GRASSMANN GEOMETRIES AND INTEGRABLE SYSTEMS

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Abstract. We describe how the loop group maps corresponding to special submanifolds associated to integrable systems may be thought of as certain Grassmann submanifolds of infinite dimensional homogeneous spaces. In general, the associated families of special submanifolds are certain Grassmann submanifolds. An example is given from the recent article [2].

1. Introduction

This article discusses some of the ideas in the article [2], where solutions to a certain loop group problem were studied. The emphasis here is on the geometric interpretation of the solutions, rather than the techniques for producing solutions.

In 1996, Ferus and Pedit [5] defined an integrable system involving a 3-involution loop group, solutions of which are isometric immersions between space forms of different non-zero sectional curvature. They modified the Adler-Kostant-Symes (AKS) theory (described in [4]) to show how to produce many solutions by solving commuting ODEs on a finite dimensional vector space.

The present author later studied this system in [1] and [3]: it had several interesting properties, including a relationship with pluriharmonic maps.

Goal here: generalize the system to arbitrary commuting involutions of any Lie group and identify the associated special submanifolds.

Results: briefly, we obtained:

- Generalizations, to all reflective submanifolds, of results concerning isometric immersions of space forms;
- In case of previous results, new proofs;
- And other new special submanifolds as integrable systems.

1.1. Motivation.

Other special submanifolds that have been studied with loop groups, (e.g. harmonic maps into symmetric spaces, CMC surfaces, special Lagrangian surfaces etc), are associated to loop groups with only

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two involutions. Therefore, it seemed that a system in a loop group with three involutions might have some interesting properties peculiar to this situation.

One such property, studied in [1], is as follows: solutions to three *distinct* problems are obtained from the *same* loop group map, by evaluating the map within different ranges of the loop parameter λ. This amounts to a kind of Lawson correspondence between solutions of these problems, and shows that the problems of obtaining complete immersions are equivalent for the three cases.

Parameter range	Induced sectional curvature	Target space
$\lambda \in i\mathbf{R}^*$	$c_\lambda \in (-\infty, 0)$	S^{m+k}
$\lambda \in \mathbf{R}^*$	$c_\lambda \in [-1, 0)$	H_k^{m+k}
$\lambda \in S^1$	$c_\lambda \in (-\infty, -1]$	H^{m+k}

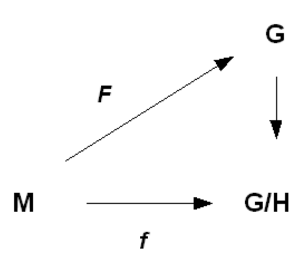
The table shows three different constant curvature Riemannian submanifolds of three different space forms obtained by evaluating the *same* loop group map for values of the spectral parameter in $\mathbf{R}$, $i\mathbf{R}$ and S^1 [1].

2. Special Submanifolds and Loop Groups

We first present an outline of how certain special submanifolds are associated to maps into loop groups.

2.1. **Moving Frame Method.** The basic concept of the moving frame method is encapsulated as follows:

- Given $f : M \to G/H$, an immersed submanifold of a homogeneous space.
- Lift, $F : M \to G$, a **frame** for f.
- **Idea:** Choose F which is adapted in some way to the geometry of f.



Example: We illustrate this with a simple example.
Special submanifold: a flat immersion,

$$f : M = \mathbf{R}^2 \to S^3,$$

Adapted frame: $F : \mathbf{R}^2 \to SO(4)$,

$$F := \begin{bmatrix} e_1 & e_2 & n & f \end{bmatrix},$$

where e_i are an orthonormal basis for the tangent space to the immersion.

2.2. **The Maurer-Cartan Form.** Given a frame $F : M \to G$, for $f : M \to G/H$, the *Maurer-Cartan form*, $\alpha = F^{-1}\mathrm{d}F \in \mathfrak{g} \otimes \Omega(M)$, is the pull-back to M of the Maurer-Cartan form of G. It is necessary that α satisfies the *Maurer-Cartan equation*

$$\mathrm{d}\alpha + \alpha \wedge \alpha = 0. \tag{2.1}$$

Conversely, if any $\alpha \in \mathfrak{g}\otimes\Omega(M)$, satisfies (2.1) then it is a basic fact from the theory of Lie groups that we can integrate α to obtain a map $F : M \to G$, whose Maurer-Cartan form is α. The map F is determined up to an initial condition $F_0 \in G$. Changing this initial condition amounts to left multiplication by an element of G, which is to say an isometry of the homogeneous space G/H, and consequently we have the
Fundamental point: α contains all geometric information about f.

