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Spectral action on noncommutative torus

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Dedicated to Alain Connes on the occasion of his 60th birthday

Abstract

The spectral action on noncommutative torus is obtained, using a Chamseddine–Connes formula via computations of zeta functions. The importance of a Diophantine condition is outlined. Several results on holomorphic continuation of series of holomorphic functions are obtained in this context.

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1 Introduction

The spectral action introduced by Chamseddine–Connes plays an important role [3] in noncommutative geometry. More precisely, given a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where $\mathcal{A}$ is an algebra acting on the Hilbert space $\mathcal{H}$ and $\mathcal{D}$ is a Dirac-like operator (see [8, 23]), they proposed a physical action depending only on the spectrum of the covariant Dirac operator

$$\mathcal{D}_A := \mathcal{D} + A + \epsilon\, JAJ^{-1} \tag{1}$$

where A is a one-form represented on $\mathcal{H}$, so has the decomposition

$$A = \sum_i a_i[\mathcal{D},\, b_i], \tag{2}$$

with $a_i,\ b_i \in \mathcal{A}$, J is a real structure on the triple corresponding to charge conjugation and $\epsilon \in \{\, 1, -1 \,\}$ depending on the dimension of this triple and comes from the commutation relation

$$J\mathcal{D} = \epsilon\, \mathcal{D}J. \tag{3}$$

This action is defined by

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) := \operatorname{Tr}\big(\Phi(\mathcal{D}_A/\Lambda)\big) \tag{4}$$

where Φ is any even positive cut-off function which could be replaced by a step function up to some mathematical difficulties investigated in [16]. This means that Φ counts the spectral values of $|\mathcal{D}_A|$ less than the mass scale Λ (note that the resolvent of $\mathcal{D}_A$ is compact since, by assumption, the same is true for $\mathcal{D}$, see Lemma 3.1 below).
In [18], the spectral action on NC-tori has been computed only for operators of the form $\mathcal{D} + A$ and computed for $\mathcal{D}_A$ in [20]. It appears that the implementation of the real structure via J, does change the spectral action, up to a coefficient when the torus has dimension 4. Here we prove that this can be also directly obtained from the Chamseddine–Connes analysis of [4] what we follow quite closely. Actually,

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) \;=\; \sum_{0<k\in Sd^+} \Phi_k\, \Lambda^k {-\!\!\!\!\!\!\int} |D_A|^{-k} + \Phi(0)\, \zeta_{D_A}(0) + \mathcal{O}(\Lambda^{-1}) \tag{5}$$

where $D_A = \mathcal{D}_A + P_A$, P_A the projection on $\operatorname{Ker} \mathcal{D}_A$, $\Phi_k = \frac{1}{2}\int_0^\infty \Phi(t)\, t^{k/2-1}\, dt$ and Sd^+ is the strictly positive part of the dimension spectrum of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. As we will see, $Sd^+ = \{\, 1, 2, \cdots, n \,\}$ and ${-\!\!\!\!\!\int} |D_A|^{-n} = {-\!\!\!\!\!\int} |D|^{-n}$. Moreover, the coefficient $\zeta_{D_A}(0)$ related to the constant term in (5) can be computed from the unpertubated spectral action since it has been proved in [4] (with an invertible Dirac operator and a 1-form A such that $\mathcal{D} + A$ is also invertible) that

$$\zeta_{\mathcal{D}+A}(0) - \zeta_{\mathcal{D}}(0) = \sum_{q=1}^{n} \tfrac{(-1)^q}{q} {-\!\!\!\!\!\!\int} (A\mathcal{D}^{-1})^q, \tag{6}$$

using $\zeta_X(s) = \operatorname{Tr}(|X|^{-s})$. We will see how this formula can be extended to the case a noninvertible Dirac operator and noninvertible perturbation of the form $\mathcal{D} + \widetilde{A}$.
All this results on spectral action are quite important in physics, especially in quantum field theory and particle physics, where one adds to the effective action some counterterms explicitly given by (6), see for instance [2–5, 17, 18, 20, 22, 28, 35–38].

Since the computation of zeta functions is crucial here, we investigate in section 2 residues of series and integrals. This section contains independent interesting results on the holomorphy of series of holomorphic functions. In particular, the necessity of a Diophantine constraint is naturally emphasized.
In section 3, we revisit the notions of pseudodifferential operators and their associated zeta functions and of dimension spectrum. The reality operator J is incorporated and we pay a particular attention to kernels of operators which can play a role in the constant term of (5). This section concerns general spectral triple with simple dimension spectrum.
Section 4 is devoted to the example of the noncommutative torus. It is shown that it has a vanishing tadpole.
In section 5, all previous technical points are then widely used for the computation of terms in (5) or (6).
Finally, the spectral action (6) is obtained in section 6 and we conjecture that the noncommutative spectral action of $\mathcal{D}_A$ has terms proportional to the spectral action of $\mathcal{D}+A$ on the commutative torus.

2 Residues of series and integral, holomorphic continuation, etc

Notations:
In the following, the prime in $\sum'$ means that we omit terms with division by zero in the summand.
B^n (resp. S^{n-1}) is the closed ball (resp. the sphere) of $\mathbb{R}^n$ with center 0 and radius 1 and the Lebesgue measure on S^{n-1} will be noted dS.
For any $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ we denote by $|x|=\sqrt{x_1^2+\cdots+x_n^2}$ the euclidean norm and $|x|_1:=|x_1|+\cdots+|x_n|$.
$\mathbb{N}=\{1,2,\dots\}$ is the set of positive integers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$ the set of non negative integers.
By $f(x,y)\ll_y g(x)$ uniformly in x, we mean that $|f(x,y)|\leq a(y)\,|g(x)|$ for all x and y for some $a(y)>0$.

2.1 Residues of series and integral

In order to be able to compute later the residues of certain series, we prove here the following

Theorem 2.1. *Let $P(X)=\sum_{j=0}^d P_j(X)\in\mathbb{C}[X_1,\cdots,X_n]$ be a polynomial function where P_j is the homogeneous part of P of degree j. The function*

$$\zeta^P(s):=\sum\nolimits'_{k\in\mathbb{Z}^n}\tfrac{P(k)}{|k|^s},\ \ s\in\mathbb{C}$$

has a meromorphic continuation to the whole complex plane $\mathbb{C}$.
Moreover $\zeta^P(s)$ is not entire if and only if $\mathcal{P}_P:=\{j\ :\ \int_{u\in S^{n-1}}P_j(u)\,dS(u)\neq 0\}\neq\varnothing$. In that case, ζ^P has only simple poles at the points $j+n$, $j\in\mathcal{P}_P$, with

$$\operatorname*{Res}_{s=j+n}\zeta^P(s)=\int_{u\in S^{n-1}}P_j(u)\,dS(u).$$

The proof of this theorem is based on the following lemmas.

Lemma 2.2. *For any polynomial $P\in\mathbb{C}[X_1,\dots,X_n]$ of total degree $\delta(P):=\sum_{i=1}^n deg_{X_i}P$ and any $\alpha\in\mathbb{N}_0^n$, we have*

$$\partial^\alpha\left(P(x)|x|^{-s}\right)\ll_{P,\alpha,n}(1+|s|)^{|\alpha|_1}|x|^{-\sigma-|\alpha|_1+\delta(P)}$$

uniformly in $x \in \mathbb{R}^n$ verifying $|x| \geq 1$, where $\sigma = \Re(s)$.

Proof. By linearity, we may assume without loss of generality that $P(X) = X^\gamma$ is a monomial. It is easy to prove (for example by induction on $|\alpha|_1$) that for all $\alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n \setminus \{0\}$:

$$\partial^\alpha \left(|x|^{-s}\right) = \alpha! \sum_{\substack{\beta,\mu \in \mathbb{N}_0^n \\ \beta+2\mu=\alpha}} \tbinom{-s/2}{|\beta|_1+|\mu|_1} \tfrac{(|\beta|_1+|\mu|_1)!}{\beta!\ \mu!} \tfrac{x^\beta}{|x|^{\sigma+2(|\beta|_1+|\mu|_1)}}.$$

It follows that for all $\alpha \in \mathbb{N}_0^n$, we have uniformly in $x \in \mathbb{R}^n$ verifying $|x| \geq 1$:

$$\partial^\alpha \left(|x|^{-s}\right) \ll_{\alpha,n} (1+|s|)^{|\alpha|_1} |x|^{-\sigma-|\alpha|_1} \tag{7}$$

By Leibniz formula and (7), we have uniformly in $x \in \mathbb{R}^n$ verifying $|x| \geq 1$:

$$\begin{aligned}
\partial^\alpha \left(x^\gamma |x|^{-s}\right) &= \sum_{\beta \leq \alpha} \tbinom{\alpha}{\beta} \partial^\beta (x^\gamma)\ \partial^{\alpha-\beta}\left(|x|^{-s}\right) \\
&\ll_{\gamma,\alpha,n} \sum_{\beta \leq \alpha; \beta \leq \gamma} x^{\gamma-\beta}\ (1+|s|)^{|\alpha|_1-|\beta|_1}\ |x|^{-\sigma-|\alpha|_1+|\beta|_1} \\
&\ll_{\gamma,\alpha,n} (1+|s|)^{|\alpha|_1}\ |x|^{-\sigma-|\alpha|_1+|\gamma|_1}.
\end{aligned}$$

□

Lemma 2.3. *Let $P \in \mathbb{C}[X_1, \ldots, X_n]$ be a polynomial of degree d. Then, the difference*

$$\Delta_P(s) := \sum\nolimits'_{k \in \mathbb{Z}^n} \tfrac{P(k)}{|k|^s} - \int_{\mathbb{R}^n \setminus B^n} \tfrac{P(x)}{|x|^s}\, dx$$

which is defined for $\Re(s) > d + n$, extends holomorphically on the whole complex plane $\mathbb{C}$.

Proof. We fix in the sequel a function $\psi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ verifying for all $x \in \mathbb{R}^n$

$$0 \leq \psi(x) \leq 1, \quad \psi(x) = 1 \text{ if } |x| \geq 1 \quad \text{and} \quad \psi(x) = 0 \text{ if } |x| \leq 1/2.$$

The function $f(x,s) := \psi(x)\ P(x)\ |x|^{-s}$, $x \in \mathbb{R}^n$ and $s \in \mathbb{C}$, is in $\mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{C})$ and depends holomorphically on s.
Lemma 2.2 above shows that f is a "gauged symbol" in the terminology of [24, p. 4]. Thus [24, Theorem 2.1] implies that $\Delta_P(s)$ extends holomorphically on the whole complex plane $\mathbb{C}$. However, to be complete, we will give here a short proof of Lemma 2.3:
It follows from the classical Euler–Maclaurin formula that for any function $h : \mathbb{R} \to \mathbb{C}$ of class $\mathcal{C}^{N+1}$ verifying $\lim_{|t|\to+\infty} h^{(k)}(t) = 0$ and $\int_{\mathbb{R}} |h^{(k)}(t)|\, dt < +\infty$ for any $k = 0 \ldots, N+1$, that we have

$$\sum_{k \in \mathbb{Z}} h(k) = \int_{\mathbb{R}} h(t) + \tfrac{(-1)^N}{(N+1)!} \int_{\mathbb{R}} B_{N+1}(t)\ h^{(N+1)}(t)\ dt$$

where B_{N+1} is the Bernoulli function of order $N+1$ (it is a bounded periodic function.)
Fix $m' \in \mathbb{Z}^{n-1}$ and $s \in \mathbb{C}$. Applying this to the function $h(t) := \psi(m',t)\ P(m',t)\ |(m',t)|^{-s}$ (we use Lemma 2.2 to verify hypothesis), we obtain that for any $N \in \mathbb{N}_0$:

$$\sum_{m_n \in \mathbb{Z}} \psi(m', m_n)\ P(m', m_n)\ |(m', m_n)|^{-s} = \int_{\mathbb{R}} \psi(m',t)\ P(m',t)\ |(m',t)|^{-s}\ dt + \mathcal{R}_N(m'; s) \tag{8}$$

where $\mathcal{R}_N(m';s) := \frac{(-1)^N}{(N+1)!}\int_{\mathbb{R}} B_{N+1}(t)\ \frac{\partial^{N+1}}{\partial {x_n}^{N+1}}\left(\psi(m',t)\ P(m',t)\ |(m',t)|^{-s}\right)\ dt$.
By Lemma 2.2,

$$\int_{\mathbb{R}}\left|B_{N+1}(t)\ \frac{\partial^{N+1}}{\partial {x_n}^{N+1}}\left(\psi(m',t)\ P(m',t)\ |(m',t)|^{-s}\right)\right| dt \ll_{P,n,N} (1+|s|)^{N+1}\ (|m'|+1)^{-\sigma-N+\delta(P)}.$$

Thus $\sum_{m'\in\mathbb{Z}^{n-1}}\mathcal{R}_N(m';s)$ converges absolutely and define a holomorphic function in the half plane $\{\sigma=\Re(s)>\delta(P)+n-N\}$.
Since N is an arbitrary integer, by letting $N\to\infty$ and using (8) above, we conclude that:

$$s\mapsto \sum_{(m',m_n)\in\mathbb{Z}^{n-1}\times\mathbb{Z}} \psi(m',m_n)\ P(m',m_n)\ |(m',m_n)|^{-s} - \sum_{m'\in\mathbb{Z}^{n-1}}\int_{\mathbb{R}}\psi(m',t)\ P(m',t)\ |(m',t)|^{-s}\ dt$$

has a holomorphic continuation to the whole complex plane $\mathbb{C}$.
After n iterations, we obtain that

$$s\mapsto \sum\nolimits_{m\in\mathbb{Z}^n}\psi(m)\ P(m)\ |m|^{-s} - \int_{\mathbb{R}^n}\psi(x)\ P(x)\ |x|^{-s}\ dx$$

has a holomorphic continuation to the whole $\mathbb{C}$.
To finish the proof of Lemma 2.3, it is enough to notice that:

- $\psi(0)=0$ and $\psi(m)=1$, $\forall m\in\mathbb{Z}^n\setminus\{0\}$;
- $s\mapsto\int_{B^n}\psi(x)\ P(x)\ |x|^{-s}\ dx = \int_{\{x\in\mathbb{R}^n:1/2\le|x|\le 1\}}\psi(x)\ P(x)\ |x|^{-s}\ dx$ is a holomorphic function on $\mathbb{C}$. □

Proof of Theorem 2.1. Using the polar decomposition of the volume form $dx=\rho^{n-1}\,d\rho\,dS$ in $\mathbb{R}^n$, we get for $\Re(s)>d+n$,

$$\int_{\mathbb{R}^n\setminus B^n}\frac{P_j(x)}{|x|^s}dx = \int_1^\infty\frac{\rho^{j+n-1}}{\rho^s}\int_{S^{n-1}}P_j(u)\,dS(u) = \frac{1}{j+n-s}\int_{S^{n-1}}P_j(u)\,dS(u).$$

Lemma 2.3 now gives the result. □

2.2 Holomorphy of certain series

Before stating the main result of this section, we give first in the following some preliminaries from Diophantine approximation theory:

Definition 2.4. *(i) Let $\delta>0$. A vector $a\in\mathbb{R}^n$ is said to be $\delta-$badly approximable if there exists $c>0$ such that $|q.a-m|\ge c\,|q|^{-\delta}$, $\forall q\in\mathbb{Z}^n\setminus\{\,0\,\}$ and $\forall m\in\mathbb{Z}$.*
We note $\mathcal{BV}(\delta)$ the set of $\delta-$badly approximable vectors and $\mathcal{BV}:=\cup_{\delta>0}\mathcal{BV}(\delta)$ the set of badly approximable vectors.
(ii) A matrix $\Theta\in\mathcal{M}_n(\mathbb{R})$ (real $n\times n$ matrices) will be said to be badly approximable if there exists $u\in\mathbb{Z}^n$ such that ${}^t\Theta(u)$ is a badly approximable vector of $\mathbb{R}^n$.

Remark. A classical result from Diophantine approximation asserts that for all $\delta>n$, the Lebesgue measure of $\mathbb{R}^n\setminus\mathcal{BV}(\delta)$ is zero (i.e almost any element of $\mathbb{R}^n$ is $\delta-$badly approximable.)
Let $\Theta\in\mathcal{M}_n(\mathbb{R})$. If its row of index i is a badly approximable vector of $\mathbb{R}^n$ (i.e. if $L_i\in\mathcal{BV}$) then ${}^t\Theta(e_i)\in\mathcal{BV}$ and thus Θ is a badly approximable matrix. It follows that almost any matrix of $\mathcal{M}_n(\mathbb{R})\approx\mathbb{R}^{n^2}$ is badly approximable.

