The Non-Anticipation of the Asynchronous Systems

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Abstract

The asynchronous systems are the models of the asynchronous circuits from the digital electrical engineering and non-anticipation is one of the most important properties in systems theory. Our present purpose is to introduce several concepts of non-anticipation of the asynchronous systems.

1   Introduction. Bibliography-Related Remarks

The asynchronous systems are the mathematical models of the asynchronous circuits from the digital electrical engineering and our challenge is the construction of an asynchronous systems theory. The insufficient bibliography that we have at disposal is probably influenced by the great importance of the topic (researchers do not publish) and it consists of:

a) mathematical studies in switching theory from the 60's and we mention here the name of Grigore Moisil that used the discrete time modeling of the asynchronous circuits;

b) engineering studies, which are always non-formalized and dedicated to applications. Such a literature creates intuition, but it does not give acceptable models or tools of investigation;

c) mathematical literature that can produce analogies.

The study of the asynchronous systems is closely connected to the notion of signal, meaning a 'nice' $\mathbf{R} \to \{0,1\}^n$ function. In this context we mention the apparent total absence of the mathematicians' interest in the study of the $\mathbf{R} \to \{0,1\}$ functions, that should be an interesting direction of investigation in temporal logic too.

The paper is dedicated to non-anticipation, one of the most important properties of the systems. We exemplify its use by showing how 0 can be chosen as initial time, how the transfers of the systems are composed and how the asynchronous real time non-deterministic systems behave in certain circumstances in a synchronous discrete time deterministic way.

2 Preliminaries

Definition 2.1 $\mathbf{B} = \{0,1\}$ *endowed with the laws* $-, \cup, \cdot, \oplus$ *is called the* ***binary Boole algebra***.

Notation 2.2 $Seq = \{(t_k)|t_k \in \mathbf{R}, k \in \mathbf{N}, t_0 < t_1 < t_2 < ...$ *unbounded from above*$\}$.

Notation 2.3 *The restriction of the function* $x : \mathbf{R} \to \mathbf{B}^n$ *at the interval* $I \subset \mathbf{R}$ *is denoted by* $x_{|I}$.

Definition 2.4 *The* ***initial value*** $x(-\infty+0) \in \mathbf{B}^n$ *and the* ***final value*** $x(\infty-0) \in \mathbf{B}^n$ *of* $x : \mathbf{R} \to \mathbf{B}^n$ *are defined by*

$$\exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = x(-\infty + 0),$$

$$\exists t_f \in \mathbf{R}, \forall t > t_f, x(t) = x(\infty - 0).$$

Notation 2.5 *Denote by* $\chi_A : \mathbf{R} \to \mathbf{B}$ *the characteristic function of the set* $A \subset \mathbf{R}$.

Definition 2.6 *The function* $x : \mathbf{R} \to \mathbf{B}^n$ *is called* ***signal*** *if* $(t_k) \in Seq$ *exists such that* $\forall t \in \mathbf{R}$,

$$x(t) = x(-\infty + 0) \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus x(t_k) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

We have used the same symbols $\cdot, \oplus$ *for the laws that are induced by those of* $\mathbf{B}$. *The set of the signals is denoted by* $S^{(n)}$ *and instead of* $S^{(1)}$ *we usually write* S.

Notation 2.7 $P^*(S^{(n)}) = \{X|X \subset S^{(n)}, X \neq \emptyset\}$.

Definition 2.8 *The* ***left limit*** $x(t-0)$ *and the* ***left derivative*** $Dx(t)$ *of* $x \in S^{(n)}$ *expressed like in Definition 2.6 are defined by*

$$x(t-0) = x(-\infty+0) \cdot \chi_{(-\infty,t_0]}(t) \oplus x(t_0) \cdot \chi_{(t_0,t_1]}(t) \oplus ... \oplus x(t_k) \cdot \chi_{(t_k,t_{k+1}]}(t) \oplus ...$$

$$Dx(t) = x(t-0) \oplus x(t).$$

Definition 2.9 *A multi-valued function* $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ *is called* ***(asynchronous) system***. *Any* $u \in U$ *is called* ***(admissible) input*** *and the functions* $x \in f(u)$ *are called* ***(possible) states***.

Definition 2.10 *If* $\forall u \in U, f(u)$ *has exactly one element, then* f *is called* ***deterministic*** *and we use the notation* $f : U \to S^{(n)}$ *of the uni-valued functions.*

Definition 2.11 *If* $g : V \to P^*(S^{(n)}), V \in P^*(S^{(m)})$ *is a system, then any system* $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ *with*

$$U \subset V \text{ and } \forall u \in U, f(u) \subset g(u)$$

is called a ***subsystem*** *of* g *and the notation is* $f \subset g$.

Remark 2.12 *The concept of system originates in the modeling of the asynchronous circuits. The multi-valued character of the cause-effect association is due to statistical fluctuations in the fabrication process, the variations in the ambiental temperature, the power supply etc.*

Sometimes the systems are given by equations and/or inequalities. In this case, determinism means that their solution is unique.

If the system g models a circuit, then the system $f \subset g$ models the same circuit more precisely, by restricting the set of the inputs perhaps.

3   The first concept of non-anticipation

Definition 3.1 *The system $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ is **non-anticipatory** if for all $u \in U$ and all $x \in f(u)$ it satisfies one of the following statements:*
a) x is constant;
b) u, x are both variable and we have

$$\min\{t | u(t-0) \neq u(t)\} \leq \min\{t | x(t-0) \neq x(t)\}, \tag{1}$$

i.e. the first input switch is prior to the first output switch.