Example: Returning to our previous example of flat surfaces in S^3, we compute the Maurer-Cartan form of $F := \begin{bmatrix} e_1 & e_2 & n & f \end{bmatrix}$,

$$\begin{aligned} \alpha = F^{-1}\mathrm{d}F &= \begin{bmatrix} e_1^T \\ e_2^T \\ n^T \\ f^T \end{bmatrix} \cdot \begin{bmatrix} \mathrm{d}e_1 & \mathrm{d}e_2 & \mathrm{d}n & \mathrm{d}f \end{bmatrix} \\ &= \begin{bmatrix} \omega & \beta & \theta \\ -\beta^t & 0 & 0 \\ -\theta^t & 0 & 0 \end{bmatrix}, \end{aligned}$$

where the 2×2 matrix ω is the connection on the tangent bundle for f, the 2×1 vector β is the second fundamental form, and the 2×1 vector θ is the coframe.

Computing the Maurer-Cartan equation $\mathrm{d}\alpha + \alpha \wedge \alpha = 0$, the three components above give the following three equations:

$$\mathrm{d}\omega + \omega \wedge \omega - \beta \wedge \beta^t - \theta \wedge \theta^t = 0, \tag{2.2}$$

$$\mathrm{d}\beta + \omega \wedge \beta = 0, \tag{2.3}$$

$$\mathrm{d}\theta + \omega \wedge \theta = 0. \tag{2.4}$$

The assumption that the induced metric is flat is given by a further equation,
Flatness:

$$\mathrm{d}\omega + \omega \wedge \omega = 0.$$

2.3. **Parameterised Families of Frames.** Now suppose we introduce a complex parameter λ in our example by setting:

$$\alpha_\lambda = \begin{bmatrix} \omega & \lambda\beta & \lambda\theta \\ -\lambda\beta^t & 0 & 0 \\ -\lambda\theta^t & 0 & 0 \end{bmatrix} = a_0 + a_1\lambda.$$

Then $\mathrm{d}\alpha_\lambda + \alpha_\lambda \wedge \alpha_\lambda = 0 \Leftrightarrow \mathrm{d}\omega + \omega \wedge \omega - \lambda^2(\beta \wedge \beta^t + \theta \wedge \theta^t) = 0$, plus (2.3) and (2.4). It follows that we have the following equivalence:

$$\mathrm{d}\alpha_\lambda + \alpha_\lambda \wedge \alpha_\lambda = 0 \text{ for all } \lambda \quad \Leftrightarrow \quad \text{(2.2), (2.3) and (2.4) plus flatness.}$$

For each real value of λ we can integrate α_λ to obtain a frame for a flat immersion. Thus the flatness condition can be encoded by assuming that we have such a 1-parameter *family* of frames.

In general, let G be a complex semisimple Lie group, and suppose we have the following ingredients:

(1) for $\lambda \in \mathbb{C}^*$, a 1-parameter family of 1-forms, $\alpha_\lambda \in \mathfrak{g} \otimes \Omega(M)$.
(2) α_λ is a Laurent polynomial in λ,

$$\alpha_\lambda = \sum_{i=a}^{b} a_i \lambda^i, \qquad a_i \in \mathfrak{g} \otimes \Omega(M).$$

(3) α_λ satisfies the Maurer-Cartan equation for all $\lambda \in \mathbb{C}^*$.

Then we can integrate to obtain family $F_\lambda : M \to G$, and project to obtain a family of special submanifolds $f_\lambda : M \to G/H$, where H is some subgroup of G.

Interesting question: what are the special submanifolds corresponding to the projections f_λ?