The goal of this section is to show the following

Theorem 2.5. *Let $P \in \mathbb{C}[X_1, \cdots, X_n]$ be a homogeneous polynomial of degree d and let b be in $\mathcal{S}(\mathbb{Z}^n \times \cdots \times \mathbb{Z}^n)$ (q times, $q \in \mathbb{N}$). Then,*
(i) Let $a \in \mathbb{R}^n$. We define $f_a(s) := \sum'_{k\in\mathbb{Z}^n} \frac{P(k)}{|k|^s} e^{2\pi i k.a}$.
1. If $a \in \mathbb{Z}^n$, then f_a has a meromorphic continuation to the whole complex plane $\mathbb{C}$. Moreover if S is the unit sphere and dS its Lebesgue measure, then f_a is not entire if and only if $\int_{u\in S^{n-1}} P(u)\, dS(u) \neq 0$. In that case, f_a has only a simple pole at the point $d+n$, with $\operatorname{Res}_{s=d+n} f_a(s) = \int_{u\in S^{n-1}} P(u)\, dS(u)$.
2. If $a \in \mathbb{R}^n \setminus \mathbb{Z}^n$, then $f_a(s)$ extends holomorphically to the whole complex plane $\mathbb{C}$.
(ii) Suppose that $\Theta \in \mathcal{M}_n(\mathbb{R})$ is badly approximable. For any $(\varepsilon_i)_i \in \{-1, 0, 1\}^q$, the function

$$g(s) := \sum_{l\in(\mathbb{Z}^n)^q} b(l)\, f_{\Theta \sum_i \varepsilon_i l_i}(s)$$

extends meromorphically to the whole complex plane $\mathbb{C}$ with only one possible pole on $s = d+n$. Moreover, if we set $\mathcal{Z} := \{l \in (\mathbb{Z}^n)^q \ :\ \sum_{i=1}^q \varepsilon_i l_i = 0\}$ and $V := \sum_{l\in\mathcal{Z}} b(l)$, then
1. If $V \int_{S^{n-1}} P(u)\, dS(u) \neq 0$, then $s = d+n$ is a simple pole of $g(s)$ and

$$\operatorname{Res}_{s=d+n} g(s) = V \int_{u\in S^{n-1}} P(u)\, dS(u).$$

2. If $V \int_{S^{n-1}} P(u)\, dS(u) = 0$, then $g(s)$ extends holomorphically to the whole complex plane $\mathbb{C}$.
(iii) Suppose that $\Theta \in \mathcal{M}_n(\mathbb{R})$ is badly approximable. For any $(\varepsilon_i)_i \in \{-1, 0, 1\}^q$, the function

$$g_0(s) := \sum_{l\in(\mathbb{Z}^n)^q\setminus\mathcal{Z}} b(l)\, f_{\Theta \sum_{i=1}^q \varepsilon_i l_i}(s)$$

where $\mathcal{Z} := \{l \in (\mathbb{Z}^n)^q \ :\ \sum_{i=1}^q \varepsilon_i l_i = 0\}$ extends holomorphically to the whole complex plane $\mathbb{C}$.

Proof of Theorem 2.5: First we remark that

If $a \in \mathbb{Z}^n$ then $f_a(s) = \sum'_{k\in\mathbb{Z}^n} \frac{P(k)}{|k|^s}$. So, the point $(i.1)$ follows from Theorem 2.1;

$g(s) := \sum_{l\in(\mathbb{Z}^n)^q\setminus\mathcal{Z}} b(l)\, f_{\Theta\sum_i \varepsilon_i l_i}(s) + \big(\sum_{l\in\mathcal{Z}} b(l)\big) \sum'_{k\in\mathbb{Z}^n} \frac{P(k)}{|k|^s}$. Thus, the point (ii) rises easily from (iii) and Theorem 2.1.

So, to complete the proof, it remains to prove the items $(i.2)$ and (iii).
The direct proof of $(i.2)$ is easy but is not sufficient to deduce (iii) of which the proof is more delicate and requires a more precise (i.e. more effective) version of $(i.2)$. The next lemma gives such crucial version, but before, let us give some notations:

$$\mathcal{F} := \{\tfrac{P(X)}{(X_1^2+\cdots+X_n^2+1)^{r/2}} \ :\ P(X) \in \mathbb{C}[X_1, \ldots, X_n] \text{ and } r \in \mathbb{N}_0\}.$$

We set $g = \deg(G) = \deg(P) - r \in \mathbb{Z}$, the degree of $G = \frac{P(X)}{(X_1^2+\cdots+X_n^2+1)^{r/2}} \in \mathcal{F}$.

By convention we set $\deg(0) = -\infty$.

Lemma 2.6. *Let $a \in \mathbb{R}^n$. We assume that $d(a.u, \mathbb{Z}) := \inf_{m\in\mathbb{Z}} |a.u - m| > 0$ for some $u \in \mathbb{Z}^n$. For all $G \in \mathcal{F}$, we define formally,*

$$F_0(G; a; s) := \sum'_{k\in\mathbb{Z}^n} \frac{G(k)}{|k|^s} e^{2\pi i\, k.a} \quad \text{and} \quad F_1(G; a; s) := \sum_{k\in\mathbb{Z}^n} \frac{G(k)}{(|k|^2+1)^{s/2}} e^{2\pi i\, k.a}.$$

Then for all $N \in \mathbb{N}$, all $G \in \mathcal{F}$ and all $i \in \{0, 1\}$, there exist positive constants $C_i := C_i(G, N, u)$, $B_i := B_i(G, N, u)$ and $A_i := A_i(G, N, u)$ such that $s \mapsto F_i(G; \alpha; s)$ extends holomorphically to the half-plane $\{\Re(s) > -N\}$ and verifies in it:

$$F_i(G; a; s) \le C_i (1 + |s|)^{B_i} \big(d(a.u, \mathbb{Z})\big)^{-A_i}.$$

Remark 2.7. *The important point here is that we obtain an explicit bound of* $F_i(G;\alpha;s)$ *in* $\{\Re(s) > -N\}$ *which depends on the vector* a *only through* $d(a.u,\mathbb{Z})$*, so depends on* u *and indirectly on* a *(in the sequel,* a *will vary.) In particular the constants* $C_i := C_i(G,N,u)$*,* $B_i = B_i(G,N)$ *and* $A_i := A_i(G,N)$ *do not depend on the vector* a *but only on* u*. This is crucial for the proof of items (ii) and (iii) of Theorem 2.5!*

2.2.1 Proof of Lemma 2.6 for $i = 1$:

Let $N \in \mathbb{N}_0$ be a fixed integer, and set $g_0 := n + N + 1$.
We will prove Lemma 2.6 by induction on $g = \deg(G) \in \mathbb{Z}$. More precisely, in order to prove case $i = 1$, it suffices to prove that:

Lemma 2.6 is true for all $G \in \mathcal{F}$ verifying $\deg(G) \le -g_0$.

Let $g \in \mathbb{Z}$ with $g \ge -g_0 + 1$. If Lemma 2.6 is true for all $G \in \mathcal{F}$ such that $\deg(G) \le g-1$, then it is also true for all $G \in \mathcal{F}$ satisfying $\deg(G) = g$.

• Step 1: Checking Lemma 2.6 for $\deg(G) \le -g_0 := -(n + N + 1)$.
Let $G(X) = \frac{P(X)}{(X_1^2+\cdots+X_n^2+1)^{r/2}} \in \mathcal{F}$ verifying $\deg(G) \le -g_0$. It is easy to see that we have uniformly in $s = \sigma + i\tau \in \mathbb{C}$ and in $k \in \mathbb{Z}^n$:

$$\frac{|G(k)\, e^{2\pi i\, k.a}|}{(|k|^2+1)^{\sigma/2}} = \frac{|P(k)|}{(|k|^2+1)^{(r+\sigma)/2}} \ll_G \frac{1}{(|k|^2+1)^{(r+\sigma-deg(P))/2}} \ll_G \frac{1}{(|k|^2+1)^{(\sigma-deg(G))/2}} \ll_G \frac{1}{(|k|^2+1)^{(\sigma+g_0)/2}}.$$

It follows that $F_1(G;a;s) = \sum_{k\in\mathbb{Z}^n} \frac{G(k)}{(|k|^2+1)^{s/2}}\, e^{2\pi i\, k.a}$ converges absolutely and defines a holomorphic function in the half plane $\{\sigma > -N\}$. Therefore, we have for any $s \in \{\Re(s) > -N\}$:

$$|F_1(G;a;s)| \ll_G \sum_{k\in\mathbb{Z}^n} \frac{1}{(|k|^2+1)^{(-N+g_0)/2}} \ll_G \sum_{k\in\mathbb{Z}^n} \frac{1}{(|k|^2+1)^{(n+1)/2}} \ll_G 1.$$

Thus, Lemma 2.6 is true when $\deg(G) \le -g_0$.
• Step 2: Induction.
Now let $g \in \mathbb{Z}$ satisfying $g \ge -g_0+1$ and suppose that Lemma 2.6 is valid for all $G \in \mathcal{F}$ verifying $\deg(G) \le g-1$. Let $G \in \mathcal{F}$ with $\deg(G) = g$. We will prove that G also verifies conclusions of Lemma 2.6:
There exist $P \in \mathbb{C}[X_1,\dots,X_n]$ of degree $d \ge 0$ and $r \in \mathbb{N}_0$ such that $G(X) = \frac{P(X)}{(X_1^2+\cdots+X_n^2+1)^{r/2}}$ and $g = \deg(G) = d - r$.
Since $G(k) \ll (|k|^2+1)^{g/2}$ uniformly in $k \in \mathbb{Z}^n$, we deduce that $F_1(G;a;s)$ converges absolutely in $\{\sigma = \Re(s) > n + g\}$.
Since $k \mapsto k + u$ is a bijection from $\mathbb{Z}^n$ into $\mathbb{Z}^n$, it follows that we also have for $\Re(s) > n + g$

$$\begin{aligned}
F_1(G;a;s) &= \sum_{k\in\mathbb{Z}^n} \frac{P(k)}{(|k|^2+1)^{(s+r)/2}}\, e^{2\pi i\, k.a} = \sum_{k\in\mathbb{Z}^n} \frac{P(k+u)}{(|k+u|^2+1)^{(s+r)/2}}\, e^{2\pi i\,(k+u).a} \\
&= e^{2\pi i\, u.a} \sum_{k\in\mathbb{Z}^n} \frac{P(k+u)}{(|k|^2+2k.u+|u|^2+1)^{(s+r)/2}}\, e^{2\pi i\, k.a} \\
&= e^{2\pi i\, u.a} \sum_{\alpha\in\mathbb{N}_0^n;|\alpha|_1=\alpha_1+\cdots+\alpha_n\le d} \frac{u^\alpha}{\alpha!} \sum_{k\in\mathbb{Z}^n} \frac{\partial^\alpha P(k)}{(|k|^2+2k.u+|u|^2+1)^{(s+r)/2}}\, e^{2\pi i\, k.a} \\
&= e^{2\pi i\, u.a} \sum_{|\alpha|_1\le d} \frac{u^\alpha}{\alpha!} \sum_{k\in\mathbb{Z}^n} \frac{\partial^\alpha P(k)}{(|k|^2+1)^{(s+r)/2}} \Big(1 + \frac{2k.u+|u|^2}{(|k|^2+1)}\Big)^{-(s+r)/2} e^{2\pi i\, k.a}.
\end{aligned}$$

Let $M := \sup(N+n+g,0) \in \mathbb{N}_0$. We have uniformly in $k \in \mathbb{Z}^n$

$$\left(1+\tfrac{2k.u+|u|^2}{(|k|^2+1)}\right)^{-(s+r)/2} = \sum_{j=0}^{M} \binom{-(s+r)/2}{j} \tfrac{\left(2k.u+|u|^2\right)^j}{(|k|^2+1)^j} + O_{M,u}\left(\tfrac{(1+|s|)^{M+1}}{(|k|^2+1)^{(M+1)/2}}\right).$$

Thus, for $\sigma = \Re(s) > n+d$,

$$\begin{aligned} F_1(G;a;s) &= e^{2\pi i\, u.a} \sum_{|\alpha|_1 \le d} \tfrac{u^\alpha}{\alpha!} \sum_{k\in\mathbb{Z}^n} \tfrac{\partial^\alpha P(k)}{(|k|^2+1)^{(s+r)/2}} \left(1+\tfrac{2k.u+|u|^2}{(|k|^2+1)}\right)^{-(s+r)/2} e^{2\pi i\, k.a} \\ &= e^{2\pi i\, u.a} \sum_{|\alpha|_1 \le d} \sum_{j=0}^{M} \tfrac{u^\alpha}{\alpha!} \binom{-(s+r)/2}{j} \sum_{k\in\mathbb{Z}^n} \tfrac{\partial^\alpha P(k)\left(2k.u+|u|^2\right)^j}{(|k|^2+1)^{(s+r+2j)/2}} e^{2\pi i\, k.a} \\ &\quad + O_{G,M,u}\left((1+|s|)^{M+1} \sum_{k\in\mathbb{Z}^n} \tfrac{1}{(|k|^2+1)^{(\sigma+M+1-g)/2}}\right). \end{aligned} \tag{9}$$

Set $I := \{(\alpha,j) \in \mathbb{N}_0^n \times \{0,\ldots,M\} \mid |\alpha|_1 \le d\}$ and $I^* := I \setminus \{\,(0,0)\,\}$.
Set also $G_{(\alpha,j);u}(X) := \frac{\partial^\alpha P(X)\left(2X.u+|u|^2\right)^j}{(|X|^2+1)^{(r+2j)/2}} \in \mathcal{F}$ for all $(\alpha,j) \in I^*$.
Since $M \ge N+n+g$, it follows from (9) that

$$(1-e^{2\pi i\, u.a})\, F_1(G;a;s) = e^{2\pi i\, u.a} \sum_{(\alpha,j)\in I^*} \tfrac{u^\alpha}{\alpha!} \binom{-(s+r)/2}{j} F_1\left(G_{(\alpha,j);u};\alpha;s\right) + R_N(G;a;u;s) \tag{10}$$

where $s \mapsto R_N(G;a;u;s)$ is a holomorphic function in the half plane $\{\sigma = \Re(s) > -N\}$, in which it satisfies the bound $R_N(G;a;u;s) \ll_{G,N,u} 1$.
Moreover it is easy to see that, for any $(\alpha,j) \in I^*$,

$$\deg\left(G_{(\alpha,j);u}\right) = \deg(\partial^\alpha P) + j - (r+2j) \le d - |\alpha|_1 + j - (r+2j) = g - |\alpha|_1 - j \le g-1.$$

Relation (10) and the induction hypothesis imply then that

$$(1-e^{2\pi i\, u.a})\, F_1(G;a;s) \text{ verifies the conclusions of Lemma 2.6.} \tag{11}$$

Since $|1-e^{2\pi i\, u.a}| = 2|\sin(\pi u.a)| \ge d\,(u.a,\mathbb{Z})$, then (11) implies that $F_1(G;a;s)$ satisfies conclusions of Lemma 2.6. This completes the induction and the proof for $i=1$.

2.2.2 Proof of Lemma 2.6 for $i = 0$:

Let $N \in \mathbb{N}$ be a fixed integer. Let $G(X) = \frac{P(X)}{(X_1^2+\cdots+X_n^2+1)^{r/2}} \in \mathcal{F}$ and $g = \deg(G) = d-r$ where $d \ge 0$ is the degree of the polynomial P. Set also $M := \sup(N+g+n,0) \in \mathbb{N}_0$.
Since $P(k) \ll |k|^d$ for $k \in \mathbb{Z}^n \setminus \{\,0\,\}$, it follows that $F_0(G;a;s)$ and $F_1(G;a;s)$ converge absolutely in the half plane $\{\sigma = \Re(s) > n+g\}$.