Remark 3.2 *The non-anticipation means that the system f is in equilibrium, as represented by the existence of the time interval $(-\infty, t_0)$, where u and x are constant: $u_{|(-\infty,t_0)} = u(t_0 - 0)$ and $x_{|(-\infty,t_0)} = x(t_0 - 0)$; then the only possibility to get out of this situation is the switch of the input.*

Moisil presumes implicitly in his works [1], [2] that the models are non-anticipatory in the sense of Definition 3.1. However his 'equilibrium', called 'rest position', is defined in the presence of a 'network function' that does not exist here.

Example 3.3 *Any Boolean function $F : \mathbf{B}^m \to \mathbf{B}^n$ defines for $d \geq 0$ a system $F_d : S^{(m)} \to S^{(n)}$, called 'ideal combinational':*

$$\forall u \in S^{(m)}, \forall t \in \mathbf{R}, F_d(u)(t) = F(u(t-d))$$

which is non-anticipatory. First, $\forall u \in S^{(m)}$ we have $F_d(u) \in S^{(n)}$ indeed. Second, for any $d, u, t_0, u_{|(-\infty,t_0)} = u(t_0-0)$ implies $F_d(u)_{|(-\infty,t_0)} = F_d(u)(t_0 - 0)$ (we have in this situation $u(t_0 - 0) = u(-\infty + 0)$ and $F_d(u)(t_0 - 0) = F_d(u)(-\infty + 0)$) and if $u(t_0 - 0) \neq u(t_0)$, then $F(u(t_0 - d - 0))$, $F(u(t_0 - d))$ represent two values that may be equal or different. We infer that $x = F_d(u)$ fulfills one of a), b) from Definition 3.1.

Example 3.4 *The system $f : S \to S$,*

$$\forall u \in S, f(u) = \chi_{[0,\infty)}$$

is anticipatory.

Lemma 3.5 *If $g : V \to P^*(S^{(n)}), V \in P^*(S^{(m)})$ is a non-anticipatory system, then any system $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ with $f \subset g$ is non-anticipatory.*

Proof. Let be $u \in U$. We have the following possibilities:

i) u is constant. From Definition 3.1 we have that $\forall x \in g(u), x$ is constant, in particular $\forall x \in f(u), x$ is constant. Therefore, f is non-anticipatory;

ii) u is variable. Let $x \in f(u)$ be arbitrary. Then

ii.1) x is constant implies that f is non-anticipatory, by Definition 3.1 a);

ii.2) x is variable. As element of $g(u), x$ satisfies (1) and, by Definition 3.1 b), f is non-anticipatory. ∎

4   Choosing 0 as initial time instant

Notation 4.1 *$\forall d \in \mathbf{R}$, the function $\tau^d : \mathbf{R} \to \mathbf{R}$ is the translation with d, thus for any $x \in S^{(n)}$ we denote by $x \circ \tau^d$ the function*

$$\forall t \in \mathbf{R}, (x \circ \tau^d)(t) = x(t - d).$$

Definition 4.2 *The system f is **time invariant** if $\forall d \in \mathbf{R}, \forall u \in U$,*

$$u \circ \tau^d \in U,$$

$$\forall x \in f(u), x \circ \tau^d \in f(u \circ \tau^d).$$

Notation 4.3 *We use the notation*

$$S_0^{(m)} = \{u | u \in S^{(m)}, \forall t < 0, u(t) = u(-\infty + 0)\}.$$

Theorem 4.4 *We state the following properties relative to some system $\widehat{f} : \widehat{U} \to P^*(S^{(n)}), \widehat{U} \in P^*(S^{(m)}) :$*

i) $\widehat{U} \subset S_0^{(m)}$;

ii) $\forall u \in \widehat{U}, \widehat{f}(u) \subset S_0^{(n)}$;

iii) $\forall d \in \mathbf{R}, \forall u \in \widehat{U}, \forall x,$

$$(x \in \widehat{f}(u) \text{ and } u \circ \tau^d \in \widehat{U}) \Longrightarrow x \circ \tau^d \in \widehat{f}(u \circ \tau^d).$$

a) The time-invariant non-anticipatory system $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ is given. We define the system $\widehat{f} : \widehat{U} \to P^*(S^{(n)})$ by

$$\widehat{U} = \{u | u \in U \cap S_0^{(m)} \text{ and } f(u) \cap S_0^{(n)} \neq \emptyset\},$$

$$\forall u \in \widehat{U}, \widehat{f}(u) = f(u) \cap S_0^{(n)}. \tag{2}$$

The system $\widehat{f}$ fulfills i), ii), iii) and is also non-anticipatory.

b) Let be the system $\widehat{f} : \widehat{U} \to P^*(S^{(n)})$ satisfying the properties i), ii), iii) and non-anticipation. The system $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ defined by

$$U = \{u \circ \tau^d | d \in \mathbf{R}, u \in \widehat{U}\},$$

$$\forall d \in \mathbf{R}, \forall u \in \widehat{U}, f(u \circ \tau^d) = \{x \circ \tau^d | x \in \widehat{f}(u)\}$$

is time invariant and non-anticipatory.

Proof. a) We show that $U \cap S_0^{(m)} \neq \emptyset$. Let be $u \in U$. We have the possibilities:

1) u is constant. Then $u \in S_0^{(m)}$, thus $u \in U \cap S_0^{(m)}$;

2) u is variable.

We denote $d = \min\{t | u(t-0) \neq u(t)\}$. If $d \geq 0$, then $u \in S_0^{(m)}$ and $u \in U \cap S_0^{(m)}$ are true. If $d < 0$, then for any $d' \geq -d$, $u \circ \tau^{d'} \in U$ is true because U is invariant to translations and $u \circ \tau^{d'} \in S_0^{(m)}$ holds true also, making $u \circ \tau^{d'} \in U \cap S_0^{(m)}$ true.