2.4. **The Connection to Special PDE.** The existence of a 1-parameter family of integrable Maurer-Cartan forms (corresponding to flat connections with values in a loop algebra) is well known to be an essential characteristic of soliton equations and other so-called integrable systems. This aspect manifests itself in the following way: given a family of 1-forms $\alpha_\lambda = \sum_{i=a}^{b} a_i \lambda^i$, $a_i \in \mathfrak{g} \otimes \Omega(M)$, as above, it is easy to see that

$$\mathrm{d}\alpha_\lambda + \alpha_\lambda \wedge \alpha_\lambda = 0, \quad \text{for all } \lambda$$

if and only if

$$\mathrm{d}a_k + \sum_{i+j=k} a_i \wedge a_j = 0.$$

This is a system of PDE (after choosing some coordinates).

Example: We return once more to our example of flat immersions into S^3. The Gauss equation: $\mathrm{d}\omega + \omega \wedge \omega - \beta \wedge \beta^t - \theta \wedge \theta^t = 0$, together with the flatness condition $\mathrm{d}\omega + \omega \wedge \omega = 0$, turn out to reduce to one equation, in special coordinates:

$$\phi_{xy} = 0,$$

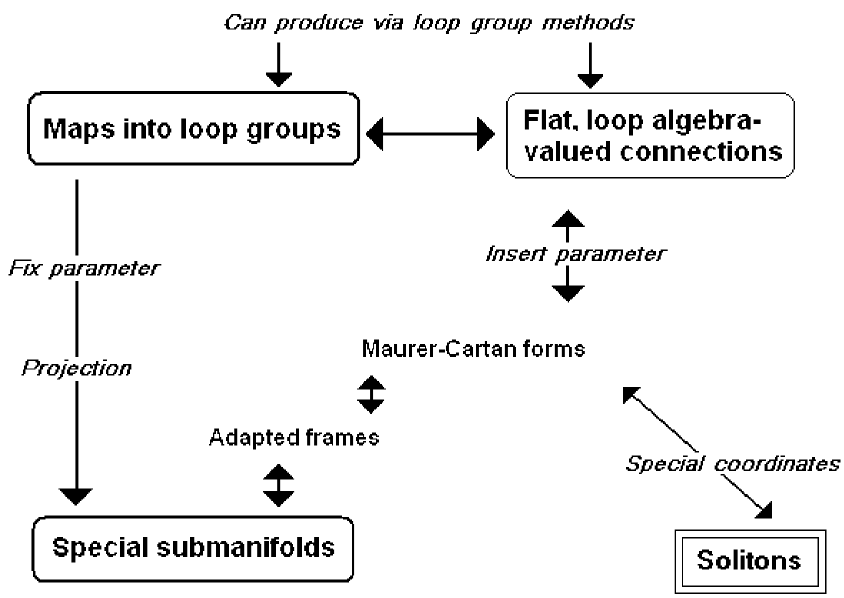


FIGURE 1. The relations between maps into loop groups, flat connections, special submanifolds and special PDE.

namely, the wave equation.

3. GRASSMANN GEOMETRIES

The methods from loop groups used here produce submanifolds which are, or are related to, Grassmann submanifolds in homogeneous spaces. This point has perhaps not been emphasized in the past, because the majority of applications studied were in space forms, where the Grassmann submanifold condition (arising from orbits of the action of the isometry group in the symmetric space representation) is satisfied by any submanifold.

The concept of a Grassmann submanifold was introduced by Harvey and Lawson in [6], as follows: let $\bar{N}$ be a manifold and take any subset, $\mathcal{V}$, of the Grassmann bundle over $\bar{N}$ consisting of tangential s-planes, $Gr_s(T\bar{N}) = \cup_{x\in\bar{N}} Gr_s(T_x\bar{N})$. A *$\mathcal{V}$-submanifold*, N, of $\bar{N}$, is an s-dimensional connected submanifold such that $T_xN \in \mathcal{V}$ for each $x \in N$. The set of such submanifolds, N, is called the *$\mathcal{V}$-geometry.*

In this article, $\bar{N}$ will always be a homogeneous space, G/H, with G a connected Lie group, and $\mathcal{V}$ an orbit of the action of G on $Gr_s(T\bar{N})$. In such a case, the geometry $\mathcal{V}$ is determined by an s-dimensional vector subspace of the tangent space at the origin, H, of of G/H. A special case is when $\bar{N} = \bar{U}/\bar{K}$ is a *symmetric space*. Then we have the canonical decomposition of the Lie algebra $\bar{\mathfrak{u}} = \bar{\mathfrak{k}} \oplus \bar{\mathfrak{p}}$, and the tangent space at the origin is $T_0\bar{N} = \bar{\mathfrak{p}}$.