Moreover, we have for $s=\sigma+i\tau\in\mathbb{C}$ verifying $\sigma>n+g$:

$$\begin{aligned}
F_0(G;a;s) &= \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \tfrac{G(k)}{(|k|^2+1-1)^{s/2}}\, e^{2\pi i\, k.a} = \sum_{k\in\mathbb{Z}^n}{}' \tfrac{G(k)}{(|k|^2+1)^{s/2}} \Big(1-\tfrac{1}{|k|^2+1}\Big)^{-s/2} e^{2\pi i\, k.a}\\
&= \sum_{k\in\mathbb{Z}^n}{}' \sum_{j=0}^{M} \tbinom{-s/2}{j}(-1)^j \tfrac{G(k)}{(|k|^2+1)^{(s+2j)/2}}\, e^{2\pi i\, k.a}\\
&\qquad\qquad + O_M\big((1+|s|)^{M+1}\sum_{k\in\mathbb{Z}^n}{}' \tfrac{|G(k)|}{(|k|^2+1)^{(\sigma+2M+2)/2}}\big)\\
&= \sum_{j=0}^{M} \tbinom{-s/2}{j}(-1)^j F_1(G;a;s+2j)\\
&\qquad\qquad + O_M\big[(1+|s|)^{M+1}\big(1+\sum_{k\in\mathbb{Z}^n}{}' \tfrac{|G(k)|}{(|k|^2+1)^{(\sigma+2M+2)/2}}\big)\big]. \qquad (12)
\end{aligned}$$

In addition we have uniformly in $s=\sigma+i\tau\in\mathbb{C}$ verifying $\sigma>-N$,

$$\sum_{k\in\mathbb{Z}^n}{}' \tfrac{|G(k)|}{(|k|^2+1)^{(\sigma+2M+2)/2}} \ll \sum_{k\in\mathbb{Z}^n}{}' \tfrac{|k|^g}{(|k|^2+1)^{(-N+2M+2)/2}} \ll \sum_{k\in\mathbb{Z}^n}{}' \tfrac{1}{|k|^{n+1}} < +\infty.$$

So (12) and Lemma 2.6 for $i=1$ imply that Lemma 2.6 is also true for $i=0$. This completes the proof of Lemma 2.6. □

2.2.3 Proof of item $(i.2)$ of Theorem 2.5:

Since $a\in\mathbb{R}^n\setminus\mathbb{Z}^n$, there exists $i_0\in\{1,\dots,n\}$ such that $a_{i_0}\notin\mathbb{Z}$. In particular $d(a.e_{i_0},\mathbb{Z})=d(a_{i_0},\mathbb{Z})>0$. Therefore, a satisfies the assumption of Lemma 2.6 with $u=e_{i_0}$. Thus, for all $N\in\mathbb{N}$, $s\mapsto f_a(s)=F_0(P;a;s)$ has a holomorphic continuation to the half-plane $\{\Re(s)>-N\}$. It follows, by letting $N\to\infty$, that $s\mapsto f_a(s)$ has a holomorphic continuation to the whole complex plane $\mathbb{C}$.

2.2.4 Proof of item (iii) of Theorem 2.5:

Let $\Theta\in\mathcal{M}_n(\mathbb{R})$, $(\varepsilon_i)_i\in\{-1,0,1\}^q$ and $b\in\mathcal{S}(\mathbb{Z}^n\times\mathbb{Z}^n)$. We assume that Θ is a badly approximable matrix. Set $\mathcal{Z}:=\{\,l=(l_1,\dots,l_q)\in(\mathbb{Z}^n)^q : \sum_i\varepsilon_i l_i=0\,\}$ and $P\in\mathbb{C}[X_1,\dots,X_n]$ of degree $d\geq 0$.
It is easy to see that for $\sigma>n+d$:

$$\sum_{l\in(\mathbb{Z}^n)^q\setminus\mathcal{Z}} |b(l)| \sum_{k\in\mathbb{Z}^n}{}' \tfrac{|P(k)|}{|k|^\sigma}\, |e^{2\pi i\, k.\Theta\sum_i\varepsilon_i l_i}| \ll_P \sum_{l\in(\mathbb{Z}^n)^q\setminus\mathcal{Z}} |b(l)| \sum_{k\in\mathbb{Z}^n}{}' \tfrac{1}{|k|^{\sigma-d}} \ll_{P,\sigma} \sum_{l\in(\mathbb{Z}^n)^q\setminus\mathcal{Z}} |b(l)| < +\infty.$$

So

$$g_0(s):=\sum_{l\in(\mathbb{Z}^n)^q\setminus\mathcal{Z}} b(l)\, f_{\Theta\sum_i\varepsilon_i l_i}(s) = \sum_{l\in(\mathbb{Z}^n)^q\setminus\mathcal{Z}} b(l) \sum_{k\in\mathbb{Z}^n}{}' \tfrac{P(k)}{|k|^s} e^{2\pi i\, k.\Theta\sum_i\varepsilon_i l_i}$$

converges absolutely in the half plane $\{\Re(s)>n+d\}$.
Moreover with the notations of Lemma 2.6, we have for all $s=\sigma+i\tau\in\mathbb{C}$ verifying $\sigma>n+d$:

$$g_0(s)=\sum_{l\in(\mathbb{Z}^n)^q\setminus\mathcal{Z}} b(l) f_{\Theta\sum_i\varepsilon_i l_i}(s) = \sum_{l\in(\mathbb{Z}^n)^q\setminus\mathcal{Z}} b(l) F_0(P;\Theta\sum_i\varepsilon_i l_i;s) \qquad (13)$$

But Θ is badly approximable, so there exists $u \in \mathbb{Z}^n$ and $\delta, c > 0$ such

$$|q.{}^t\Theta u - m| \geq c\,(1+|q|)^{-\delta},\ \forall q \in \mathbb{Z}^n \setminus \{\,0\,\},\ \forall m \in \mathbb{Z}.$$

We deduce that $\forall l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}$,

$$|\big(\Theta \sum\nolimits_i \varepsilon_i l_i\big).u - m| = |\big(\sum\nolimits_i \varepsilon_i l_i\big).{}^t\Theta u - m| \geq c\,\big(1+|\sum\nolimits_i \varepsilon_i l_i|\big)^{-\delta} \geq c\,(1+|l|)^{-\delta}.$$

It follows that there exists $u \in \mathbb{Z}^n$, $\delta > 0$ and $c > 0$ such that

$$\forall l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}, \quad d\big((\Theta \sum\nolimits_i \varepsilon_i l_i).u; \mathbb{Z}\big) \geq c\,(1+|l|)^{-\delta}. \tag{14}$$

Therefore, for any $l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}$, the vector $a = \Theta \sum_i \varepsilon_i l_i$ verifies the assumption of Lemma 2.6 with the same u. Moreover δ and c in (14) are also independent on l.

We fix now $N \in \mathbb{N}$. Lemma 2.6 implies that there exist positive constants $C_0 := C_0(P,N,u)$, $B_0 := B_i(P,N,u)$ and $A_0 := A_0(P,N,u)$ such that for all $l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}$, $s \mapsto F_0(P; \Theta \sum_i \varepsilon_i l_i; s)$ extends *holomorphically* to the half plane $\{\Re(s) > -N\}$ and verifies in it the bound

$$F_0(P; \Theta \sum_i \varepsilon_i l_i; s) \leq C_0\,(1+|s|)^{B_0}\ d\big((\Theta \sum\nolimits_i \varepsilon_i l_i).u; \mathbb{Z}\big)^{-A_0}.$$

This and (14) imply that for any compact set K included in the half plane $\{\Re(s) > -N\}$, there exist two constants $C := C(P,N,c,\delta,u,K)$ and $D := D(P,N,c,\delta,u)$ (independent on $l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}$) such that

$$\forall s \in K \text{ and } \forall l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}, \quad F_0(P; \Theta \sum_i \varepsilon_i l_i; s) \leq C\,(1+|l|)^D\,. \tag{15}$$

It follows that $s \mapsto \sum_{l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}} b(l) F_0(P; \Theta \sum_i \varepsilon_i l_i; s)$ has a holomorphic continuation to the half plane $\{\Re(s) > -N\}$.

This and (13) imply that $s \mapsto g_0(s) = \sum_{l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}} b(l) f_{\Theta \sum_i \varepsilon_i l_i}(s)$ has a holomorphic continuation to $\{\Re(s) > -N\}$. Since N is an arbitrary integer, by letting $N \to \infty$, it follows that $s \mapsto g_0(s)$ has a holomorphic continuation to the whole complex plane $\mathbb{C}$ which completes the proof of the theorem. □

Remark 2.8. *By equation (11), we see that a Diophantine condition is sufficient to get Lemma 2.6. Our Diophantine condition appears also (in equivalent form) in Connes [7, Prop. 49] (see Remark 4.2 below). The following heuristic argument shows that our condition seems to be necessary in order to get the result of Theorem 2.5:*
For simplicity we assume $n = 1$ (but the argument extends easily to any n).
Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. We know (see this reflection formula in [15, p. 6]) that for any $l \in \mathbb{Z} \setminus \{0\}$,

$$g_{\theta l}(s) := \sum_{k\in\mathbb{Z}}{}' \tfrac{e^{2\pi i \theta l k}}{|k|^s} = \tfrac{\pi^{s-1/2}}{\Gamma(\frac{1-s}{2})} \Gamma(\tfrac{s}{2})\ h_{\theta l}(1-s) \ \textit{where}\ h_{\theta l}(s) := \sum_{k\in\mathbb{Z}}{}' \tfrac{1}{|\theta l + k|^s}.$$

So, for any $(a_l) \in \mathcal{S}(\mathbb{Z})$, the existence of meromorphic continuation of $g_0(s) := \sum'_{l\in\mathbb{Z}} a_l\, g_{\theta l}(s)$ is equivalent to the existence of meromorphic continuation of

$$h_0(s) := \sum_{l\in\mathbb{Z}}{}' a_l\, h_{\theta l}(s) = \sum_{l\in\mathbb{Z}}{}' a_l \sum_{k\in\mathbb{Z}}{}' \tfrac{1}{|\theta l + k|^s}.$$

So, for at least one $\sigma_0 \in \mathbb{R}$, we must have $\frac{|a_l|}{|\theta l + k|^{\sigma_0}} = O(1)$ uniformly in $k, l \in \mathbb{Z}^*$.
It follows that for any $(a_l) \in \mathcal{S}(\mathbb{Z})$, $|\theta l + k| \gg |a_l|^{1/\sigma_0}$ uniformly in $k, l \in \mathbb{Z}^*$. Therefore, our Diophantine condition seems to be necessary.

2.2.5 Commutation between sum and residue

Let $p \in \mathbb{N}$. Recall that $\mathcal{S}((\mathbb{Z}^n)^p)$ is the set of the Schwartz sequences on $(\mathbb{Z}^n)^p$. In other words, $b \in \mathcal{S}((\mathbb{Z}^n)^p)$ if and only if for all $r \in \mathbb{N}_0$, $(1+|l_1|^2+\cdots|l_p|^2)^r\,|b(l_1,\cdots,l_p)|^2$ is bounded on $(\mathbb{Z}^n)^p$. We note that if $Q \in \mathbb{R}[X_1,\cdots,X_{np}]$ is a polynomial, $(a_j) \in \mathcal{S}(\mathbb{Z}^n)^p$, $b \in \mathcal{S}(\mathbb{Z}^n)$ and ϕ a real-valued function, then $l := (l_1,\cdots,l_p) \mapsto \widetilde{a}(l)\, b(-\widehat{l_p})\, Q(l)\, e^{i\phi(l)}$ is a Schwartz sequence on $(\mathbb{Z}^n)^p$, where

$$\begin{aligned}\widetilde{a}(l) &:= a_1(l_1)\cdots a_p(l_p),\\ \widehat{l_i} &:= l_1+\ldots+l_i.\end{aligned}$$

In the following, we will use several times the fact that for any $(k,l) \in (\mathbb{Z}^n)^2$ such that $k \neq 0$ and $k \neq -l$, we have

$$\frac{1}{|k+l|^2} = \frac{1}{|k|^2} - \frac{2k.l+|l|^2}{|k|^2|k+l|^2}\,. \tag{16}$$

Lemma 2.9. *There exists a polynomial $P \in \mathbb{R}[X_1,\cdots,X_p]$ of degree $4p$ and with positive coefficients such that for any $k \in \mathbb{Z}^n$, and $l := (l_1,\cdots,l_p) \in (\mathbb{Z}^n)^p$ such that $k \neq 0$ and $k \neq -\widehat{l_i}$ for all $1 \le i \le p$, the following holds:*

$$\frac{1}{|k+\widehat{l_1}|^2\ldots|k+\widehat{l_p}|^2} \le \frac{1}{|k|^{2p}}\;P(|l_1|,\cdots,|l_p|).$$

Proof. Let's fix i such that $1 \le i \le p$. Using two times (16), Cauchy–Schwarz inequality and the fact that $|k+\widehat{l_i}|^2 \ge 1$, we get

$$\begin{aligned}\tfrac{1}{|k+\widehat{l_i}|^2} &\le \tfrac{1}{|k|^2} + \tfrac{2|k||\widehat{l_i}|+|\widehat{l_i}|^2}{|k|^4} + \tfrac{(2|k||\widehat{l_i}|+|\widehat{l_i}|^2)^2}{|k|^4|k+\widehat{l_i}|^2}\\ &\le \tfrac{1}{|k|^2} + \tfrac{2}{|k|^3}|\widehat{l_i}| + \big(\tfrac{1}{|k|^4}+\tfrac{4}{|k|^2}\big)|\widehat{l_i}|^2 + \tfrac{4}{|k|^3}|\widehat{l_i}|^3 + \tfrac{1}{|k|^4}|\widehat{l_i}|^4.\end{aligned}$$

Since $|k| \ge 1$, and $|\widehat{l_i}|^j \le |\widehat{l_i}|^4$ if $1 \le j \le 4$, we find

$$\begin{aligned}\tfrac{1}{|k+\widehat{l_i}|^2} &\le \tfrac{5}{|k|^2}\sum\nolimits_{j=0}^4 |\widehat{l_i}|^j \le \tfrac{5}{|k|^2}\big(1+4|\widehat{l_i}|^4\big) \le \tfrac{5}{|k|^2}\big(1+4(\sum\nolimits_{j=1}^p |l_j|)^4\big),\\ \tfrac{1}{|k+\widehat{l_1}|^2\ldots|k+\widehat{l_p}|^2} &\le \tfrac{5^p}{|k|^{2p}}\big(1+4(\sum\nolimits_{j=1}^p |l_j|)^4\big)^p.\end{aligned}$$

Taking $P(X_1,\cdots,X_p) := 5^p\big(1+4(\sum_{j=1}^p X_j)^4\big)^p$ now gives the result. ∎

Lemma 2.10. *Let $b \in \mathcal{S}((\mathbb{Z}^n)^p)$, $p \in \mathbb{N}$, $P_j \in \mathbb{R}[X_1,\cdots,X_n]$ be a homogeneous polynomial function of degree j, $k \in \mathbb{Z}^n$, $l := (l_1,\cdots,l_p) \in (\mathbb{Z}^n)^p$, $r \in \mathbb{N}_0$, ϕ be a real-valued function on $\mathbb{Z}^n \times (\mathbb{Z}^n)^p$ and*

$$h(s,k,l) := \frac{b(l)\,P_j(k)\,e^{i\phi(k,l)}}{|k|^{s+r}|k+\widehat{l_1}|^2\cdots|k+\widehat{l_p}|^2}\,,$$

with $h(s,k,l) := 0$ if, for $k \neq 0$, one of the denominators is zero.
For all $s \in \mathbb{C}$ such that $\Re(s) > n+j-r-2p$, the series

$$H(s) := {\sum}'_{(k,l)\in(\mathbb{Z}^n)^{p+1}} h(s,k,l)$$

is absolutely summable. In particular,

$${\sum_{k\in\mathbb{Z}^n}}' \sum_{l\in(\mathbb{Z}^n)^p} h(s,k,l) = \sum_{l\in(\mathbb{Z}^n)^p} {\sum_{k\in\mathbb{Z}^n}}' h(s,k,l)\,.$$

Proof. Let $s = \sigma + i\tau \in \mathbb{C}$ such that $\sigma > n + j - r - 2p$. By Lemma 2.9 we get, for $k \neq 0$,

$$|h(s,k,l)| \leq |b(l)\, P_j(k)|\, |k|^{-r-\sigma-2p}\, P(l),$$

where $P(l) := P(|l_1|, \cdots, |l_p|)$ and P is a polynomial of degree $4p$ with positive coefficients. Thus, $|h(s,k,l)| \leq F(l)\, G(k)$ where $F(l) := |b(l)|\, P(l)$ and $G(k) := |P_j(k)||k|^{-r-\sigma-2p}$. The summability of $\sum_{l\in(\mathbb{Z}^n)^p} F(l)$ is implied by the fact that $b \in \mathcal{S}((\mathbb{Z}^n)^p)$. The summability of $\sum'_{k\in\mathbb{Z}^n} G(k)$ is a consequence of the fact that $\sigma > n + j - r - 2p$. Finally, as a product of two summable series, $\sum_{k,l} F(l)G(k)$ is a summable series, which proves that $\sum_{k,l} h(s,k,l)$ is also absolutely summable. ☐

Definition 2.11. *Let f be a function on $D \times (\mathbb{Z}^n)^p$ where D is an open neighborhood of 0 in $\mathbb{C}$. We say that f satisfies (H1) if and only if there exists $\rho > 0$ such that*

(i) for any l, $s \mapsto f(s,l)$ extends as a holomorphic function on U_ρ, where U_ρ is the open disk of center 0 and radius ρ,

(ii) the series $\sum_{l\in(\mathbb{Z}^n)^p} \|H(\cdot,l)\|_{\infty,\rho}$ is summable,where $\|H(\cdot,l)\|_{\infty,\rho} := \sup_{s\in U_\rho} |H(s,l)|$.