We show that $\widehat{U} \neq \emptyset$. We take some arbitrary $u \in U \cap S_0^{(m)}$. If $f(u) \cap S_0^{(n)} \neq \emptyset$, the property is true, otherwise let be some $x \in f(u)$. The fact that $x \notin S_0^{(n)}$ shows that it is variable and if we denote by $d = \min\{t | x(t-0) \neq x(t)\}$, we have $d < 0$. Remark that for all $d' \geq -d$, $u \circ \tau^{d'} \in U$, $u \circ \tau^{d'} \in S_0^{(m)}$, $x \circ \tau^{d'} \in f(u \circ \tau^{d'})$ and $x \circ \tau^{d'} \in S_0^{(n)}$ take place. In other words $u \circ \tau^{d'} \in \widehat{U}$.

This shows that $\widehat{f}$ is well defined, in the sense that $\widehat{U} \neq \emptyset$ and $\forall u \in \widehat{U}, \widehat{f}(u) \neq \emptyset$. Moreover, i) and ii) are obviously satisfied.

We show now the truth of iii). We take $d \in \mathbf{R}$, $u \in \widehat{U}$, x arbitrary with $x \in \widehat{f}(u)$ and $u \circ \tau^d \in \widehat{U}$ true. We have the possibilities:

j) x is constant. Then $x \circ \tau^d = x$ is constant and $x \circ \tau^d \in S_0^{(n)}$;

jj) x is variable. Because $x \in f(u)$, from the non-anticipation of f we have that u is variable and

$$0 \leq \min\{t | (u \circ \tau^d)(t-0) \neq (u \circ \tau^d)(t)\} \leq \min\{t | (x \circ \tau^d)(t-0) \neq (x \circ \tau^d)(t)\},$$

showing that $x \circ \tau^d \in S_0^{(n)}$.

In both cases j), jj), $x \in \widehat{f}(u)$ has implied $x \in f(u)$ and, furthermore, $x \circ \tau^d \in f(u \circ \tau^d)$ from the time invariance of f and, eventually, $x \circ \tau^d \in \widehat{f}(u \circ \tau^d)$ $(= f(u \circ \tau^d) \cap S_0^{(n)})$.

Because $\widehat{f} \subset f$, the non-anticipation of $\widehat{f}$ is a consequence of Lemma 3.5.

b) We show that f is well defined in the sense that if $d, d' \in \mathbf{R}$ and $u, v \in \widehat{U}$ satisfy $u \circ \tau^d = v \circ \tau^{d'}$, we get $f(u \circ \tau^d) = f(v \circ \tau^{d'})$. Let be $x \circ \tau^d \in f(u \circ \tau^d)$. We infer that $x \in \widehat{f}(u)$ and $v = u \circ \tau^{d-d'} \in \widehat{U}$. From iii) we have that $x \circ \tau^{d-d'} \in \widehat{f}(v)$, i.e. $x \circ \tau^d = x \circ \tau^{d-d'} \circ \tau^{d'} \in f(v \circ \tau^{d'})$. We have obtained that $f(u \circ \tau^d) \subset f(v \circ \tau^{d'})$ and the inverse inclusion is shown similarly.

We show that U is invariant to translations. Let be $v \in U$. Then there are some $d \in \mathbf{R}$ and $u \in \widehat{U}$ such that $v = u \circ \tau^d$. For an arbitrary $d' \in \mathbf{R}$, as $v \circ \tau^{d'} = u \circ \tau^{d+d'}$, we infer $v \circ \tau^{d'} \in U$.

5

We show that f is time invariant. Let be $v \in U$ and $y \in f(v)$, meaning the existence of $u \in \widehat{U}$ and $d \in \mathbf{R}$ with $v = u \circ \tau^d$. We get $y \in f(u \circ \tau^d) = \{x \circ \tau^d | x \in \widehat{f}(u)\}$. In other words $\exists x, y = x \circ \tau^d$ and $x \in \widehat{f}(u)$. We take an arbitrary $d' \in \mathbf{R}$ for which $y \circ \tau^{d'} = x \circ \tau^{d+d'}$, $y \circ \tau^{d'} \in \{x \circ \tau^{d+d'} | x \in \widehat{f}(u)\} = f(u \circ \tau^{d+d'}) = f(v \circ \tau^{d'})$.

We show now that f is non-anticipatory. Let us take, like previously, $v \in U$ and $y \in f(v)$, for which there are $u \in \widehat{U}, x \in \widehat{f}(u)$ and $d \in \mathbf{R}$ such that $v = u \circ \tau^d$ and $y = x \circ \tau^d$. We have the possibilities:

I) y is constant. Then f is non-anticipatory;

II) y is variable. Then $x \in \widehat{f}(u)$ is variable and the hypothesis concerning the non-anticipation of $\widehat{f}$ states that u is variable and

$$\min\{t | u(t-0) \neq u(t)\} \leq \min\{t | x(t-0) \neq x(t)\}.$$

We obtain

$$\min\{t | v(t-0) \neq v(t)\} = \min\{t | (u \circ \tau^d)(t-0) \neq (u \circ \tau^d)(t)\} =$$

$$= d + \min\{t | u(t-0) \neq u(t)\} \leq d + \min\{t | x(t-0) \neq x(t)\} =$$

$$= \min\{t | (x \circ \tau^d)(t-0) \neq (x \circ \tau^d)(t)\} = \min\{t | y(t-0) \neq y(t)\}.$$