So for symmetric spaces we have the correspondence:

$$\{s\text{-Dim } \mathcal{V}\text{-geometries}\} \leftrightarrow \{s\text{-Dim subspaces } \mathfrak{p} \subset \bar{\mathfrak{p}}\}.$$

Given $\mathfrak{p} \subset \bar{\mathfrak{p}}$, we will call the associated geometry the $\mathcal{V}_{\mathfrak{p}}$-geometry.

If $Ad_{\bar{K}}\mathfrak{p} \subset \mathfrak{p}$ then the $\mathcal{V}_{\mathfrak{p}}$-geometry consists of integral submanifolds of a *distribution* determined by $\mathfrak{p}$, but otherwise it is a more general concept.

3.1. **Examples.** For space forms, *any* s-dimensional submanifold is a $\mathcal{V}_{\mathfrak{p}}$-submanifold for any s-dim subspace $\mathfrak{p} \subset \bar{\mathfrak{p}}$. We demonstrate this for curves in $\bar{N} = SO(3)/SO(2) = S^2$. We have the canonical decomposition:

$$\mathfrak{so}(3) = \bar{\mathfrak{k}} \oplus \bar{\mathfrak{p}} = \left\{ \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \oplus \left\{ \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{bmatrix} \right\}.$$

For a 1-dimensional subspace $\mathfrak{p} \subset \bar{\mathfrak{p}}$, we can, with no loss of generality, take $\mathfrak{p} = \left\{ \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ * & 0 & 0 \end{bmatrix} \right\}$.

Let $f : \mathbf{R} \to S^2$ be any curve. The $\mathcal{V}_{\mathfrak{p}}$-geometry is determined by the left action of $SO(3)$ on $Gr_1(TS^2)$, and to show that a curve in S^3 is a $\mathcal{V}_{\mathfrak{p}}$-submanifold, we need to show there exists frame $F \in SO(3)$ for f, such that the projection onto $\bar{\mathfrak{p}}$ of $F^{-1}\mathrm{d}F$ lies in $\mathfrak{p}$. This is achieved by choosing an *adapted frame* $F : \mathbf{R} \to SO(3)$,

$$F = [e, n, f], \qquad e \text{ tangent}, \ n \text{ normal},$$

$$F^{-1}\mathrm{d}F = \begin{bmatrix} e^t \\ n^t \\ f^t \end{bmatrix} \begin{bmatrix} \mathrm{d}e & \mathrm{d}n & \mathrm{d}f \end{bmatrix} = \begin{bmatrix} 0 & e^t\mathrm{d}n & e^t\mathrm{d}f \\ n^t\mathrm{d}e & 0 & n^t\mathrm{d}f \\ f^t\mathrm{d}e & f^t\mathrm{d}n & 0 \end{bmatrix}.$$

The $\bar{\mathfrak{p}}$ part is $\begin{bmatrix} 0 & 0 & e^t\mathrm{d}f \\ 0 & 0 & n^t\mathrm{d}f \\ f^t\mathrm{d}e & f^t\mathrm{d}n & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & e^t\mathrm{d}f \\ 0 & 0 & 0 \\ f^t\mathrm{d}e & 0 & 0 \end{bmatrix} \in \mathfrak{p}.$

More meaningful examples of Grassmann submanifolds are Lagrangian submanifolds of $\mathbf{C}P^n$ and almost complex and totally real submanifolds of S^6. The latter arise with respect to the action of G_2 on the homogeneous space $S^6 = G_2/SU(3)$, which is not a symmetric space representation of S^6; hence there is no conflict with the above comment concerning space forms.

4. Grassmann geometries associated to loop groups

Loop group techniques (AKS-theory, DPW, etc) produce maps into a subgroup of a loop group which are characterized by the fact that the Maurer-Cartan form is a Laurent polynomial of fixed degree in the loop parameter, λ. Solutions are determined modulo the action of the constant subgroup - hence actually frames for maps into a homogeneous space.