We say that f satisfies (H2) if and only if there exists $\rho > 0$ such that

(i) for any l, $s \mapsto f(s,l)$ extends as a holomorphic function on $U_\rho - \{0\}$,

(ii) for any δ such that $0 < \delta < \rho$, the series $\sum_{l\in(\mathbb{Z}^n)^p} \|H(\cdot,l)\|_{\infty,\delta,\rho}$ is summable, where $\|H(\cdot,l)\|_{\infty,\delta,\rho} := \sup_{\delta<|s|<\rho} |H(s,l)|$.

Remark 2.12. *Note that (H1) implies (H2). Moreover, if f satisfies (H1) (resp. (H2) for $\rho > 0$, then it is straightforward to check that $f : s \mapsto \sum_{l\in(\mathbb{Z}^n)^p} f(s,l)$ extends as an holomorphic function on U_ρ (resp. on $U_\rho \setminus \{\,0\,\}$).*

Corollary 2.13. *With the same notations of Lemma 2.10, suppose that $r + 2p - j > n$, then, the function $H(s,l) := \sum'_{k\in\mathbb{Z}^n} h(s,k,l)$ satisfies (H1).*

Proof. (i) Let's fix $\rho > 0$ such that $\rho < r + 2p - j - n$. Since $r + 2p - j > n$, U_ρ is inside the half-plane of absolute convergence of the series defined by $H(s,l)$. Thus, $s \mapsto H(s,l)$ is holomorphic on U_ρ.
(ii) Since $\big||k|^{-s}\big| \leq |k|^\rho$ for all $s \in U_\rho$ and $k \in \mathbb{Z}^n \setminus \{\,0\,\}$, we get as in the above proof

$$|h(s,k,l)| \leq |b(l)\, P_j(k)|\, |k|^{-r+\rho-2p}\, P(|l_1|, \cdots, |l_p|).$$

Since $\rho < r + 2p - j - n$, the series $\sum'_{k\in\mathbb{Z}^n} |P_j(k)||k|^{-r+\rho-2p}$ is summable.
Thus, $\|H(\cdot,l)\|_{\infty,\rho} \leq K\, F(l)$ where $K := {\sum_k}' |P_j(k)||k|^{-r+\rho-2p} < \infty$. We have already seen that the series $\sum_l F(l)$ is summable, so we get the result. ☐

We note that if f and g both satisfy (H1) (or (H2)), then so does $f + g$. In the following, we will use the equivalence relation

$$f \sim g \Longleftrightarrow f - g \text{ satisfies (H1)}.$$

Lemma 2.14. *Let f and g be two functions on $D \times (\mathbb{Z}^n)^p$ where D is an open neighborhood of 0 in $\mathbb{C}$, such that $f \sim g$ and such that g satisfies (H2). Then*

$$\operatorname*{Res}_{s=0} \sum_{l\in(\mathbb{Z}^n)^p} f(s,l) = \sum_{l\in(\mathbb{Z}^n)^p} \operatorname*{Res}_{s=0} g(s,l)\,.$$

Proof. Since $f \sim g$, f satisfies (H2) for a certain $\rho > 0$. Let's fix η such that $0 < \eta < \rho$ and define C_η as the circle of center 0 and radius η. We have

$$\operatorname*{Res}_{s=0} g(s,l) = \operatorname*{Res}_{s=0} f(s,l) = \tfrac{1}{2\pi i} \oint_{C_\eta} f(s,l)\, ds = \int_I u(t,l) dt\,.$$

where $I = [0, 2\pi]$ and $u(t,l) := \frac{1}{2\pi} \eta e^{it} f(\eta\, e^{it}, l)$. The fact that f satisfies (H2) entails that the series $\sum_{l\in(\mathbb{Z}^n)^p} \|f(\cdot,l)\|_{\infty,C_\eta}$ is summable. Thus, since $\|u(\cdot,l)\|_\infty = \frac{1}{2\pi}\eta\, \|f(\cdot,l)\|_{\infty,C_\eta}$, the series $\sum_{l\in(\mathbb{Z}^n)^p} \|u(\cdot,l)\|_\infty$ is summable, so, as a consequence, $\int_I \sum_{l\in(\mathbb{Z}^n)^p} u(t,l)dt = \sum_{l\in(\mathbb{Z}^n)^p} \int_I u(t,l)dt$ which gives the result. □

2.3 Computation of residues of zeta functions

Since, we will have to compute residues of series, let us introduce the following

Definition 2.15.

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s},$$

$$Z_n(s) := \sum_{k\in\mathbb{Z}^n}{}' \, |k|^{-s},$$

$$\zeta_{p_1,\dots,p_n}(s) := \sum_{k\in\mathbb{Z}^n}{}' \, \frac{k_1^{p_1}\cdots k_n^{p_n}}{|k|^s} \quad \text{, for } p_i \in \mathbb{N},$$

where $\zeta(s)$ is the Riemann zeta function (see [25] or [14]).
By the symmetry $k \to -k$, it is clear that these functions $\zeta_{p_1,\dots,p_n}$ all vanish for odd values of p_i.
Let us now compute $\zeta_{0,\cdots,0,1_i,0\cdots,0,1_j,0\cdots,0}(s)$ in terms of $Z_n(s)$:
Since $\zeta_{0,\cdots,0,1_i,0\cdots,0,1_j,0\cdots,0}(s) = A_i(s)\,\delta_{ij}$, exchanging the components k_i and k_j, we get

$$\zeta_{0,\cdots,0,1_i,0\cdots,0,1_j,0\cdots,0}(s) = \tfrac{\delta_{ij}}{n}\, Z_n(s-2).$$

Similarly,

$$\textstyle\sum'_{\mathbb{Z}^n} \frac{k_1^2 k_2^2}{|k|^{s+8}} = \frac{1}{n(n-1)} Z_n(s+4) - \frac{1}{n-1} \sum'_{\mathbb{Z}^n} \frac{k_1^4}{|k|^{s+8}}$$

but it is difficult to write explicitly $\zeta_{p_1,\dots,p_n}(s)$ in terms of $Z_n(s-4)$ and other $Z_n(s-m)$ when at least four indices p_i are non zero.
When all p_i are even, $\zeta_{p_1,\dots,p_n}(s)$ is a nonzero series of fractions $\frac{P(k)}{|k|^s}$ where P is a homogeneous polynomial of degree $p_1 + \cdots + p_n$. Theorem 2.1 now gives us the following

Proposition 2.16. *$\zeta_{p_1,\dots,p_n}$ has a meromorphic extension to the whole plane with a unique pole at $n + p_1 + \cdots + p_n$. This pole is simple and the residue at this pole is*

$$\operatorname*{Res}_{s=n+p_1+\cdots+p_n} \zeta_{p_1,\dots,p_n}(s) = 2\, \frac{\Gamma(\frac{p_1+1}{2})\cdots\Gamma(\frac{p_n+1}{2})}{\Gamma(\frac{n+p_1+\cdots+p_n}{2})} \tag{17}$$

when all p_i are even or this residue is zero otherwise.
In particular, for $n = 2$,

$$\operatorname*{Res}_{s=0} \sum_{k\in\mathbb{Z}^2}{}' \, \tfrac{k_i k_j}{|k|^{s+4}} = \delta_{ij}\,\pi\,, \tag{18}$$

and for $n = 4$,

$$\operatorname*{Res}_{s=0} \sum_{k\in\mathbb{Z}^4}{}' \tfrac{k_ik_j}{|k|^{s+6}} = \delta_{ij}\,\tfrac{\pi^2}{2}\,,$$

$$\operatorname*{Res}_{s=0} \sum_{k\in\mathbb{Z}^4}{}' \tfrac{k_ik_jk_lk_m}{|k|^{s+8}} = (\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})\,\tfrac{\pi^2}{12}\,. \tag{19}$$

Proof. Equation (17) follows from Theorem (2.1)

$$\operatorname*{Res}_{s=n+p_1+\cdots+p_n} \zeta_{p_1,\ldots,p_n}(s) = \int_{k\in S^{n-1}} k_1^{p_1}\cdots k_n^{p_n}\, dS(k)$$

and standard formulae (see for instance [32, VIII,1;22]). Equation (18) is a straightforward consequence of Equation (17). Equation (19) can be checked for the cases $i = j \neq l = m$ and $i = j = l = m$. □

Note that $Z_n(s)$ is an Epstein zeta function associated to the quadratic form $q(x) := x_1^2+...+x_n^2$, so Z_n satisfies the following functional equation

$$Z_n(s) = \pi^{s-n/2}\Gamma(n/2-s/2)\Gamma(s/2)^{-1}\, Z_n(n-s).$$

Since $\pi^{s-n/2}\Gamma(n/2-s/2)\,\Gamma(s/2)^{-1} = 0$ for any negative even integer n and $Z_n(s)$ is meromorphic on $\mathbb{C}$ with only one pole at $s = n$ with residue $2\pi^{n/2}\Gamma(n/2)^{-1}$ according to previous proposition, so we get $Z_n(0) = -1$. We have proved that

$$\operatorname*{Res}_{s=0} Z_n(s+n) = 2\pi^{n/2}\,\Gamma(n/2)^{-1}, \tag{20}$$

$$Z_n(0) = -1. \tag{21}$$

2.4 Meromorphic continuation of a class of zeta functions

Let $n, q \in \mathbb{N}$, $q \geq 2$, and $p = (p_1, \ldots, p_{q-1}) \in \mathbb{N}_0^{q-1}$.
Set $I := \{i \mid p_i \neq 0\}$ and assume that $I \neq \emptyset$ and

$$\mathcal{I} := \{\alpha = (\alpha_i)_{i\in I} \mid \forall i \in I\ \alpha_i = (\alpha_{i,1}, \ldots, \alpha_{i,p_i}) \in \mathbb{N}_0^{p_i}\} = \prod_{i\in I} \mathbb{N}_0^{p_i}.$$

We will use in the sequel also the following notations:

- for $x = (x_1, \ldots, x_t) \in \mathbb{R}^t$ recall that $|x|_1 = |x_1| + \cdots + |x_t|$ and $|x| = \sqrt{x_1^2 + \cdots + x_t^2}$;
- for all $\alpha = (\alpha_i)_{i\in I} \in \mathcal{I} = \prod_{i\in I} \mathbb{N}_0^{p_i}$,

$$|\alpha|_1 = \sum_{i\in I} |\alpha_i|_1 = \sum_{i\in I}\sum_{j=1}^{p_i} |\alpha_{i,j}| \text{ and } \tbinom{1/2}{\alpha} = \prod_{i\in I} \tbinom{1/2}{\alpha_i} = \prod_{i\in I}\prod_{j=1}^{p_i} \tbinom{1/2}{\alpha_{i,j}}.$$

2.4.1 A family of polynomials

In this paragraph we define a family of polynomials which plays an important role later.
Consider first the variables:

- for $X_1, \ldots, X_n$ we set $X = (X_1, \ldots, X_n)$;
- for any $i = 1, \ldots, 2q$, we consider the variables $Y_{i,1}, \ldots, Y_{i,n}$ and set $Y_i := (Y_{i,1}, \ldots, Y_{i,n})$ and $Y := (Y_1, \ldots, Y_{2q})$;

- for $Y=(Y_1,\dots,Y_{2q})$, we set for any $1\le j\le q$, $\widetilde{Y}_j := Y_1+\cdots+Y_j+Y_{q+1}+\cdots+Y_{q+j}$.
We define for all $\alpha=(\alpha_i)_{i\in I}\in\mathcal{I}=\prod_{i\in I}\mathbb{N}_0^{p_i}$ the polynomial

$$P_\alpha(X,Y) := \prod_{i\in I}\prod_{j=1}^{p_i}(2\langle X,\widetilde{Y}_i\rangle + |\widetilde{Y}_i|^2)^{\alpha_{i,j}}. \tag{22}$$

It is clear that $P_\alpha(X,Y)\in\mathbb{Z}[X,Y]$, $\deg_X P_\alpha \le |\alpha|_1$ and $\deg_Y P_\alpha \le 2|\alpha|_1$.
Let us fix a polynomial $Q\in\mathbb{R}[X_1,\cdots,X_n]$ and note $d:=\deg Q$. For $\alpha\in\mathcal{I}$, we want to expand $P_\alpha(X,Y)\,Q(X)$ in homogeneous polynomials in X and Y so defining

$$L(\alpha) := \{\,\beta\in\mathbb{N}_0^{(2q+1)n} \,:\, |\beta|_1-d_\beta\le 2|\alpha|_1 \text{ and } d_\beta\le|\alpha|_1+d\,\}$$

where $d_\beta := \sum_1^n \beta_i$, we set

$$\tbinom{1/2}{\alpha}P_\alpha(X,Y)\,Q(X) =: \sum_{\beta\in L(\alpha)} c_{\alpha,\beta}\,X^\beta Y^\beta$$

where $c_{\alpha,\beta}\in\mathbb{R}$, $X^\beta := X_1^{\beta_1}\cdots X_n^{\beta_n}$ and $Y^\beta := Y_{1,1}^{\beta_{n+1}}\cdots Y_{2q,n}^{\beta_{2(q+1)n}}$. By definition, X^β is a homogeneous polynomial of degree in X equals to d_β. We note

$$M_{\alpha,\beta}(Y) := c_{\alpha,\beta}\,Y^\beta.$$

2.4.2 Residues of a class of zeta functions

In this section we will prove the following result, used in Proposition 5.4 for the computation of the spectrum dimension of the noncommutative torus:

Theorem 2.17. *(i) Let $\frac{1}{2\pi}\Theta$ be a badly approximable matrix, and $\widetilde{a}\in\mathcal{S}\big((\mathbb{Z}^n)^{2q}\big)$. Then*

$$s\mapsto f(s) := \sum_{l\in[(\mathbb{Z}^n)^q]^2}\widetilde{a}_l \sum_{k\in\mathbb{Z}^n}{}' \prod_{i=1}^{q-1}|k+\widetilde{l}_i|^{p_i}|k|^{-s}\,Q(k)\,e^{ik.\Theta\sum_1^q l_j}$$

has a meromorphic continuation to the whole complex plane $\mathbb{C}$ with at most simple possible poles at the points $s=n+d+|p|_1-m$ where $m\in\mathbb{N}_0$.
(ii) Let $m\in\mathbb{N}_0$ and set $I(m) := \{\,(\alpha,\beta)\in\mathcal{I}\times\mathbb{N}_0^{(2q+1)n} \,:\, \beta\in L(\alpha)$ and $m=2|\alpha|_1-d_\beta+d\,\}$. Then $I(m)$ is a finite set and $s=n+d+|p|_1-m$ is a pole of f if and only if

$$C(f,m) := \sum_{l\in Z}\widetilde{a}_l\sum_{(\alpha,\beta)\in I(m)} M_{\alpha,\beta}(l)\int_{u\in S^{n-1}} u^\beta\,dS(u)\ne 0,$$

with $Z:=\{l \,:\, \sum_1^q l_j=0\}$ and the convention $\sum_\emptyset=0$. In that case $s=n+d+|p|_1-m$ is a simple pole of residue $\operatorname*{Res}_{s=n+d+|p|_1-m} f(s) = C(f,m)$.

In order to prove the theorem above we need the following

Lemma 2.18. *For all $N\in\mathbb{N}$ we have*

$$\prod_{i=1}^{q-1}|k+\widetilde{l}_i|^{p_i} = \sum_{\alpha=(\alpha_i)_{i\in I}\in\prod_{i\in I}\{0,\dots,N\}^{p_i}} \tbinom{1/2}{\alpha}\,\tfrac{P_\alpha(k,l)}{|k|^{2|\alpha|_1-|p|_1}} + \mathcal{O}_N(|k|^{|p|_1-(N+1)/2})$$

uniformly in $k\in\mathbb{Z}^n$ and $l\in(\mathbb{Z}^n)^{2q}$ verifying $|k|>U(l):=36\,(\sum_{i=1,\,i\ne q}^{2q-1}|l_i|)^4$.