∎

Remark 4.5 $S_0^{(m)}$ *consists in these signals* $u \in S^{(m)}$ *that accept the 'initial time instant'* t_0 *be 0. Items i), ii) of Theorem 4.4 mean that the inputs and the states of* $\widehat{f}$ *accept the initial time instant be* 0 *and item iii) of that theorem represents time invariance adapted to the situation when* $\widehat{U}$ *is not closed to translations (any* $\widehat{U} \subset S_0^{(m)}$ *that contains non-constant signals is not invariant to translations).*

The possibility of choosing 0 *as initial time instant simplifies a little the study of the asynchronous systems.*

5 Non-Anticipation, the Second Definition

Definition 5.1 *Let the system* $f : U \to P^*(S^{(n)})$ *be given,* $U \in P^*(S^{(m)})$. *It is called* **non-anticipatory** *if* $\forall t \in \mathbf{R}$, $\forall u \in U$, $\forall v \in U$,

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \Longrightarrow \{x_{|(-\infty,t]} | x \in f(u)\} = \{y_{|(-\infty,t]} | y \in f(v)\}.$$

Remark 5.2 *Definition 5.1 states that the history of all the possible states until the present moment, including the present depends only on the history of the input and it does not depend on the present and the future values of the input. The definition means that* $\forall t \in \mathbf{R}$ *a function* f_t *exists that associates* $\forall u \in U$ *to* $u_{|(-\infty,t)}$ *the set*

$$f_t(u_{|(-\infty,t)}) = \{x_{|(-\infty,t]} | x \in f(u)\}.$$

Definition 5.1 represents a perspective of non-anticipation, other than the previous one and the two properties are logically independent.

Example 5.3 *The deterministic system $f : S^{(m)} \to S$,*

$$\forall u \in S^{(m)}, f(u) = \chi_{[0,1)} \oplus (u_1 \circ \tau^1) \cdot \chi_{[1,\infty)}$$

is non-anticipatory in the sense of Definition 5.1. The system f is anticipatory in the sense of Definition 3.1 because for $u_1 = \chi_{[2,\infty)}$, $u_2 = ... = u_m = 0$ the contradiction $\min\{t|u(t-0) \neq u(t)\} = 2 > 0 = \min\{t|x(t-0) \neq x(t)\}$ is obtained.

Example 5.4 *The deterministic system $f : S \to S$,*

$$\forall u \in S, f(u) = \left\{ \begin{array}{l} 1, if\ u = \chi_{[0,\infty)} \\ u,\ otherwise \end{array} \right. ,$$

is anticipatory in the sense of Definition 5.1 because for $t = 1$, $u = \chi_{[0,\infty)}$, $v = \chi_{[0,2)}$ we have $u_{|(-\infty,1)} = v_{|(-\infty,1)}$ but $1_{|(-\infty,1]} \neq \chi_{[0,2)_{|(-\infty,1]}}$. However it is non-anticipatory in the sense of Definition 3.1.

Example 5.5 *The deterministic system $f : S \to S$*

$$\forall u \in S, f(u) = \left\{ \begin{array}{l} 1, if\ u = \chi_{[0,\infty)} \\ u \circ \tau^{-1},\ otherwise \end{array} \right.$$

is anticipatory in the sense of both Definitions 3.1 and 5.1.

Example 5.6 *The deterministic system*

$$Dx(t) = (x(t-0) \oplus u(t-0)) \cdot \overline{\bigcup_{\xi \in (t-d,t)} Du(\xi)}$$

$u, x \in S, d > 0$ is non-anticipatory in the sense of both Definitions 3.1, 5.1. The idea expressed by such an equation is: x switches ($Dx(t) = 1$) at these time instants when u has indicated the necessity of such a switch ($x(t-0) \oplus u(t-0) = 1$) for d time units ($u_{|[t-d,t)}$ is the constant function, with null derivative in the interval $(t-d,t)$). This equation models the delay circuit.

6 Other Definitions of Non-Anticipation. Non-Anticipation*

Definition 6.1 *Let be the system $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$. It is called **non-anticipatory** if it satisfies one of the following conditions, called conditions of non-anticipation:*
   i) $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U,$

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\};$$

   ii) $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U, \exists d > 0,$

$$u_{|[t-d,t)} = v_{|[t-d,t)} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\};$$

iii) $\forall t \in \mathbf{R}, \exists d > 0, \forall u \in U, \forall v \in U,$

$$u_{|[t-d,t)} = v_{|[t-d,t)} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\};$$

iv) $\exists d > 0, \forall t \in \mathbf{R}, \forall u \in U, \forall v \in U,$

$$u_{|[t-d,t)} = v_{|[t-d,t)} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\};$$

v) $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U,$

$$u_{|(-\infty,t]} = v_{|(-\infty,t]} \Longrightarrow \{x_{|(-\infty,t]}|x \in f(u)\} = \{y_{|(-\infty,t]}|y \in f(v)\};$$

vi) $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U,$

$$u_{|(-\infty,t]} = v_{|(-\infty,t]} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\};$$

vii) $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U, \exists d, \exists d', 0 \le d \le d'$ *and*

$$u_{|[t-d',t-d]} = v_{|[t-d',t-d]} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\};$$

viii) $\forall t \in \mathbf{R}, \exists d, \exists d', 0 \le d \le d'$ *and* $\forall u \in U, \forall v \in U,$

$$u_{|[t-d',t-d]} = v_{|[t-d',t-d]} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\};$$

ix) $\exists d, \exists d', 0 \le d \le d'$ *and* $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U,$

$$u_{|[t-d',t-d]} = v_{|[t-d',t-d]} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\}.$$

Theorem 6.2 *If $f : U \to S^{(n)}$ is a deterministic system, then Definition 6.1 v) and Definition 6.1 vi) are equivalent. We have that Definition 5.1 and Definition 6.1 i) are equivalent in this case too.*