We formulate this in the language of Grassmann geometries: Let G be a complex semisimple Lie group, and define the loop group

$$\Lambda G := \{\gamma : S^1 \to G\},$$

where the maps have some convergence condition, such as the Wiener topology, which makes ΛG a Banach Lie group. Let $\mathcal{H}$ be a Banach subgroup of ΛG, and denote by $\mathcal{H}^0 := \mathcal{H} \cap G$, the subgroup of constant loops. Then the left coset space $\mathcal{H}/\mathcal{H}^0$ is a homogeneous space on which $\mathcal{H}$ acts on the left.

To define Grassmann geometries on $\mathcal{H}/\mathcal{H}^0$, we need to describe its tangent space at the origin. The Lie algebra of ΛG is $\Lambda\mathfrak{g} = \{\sum_{i-\infty}^{\infty} a_i\lambda^i \mid a_i \in \mathfrak{g}\}$, and Lie($\mathcal{H}$) is a vector subspace of $\Lambda\mathfrak{g}$. Clearly Lie($\mathcal{H}^0$) = { constant polynomials in Lie($\mathcal{H}$)}, from which it follows that

$$T_0\frac{\mathcal{H}}{\mathcal{H}^0} = \{\sum_{i\neq 0} a_i\lambda^i\} \subset \mathrm{Lie}(\mathcal{H}).$$

For integers $a < b$, define $W_a^b \subset T_0\frac{\mathcal{H}}{\mathcal{H}^0}$ by

$$W_a^b = \{x \in T_0\frac{\mathcal{H}}{\mathcal{H}^0} \mid \sum_{i=a}^{b} a_i\lambda^i\}.$$

Now set $\mathcal{V}_a^b$ to be the distribution given by the orbit of W_a^b under the action of $\mathcal{H}$ on $Gr_{b-a}\big(T\frac{\mathcal{H}}{\mathcal{H}^0}\big)$.

The basic object we can construct, using the techniques described here, are $\mathcal{V}_a^b$-compatible (immersed) submanifolds of $\mathcal{H}/\mathcal{H}^0$, i.e. maps $f : M \to \mathcal{H}/\mathcal{H}^0$ for which there exists frames $F : M \to \mathcal{H}$ with $F^{-1}\mathrm{d}F = \sum_{i=a}^{b} \alpha_i\lambda^i$.

5. Special submanifolds from loop group maps

A $\mathcal{V}_a^b$-immersion $f : M \to \mathcal{H}/\mathcal{H}^0$, leads naturally to families of special submanifolds as follows: Evaluate f at some λ_0, to get a map $f_{\lambda_0} : M \to G/\mathcal{H}^0$. The subgroup $\mathcal{H}$ together with the $\mathcal{V}_a^b$ condition make f_{λ_0} a certain Grassman submanifold.

Since f is a $\mathcal{V}_a^b$-immersion, by definition, there exists a lift $F : M \to \mathcal{H}$, such that $\alpha := F^{-1}\mathrm{d}F = \sum_{i=a}^{b} \alpha_i\lambda^i$. An essential point is: α must satisfy the Maurer-Cartan equation,

$$\mathrm{d}\alpha + \alpha \wedge \alpha = 0,$$

for *all* values of λ. This is equivalent to some conditions on α_i,

$$\mathrm{d}\alpha_k + \sum_{i+j=k} \alpha_i \wedge \alpha_j = 0, \tag{5.1}$$

independent of λ.

The equations (5.1) give some *extra conditions*, usually on the (tangent and normal) curvature of the submanifold. This will be illustrated by our example below.

6. The three involution loop group

Now we define the generalization of the loop group construction of [5]. Let G be a complex semisimple Lie group and $\bar{\tau}$, $\hat{\sigma}$, ρ commuting involutions of G, where ρ is $\mathbf{C}$-antilinear. The fixed point subgroup with respect to ρ, $\bar{U} := G_\rho$, is a real form of the group.