Proof. For $i = 1, \dots, q-1$, we have uniformly in $k \in \mathbb{Z}^n$ and $l \in (\mathbb{Z}^n)^{2q}$ verifying $|k| > U(l)$,

$$\tfrac{\big|2\langle k,\widetilde{l}_i\rangle + |\widetilde{l}_i|^2\big|}{|k|^2} \leq \tfrac{\sqrt{U(l)}}{2|k|} < \tfrac{1}{2\sqrt{|k|}}. \tag{23}$$

In that case,

$$|k+\widetilde{l}_i| \;=\; \big(|k|^2 + 2\langle k,\widetilde{l}_i\rangle + |\widetilde{l}_i|^2\big)^{1/2} = |k|\big(1 + \tfrac{2\langle k,\widetilde{l}_i\rangle + |\widetilde{l}_i|^2}{|k|^2}\big)^{1/2} = \sum_{u=0}^{\infty} \tbinom{1/2}{u} \tfrac{1}{|k|^{2u-1}} P_u^i(k,l)$$

where for all $i = 1, \dots, q-1$ and for all $u \in \mathbb{N}_0$,

$$P_u^i(k,l) := \big(2\langle k,\widetilde{l}_i\rangle + |\widetilde{l}_i|^2\big)^u,$$

with the convention $P_0^i(k,l) := 1$.
In particular $P_u^i(k,l) \in \mathbb{Z}[k,l]$, $\deg_k P_u^i \leq u$ and $\deg_l P_u^i \leq 2u$. Inequality (23) implies that for all $i = 1, \dots, q-1$ and for all $u \in \mathbb{N}$,

$$\tfrac{1}{|k|^{2u}}\, |P_u^i(k,l)| \leq \big(2\sqrt{|k|}\big)^{-u}$$

uniformly in $k \in \mathbb{Z}^n$ and $l \in (\mathbb{Z}^n)^{2q}$ verifying $|k| > U(l)$.
Let $N \in \mathbb{N}$. We deduce from the previous that for any $k \in \mathbb{Z}^n$ and $l \in (\mathbb{Z}^n)^{2q}$ verifying $|k| > U(l)$ and for all $i = 1, \dots, q-1$, we have

$$\begin{aligned} |k+\widetilde{l}_i| \;&=\; \sum_{u=0}^{N} \tbinom{1/2}{u} \tfrac{1}{|k|^{2u-1}} P_u^i(k,l) + \mathcal{O}\big(\sum_{u>N} |k|\, |\tbinom{1/2}{u}|\, (2\sqrt{|k|})^{-u}\big) \\ &=\; \sum_{u=0}^{N} \tbinom{1/2}{u} \tfrac{1}{|k|^{2u-1}} P_u^i(k,l) + \mathcal{O}_N\big(\tfrac{1}{|k|^{(N-1)/2}}\big). \end{aligned}$$

It follows that for any $N \in \mathbb{N}$, we have uniformly in $k \in \mathbb{Z}^n$ and $l \in (\mathbb{Z}^n)^{2q}$ verifying $|k| > U(l)$ and for all $i \in I$,

$$|k+\widetilde{l}_i|^{p_i} = \sum_{\alpha_i \in \{0,\dots,N\}^{p_i}} \tbinom{1/2}{\alpha_i} \tfrac{1}{|k|^{2|\alpha_i|_1 - p_i}} P_{\alpha_i}^i(k,l) + \mathcal{O}_N\left(\tfrac{1}{|k|^{(N+1)/2 - p_i}}\right)$$

where $P_{\alpha_i}^i(k,l) = \prod_{j=1}^{p_i} P_{\alpha_{i,j}}^i(k,l)$ for all $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,p_i}) \in \{0,\dots,N\}^{p_i}$ and

$$\prod_{i\in I} |k+\widetilde{l}_i|^{p_i} = \sum_{\alpha=(\alpha_i)\in\prod_{i\in I}\{0,\dots,N\}^{p_i}} \tbinom{1/2}{\alpha} \tfrac{1}{|k|^{2|\alpha|_1 - |p|_1}} P_\alpha(k,l) + \mathcal{O}_N\big(\tfrac{1}{|k|^{(N+1)/2 - |p|_1}}\big)$$

where $P_\alpha(k,l) = \prod_{i\in I} P_{\alpha_i}^i(k,l) = \prod_{i\in I}\prod_{j=1}^{p_i} P_{\alpha_{i,j}}^i(k,l)$. □

Proof of Theorem 2.17.
(i) All n, q, $p = (p_1, \dots, p_{q-1})$ and $\widetilde{a} \in \mathcal{S}\left((\mathbb{Z}^n)^{2q}\right)$ are fixed as above and we define formally for any $l \in (\mathbb{Z}^n)^{2q}$

$$F(l,s) := \sum_{k\in\mathbb{Z}^n}{}' \prod_{i=1}^{q-1} |k+\widetilde{l}_i|^{p_i}\, Q(k)\, e^{ik.\Theta\sum_1^q l_j}\, |k|^{-s}. \tag{24}$$

Thus, still formally,

$$f(s) := \sum_{l\in(\mathbb{Z}^n)^{2q}} \widetilde{a}_l\ F(l,s). \tag{25}$$

It is clear that $F(l,s)$ converges absolutely in the half plane $\{\sigma = \Re(s) > n + d + |p|_1\}$ where $d = \deg Q$.

Let $N \in \mathbb{N}$. Lemma 2.18 implies that for any $l \in (\mathbb{Z}^n)^{2q}$ and for $s \in \mathbb{C}$ such that $\sigma > n+|p|_1+d$,

$$\begin{aligned} F(l,s) = &\sum_{|k|\le U(l)}{}' \prod_{i=1}^{q-1} |k+\widetilde{l}_i|^{p_i}\, Q(k)\, e^{ik.\Theta\sum_1^q l_j}\, |k|^{-s} \\ &+ \sum_{\alpha=(\alpha_i)_{i\in I}\in\prod_{i\in I}\{0,\dots,N\}^{p_i}} \tbinom{1/2}{\alpha} \sum_{|k|>U(l)} \tfrac{1}{|k|^{s+2|\alpha|_1-|p|_1}} P_\alpha(k,l) Q(k)\, e^{ik.\Theta\sum_1^q l_j} + G_N(l,s). \end{aligned}$$

where $s \mapsto G_N(l,s)$ is a holomorphic function in the half-plane $D_N := \{\sigma > n + d + |p|_1 - \frac{N+1}{2}\}$ and verifies in it the bound $G_N(l,s) \ll_{N,\sigma} 1$ uniformly in l.

It follows that

$$F(l,s) = \sum_{\alpha=(\alpha_i)_{i\in I}\in\prod_{i\in I}\{0,\dots,N\}^{p_i}} H_\alpha(l,s) + R_N(l,s), \tag{26}$$

where

$$\begin{aligned} H_\alpha(l,s) &:= \sum_{|k|\le U(l)}{}' \tbinom{1/2}{\alpha} \tfrac{1}{|k|^{s+2|\alpha|_1-|p|_1}} P_\alpha(k,l)\, Q(k)\, e^{ik.\Theta\sum_1^q l_j}, \\ R_N(l,s) &:= \sum_{|k|\le U(l)}{}' \prod_{i=1}^{q-1} |k+\widetilde{l}_i|^{p_i}\, Q(k)\, e^{ik.\Theta\sum_1^q l_j}\, |k|^{-s} \\ &\quad - \sum_{|k|\le U(l)}{}' \sum_{\alpha=(\alpha_i)_{i\in I}\in\prod_{i\in I}\{0,\dots,N\}^{p_i}} \tbinom{1/2}{\alpha} \tfrac{P_\alpha(k,l)}{|k|^{s+2|\alpha|_1-|p|_1}} Q(k)\, e^{ik.\Theta\sum_1^q l_j} + G_N(l,s). \end{aligned}$$

In particular there exists $A(N) > 0$ such that $s \mapsto R_N(l,s)$ extends holomorphically to the half-plane D_N and verifies in it the bound $R_N(l,s) \ll_{N,\sigma} 1 + |l|^{A(N)}$ uniformly in l.

Let us note formally

$$h_\alpha(s) := \sum_l \widetilde{a}_l\, H_\alpha(l,s).$$

Equation (26) and $R_N(l,s) \ll_{N,\sigma} 1 + |l|^{A(N)}$ imply that

$$f(s) \sim_N \sum_{\alpha=(\alpha_i)_{i\in I}\in\prod_{i\in I}\{0,\dots,N\}^{p_i}} h_\alpha(s), \tag{27}$$

where $\sim_N$ means modulo a holomorphic function in D_N.

Recall the decomposition $\binom{1/2}{\alpha} P_\alpha(k,l)\, Q(k) = \sum_{\beta\in L(\alpha)} M_{\alpha,\beta}(l)\, k^\beta$ and we decompose similarly $h_\alpha(s) = \sum_{\beta\in L(\alpha)} h_{\alpha,\beta}(s)$. Theorem 2.5 now implies that for all $\alpha = (\alpha_i)_{i\in I} \in \prod_{i\in I}\{0,\dots,N\}^{p_i}$ and $\beta \in L(\alpha)$,

- the map $s \mapsto h_{\alpha,\beta}(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ with only one simple possible pole at $s = n + |p|_1 - 2|\alpha|_1 + d_\beta$,

- the residue at this point is equal to

$$\operatorname*{Res}_{s=n+|p|_1-2|\alpha|_1+d_\beta} h_{\alpha,\beta}(s) = \sum_{l\in\mathcal{Z}} \widetilde{a}_l\, M_{\alpha,\beta}(l) \int_{u\in S^{n-1}} u^\beta dS(u) \tag{28}$$

where $\mathcal{Z} := \{l \in (\mathbb{Z})^n)^{2q} \,:\, \sum_1^q l_j = 0\}$. If the right hand side is zero, $h_{\alpha,\beta}(s)$ is holomorphic on $\mathbb{C}$.
By (27), we deduce therefore that $f(s)$ has a meromorphic continuation on the halfplane D_N, with only simple possible poles in the set $\{\, n+|p|_1+k \,:\, -2N|p|_1 \leq k \leq d\,\}$. Taking now $N \to \infty$ yields the result.
(ii) Let $m \in \mathbb{N}_0$ and set $I(m) := \{\, (\alpha,\beta) \in \mathcal{I} \times \mathbb{N}_0^{(2q+1)n} \,:\, \beta \in L(\alpha) \text{ and } m = 2|\alpha|_1 - d_\beta + d\,\}$. If $(\alpha,\beta) \in I(m)$, then $|\alpha|_1 \leq m$ and $|\beta|_1 \leq 3m+d$, so $I(m)$ is finite.
With a chosen N such that $2N|p|_1 + d > m$, we get by (27) and (28)

$$\underset{s=n+d+|p|_1-m}{\mathrm{Res}} f(s) = \sum_{l \in \mathcal{Z}} \widetilde{a}_l \sum_{(\alpha,\beta)\in I(m)} M_{\alpha,\beta}(l) \int_{u \in S^{n-1}} u^\beta \, dS(u) = C(f,m)$$

with the convention $\sum_\emptyset = 0$. Thus, $n+d+|p|_1-m$ is a pole of f if and only if $C(f,m) \neq 0$. ☐

3 Noncommutative integration on a simple spectral triple

In this section, we revisit the notion of noncommutative integral pioneered by Alain Connes, paying particular attention to the reality (Tomita–Takesaki) operator J and to kernels of perturbed Dirac operators by symmetrized one-forms.

3.1 Kernel dimension

We will have to compare here the kernels of $\mathcal{D}$ and $\mathcal{D}_A$ which are both finite dimensional:

Lemma 3.1. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a spectral triple with a reality operator J and chirality χ. If $A \in \Omega^1_{\mathcal{D}}$ is a one-form, the fluctuated Dirac operator*

$$\mathcal{D}_A := \mathcal{D} + A + \epsilon JAJ^{-1}$$

(where $\mathcal{D}J = \epsilon\, J\mathcal{D}$, $\epsilon = \pm 1$) is an operator with compact resolvent, and in particular its kernel $\mathrm{Ker}\,\mathcal{D}_A$ *is a finite dimensional space. This space is invariant by J and χ.*

Proof. Let T be a bounded operator and let z be in the resolvent of $\mathcal{D}+T$ and z' be in the resolvent of $\mathcal{D}$. Then

$$(\mathcal{D}+T-z)^{-1} = (\mathcal{D}-z')^{-1}\,[1-(X+z'-z)(\mathcal{D}+T-z)^{-1}].$$

Since $(\mathcal{D}-z')^{-1}$ is compact by hypothesis and since the term in bracket is bounded, $\mathcal{D}+T$ has a compact resolvent. Applying this to $T = A+\epsilon JAJ^{-1}$, $\mathcal{D}_A$ has a finite dimensional kernel (see for instance [27, Theorem 6.29]).
Since according to the dimension, $J^2 = \pm 1$, J commutes or anticommutes with χ, χ commutes with the elements in the algebra $\mathcal{A}$ and $\mathcal{D}\chi = -\chi\mathcal{D}$ (see [10] or [23, p. 405]), we get $\mathcal{D}_A\chi = -\chi\mathcal{D}_A$ and $\mathcal{D}_A J = \pm J\mathcal{D}_A$ which gives the result. ☐

3.2 Pseudodifferential operators

Let $(A,\mathcal{D},\mathcal{H})$ be a given real regular spectral triple of dimension n.
We note

$$P_0 \text{ the projection on } \mathrm{Ker}\,\mathcal{D}\,, P_A \text{ the projection on } \mathrm{Ker}\,\mathcal{D}_A\,,$$
$$D := \mathcal{D}+P_0\,, D_A := \mathcal{D}_A + P_A\,.$$

P_0 and P_A are thus finite-rank selfadjoint bounded operators. We remark that D and D_A are selfadjoint invertible operators with compact inverses.

Remark 3.2. *Since we only need to compute the residues and the value at 0 of the ζ_D, ζ_{D_A} functions, it is not necessary to define the operators $\mathcal{D}^{-1}$ or $\mathcal{D}_A^{-1}$ and the associated zeta functions. However, we can remark that all the work presented here could be done using the process of Higson in [26] which proves that we can add any smoothing operator to $\mathcal{D}$ or $\mathcal{D}_A$ such that the result is invertible without changing anything to the computation of residues.*

Define for any $\alpha \in \mathbb{R}$

$$OP^0 := \{T \,:\, t \mapsto F_t(T) \in C^\infty\big(\mathbb{R}, \mathcal{B}(\mathcal{H})\big)\},$$
$$OP^\alpha := \{\,T \,:\, T|D|^{-\alpha} \in OP^0\,\}.$$

where $F_t(T) := e^{it|D|}\,T\,e^{-it|D|} = e^{it|\mathcal{D}|}\,T\,e^{-it|\mathcal{D}|}$ since $|D| = |\mathcal{D}| + P_0$. Define

$$\delta(T) := [|D|, T],$$
$$\nabla(T) := [\mathcal{D}^2, T],$$
$$\sigma_s(T) := |D|^s T |D|^{-s},\ s \in \mathbb{C}.$$

It has been shown in [13] that $OP^0 = \bigcap_{p\geq 0} \mathrm{Dom}(\delta^p)$, for $p \in \mathbb{N}_0$. In particular, OP^0 is a subalgebra of $\mathcal{B}(\mathcal{H})$ (while elements of OP^α are not necessarily bounded for $\alpha > 0$) and $\mathcal{A} \subseteq OP^0$, $J\mathcal{A}J^{-1} \subseteq OP^0$, $[\mathcal{D}, \mathcal{A}] \subseteq OP^0$. Note that $P_0 \in OP^{-\infty}$ and $\delta(OP^0) \subseteq OP^0$.
For any $t > 0$, $\mathcal{D}^t$ and and $|\mathcal{D}|^t$ are in OP^t and for any $\alpha \in \mathbb{R}, D^\alpha$ and $|D|^\alpha$ are in OP^α. By hypothesis, $|D|^{-n} \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ so for any $\alpha > n$, $OP^{-\alpha} \subseteq \mathcal{L}^1(\mathcal{H})$.

Lemma 3.3. *[13]*
(i) For any $T \in OP^0$ and $s \in \mathbb{C}$, $\sigma_s(T) \in OP^0$.
(ii) For any $\alpha, \beta \in \mathbb{R}$, $OP^\alpha OP^\beta \subseteq OP^{\alpha+\beta}$.
(iii) If $\alpha \leq \beta$, $OP^\alpha \subseteq OP^\beta$.
(iv) For any α, $\delta(OP^\alpha) \subseteq OP^\alpha$.
(v) For any α and $T \in OP^\alpha$, $\nabla(T) \in OP^{\alpha+1}$.