Proof. We prove the first statement. Because v)$\Longrightarrow$vi) is obvious, we prove vi)$\Longrightarrow$v). Let us suppose against all reason that v) is not true, i.e. $\exists t \in \mathbf{R}, \exists u \in U, \exists v \in U, u_{|(-\infty,t]} = v_{|(-\infty,t]}$ and $f(u)_{|(-\infty,t]} \ne f(v)_{|(-\infty,t]}$. This means the existence of $t' \le t$ such that $u_{|(-\infty,t']} = v_{|(-\infty,t']}$ and $f(u)(t') \ne f(v)(t')$, contradiction with vi). ∎

Remark 6.3 *In Definition 6.1, all of i),...,ix) express the same idea like Definition 5.1, namely that the present depends on the past only and it is independent on the future. The implications are:*

$$
\begin{array}{ccccccccc}
iv) & \Longrightarrow & iii) & \Longrightarrow & ii) & \Longrightarrow & i) & \Longleftarrow & Definition\ 5.1 \\
 & & & & & & \Downarrow & & \Downarrow \\
ix) & \Longrightarrow & viii) & \Longrightarrow & vii) & \Longrightarrow & vi) & \Longleftarrow & v)
\end{array}
$$

In ii),...,iv), vii),...,ix) the boundness of the memory occurs: these are systems whose states do not depend on all the input segment $u_{|(-\infty,t)}$, but on the last d time units $u_{|[t-d,t)}$ only and similarly for $u_{|(-\infty,t]}$ and $u_{|[t-d',t-d]}$.

Now have a look at the non-anticipation property iv). We note that if $d > 0$ is a number for which it is fulfilled, then any number $d' \geq d$ fulfills it also: $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,

$$u_{|[t-d',t)} = v_{|[t-d',t)} \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\}.$$

Our problem is whether, for a system f, the set of those d satisfying implication iv) is bounded from below by some $d'' > 0$, because we have a non-anticipation property

$\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,

$$u(t-0) = v(t-0) \Longrightarrow \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\}$$

also, like in the example

$$u(t-0) \cdot x(t) = 0$$

where $u, x \in S$. If this lower bound exists, we obtain a new shading of that concept of non-anticipation. The problem of the existence of such bounds is, in principle, the same if d is variable like in ii), iii) or if instead of one parameter d we have two parameters d, d' and two bounds, like in vii), viii), ix).

Remark that the reasoning of Theorem 6.2 is impossible to use if f is non-deterministic. We suppose, for this, that the system $f : S \to P^*(S)$ satisfies $f(0) = \{0, 1\}$, $f(\chi_{[2,\infty)}) = \{\chi_{(-\infty,0)}, \chi_{[0,\infty)}\}$, where $0, 1 \in S$ are the constant functions. We have $\forall t \in [0, 2)$,

$$0_{|(-\infty,t]} = \chi_{[2,\infty)|(-\infty,t]} \ \ and$$

$$and \ \{0_{|(-\infty,t]}, 1_{|(-\infty,t]}\} \neq \{\chi_{(-\infty,0)|(-\infty,t]}, \chi_{[0,\infty)|(-\infty,t]}\} \ \ and$$

$$and \ \{x(t)|x \in f(0)\} = \{0, 1\} = \{y(t)|y \in f(\chi_{[2,\infty)})\}.$$

Example 6.4 *The system $I_d : S \to S$ called the 'pure delay model' of the delay circuit, defined by $\forall u \in S, x(t) = I_d(u)(t) = u(t - d)$, satisfies for $d > 0$ all the non-anticipation properties i),...,ix) from Definition 6.1.*

Example 6.5 *Let the system $f : S \to P^*(S)$ (version of the 'bounded delay model' of the delay circuit) be defined by the inequalities*

$$\bigcap_{\xi \in [t-d_r,t)} u(\xi) \leq x(t) \leq \bigcup_{\xi \in [t-d_f,t)} u(\xi),$$

where $d_r > 0, d_f > 0$. It satisfies all the non-anticipation properties i),...,ix) from Definition 6.1.

Example 6.6 *The system $f : S \to P^*(S)$ (version of the 'bounded delay model' of the delay circuit) described by the inequalities*

$$\bigcap_{\xi \in [t-d',t-d]} u(\xi) \leq x(t) \leq \bigcup_{\xi \in [t-d',t-d]} u(\xi),$$

where $0 \leq d \leq d'$ satisfies the non-anticipation properties v),...,ix) from Definition 6.1.

Example 6.7 *The system $f : S \to P^*(S)$ defined by*

$$\int_{-\infty}^{t} Du \leq x(t)$$

where

$$\int_{-\infty}^{t} Du = \begin{cases} 1, & if \ |suppDu \cap (-\infty, t]| \ is \ odd \\ 0, & if \ |suppDu \cap (-\infty, t]| \ is \ even \end{cases}$$

satisfies the non-anticipation properties v), vi) of Definition 6.1. We have denoted by $|suppDu \cap (-\infty, t]|$ the number of elements of the finite set $\{\xi | \xi \in \mathbf{R}, Du(\xi) = 1\} \cap (-\infty, t]$ and we have supposed that 0 is an even number.

Example 6.8 *Denote by $\varphi : S^{(m)} \to [0, \infty)$ the function $\forall u \in S^{(m)}$,*

$$\varphi(u) = \begin{cases} 0, & if \ u \ is \ constant \\ \max\{-\min\{t | u(t-0) \neq u(t)\}, \min\{t | u(t-0) \neq u(t)\}\}, & if \ u \ is \ variable \end{cases}$$

The deterministic system

$$x(t) = \bigcap_{\xi \in [t - 2\varphi(u), t - \varphi(u)]} u(\xi),$$

$u, x \in S$, satisfies the non-anticipation property vii) of Definition 6.1.