We extend the involutions to ΛG by the rules:

$$(\rho X)(\lambda) = \rho(X(\bar{\lambda})),$$
$$(\hat{\sigma} X)(\lambda) = \hat{\sigma}(X(-\lambda)),$$
$$(\bar{\tau} X)(\lambda) = \bar{\tau}(X(-1/\lambda)),$$

and consider the subgroup fixed by all three involutions:

$$\mathcal{H} = \Lambda G_{\rho\bar{\tau}\hat{\sigma}}.$$

Consider a $\mathcal{V}_{-1}^1$-immersion $f : M \to \mathcal{H}/\mathcal{H}^0$. For $\lambda \in \mathbf{R}^*$,

$$f_\lambda : M \to \bar{U}/\bar{U}_{\bar{\tau}\hat{\sigma}},$$

since $\mathcal{H}^0 = \bar{U}_{\bar{\tau}\hat{\sigma}} = \bar{U}_{\bar{\tau}} \cap \bar{U}_{\hat{\sigma}}$.

We can also project to obtain maps into the symmetric spaces $\bar{U}/\bar{U}_{\bar{\tau}}$ and $\bar{U}/\bar{U}_{\hat{\sigma}}$, or more generally, into any homogeneous space $\bar{U}/H$, where $\bar{U}_{\bar{\tau}} \cap \bar{U}_{\hat{\sigma}} \subset H$.

What are the special submanifolds so obtained?

7. Reflective submanifolds

We are primarily interested in the projection to $\bar{U}/\bar{U}_{\bar{\tau}}$, as this generalizes the isometric immersions of space forms studied in [5]. To describe the projections, we first need to define reflective submanifolds.

Examples In space forms, these are just the complete totally geodesic submanifolds. Other examples are Lagrangian embeddings of $\mathbf{R}P^n \subset \mathbf{C}P^n$ and $\mathbf{R}H^n \subset \mathbf{C}H^n$.

Definition: A *reflective submanifold*, N, of a Riemannian manifold, $\bar{N}$, is a totally geodesic symmetric submanifold.

For a connected symmetric space $\bar{N} = \bar{U}/\bar{K}$, we can characterize a reflective submanifold N of $\bar{N}$, by the existence of a second involution on the Lie algebra of $\bar{U}$. Specifically, $N \subset \bar{N}$ is characterized by a pair of commuting involutions, $\bar{\tau}$ and $\hat{\sigma}$, of the Lie algebra $\bar{\mathfrak{u}}$ of $\bar{U}$, and $\bar{K} = \bar{U}_{\bar{\tau}}$. That is:

$$N \subset \bar{U}/\bar{K} \quad \leftrightarrow \quad (\bar{\mathfrak{u}}, \bar{\tau}, \hat{\sigma}).$$

We have two canonical decompositions of the Lie algebra $\bar{\mathfrak{u}} = \bar{\mathfrak{k}} \oplus \bar{\mathfrak{p}} = \hat{\mathfrak{k}} \oplus \hat{\mathfrak{p}}$, into the $+1$ and -1 eigenspaces of the two involutions. Setting

$$\mathfrak{p} := \bar{\mathfrak{p}} \cap \hat{\mathfrak{p}},$$

the reflective submanifold is given by: $N = \pi_{\bar{N}} \exp(\mathfrak{p})$.

Reflective submanifolds of symmetric spaces were classified by DSP Leung (1974-1979), and there are clearly many cases.

8. Isometric immersions of space forms

The three involution loop group leads naturally to a generalization of the following results/conjectures:

An isometric immersion $f : M^k(c) \to M^n(\tilde{c})$, of space forms with constant sectional curvature c and $\tilde{c}$ respectively, has *negative extrinsic curvature* if $c < \tilde{c}$. There are two basic questions: existence of a local solution, and existence of a complete solution. For these it is known:

(1) Local solutions exist iff $n \geq 2k - 1$ (Cartan).

(2) Theorem (JD Moore): If $0 < c < 1$, there is no complete isometric immersion with flat normal bundle of $S^k(c)$ into S^n for any $k > 1$ and any n.

(3) Plausible conjecture: If $c < -1$ there is no complete isometric immersion with flat normal bundle of $H^k(c)$ into $H^n(-1)$ for any $k > 1$ and any n.

For the case $n = 2k - 1$, this is equivalent to the conjectured generalization of Hilberts's non-immersibility of H^2 into E^3.

9. The generalization to other reflective submanifolds

M a Riemannian manifold, let M_R denote the same manifold with the metric scaled by a factor $R > 0$.