Proof. See the appendix. □

Remark 3.4. *Any operator in OP^α, where $\alpha \in \mathbb{R}$, extends as a continuous linear operator from* $\mathrm{Dom}\,|D|^{\alpha+1}$ *to* $\mathrm{Dom}\,|D|$ *where the* $\mathrm{Dom}\,|D|^\alpha$ *spaces have their natural norms (see [13, 26]).*

We now introduce a definition of pseudodifferential operators in a slightly different way than in [9, 13, 26] which in particular pays attention to the reality operator J and the kernel of $\mathcal{D}$ and allows $\mathcal{D}$ and $|D|^{-1}$ to be a pseudodifferential operators. It is more in the spirit of [4].

Definition 3.5. *Let us define $\mathcal{D}(\mathcal{A})$ as the polynomial algebra generated by $\mathcal{A}$, $J\mathcal{A}J^{-1}$, $\mathcal{D}$ and $|\mathcal{D}|$.*
A pseudodifferential operator is an operator T such that there exists $d \in \mathbb{Z}$ such that for any $N \in \mathbb{N}$, there exist $p \in \mathbb{N}_0$, $P \in \mathcal{D}(\mathcal{A})$ and $R \in OP^{-N}$ (p, P and R may depend on N) such that $P\,D^{-2p} \in OP^d$ and

$$T = P\,D^{-2p} + R\,.$$

Define $\Psi(\mathcal{A})$ as the set of pseudodifferential operators and $\Psi(\mathcal{A})^k := \Psi(\mathcal{A}) \cap OP^k$.

Note that if A is a 1-form, A and JAJ^{-1} are in $\mathcal{D}(\mathcal{A})$ and moreover $\mathcal{D}(\mathcal{A}) \subseteq \cup_{p\in\mathbb{N}_0} OP^p$. Since $|\mathcal{D}| \in \mathcal{D}(\mathcal{A})$ by construction and P_0 is a pseudodifferential operator, for any $p \in \mathbb{Z}$, $|D|^p$ is a pseudodifferential operator (in OP^p.) Let us remark also that $\mathcal{D}(\mathcal{A}) \subseteq \Psi(\mathcal{A}) \subseteq \cup_{k\in\mathbb{Z}} OP^k$.

Lemma 3.6. *[9, 13] The set of all pseudodifferential operators $\Psi(\mathcal{A})$ is an algebra. Moreover, if $T \in \Psi(\mathcal{A})^d$ and $T \in \Psi(\mathcal{A})^{d'}$, then $TT' \in \Psi(\mathcal{A})^{d+d'}$.*

Proof. See the appendix. □

Due to the little difference of behavior between scalar and nonscalar pseudodifferential operators (i.e. when coefficients like $[\mathcal{D}, a]$, $a \in \mathcal{A}$ appears in P of Definition 3.5), it is convenient to also introduce

Definition 3.7. *Let $\mathcal{D}_1(\mathcal{A})$ be the algebra generated by $\mathcal{A}$, JAJ^{-1} and $\mathcal{D}$, and $\Psi_1(\mathcal{A})$ be the set of pseudodifferential operators constructed as before with $\mathcal{D}_1(\mathcal{A})$ instead of $\mathcal{D}(\mathcal{A})$. Note that $\Psi_1(\mathcal{A})$ is subalgebra of $\Psi(\mathcal{A})$.*

Remark that $\Psi_1(\mathcal{A})$ does not necessarily contain operators such as $|D|^k$ where $k \in \mathbb{Z}$ is odd. This algebra is similar to the one defined in [4].

3.3 Zeta functions and dimension spectrum

For any operator B and if X is either D or D_A, we define

$$\zeta_X^B(s) := \mathrm{Tr}\left(B|X|^{-s}\right),$$
$$\zeta_X(s) := \mathrm{Tr}\left(|X|^{-s}\right).$$

The dimension spectrum $Sd(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of a spectral triple has been defined in [9,13]. It is extended here to pay attention to the operator J and to our definition of pseudodifferential operator.

Definition 3.8. *The spectrum dimension of the spectral triple is the subset $Sd(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of all poles of the functions $\zeta_D^P := s \mapsto \mathrm{Tr}\left(P|D|^{-s}\right)$ where P is any pseudodifferential operator in OP^0. The spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is simple when these poles are all simple.*

Remark 3.9. *If $Sp(\mathcal{A}, \mathcal{H}, \mathcal{D})$ denotes the set of all poles of the functions $s \mapsto \mathrm{Tr}\left(P|D|^{-s}\right)$ where P is any pseudodifferential operator, then, $Sd(\mathcal{A}, \mathcal{H}, \mathcal{D}) \subseteq Sp(\mathcal{A}, \mathcal{H}, \mathcal{D})$.*
When $Sp(\mathcal{A}, \mathcal{H}, \mathcal{D}) = \mathbb{Z}$, $Sd(\mathcal{A}, \mathcal{H}, \mathcal{D}) = \{\, n - k \;:\; k \in \mathbb{N}_0 \,\}$: indeed, if P is a pseudodifferential operator in OP^0, and $q \in \mathbb{N}$ is such that $q > n$, $P|D|^{-s}$ is in $OP^{-\Re(s)}$ so is trace-class for s in a neighborhood of q; as a consequence, q cannot be a pole of $s \mapsto \mathrm{Tr}\left(P|D|^{-s}\right)$.

Remark 3.10. *$Sp(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is also the set of poles of functions $s \mapsto \mathrm{Tr}\left(B|D|^{-s-2p}\right)$ where $p \in \mathbb{N}_0$ and $B \in \mathcal{D}(\mathcal{A})$.*

3.4 The noncommutative integral $⨍$

We already used the one parameter group $\sigma_z(T) := |D|^z T |D|^{-z}$, $z \in \mathbb{C}$.
Introducing the notation (recall that $\nabla(T) = [\mathcal{D}^2, T]$) for an operator T,

$$\varepsilon(T) := \nabla(T) D^{-2},$$

we get from [4, (2.44)] the following expansion for $T \in OP^q$

$$\sigma_z(T) \sim \sum_{r=0}^{N} g(z,r)\, \varepsilon^r(T) \quad \bmod OP^{-N-1+q} \tag{29}$$

where $g(z,r) := \frac{1}{r!}(\frac{z}{2}) \cdots (\frac{z}{2} - (r-1)) = \binom{z/2}{r}$ with the convention $g(z,0) := 1$.
We define the noncommutative integral by

$$-\!\!\!\!\!\!\int T := \operatorname*{Res}_{s=0} \zeta_D^T(s) = \operatorname*{Res}_{s=0} \operatorname{Tr}\big(T|D|^{-s}\big).$$

Proposition 3.11. *[13] If the spectral triple is simple, $-\!\!\!\!\!\!\int$ is a trace on $\Psi(\mathcal{A})$.*

Proof. See the appendix. □

4 Residues of ζ_{D_A} for a spectral triple with simple dimension spectrum

We fix a regular spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of dimension n and a self-adjoint 1-form A.
Recall that

$$\mathcal{D}_A := \mathcal{D} + \widetilde{A} \text{ where } \widetilde{A} := A + \varepsilon J A J^{-1},$$
$$D_A := \mathcal{D}_A + P_A$$

where P_A is the projection on $\operatorname{Ker} \mathcal{D}_A$. Remark that $\widetilde{A} \in \mathcal{D}(\mathcal{A}) \cap OP^0$ and $\mathcal{D}_A \in \mathcal{D}(\mathcal{A}) \cap OP^1$.
We note

$$V_A := P_A - P_0.$$

As the following lemma shows, V_A is a smoothing operator:

Lemma 4.1. *(i)* $\bigcap_{k\geq 1} \operatorname{Dom}(\mathcal{D}_A)^k \subseteq \bigcap_{k\geq 1} \operatorname{Dom}|D|^k$
(ii) $\operatorname{Ker} \mathcal{D}_A \subseteq \bigcap_{k\geq 1} \operatorname{Dom}|D|^k$
(iii) For any $\alpha, \beta \in \mathbb{R}$, $|D|^\beta P_A |D|^\alpha$ is bounded.
(iv) $P_A \in OP^{-\infty}$.

Proof. (i) Let us define for any $p \in \mathbb{N}$, $R_p := (\mathcal{D}_A)^p - \mathcal{D}^p$, so $R_p \in OP^{p-1}$ and $R_p\big(\operatorname{Dom}|D|^p\big) \subseteq \operatorname{Dom}|D|$ (see Remark 3.4).
Let us fix $k \in \mathbb{N}$, $k \geq 2$. Since $\operatorname{Dom} \mathcal{D}_A = \operatorname{Dom} \mathcal{D} = \operatorname{Dom}|D|$, we have

$$\operatorname{Dom}(\mathcal{D}_A)^k = \{\, \phi \in \operatorname{Dom}|D| \ : \ (\mathcal{D}^j + R_j)\, \phi \in \operatorname{Dom}|D| \ , \ \forall j \ \ 1 \leq j \leq k-1 \,\}.$$

Let $\phi \in \operatorname{Dom}(\mathcal{D}_A)^k$. We prove by recurrence that for any $j \in \{\, 1, \cdots, k-1 \,\}$, $\phi \in \operatorname{Dom}|D|^{j+1}$:
We have $\phi \in \operatorname{Dom}|D|$ and $(\mathcal{D} + R_1)\, \phi \in \operatorname{Dom}|D|$. Thus, since $R_1\, \phi \in \operatorname{Dom}|D|$, $\mathcal{D}\phi \in \operatorname{Dom}|D|$, which proves that $\phi \in \operatorname{Dom}|D|^2$. Hence, case $j = 1$ is done.
Suppose now that $\phi \in \operatorname{Dom}|D|^{j+1}$ for a $j \in \{\, 1, \cdots, k-2 \,\}$. Since $(\mathcal{D}^{j+1} + R_{j+1})\, \phi \in \operatorname{Dom}|D|$, and $R_{j+1}\, \phi \in \operatorname{Dom}|D|$, we get $\mathcal{D}^{j+1}\, \phi \in \operatorname{Dom}|D|$, which proves that $\phi \in \operatorname{Dom}|D|^{j+2}$.
Finally, if we set $j = k-1$, we get $\phi \in \operatorname{Dom}|D|^k$, so $\operatorname{Dom}(\mathcal{D}_A)^k \subseteq \operatorname{Dom}|D|^k$.
(ii) follows from $\operatorname{Ker} \mathcal{D}_A \subseteq \bigcap_{k\geq 1} \operatorname{Dom}(\mathcal{D}_A)^k$ and (i).

(iii) Let us first check that $|D|^\alpha P_A$ is bounded. We define D_0 as the operator with domain $\operatorname{Dom} D_0 = \operatorname{Im} P_A \cap \operatorname{Dom} |D|^\alpha$ and such that $D_0\,\phi = |D|^\alpha\,\phi$. Since $\operatorname{Dom} D_0$ is finite dimensional, D_0 extends as a bounded operator on $\mathcal{H}$ with finite rank. We have

$$\sup_{\phi\in \operatorname{Dom}|D|^\alpha P_A,\ \|\phi\|\leq 1} \| |D|^\alpha P_A\,\phi\| \leq \sup_{\phi\in \operatorname{Dom} D_0,\ \|\phi\|\leq 1} \| |D|^\alpha\,\phi\| = \|D_0\| < \infty$$

so $|D|^\alpha P_A$ is bounded. We can remark that by (ii), $\operatorname{Dom} D_0 = \operatorname{Im} P_A$ and $\operatorname{Dom}|D|^\alpha P_A = \mathcal{H}$.
Let us prove now that $P_A|D|^\alpha$ is bounded: Let $\phi \in \operatorname{Dom} P_A|D|^\alpha = \operatorname{Dom}|D|^\alpha$. By (ii), we have $\operatorname{Im} P_A \subseteq \operatorname{Dom}|D|^\alpha$ so we get

$$\begin{aligned}\|P_A|D|^\alpha\,\phi\| &\leq \sup_{\psi\in\operatorname{Im} P_A,\ \|\psi\|\leq 1} |<\psi, |D|^\alpha\,\phi>| \leq \sup_{\psi\in\operatorname{Im} P_A,\ \|\psi\|\leq 1} |<|D|^\alpha\psi,\phi>| \\ &\leq \sup_{\psi\in\operatorname{Im} P_A,\ \|\psi\|\leq 1} \| |D|^\alpha\psi\|\,\|\phi\| = \|D_0\|\,\|\phi\|\,.\end{aligned}$$

(iv) For any $k\in\mathbb{N}_0$ and $t\in\mathbb{R}$, $\delta^k(P_A)|D|^t$ is a linear combination of terms of the form $|D|^\beta P_A|D|^\alpha$, so the result follows from (iii). $\square$

Remark 4.2. *We will see later on the noncommutative torus example how important is the difference between $\mathcal{D}_A$ and $\mathcal{D}+A$. In particular, the inclusion $\operatorname{Ker}\mathcal{D} \subseteq \operatorname{Ker}\mathcal{D}+A$ is not satisfied since A does not preserve $\operatorname{Ker}\mathcal{D}$ contrarily to $\widetilde{A}$.*

The coefficient of the nonconstant term Λ^k $(k>0)$ in the expansion (5) of the spectral action $S(\mathcal{D}_A,\Phi,\Lambda)$ is equal to the residue of $\zeta_{D_A}(s)$ at k. We will see in this section how we can compute these residues in term of noncommutative integral of certain operators.
Define for any operator T, $p\in\mathbb{N}$, $s\in\mathbb{C}$,

$$K_p(T,s) := (-\tfrac{s}{2})^p \int_{0\leq t_1\leq\cdots\leq t_p\leq 1} \sigma_{-st_1}(T)\cdots\sigma_{-st_p}(T)\,dt$$

with $dt := dt_1\cdots dt_p$.
Remark that if $T\in OP^\alpha$, then $\sigma_z(T)\in OP^\alpha$ for $z\in\mathbb{C}$ and $K_p(T,s)\in OP^{\alpha p}$.
Let us define

$$\begin{aligned} X &:= \mathcal{D}_A^2 - \mathcal{D}^2 = \widetilde{A}\mathcal{D} + \mathcal{D}\widetilde{A} + \widetilde{A}^2 \\ X_V &:= X + V_A, \end{aligned}$$

thus $X\in\mathcal{D}_1(\mathcal{A})\cap OP^1$ and by Lemma 4.1,

$$X_V \sim X \mod OP^{-\infty}. \tag{30}$$

We will use

$$Y := \log(D_A^2) - \log(D^2)$$

which makes sense since $D_A^2 = \mathcal{D}_A^2 + P_A$ is invertible for any A.
By definition of X_V, we get

$$Y = \log(D^2 + X_V) - \log(D^2).$$

Lemma 4.3. *[4]*
(i) Y is a pseudodifferential operator in OP^{-1} with the following expansion for any $N \in \mathbb{N}$

$$Y \sim \sum_{p=1}^{N} \sum_{k_1,\cdots,k_p=0}^{N-p} \tfrac{(-1)^{|k|_1+p+1}}{|k|_1+p} \nabla^{k_p}(X\nabla^{k_{p-1}}(\cdots X\nabla^{k_1}(X)\cdots))D^{-2(|k|_1+p)} \mod OP^{-N-1}.$$

(ii) For any $N \in \mathbb{N}$ and $s \in \mathbb{C}$,

$$|D_A|^{-s} \sim |D|^{-s} + \sum_{p=1}^{N} K_p(Y,s)|D|^{-s} \mod OP^{-N-1-\Re(s)}. \tag{31}$$

Proof. (i) We follow [4, Lemma 2.2]. By functional calculus, $Y = \int_0^\infty I(\lambda)\, d\lambda$, where

$$I(\lambda) \sim \sum_{p=1}^{N} (-1)^{p+1}\big((D^2+\lambda)^{-1}X_V\big)^p(D^2+\lambda)^{-1} \mod OP^{-N-3}$$

By (30), $\big((D^2+\lambda)^{-1}X_V\big)^p \sim \big((D^2+\lambda)^{-1}X\big)^p \mod OP^{-\infty}$ and we get

$$I(\lambda) \sim \sum_{p=1}^{N} (-1)^{p+1}\big((D^2+\lambda)^{-1}X\big)^p(D^2+\lambda)^{-1} \mod OP^{-N-3}$$