Definition 6.9 *The system f is called **non-anticipatory*** if it satisfies one of the following conditions, called conditions of non-anticipation* :*
i) $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,

$$(u_{|[t,\infty)} = v_{|[t,\infty)} \ and \ \{x(t) | x \in f(u)\} = \{y(t) | y \in f(v)\}) \implies$$

$$\implies \{x_{|[t,\infty)} | x \in f(u)\} = \{y_{|[t,\infty)} | y \in f(v)\};$$

ii) $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,

$$u_{|[t,\infty)} = v_{|[t,\infty)} \implies \exists t' \in \mathbf{R}, \{x_{|[t',\infty)} | x \in f(u)\} = \{y_{|[t',\infty)} | y \in f(v)\};$$

iii) $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,

$$(u_{|[t,\infty)} = v_{|[t,\infty)} \ and \ \{x_{|(-\infty,t]} | x \in f(u)\} = \{y_{|(-\infty,t]} | y \in f(v)\}) \implies$$

$$\implies \exists t' \in \mathbf{R}, \{x_{|[t',\infty)} | x \in f(u)\} = \{y_{|[t',\infty)} | y \in f(v)\}.$$

Remark 6.10 *We remark that property i) resembles somehow with fixing the initial conditions in a differential equation ($\{x(t) | x \in f(u)\} = \{y(t) | y \in f(v)\}$). The consequence is that the solution is unique ($\{x_{|[t,\infty)} | x \in f(u)\} = \{y_{|[t,\infty)} | y \in f(v)\}$) under an arbitrary given input ($u_{|[t,\infty)} = v_{|[t,\infty)}$).*

The reader is invited to write other similar properties of non-anticipation and non-anticipation*.

7 The Transfers of the Non-Anticipatory Systems

Theorem 7.1 *Let the system f satisfy the conditions:*

a) U is closed under 'concatenation' $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,

$$u \cdot \chi_{(-\infty,t)} \oplus v \cdot \chi_{[t,\infty)} \in U;$$

b) non-anticipation $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \Longrightarrow \{x_{|(-\infty,t]}|x \in f(u)\} = \{y_{|(-\infty,t]}|y \in f(v)\};$$

c) non-anticipation $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U$,*

$$(u_{|[t,\infty)} = v_{|[t,\infty)} \ and \ \{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\}) \Longrightarrow$$

$$\Longrightarrow \{x_{|[t,\infty)}|x \in f(u)\} = \{y_{|[t,\infty)}|y \in f(v)\};$$

d) time invariance $\forall d \in \mathbf{R}, \forall u \in U$,

$$u \circ \tau^d \in U,$$

$$\forall x \in f(u), x \circ \tau^d \in f(u \circ \tau^d);$$

e) $t_1, t_2 \in \mathbf{R}, u^0, u^1 \in U$ and $\mu, \mu', \mu'' \in \mathbf{B}^n$ are given such that

$$\forall x \in f(u^0), \exists t_0 < t_1, x(t_0) = \mu, \tag{3}$$

$$\forall x \in f(u^0), x(t_1) = \mu', \tag{4}$$

$$\forall x' \in f(u^1), x'(t_2) = \mu', \tag{5}$$

$$\forall x' \in f(u^1), \exists t_3 > t_2, x'(t_3) = \mu''. \tag{6}$$

Put $d = t_1 - t_2$. Then $\widetilde{u} \in U$ defined as

$$\widetilde{u} = u^0 \cdot \chi_{(-\infty,t_1)} \oplus (u^1 \circ \tau^d) \cdot \chi_{[t_1,\infty)}, \tag{7}$$

satisfies

$$\forall \widetilde{x} \in f(\widetilde{u}), \exists t_0 < t_1, \widetilde{x}(t_0) = \mu, \tag{8}$$

$$\forall \widetilde{x} \in f(\widetilde{u}), \exists t'_3 > t_1, \widetilde{x}(t'_3) = \mu''. \tag{9}$$

Proof. $\widetilde{u}$ belongs to U indeed, because of a) and d). We remark that we have

$$\widetilde{u}_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)}. \tag{10}$$

From (10) and b) we infer

$$\{\widetilde{x}_{|(-\infty,t_1]}|\widetilde{x} \in f(\widetilde{u})\} = \{x_{|(-\infty,t_1]}|x \in f(u^0)\} \tag{11}$$

and if, in addition, we take into account (3), (4), then we get the truth of (8) and of

$$\forall \widetilde{x} \in f(\widetilde{u}), \widetilde{x}(t_1) = \mu'. \tag{12}$$

Let be now some arbitrary $x'' \in f(u^1 \circ \tau^d)$. From d) we obtain the existence of $x' \in f(u^1)$, such that $x'' = x' \circ \tau^d$ (namely $x' = x'' \circ \tau^{-d}$) and we have $x''(t_1) = (x' \circ \tau^d)(t_1) = x'(t_2) = \mu'$ (we have taken into account (5)) thus

$$\forall x'' \in f(u^1 \circ \tau^d), x''(t_1) = \mu' \tag{13}$$

and, similarly,

$$\forall x'' \in f(u^1 \circ \tau^d), \exists t'_3 > t_1, x''(t'_3) = \mu''. \tag{14}$$