Problem A: Suppose given a reflective submanifold

$$N \subset \bar{N}$$

of a symmetric space. Thus, $N_R \subset \bar{N}_R$ is also a reflective submanifold. Does there exist a (local or global) isometric immersion

$$N_R \to \bar{N},$$

satisfying condition X? That is, can we shrink/stretch N within $\bar{N}$? More specifically, we ask this for:

(1) $R > 1$, if $\bar{N}$ is of compact type,
(2) $R < 1$, if $\bar{N}$ is of non-compact type.

Reflective submanifolds in other symmetric spaces do not generally have flat normal bundle. Thus, we need to replace the flat normal bundle condition with an appropriate one, which we call here *condition X*.

Condition X just says:

(1) $N_R \to \bar{N}$ is a $\mathcal{V}_{\mathfrak{p}}$-submanifold, where $N = \exp(\mathfrak{p})$.

(2) The normal bundle of $N_R \to \bar{N}$ is isomorphic (as an affine vector bundle/connection pair) with the normal bundle of $N_R \subset \bar{N}_R$.

10. Projections to $\bar{U}/\bar{U}_{\bar{\tau}}$ and $\bar{U}/\bar{U}_{\hat{\sigma}}$

Here we summarize results from [2]. In fact Proposition 10.2 is stated incorrectly in [2] - the limit as $\lambda \to \infty$ or $\lambda \to 0$ must be taken before a curved flat is obtained.

Set $\bar{K} := \bar{U}_{\bar{\tau}}$, and $\widehat{K} := \bar{U}_{\hat{\sigma}}$. Take $f : M \to \mathcal{H}/\mathcal{H}^0$ a $\mathcal{V}^1_{-1}$-immersion. Recall $f_\lambda : M \to \bar{U}/(\bar{K} \cap \widehat{K})$, for $\lambda \in \mathbf{R}$.

Proposition 10.1. *Let $\bar{f}_\lambda : M \to \bar{U}/\bar{K}$ be the projection of f_λ. Suppose that $\bar{f}_\lambda$ is regular. Then $\bar{f}_\lambda$ is a solution of Problem A (for $R > 1$). Conversely, any solution of Problem A, corresponds to such a $\mathcal{V}^1_{-1}$-immersion.*

Proposition 10.2. *Let $\hat{f}_\lambda : M \to \bar{U}/\widehat{K}$ be the projection of f_λ. Then:*

- *$\hat{f}_\lambda$ is asymptotic to a curved flat in $\bar{U}/\widehat{K}$, as $\lambda \to \infty$, or $\lambda \to 0$.*
- *If $\bar{f}_\lambda$ is regular then so is $\hat{f}_\lambda$ (but not conversely).*

*Hence, if $\bar{U}/\bar{K}$ **compact** then:*

(1) *Local regular solutions to Problem A exist* $\Rightarrow \mathrm{Dim}(\mathfrak{p}) \leq \mathrm{Rank}(\bar{U}/\widehat{K})$.

(2) *Global regular solutions to Problem A do not exist for Dim(N) $>$ 1.*

11. Consequences

Theorem 11.1. ***(Compact Case)*** *The following list contains the geometric interpretations of all possible solutions to Problem A for the case $R > 1$ and $\bar{N}$ is a simply connected, compact, irreducible, Riemannian symmetric space.*

In all cases, local solutions exist and can be constructed by loop group methods. In all cases where $\mathrm{Dim}(N_R) > 1$*, there is no solution which is geodesically complete.*

(1) *$N_R = S^k_R$ is an isometric immersion with flat normal bundle of a k-sphere of radius $\sqrt{R}$ into the unit sphere S^n, with $0 < k \leq (n+1)/2$, and $n \geq 2$.*

(2) *$N_R = S^n_R$ is an isometric totally real immersion of an n-sphere of radius $\sqrt{R}$ into complex projective space $\mathbf{C}P^n$, with $n \geq 2$.*

Note: Lagrangian immersions of a sphere into $\mathbf{C}P^n$ is a new example of a submanifold as an integrable system.

Theorem 11.2. ***(Non-Compact Case)*** *The analogue -* ***except*** *we do not obtain the global non-existence result, which remains an open problem.*

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