We set $A_p(X) := \big((D^2+\lambda)^{-1}X\big)^p(D^2+\lambda)^{-1}$ and $L := (D^2+\lambda)^{-1} \in OP^{-2}$ for a fixed λ. Since $[D^2+\lambda, X] \sim \nabla(X) \mod OP^{-\infty}$, a recurrence proves that if T is an operator in OP^r, then, for $q \in \mathbb{N}_0$,

$$A_1(T) = LTL \sim \sum_{k=0}^{q} (-1)^k \nabla^k(T)L^{k+2} \mod OP^{r-q-5}.$$

With $A_p(X) = LXA_{p-1}(X)$, another recurrence gives, for any $q \in \mathbb{N}_0$,

$$A_p(X) \sim \sum_{k_1,\cdots,k_p=0}^{q} (-1)^{|k|_1} \nabla^{k_p}(X\nabla^{k_{p-1}}(\cdots X\nabla^{k_1}(X)\cdots))L^{|k|_1+p+1} \mod OP^{-q-p-3},$$

which entails that

$$I(\lambda) \sim \sum_{p=1}^{N}(-1)^{p+1} \sum_{k_1,\cdots,k_p=0}^{N-p} (-1)^{|k|_1} \nabla^{k_p}(X\nabla^{k_{p-1}}(\cdots X\nabla^{k_1}(X)\cdots))L^{|k|_1+p+1} \mod OP^{-N-3}.$$

With $\int_0^\infty (D^2+\lambda)^{-(|k|_1+p+1)}d\lambda = \frac{1}{|k|_1+p}D^{-2(|k|_1+p)}$, we get the result provided we control the remainders. Such a control is given in [4, (2.27)].
(ii) We have $|D_A|^{-s} = e^{B-(s/2)Y}e^{-B}\,|D|^{-s}$ where $B := (-s/2)\log(D^2)$. Following [4, Theorem 2.4], we get

$$|D_A|^{-s} = |D|^{-s} + \sum_{p=1}^{\infty} K_p(Y,s)|D|^{-s}\,. \tag{32}$$

and each $K_p(Y,s)$ is in OP^{-p}. ∎

Corollary 4.4. *For any $p \in \mathbb{N}$ and $r_1,\cdots,r_p \in \mathbb{N}_0$, $\varepsilon^{r_1}(Y)\cdots\varepsilon^{r_p}(Y) \in \Psi_1(\mathcal{A})$.*

Proof. If for any $q \in \mathbb{N}$ and $k = (k_1, \cdots, k_q) \in \mathbb{N}_0^q$,

$$\Gamma_q^k(X) := \tfrac{(-1)^{|k|_1+q+1}}{|k|_1+q} \nabla^{k_q}(X\nabla^{k_{q-1}}(\cdots X\nabla^{k_1}(X)\cdots)),$$

then, $\Gamma_q^k(X) \in OP^{|k|_1+q}$. For any $N \in \mathbb{N}$,

$$Y \sim \sum_{q=1}^{N} \sum_{k_1,\cdots,k_q=0}^{N-q} \Gamma_q^k(X) D^{-2(|k|_1+q)} \mod OP^{-N-1}. \tag{33}$$

Note that the $\Gamma_q^k(X)$ are in $\mathcal{D}_1(\mathcal{A})$, which, with (33) proves that Y and thus $\varepsilon^r(Y) = \nabla^r(Y)D^{-2r}$, are also in $\Psi_1(\mathcal{A})$. ∎

We remark, as in [11], that the fluctuations leave invariant the first term of the spectral action (5). This is a generalization of the fact that in the commutative case, the noncommutative integral depends only on the principal symbol of the Dirac operator $\mathcal{D}$ and this symbol is stable by adding a gauge potential like in $\mathcal{D} + A$. Note however that the symmetrized gauge potential $A + \epsilon JAJ^{-1}$ is always zero in this case for any selfadjoint one-form A.

Lemma 4.5. *If the spectral triple is simple, formula (6) can be extended as*

$$\zeta_{D_A}(0) - \zeta_D(0) = \sum_{q=1}^{n} \tfrac{(-1)^q}{q} ⨍ (\widetilde{A}D^{-1})^q. \tag{34}$$

Proof. Since the spectral triple is simple, equation (32) entails that

$$\zeta_{D_A}(0) - \zeta_D(0) = \operatorname{Tr}(K_1(Y,s)|D|^{-s})_{|s=0}\,.$$

Thus, with (29), we get $\zeta_{D_A}(0) - \zeta_D(0) = -\frac{1}{2}⨍ Y$. Replacing A by $\widetilde{A}$, the same proof as in [4] gives

$$-\tfrac{1}{2} ⨍ Y = \sum_{q=1}^{n} \tfrac{(-1)^q}{q} ⨍ (\widetilde{A}D^{-1})^q.$$ ∎

Lemma 4.6. *For any $k \in \mathbb{N}_0$,*

$$\operatorname*{Res}_{s=n-k} \zeta_{D_A}(s) = \operatorname*{Res}_{s=n-k} \zeta_D(s) + \sum_{p=1}^{k} \sum_{r_1,\cdots,r_p=0}^{k-p} \operatorname*{Res}_{s=n-k} h(s,r,p)\ \operatorname{Tr}\big(\varepsilon^{r_1}(Y)\cdots\varepsilon^{r_p}(Y)|D|^{-s}\big),$$

where

$$h(s,r,p) := (-s/2)^p \int_{0\leq t_1\leq\cdots\leq t_p\leq 1} g(-st_1,r_1)\cdots g(-st_p,r_p)\,dt\,.$$

Proof. By Lemma 4.3 (*ii*), $|D_A|^{-s} \sim |D|^{-s} + \sum_{p=1}^{k} K_p(Y,s)|D|^{-s} \mod OP^{-(k+1)-\Re(s)}$, where the convention $\sum_\emptyset = 0$ is used. Thus, we get for s in a neighborhood of $n-k$,

$$|D_A|^{-s} - |D|^{-s} - \sum_{p=1}^{k} K_p(Y,s)|D|^{-s} \in OP^{-(k+1)-\Re(s)} \subseteq \mathcal{L}^1(\mathcal{H})$$

which gives

$$\operatorname*{Res}_{s=n-k} \zeta_{D_A}(s) = \operatorname*{Res}_{s=n-k} \zeta_D(s) + \sum_{p=1}^{k} \operatorname*{Res}_{s=n-k} \operatorname{Tr}\big(K_p(Y,s)|D|^{-s}\big). \tag{35}$$

Let us fix $1 \le p \le k$ and $N \in \mathbb{N}$. By (29) we get

$$K_p(Y,s) \sim (-\tfrac{s}{2})^p \int_{0\le t_1 \le \cdots t_p \le 1} \sum_{r_1,\cdots,r_p=0}^{N} g(-st_1, r_1)\cdots g(-st_p, r_p)$$
$$\varepsilon^{r_1}(Y)\cdots\varepsilon^{r_p}(Y)\, dt \quad \bmod OP^{-N-p-1} \tag{36}$$

If we now take $N = k-p$, we get for s in a neighborhood of $n-k$

$$K_p(Y,s)|D|^{-s} - \sum_{r_1,\cdots,r_p=0}^{k-p} h(s,r,p)\,\varepsilon^{r_1}(Y)\cdots\varepsilon^{r_p}(Y)|D|^{-s} \in OP^{-k-1-\Re(s)} \subseteq \mathcal{L}^1(\mathcal{H})$$

so (35) gives the result. □

Our operators $|D_A|^k$ are pseudodifferential operators:

Lemma 4.7. *For any $k \in \mathbb{Z}$, $|D_A|^k \in \Psi^k(\mathcal{A})$.*

Proof. Using (36), we see that $K_p(Y,s)$ is a pseudodifferential operator in OP^{-p}, so (31) proves that $|D_A|^k$ is a pseudodifferential operator in OP^k. □

The following result is quite important since it shows that one can use $⨍$ for D or D_A:

Proposition 4.8. *If the spectral triple is simple, $\operatorname*{Res}_{s=0} \operatorname{Tr}\big(P|D_A|^{-s}\big) = ⨍ P$ for any pseudodifferential operator P. In particular, for any $k \in \mathbb{N}_0$*

$$⨍ |D_A|^{-(n-k)} = \operatorname*{Res}_{s=n-k} \zeta_{D_A}(s).$$

Proof. Suppose $P \in OP^k$ with $k \in \mathbb{Z}$ and let us fix $p \ge 1$. With (36), we see that for any $N \in \mathbb{N}$,

$$PK_p(Y,s)|D|^{-s} \sim \sum_{r_1,\cdots,r_p=0}^{N} h(s,r,p)\, P\varepsilon^{r_1}(Y)\cdots\varepsilon^{r_p}(Y)|D|^{-s} \quad \bmod OP^{-N-p-1+k-\Re(s)}.$$

Thus if we take $N = n-p+k$, we get

$$\operatorname*{Res}_{s=0} \operatorname{Tr}\big(PK_p(Y,s)|D|^{-s}\big) = \sum_{r_1,\cdots,r_p=0}^{n-p+k} \operatorname*{Res}_{s=0} h(s,r,p) \operatorname{Tr}\big(P\varepsilon^{r_1}(Y)\cdots\varepsilon^{r_p}(Y)|D|^{-s}\big).$$

Since $s=0$ is a zero of the analytic function $s \mapsto h(s,r,p)$ and $s \mapsto \operatorname{Tr} P\varepsilon^{r_1}(Y)\cdots\varepsilon^{r_p}(Y)|D|^{-s}$ has only simple poles by hypothesis, we see that $\operatorname*{Res}_{s=0} h(s,r,p) \operatorname{Tr}\big(P\varepsilon^{r_1}(Y)\cdots\varepsilon^{r_p}(Y)|D|^{-s}\big) = 0$ and

$$\operatorname*{Res}_{s=0} \operatorname{Tr}\big(PK_p(Y,s)|D|^{-s}\big) = 0. \tag{37}$$

Using (31), $P|D_A|^{-s} \sim P|D|^{-s} + \sum_{p=1}^{k+n} PK_p(Y,s)|D|^{-s} \mod OP^{-n-1-\Re(s)}$ and thus,

$$\operatorname*{Res}_{s=0} \operatorname{Tr}(P|D_A|^{-s}) = ⨍ P + \sum_{p=1}^{k+n} \operatorname*{Res}_{s=0} \operatorname{Tr}\big(PK_p(Y,s)|D|^{-s}\big). \tag{38}$$

The result now follows from (37) and (38). To get the last equality, one uses the pseudodifferential operator $|D_A|^{-(n-k)}$. □

Proposition 4.9. *If the spectral triple is simple, then*

$$⨍ |D_A|^{-n} = ⨍ |D|^{-n}. \tag{39}$$

Proof. Lemma 4.6 and previous proposition for $k = 0$. □

Lemma 4.10. *If the spectral triple is simple,*

(*i*) $⨍ |D_A|^{-(n-1)} = ⨍ |D|^{-(n-1)} - (\frac{n-1}{2}) ⨍ X|D|^{-n-1}$.

(*ii*) $⨍ |D_A|^{-(n-2)} = ⨍ |D|^{-(n-2)} + \frac{n-2}{2}\big(-⨍ X|D|^{-n} + \frac{n}{4} ⨍ X^2|D|^{-2-n}\big)$.

Proof. (*i*) By (31),

$$\operatorname*{Res}_{s=n-1} \zeta_{D_A}(s) - \zeta_D(s) = \operatorname*{Res}_{s=n-1} (-s/2) \operatorname{Tr}\big(Y|D|^{-s}\big) = -\tfrac{n-1}{2} \operatorname*{Res}_{s=0} \operatorname{Tr}\big(Y|D|^{-(n-1)}|D|^{-s}\big)$$

where for the last equality we use the simple dimension spectrum hypothesis. Lemma 4.3 (*i*) yields $Y \sim XD^{-2} \mod OP^{-2}$ and $Y|D|^{-(n-1)} \sim X|D|^{-n-1} \mod OP^{-n-1} \subseteq \mathcal{L}^1(\mathcal{H})$. Thus,

$$\operatorname*{Res}_{s=0} \operatorname{Tr}\big(Y|D|^{-(n-1)}|D|^{-s}\big) = \operatorname*{Res}_{s=0} \operatorname{Tr}\big(X|D|^{-n-1}|D|^{-s}\big) = ⨍ X|D|^{-n-1}.$$

(*ii*) Lemma 4.6 (*ii*) gives

$$\operatorname*{Res}_{s=n-2} \zeta_{D_A}(s) = \operatorname*{Res}_{s=n-2} \zeta_D(s) + \operatorname*{Res}_{s=n-2} \sum_{r=0}^{1} h(s,r,1) \operatorname{Tr}\big(\varepsilon^r(Y)|D|^{-s}\big) + h(s,0,2) \operatorname{Tr}\big(Y^2|D|^{-s}\big).$$

We have $h(s,0,1) = -\frac{s}{2}$, $h(s,1,1) = \frac{1}{2}(\frac{s}{2})^2$ and $h(s,0,2) = \frac{1}{2}(\frac{s}{2})^2$. Using again Lemma 4.3 (*i*),

$$Y \sim XD^{-2} - \tfrac{1}{2}\nabla(X)D^{-4} - \tfrac{1}{2}X^2D^{-4} \mod OP^{-3}.$$

Thus,

$$\operatorname*{Res}_{s=n-2} \operatorname{Tr}\big(Y|D|^{-s}\big) = ⨍ X|D|^{-n} - \tfrac{1}{2} ⨍ (\nabla(X) + X^2)|D|^{-2-n}.$$

Moreover, using $⨍ \nabla(X)|D|^{-k} = 0$ for any $k \geq 0$ since $⨍$ is a trace,

$$\operatorname*{Res}_{s=n-2} \operatorname{Tr}\big(\varepsilon(Y)|D|^{-s}\big) = \operatorname*{Res}_{s=n-2} \operatorname{Tr}\big(\nabla(X)D^{-4}|D|^{-s}\big) = ⨍ \nabla(X)|D|^{-2-n} = 0.$$

Similarly, since $Y \sim XD^{-2} \mod OP^{-2}$ and $Y^2 \sim X^2D^{-4} \mod OP^{-3}$, we get

$$\operatorname*{Res}_{s=n-2} \operatorname{Tr}\big(Y^2|D|^{-s}\big) = \operatorname*{Res}_{s=n-2} \operatorname{Tr}\big(X^2D^{-4}|D|^{-s}\big) = ⨍ X^2|D|^{-2-n}.$$

Thus,

$$\operatorname*{Res}_{s=n-2} \zeta_{D_A}(s) = \operatorname*{Res}_{s=n-2} \zeta_D(s) + (-\tfrac{n-2}{2})(-\!\!\!\!\!\!\int X|D|^{-n} - \tfrac{1}{2} -\!\!\!\!\!\!\int (\nabla(X) + X^2)|D|^{-2-n}) \\ + \tfrac{1}{2}(\tfrac{n-2}{2})^2 -\!\!\!\!\!\!\int \nabla(X)|D|^{-2-n} + \tfrac{1}{2}(\tfrac{n-2}{2})^2 -\!\!\!\!\!\!\int X^2|D|^{-2-n}.$$

Finally,

$$\operatorname*{Res}_{s=n-2} \zeta_{D_A}(s) = \operatorname*{Res}_{s=n-2} \zeta_D(s) + (-\tfrac{n-2}{2})\big(-\!\!\!\!\!\!\int X|D|^{-n} - \tfrac{1}{2} -\!\!\!\!\!\!\int X^2|D|^{-2-n}\big) + \tfrac{1}{2}(\tfrac{n-2}{2})^2 -\!\!\!\!\!\!\int X^2|D|^{-2-n}$$

and the result follows from Proposition 4.8. □

Corollary 4.11. *If the spectral triple is simple and satisfies* $-\!\!\!\!\!\!\int |D|^{-(n-2)} = -\!\!\!\!\!\!\int \widetilde{A}\mathcal{D}|D|^{-n} = -\!\!\!\!\!\!\int \mathcal{D}\widetilde{A}|D|^{-n} = 0$, *then*

$$-\!\!\!\!\!\!\int |D_A|^{-(n-2)} = \tfrac{n(n-2)}{4}\big(-\!\!\!\!\!\!\int \widetilde{A}\mathcal{D}\widetilde{A}\mathcal{D}|D|^{-n-2} + \tfrac{n-2}{n} -\!\!\!\!\!\!\int \widetilde{A}^2|D|^{-n}\big).$$

Proof. By previous lemma,

$$\operatorname*{Res}_{s=n-2} \zeta_{D_A}(s) = \tfrac{n-2}{2}\big(- -\!\!\!\!\!\!\int \widetilde{A}^2|D|^{-n} + \tfrac{n}{4} -\!\!\!\!\!\!\int (\, \widetilde{A}\mathcal{D}\widetilde{A}\mathcal{D} + \mathcal{D}\widetilde{A}\mathcal{D}\widetilde{A} + \widetilde{A}\mathcal{D}^2\widetilde{A} + \mathcal{D}\widetilde{A}^2\mathcal{D}\,)|D|^{-n-2}\big).$$

Since $\nabla(\widetilde{A}) \in OP^1$, the trace property of $-\!\!\!\!\!\!\int$ yields the result. □