We see that

$$\widetilde{u}_{|[t_1,\infty)} = (u^1 \circ \tau^d)_{|[t_1,\infty)}. \tag{15}$$

The hypothesis of c) is fulfilled by t_1, $\widetilde{u}$ and $u^1 \circ \tau^d$, as follows from (12), (13) and (15). The conclusion of c) expresses the fact that

$$\{\widetilde{x}_{|[t_1,\infty)} | \widetilde{x} \in f(\widetilde{u})\} = \{x''_{|[t_1,\infty)} | x'' \in f(u^1 \circ \tau^d)\} \tag{16}$$

and, by (14), we get the truth of (9). ∎

Remark 7.2 *The relations (3), (6), (8), (9) show the asynchronous access (weaker, the time instant when the access happens depends on x) of the states of f to the values μ, μ'' and the relations (4), (5) represent the synchronous access (stronger, the time instant when the access happens is the same for all x; these two accesses must match) of the states of f to the value μ'. The theorem states that if $f(u^0)$ transfers μ in μ' and $f(u^1)$ transfers μ' in μ'', then $f(\widetilde{u})$ transfers μ in μ''.*

Several versions of this theorem are obtained if we take $t_1 = t_2$ (then time invariance disappears from the hypothesis), if we have in the hypothesis countable many transfers (instead of two; these transfers must have synchronous accesses), if we state in the hypothesis a controllability/accessibility request etc.

8   The Fundamental Mode

Definition 8.1 *Consider the system f supposed to be non-anticipatory (Definition 5.1) and let $u \in U$ be a fixed input. If there are $(t_k) \in Seq$, $(u^k) \in U$ and $(\mu^k) \in \mathbf{B}^n$ such that*

$$\forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0 \text{ and } x_{|[t_1,\infty)} = \mu^1,$$

$$u_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)}, \ u_{|(-\infty,t_2)} = u^1_{|(-\infty,t_2)}, \ u_{|(-\infty,t_3)} = u^2_{|(-\infty,t_3)}, \dots$$

$$\forall x \in f(u^1), x_{|[t_2,\infty)} = \mu^2, \ \forall x \in f(u^2), x_{|[t_3,\infty)} = \mu^3, \ \forall x \in f(u^3), x_{|[t_4,\infty)} = \mu^4, \dots$$

*then the input u is called a **fundamental (operating) mode (of f)**.*

Remark 8.2 *The evolution of f under the fundamental mode u may be inter-preted as the discrete time evolution of a deterministic system of the form*

$$\mu^0 = x(0) \overset{u^0}{\to} \mu^1 = x(1) \overset{u^1}{\to} ... \overset{u^k}{\to} \mu^{k+1} = x(k+1) \overset{u^{k+1}}{\to} ...$$

To be remarked the appearance of the next state partial function $\forall k \in \mathbf{N}, \mathbf{B}^n \times U \ni (\mu^k, u^k) \to \mu^{k+1} \in \mathbf{B}^n$. If $\exists k \in \mathbf{N}$ such that $u^k = u^{k+1} = ...$ and $\mu^{k+1} = \mu^{k+2} = ...$, then the evolution may be considered to be given by a finite sequence

$$\mu^0 = x(0) \overset{u^0}{\to} \mu^1 = x(1) \overset{u^1}{\to} ... \overset{u^k}{\to} \mu^{k+1} = x(k+1).$$

Example 8.3 *We get back to the system f from the Example 6.5, that fulfills the non-anticipation property from Definition 5.1. We suppose that the sequence $(t_k) \in Seq$ satisfies $\forall k \in \mathbf{N}$,*

$$t_{2k+1} \geq t_{2k} + d_r, \ t_{2k+2} \geq t_{2k+1} + d_f$$

and let be the sequences $(u^k) \in S, (\mu^k) \in \mathbf{B}$,

$$u^0(t) = \chi_{[t_0,\infty)}(t), u^2(t) = \chi_{[t_0,t_1)}(t) \oplus \chi_{[t_2,\infty)}(t),$$

$$u^4(t) = \chi_{[t_0,t_1)}(t) \oplus \chi_{[t_2,t_3)}(t) \oplus \chi_{[t_4,\infty)}(t), ...$$

$$u^1(t) = \chi_{[t_0,t_1)}(t), u^3(t) = \chi_{[t_0,t_1)}(t) \oplus \chi_{[t_2,t_3)}(t), ...$$

$$\mu^0 = \mu^2 = \mu^4 = ... = 0, \mu^1 = \mu^3 = \mu^5 = ... = 1.$$

For

$$u(t) = \chi_{[t_0,t_1)}(t) \oplus \chi_{[t_2,t_3)}(t) \oplus \chi_{[t_4,t_5)}(t) \oplus ...$$

we have

$$\forall x \in f(u^0), x(t) = x(t) \cdot \chi_{(t_0,t_0+d_r)}(t) \oplus \chi_{[t_0+d_r,\infty)}(t),$$

$$\forall x \in f(u^1), x(t) = x(t) \cdot \chi_{(t_0,t_0+d_r)}(t) \oplus \chi_{[t_0+d_r,t_1]}(t) \oplus x(t) \cdot \chi_{(t_1,t_1+d_f)}(t),$$

$$\forall x \in f(u^2), x(t) = x(t) \cdot \chi_{(t_0,t_0+d_r)}(t) \oplus \chi_{[t_0+d_r,t_1]}(t) \oplus x(t) \cdot \chi_{(t_1,t_1+d_f)}(t) \oplus$$

$$\oplus x(t) \cdot \chi_{(t_2,t_2+d_r)}(t) \oplus \chi_{[t_2+d_r,\infty)}(t),$$

$$\forall x \in f(u^3), x(t) = x(t) \cdot \chi_{(t_0,t_0+d_r)}(t) \oplus \chi_{[t_0+d_r,t_1]}(t) \oplus x(t) \cdot \chi_{(t_1,t_1+d_f)}(t) \oplus$$

