $(N_{T+t} - N_T\,; t \geq 0\,)$ is independent from $(N_t, t \leq T)$ and has the same law as N, we obtain :

$$A_\infty(h) \stackrel{d}{=} h(0)T + A_\infty(h_1)\,,$$

where $h_1(x) = h(x+1)$ and on the right hand side the random variables are assumed to be independent. Taking Laplace transforms gives

$$\mathbb{E}\left[e^{-\lambda A_\infty(h)}\right] = (1 + \lambda \frac{h(0)}{\theta})^{-1}\, \mathbb{E}\left[e^{-\lambda A_\infty(h_1)}\right] \qquad (\lambda > 0)$$

from which we derive (i).
(ii) The function

$$L(z) = \prod_{p \geq 0} (1 + \frac{z}{\theta} h(p))^{-1}\,,$$

is a meromorphic function of z which coincides with the Laplace transform of $A_\infty(h)$ for $z = \lambda > 0$, and has poles at $z_p = -\theta/h(p)$. The

Laplace inversion formula and the Cauchy integral formula imply that $k(t)$ is the sum of the residues of the function $z \to e^{zt}L(z)$.
(iii) is derived from (i). □

The function $h(x) = e^{ax}$ (with $a < 0$) is not the only one to give interesting results :

Corollary 6.6. *Let $a > 0$ and $h_a(x) = 2\theta(x+a)^{-2}$.*
(i) The random variable $A_\infty(h_a) = 2\theta\int_0^\infty \frac{ds}{(N_s+a)^2}$ has density

$$k_a(t) = ?\,(a)^{-2}\sum_{n\geq 0}(-1)^n \frac{?\,(n+2a)}{n!}(n+a)\exp(-\frac{t}{2}(n+a)^2) \quad (t>0)$$

and Laplace transform :

$$\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}A_\infty(h_a)\right)\right] = \frac{|?\,(a+i\lambda)|^2}{?\,(a)^2}.$$

(ii) Let N be a standard normal random variable independent from $A_\infty(h_a)$ and let Z_a, Z'_a be two independent gamma variables of shape parameter a. Then :

$$N\sqrt{A_\infty(h_a)} \stackrel{d}{=} \log(Z_a/Z'_a).$$

Considering $a = \frac{1}{2}$ and $a = 1$, we get :

Corollary 6.7.

$$\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}A_\infty(h_{\frac{1}{2}})\right)\right] = \frac{1}{\cosh(\pi\lambda)} \qquad \mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}A_\infty(h_1)\right)\right] = \frac{\pi\lambda}{\sinh(\pi\lambda)},$$

that is, the identities in law :

$$A_\infty(h_{\frac{1}{2}}) \stackrel{d}{=} S_\pi \qquad \textit{and} \qquad A_\infty(h_1) \stackrel{d}{=} S_\pi - g_{S_\pi},$$

where $S_\pi = \inf\{t : |B_t| = \pi\}$, $g_{S_\pi} = \sup\{t < S_\pi : B_t = 0\}$, and $(B_t\,; t \geq 0)$ is a standard Brownian motion.

Remark 6.8. (1) Observe that, on one hand

$$\frac{1}{\cosh(\pi\lambda)} = \frac{\pi\lambda}{\sinh\pi\lambda}\frac{\tanh\pi\lambda}{\pi\lambda},$$

and on the other hand : $A_\infty(h_{\frac{1}{2}}) = A_\infty(h_1) + A_\infty(h_{\frac{1}{2}} - h_1)$. One is tempted to think that the random variables $A_\infty(h_1)$ and $A_\infty(h_{\frac{1}{2}} - h_1)$ are independent, and that the latter has Laplace transform $\frac{\tanh\pi\lambda}{\pi\lambda}$. This is not the case for the following reasons :

- From Proposition 6.5 (i) we deduce that the random variables $A_\infty(f)$ and $A_\infty(h)$ are independent iff f and h have disjoint supports.

- The explicit density of the random variable which has Laplace transform in $\frac{\lambda^2}{2}$: $\dfrac{\tanh \pi\lambda}{\pi\lambda}$ cannot be the density of $A_\infty(h)$ for a function h.

The joint law of $(A_\infty(h_a), A_\infty(h_b))$ may be described by the Laplace transform :

$$\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}A_\infty(h_a) - \frac{\mu^2}{2}A_\infty(h_b)\right)\right] = ?\,(a)^{-2}?\,(b)^{-2}\prod_{i=1}^{4} ?\,(-x_i)$$

where $(x_i, 1 \le i \le 4)$ denote the roots of :
$0 = (x+a)^2(x+b)^2 + \lambda^2(x+b)^2 + \mu^2(x+a)^2$.
(2) If $X_a := \int_0^1 R_s(4a)\, ds$, where $(R_s(\delta), s \ge 0)$ is the square of a δ-dimensional Bessel process starting from 0 (see Yor [32]), then :

$$\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}A_\infty(h_a)\right)\right] = c_a\, \mathbb{E}\left[X_a^{-\frac{1}{2}}\, \exp(-\frac{2\lambda^2}{X_a})\right].$$

This equation provides another way of obtaining the density of X_a.

6.4. **The relationship with Azéma-Emery martingales.** Lamperti [21] showed that to a Lévy process $(\xi_t\,; t \ge 0\,)$ we may associate a unique 1-self similar Markov process X by :

$$e^{\xi_t} = X_{A_t} \quad \text{and} \quad A_t = \int_0^t e^{\xi_s}\, ds\,.$$

Furthermore, if

$$T_0 = T_0(X) = \inf\{u : X_u = 0\, or\, X_{u-} = 0\}\,,$$

then $T_0 = A_\infty$ and ξ may be recovered from X by :

$$X_u = e^{\xi_{C_u}}, \quad \text{and} \quad C_u = \inf\{t : A_t > u\} = \int_0^u \frac{ds}{X_s} \qquad (u < T_0).$$

Consider the structure equation :

$$d[X, X]_t = dt + \beta X_{t-}\, dX_t$$

the solution of which is the Azéma-Emery martingale with parameter β (see [13]) :

$$Y_t = (1+\beta)^{N_{\gamma_t}}\, e^{-\beta\gamma_t} \;= exp\{Log(1+\beta)N_{\gamma_t} - \beta\gamma_t\},$$
$$\gamma_t = \beta^{-2}\int_0^t \frac{ds}{Y_s^2}, \qquad (t < T_0(Y)),$$

where $(N_t\,; t \ge 0\,)$ is a standard Poisson process.
Assume $1+\beta > 0$, and consider the Poisson process with drift $(\xi_t = -ct + aN_t\,; t \ge 0\,)$, where $a = 2\mathrm{Log}(1+\beta)$ and $c = 2\beta$. Since

$\mathbb{E}[\xi_1] < 0$ we have $A_\infty(\xi) < \infty$, and the self-similar process associated to ξ by Lamperti's construction is :

$$X_t = Y^2_{t\beta^2} \qquad (t \geq 0)$$

Hence, we have :

Lemma 6.9. *The first hitting time of* 0 *by the Azéma-Emery martingale satisfies*

$$T_0(Y) \stackrel{d}{=} \beta^2 A_\infty(\xi),$$

where $\xi = (-ct + aN_t\,; t \geq 0)$ *is a Poisson process of parameter* $\theta = 1$, *with drift* $c = 2\beta$, *and* $a = 2Log(1+\beta)$.

So, when $\beta > 0$, the law of $T_0(Y)$ is determined, since we have already obtained the law of $A_\infty(\xi)$ for $c > a\theta > 0$.

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