$$\oplus x(t) \cdot \chi_{(t_2,t_2+d_r)}(t) \oplus \chi_{[t_2+d_r,t_3]}(t) \oplus x(t) \cdot \chi_{(t_3,t_3+d_f)}(t)$$

$$...$$

We conclude that u is a fundamental mode of f:

$$\forall x \in f(u^0), x_{|(-\infty,t_0)} = 0 \ and \ x_{|[t_1,\infty)} = 1,$$

$$u_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)}, \ u_{|(-\infty,t_2)} = u^1_{|(-\infty,t_2)}, \ u_{|(-\infty,t_3)} = u^2_{|(-\infty,t_3)}, ...$$

$$\forall x \in f(u^1), x_{|[t_2,\infty)} = 0, \ \forall x \in f(u^2), x_{|[t_3,\infty)} = 1, \ \forall x \in f(u^3), x_{|[t_4,\infty)} = 0, ...$$

9 Accessibility vs fundamental mode

Theorem 9.1 *Let be the non-anticipatory system (Definition 5.1) $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ and we suppose that the following requirements are fulfilled:*

a) for any $(t_k) \in Seq$ and any $(u^k) \in U$, we have $u^0 \cdot \chi_{(-\infty,t_0)} \oplus u^1 \cdot \chi_{[t_0,t_1)} \oplus u^2 \cdot \chi_{[t_1,t_2)} \oplus ... \in U$;

b) f has race-free initial states and bounded initial time, i.e.

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t)} = \mu;$$

c) any vector from $\mathbf{B}^n$ is the common final value of the states under an input having arbitrary initial segment

$$\forall \mu \in \mathbf{B}^n, \forall u \in U, \forall t \in \mathbf{R}, \exists v \in U, \exists t' > t,$$

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \text{ and } \forall y \in f(v), y_{|[t',\infty)} = \mu.$$

Then there is some $\mu^0 \in \mathbf{B}^n$ such that for any sequence $\mu^k \in \mathbf{B}^n, k \geq 1$ of binary vectors, there are the sequences $(t_k) \in Seq, u^k \in U, k \in \mathbf{N}$ and an input $\widetilde{u} \in U$ such that

$$\forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0 \text{ and } x_{|[t_1,\infty)} = \mu^1,$$

$$\widetilde{u}_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)}, \ \widetilde{u}_{|(-\infty,t_2)} = u^1_{|(-\infty,t_2)}, \ \widetilde{u}_{|(-\infty,t_3)} = u^2_{|(-\infty,t_3)}, ...$$

$$\forall x \in f(u^1), x_{|[t_2,\infty)} = \mu^2, \ \forall x \in f(u^2), x_{|[t_3,\infty)} = \mu^3, \ \forall x \in f(u^3), x_{|[t_4,\infty)} = \mu^4, ...$$

Proof. Let $v^0 \in U$ be an arbitrary input. From b) we get the existence of $\mu^0 \in \mathbf{B}^n$ and $t_0 \in \mathbf{R}$ depending on v^0, such that

$$\forall x \in f(v^0), x_{|(-\infty,t_0)} = \mu^0. \tag{17}$$

Let us fix the sequence $\mu^k \in \mathbf{B}^n, k \geq 1$ and an arbitrary number $\delta > 0$. At this moment the property c) implies the existence of $u^0 \in U$ and $t_1 > t_0 + \delta$ such that

$$v^0_{|(-\infty,t_0)} = u^0_{|(-\infty,t_0)} \text{ and } \forall x \in f(u^0), x_{|[t_1,\infty)} = \mu^1,$$

of $u^1 \in U$ and $t_2 > t_1 + \delta$ such that

$$u^0_{|(-\infty,t_1)} = u^1_{|(-\infty,t_1)} \text{ and } \forall x \in f(u^1), x_{|[t_2,\infty)} = \mu^2,$$

of $u^2 \in U$ and $t_3 > t_2 + \delta$ such that

$$u^1_{|(-\infty,t_2)} = u^2_{|(-\infty,t_2)} \text{ and } \forall x \in f(u^2), x_{|[t_3,\infty)} = \mu^3,$$

$$...$$

The way that (t_k) was constructed guarantees the fact that this sequence belongs to Seq. Thus, by a), the input $\widetilde{u}$ defined as

$$\widetilde{u} = u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,t_2)} \oplus u^2 \cdot \chi_{[t_2,t_3)} \oplus ...$$

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belongs to U. We have

$$\widetilde{u}_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)}, \ \widetilde{u}_{|(-\infty,t_2)} = u^1_{|(-\infty,t_2)}, \ \widetilde{u}_{|(-\infty,t_3)} = u^2_{|(-\infty,t_3)}, ...$$

∎

Remark 9.2 *The request b) of the theorem, that the initial states μ are race-free and the initial time t is bounded shows the order of the four quantifiers $\forall u, \exists \mu, \exists t, \forall x$, that is the two existential quantifiers are in the middle (the total number of possibilities is $3 \times 3 = 9$).*

The key request in the hypothesis of the theorem is however that of controllability and accessibility from item c). We make the terminological remark that due to the frequent confusion that exists in the literature generated by the concepts of controllability and accessibility, we prefer to call all such requests 'accessibility'.

The theorem states that, in certain conditions (which are fulfilled by the system f from Example 6.5 and by many other systems), the initial state μ^0 exists such that for any sequence $\mu^k \in \mathbf{B}^n, k \geq 1$, the fundamental mode $\widetilde{u} \in U$ exists, making $f(\widetilde{u})$ access the values $\mu^0, \mu^1, \mu^2, ...$ synchronously, in this order.

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