5 The noncommutative torus

5.1 Notations

Let $C^\infty(\mathbb{T}^n_\Theta)$ be the smooth noncommutative n-torus associated to a non-zero skew-symmetric deformation matrix $\Theta \in M_n(\mathbb{R})$ (see [6], [30]). This means that $C^\infty(\mathbb{T}^n_\Theta)$ is the algebra generated by n unitaries u_i, $i = 1, \dots, n$ subject to the relations

$$u_i\, u_j = e^{i\Theta_{ij}}\, u_j\, u_i, \tag{40}$$

and with Schwartz coefficients: an element $a \in C^\infty(\mathbb{T}^n_\Theta)$ can be written as $a = \sum_{k\in\mathbb{Z}^n} a_k\, U_k$, where $\{a_k\} \in \mathcal{S}(\mathbb{Z}^n)$ with the Weyl elements defined by $U_k := e^{-\frac{i}{2}k.\chi k}\, u_1^{k_1} \cdots u_n^{k_n}$, $k \in \mathbb{Z}^n$, relation (40) reads

$$U_kU_q = e^{-\frac{i}{2}k.\Theta q}\, U_{k+q}, \text{ and } U_kU_q = e^{-ik.\Theta q}\, U_qU_k \tag{41}$$

where χ is the matrix restriction of Θ to its upper triangular part. Thus unitary operators U_k satisfy $U_k^* = U_{-k}$ and $[U_k, U_l] = -2i\sin(\frac{1}{2}k.\Theta l)\, U_{k+l}$.
Let τ be the trace on $C^\infty(\mathbb{T}^n_\Theta)$ defined by $\tau\big(\sum_{k\in\mathbb{Z}^n} a_k\, U_k\big) := a_0$ and $\mathcal{H}_\tau$ be the GNS Hilbert space obtained by completion of $C^\infty(\mathbb{T}^n_\Theta)$ with respect of the norm induced by the scalar product $\langle a, b\rangle := \tau(a^*b)$. On $\mathcal{H}_\tau = \{\sum_{k\in\mathbb{Z}^n} a_k\, U_k \;:\; \{a_k\}_k \in l^2(\mathbb{Z}^n)\,\}$, we consider the left and right regular representations of $C^\infty(\mathbb{T}^n_\Theta)$ by bounded operators, that we denote respectively by $L(.)$ and $R(.)$.
Let also δ_μ, $\mu \in \{\,1, \dots, n\,\}$, be the n (pairwise commuting) canonical derivations, defined by

$$\delta_\mu(U_k) := ik_\mu U_k. \tag{42}$$

We need to fix notations: let $\mathcal{A}_\Theta := C^\infty(\mathbb{T}^n_\Theta)$ acting on $\mathcal{H} := \mathcal{H}_\tau \otimes \mathbb{C}^{2^m}$ with $n = 2m$ or $n = 2m+1$ (i.e., $m = \lfloor \frac{n}{2} \rfloor$ is the integer part of $\frac{n}{2}$), the square integrable sections of the trivial spin bundle over $\mathbb{T}^n$.
Each element of $\mathcal{A}_\Theta$ is represented on $\mathcal{H}$ as $L(a) \otimes 1_{2^m}$ where L (resp. R) is the left (resp. right) multiplication. The Tomita conjugation $J_0(a) := a^*$ satisfies $[J_0, \delta_\mu] = 0$ and we define $J := J_0 \otimes C_0$ where C_0 is an operator on $\mathbb{C}^{2^m}$. The Dirac operator is given by

$$\mathcal{D} := -i\,\delta_\mu \otimes \gamma^\mu,$$

where we use hermitian Dirac matrices γ. It is defined and symmetric on the dense subset of $\mathcal{H}$ given by $C^\infty(\mathbb{T}^n_\Theta) \otimes \mathbb{C}^{2^m}$. We still note $\mathcal{D}$ its selfadjoint extension. This implies

$$C_0 \gamma^\alpha = -\varepsilon \gamma^\alpha C_0, \tag{43}$$

and

$$\mathcal{D}\; U_k \otimes e_i = k_\mu U_k \otimes \gamma^\mu e_i,$$

where (e_i) is the canonical basis of $\mathbb{C}^{2^m}$. Moreover, $C_0^2 = \pm 1_{2^m}$ depending on the parity of m. Finally, one introduces the chirality (which in the even case is $\chi := id \otimes (-i)^m \gamma^1 \cdots \gamma^n$) and this yields that $(\mathcal{A}_\Theta, \mathcal{H}, \mathcal{D}, J, \chi)$ satisfies all axioms of a spectral triple, see [8, 23].
The perturbed Dirac operator $V_u \mathcal{D} V_u^*$ by the unitary

$$V_u := \big(L(u) \otimes 1_{2^m}\big) J \big(L(u) \otimes 1_{2^m}\big) J^{-1},$$

defined for every unitary $u \in \mathcal{A}$, $uu^* = u^*u = U_0$, must satisfy condition (3) (which is equivalent to $\mathcal{H}$ being endowed with a structure of $\mathcal{A}_\Theta$-bimodule). This yields the necessity of a symmetrized covariant Dirac operator:

$$\mathcal{D}_A := \mathcal{D} + A + \epsilon J\, A\, J^{-1}$$

since $V_u \mathcal{D} V_u^* = \mathcal{D}_{L(u) \otimes 1_{2^m} [\mathcal{D}, L(u^*) \otimes 1_{2^m}]}$: in fact, for $a \in \mathcal{A}_\Theta$, using $J_0 L(a) J_0^{-1} = R(a^*)$, we get

$$\epsilon J \big(L(a) \otimes \gamma^\alpha\big) J^{-1} = -R(a^*) \otimes \gamma^\alpha$$

and that the representation L and the anti-representation R are $\mathbb{C}$-linear, commute and satisfy

$$[\delta_\alpha, L(a)] = L(\delta_\alpha a), \quad [\delta_\alpha, R(a)] = R(\delta_\alpha a).$$

This induces some covariance property for the Dirac operator: one checks that for all $k \in \mathbb{Z}^n$,

$$L(U_k) \otimes 1_{2^m} [\mathcal{D}, L(U_k^*) \otimes 1_{2^m}] = 1 \otimes (-k_\mu \gamma^\mu), \tag{44}$$

so with (43), we get $U_k[\mathcal{D}, U_k^*] + \epsilon J U_k [\mathcal{D}, U_k^*] J^{-1} = 0$ and

$$V_{U_k}\, \mathcal{D}\, V_{U_k}^* = \mathcal{D} = \mathcal{D}_{L(U_k) \otimes 1_{2^m} [\mathcal{D}, L(U_k^*) \otimes 1_{2^m}]}. \tag{45}$$

Moreover, we get the gauge transformation:

$$V_u \mathcal{D}_A V_u^* = \mathcal{D}_{\gamma_u(A)} \tag{46}$$

where the gauged transform one-form of A is

$$\gamma_u(A) := u[\mathcal{D}, u^*] + uAu^*, \tag{47}$$

with the shorthand $L(u) \otimes 1_{2^m} \longrightarrow u$.
As a consequence, the spectral action is gauge invariant:

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \mathcal{S}(\mathcal{D}_{\gamma_u(A)}, \Phi, \Lambda).$$

An arbitrary selfadjoint one-form A, can be written as

$$A = L(-iA_\alpha) \otimes \gamma^\alpha, \;\; A_\alpha = -A_\alpha^* \in \mathcal{A}_\Theta, \tag{48}$$

thus

$$\mathcal{D}_A = -i\left(\delta_\alpha + L(A_\alpha) - R(A_\alpha)\right) \otimes \gamma^\alpha. \tag{49}$$

Defining

$$\tilde{A}_\alpha := L(A_\alpha) - R(A_\alpha),$$

we get $\mathcal{D}_A^2 = -g^{\alpha_1\alpha_2}(\delta_{\alpha_1} + \tilde{A}_{\alpha_1})(\delta_{\alpha_2} + \tilde{A}_{\alpha_2}) \otimes 1_{2^m} - \frac{1}{2}\Omega_{\alpha_1\alpha_2} \otimes \gamma^{\alpha_1\alpha_2}$ where

$$\begin{aligned} \gamma^{\alpha_1\alpha_2} &:= \tfrac{1}{2}(\gamma^{\alpha_1}\gamma^{\alpha_2} - \gamma^{\alpha_2}\gamma^{\alpha_1}), \\ \Omega_{\alpha_1\alpha_2} &:= [\delta_{\alpha_1} + \tilde{A}_{\alpha_1}, \delta_{\alpha_2} + \tilde{A}_{\alpha_2}] \; = L(F_{\alpha_1\alpha_2}) - R(F_{\alpha_1\alpha_2}) \end{aligned}$$

with

$$F_{\alpha_1\alpha_2} := \delta_{\alpha_1}(A_{\alpha_2}) - \delta_{\alpha_2}(A_{\alpha_1}) + [A_{\alpha_1}, A_{\alpha_2}]. \tag{50}$$

In summary,

$$\begin{aligned} \mathcal{D}_A^2 = -\delta^{\alpha_1\alpha_2}\Big(\delta_{\alpha_1} + L(A_{\alpha_1}) - R(A_{\alpha_1})\Big)\Big(\delta_{\alpha_2} + L(A_{\alpha_2}) - R(A_{\alpha_2})\Big) \otimes 1_{2^m} \\ -\tfrac{1}{2}\left(L(F_{\alpha_1\alpha_2}) - R(F_{\alpha_1\alpha_2})\right) \otimes \gamma^{\alpha_1\alpha_2}. \end{aligned} \tag{51}$$

5.2 Kernels and dimension spectrum

We now compute the kernel of the perturbed Dirac operator:

Proposition 5.1. *(i)* $\operatorname{Ker}\mathcal{D} = U_0 \otimes \mathbb{C}^{2^m}$*, so* $\dim \operatorname{Ker}\mathcal{D} = 2^m$*.*
(ii) For any selfadjoint one-form A*,* $\operatorname{Ker}\mathcal{D} \subseteq \operatorname{Ker}\mathcal{D}_A$*.*
(iii) For any unitary $u \in \mathcal{A}$*,* $\operatorname{Ker}\mathcal{D}_{\gamma_u(A)} = V_u \operatorname{Ker}\mathcal{D}_A$*.*

Proof. (i) Let $\psi = \sum_{k,j} c_{k,j}\, U_k \otimes e_j \in \operatorname{Ker}\mathcal{D}$. Thus, $0 = \mathcal{D}^2\psi = \sum_{k,i} c_{k,j}|k|^2\, U_k \otimes e_j$ which entails that $c_{k,j}|k|^2 = 0$ for any $k \in \mathbb{Z}^n$ and $1 \leq j \leq 2^m$. The result follows.
(ii) Let $\psi \in \operatorname{Ker}\mathcal{D}$. So, $\psi = U_0 \otimes v$ with $v \in \mathbb{C}^{2^m}$ and from (49), we get

$$\mathcal{D}_A\psi = \mathcal{D}\psi + (A + \epsilon JAJ^{-1})\psi = (A + \epsilon JAJ^{-1})\psi = -i[A_\alpha, U_0] \otimes \gamma^\alpha v = 0$$

since U_0 is the unit of the algebra, which proves that $\psi \in \operatorname{Ker}\mathcal{D}_A$.
(iii) This is a direct consequence of (46). □

Corollary 5.2. *Let* A *be a selfadjoint one-form. Then* $\operatorname{Ker}\mathcal{D}_A = \operatorname{Ker}\mathcal{D}$ *in the following cases:*
(i) $A_u := L(u) \otimes 1_{2^m}[\mathcal{D}, L(u^*) \otimes 1_{2^m}]$ *when* u *is a unitary in* $\mathcal{A}$*.*
(ii) $||A|| < \frac{1}{2}$*.*
(iii) The matrix $\frac{1}{2\pi}\Theta$ *has only integral coefficients.*

Proof. (i) This follows from previous result because $V_u(U_0 \otimes v) = U_0 \otimes v$ for any $v \in \mathbb{C}^{2^m}$.
(ii) Let $\psi = \sum_{k,j} c_{k,j}\, U_k \otimes e_j$ be in $\operatorname{Ker} \mathcal{D}_A$ (so $\sum_{k,j} |c_{k,j}|^2 < \infty$) and $\phi := \sum_j c_{0,j}\, U_0 \otimes e_j$. Thus $\psi' := \psi - \phi \in \operatorname{Ker}\, \mathcal{D}_A$ since $\phi \in \operatorname{Ker} \mathcal{D} \subseteq \operatorname{Ker} \mathcal{D}_A$ and

$$|| \sum_{0 \neq k \in \mathbb{Z}^n,\, j} c_{k,j}\, k_\alpha\, U_k \otimes \gamma^\alpha e_j ||^2 = ||\mathcal{D}\psi'||^2 = || - (A + \epsilon JAJ^{-1})\psi'||^2 \leq 4||A||^2 ||\psi'||^2 < ||\psi'||^2.$$

Defining $X_k := \sum_\alpha k_\alpha \gamma_\alpha$, $X_k^2 = \sum_\alpha |k_\alpha|^2\, 1_{2^m}$ is invertible and the vectors $\{\, U_k \otimes X_k e_j \,\}_{0 \neq k \in \mathbb{Z}^n,\, j}$ are orthogonal in $\mathcal{H}$, so

$$\sum_{0 \neq k \in \mathbb{Z}^n,\, j} \big(\sum_\alpha |k_\alpha|^2 \big)\, |c_{k,j}|^2 < \sum_{0 \neq k \in \mathbb{Z}^n,\, j} |c_{k,j}|^2$$

which is possible only if $c_{k,j} = 0$, $\forall k,\, j$ that is $\psi' = 0$ et $\psi = \phi \in \operatorname{Ker}\, \mathcal{D}$.
(iii) This is a consequence of the fact that the algebra is commutative, thus $A + \epsilon JAJ^{-1} = 0$. $\square$

Note that if $\widetilde{A}_u := A_u + \epsilon J A_u J^{-1}$, then by (44), $\widetilde{A}_{U_k} = 0$ for all $k \in \mathbb{Z}^n$ and $\|A_{U_k}\| = |k|$, but for an arbitrary unitary $u \in \mathcal{A}$, $\widetilde{A}_u \neq 0$ so $\mathcal{D}_{A_u} \neq \mathcal{D}$.
Naturally the above result is also a direct consequence of the fact that the eigenspace of an isolated eigenvalue of an operator is not modified by small perturbations. However, it is interesting to compute the last result directly to emphasize the difficulty of the general case:
Let $\psi = \sum_{l \in \mathbb{Z}^n, 1 \leq j \leq 2^m} c_{l,j}\, U_l \otimes e_j \in \operatorname{Ker} \mathcal{D}_A$, so $\sum_{l \in \mathbb{Z}^n, 1 \leq j \leq 2^m} |c_{l,j}|^2 < \infty$. We have to show that $\psi \in \operatorname{Ker}\, \mathcal{D}$ that is $c_{l,j} = 0$ when $l \neq 0$.
Taking the scalar product of $\langle U_k \otimes e_i|$ with

$$0 = \mathcal{D}_A \psi = \sum_{l,\, \alpha,\, j} c_{l,\, j} (l^\alpha U_l - i[A_\alpha, U_l]) \otimes \gamma^\alpha e_j,$$

we obtain

$$0 = \sum_{l,\, \alpha,\, j} c_{l,\, j} \big(l^\alpha \delta_{k,l} - i\langle U_k, [A_\alpha, U_l]\rangle \big) \langle e_i, \gamma^\alpha e_j \rangle.$$

If $A_\alpha = \sum_{\alpha,l} a_{\alpha,l}\, U_l \otimes \gamma^\alpha$ with $\{\, a_{\alpha,l} \,\}_l \in \mathcal{S}(\mathbb{Z}^n)$, note that $[U_l, U_m] = -2i \sin(\tfrac{1}{2} l.\Theta m)\, U_{l+m}$ and

$$\langle U_k, [A_\alpha, U_l] \rangle = \sum_{l' \in \mathbb{Z}^n} a_{\alpha,l'} (-2i \sin(\tfrac{1}{2} l'.\Theta l) \langle U_k, U_{l'+l} \rangle = -2i\, a_{\alpha,k-l}\, \sin(\tfrac{1}{2} k.\Theta l).$$

Thus

$$0 = \sum_{l \in \mathbb{Z}^n} \sum_{\alpha=1}^{n} \sum_{j=1}^{2^m} c_{l,\, j} \big(l^\alpha \delta_{k,l} - 2a_{\alpha,k-l}\, \sin(\tfrac{1}{2} k.\Theta l) \big)\, \langle e_i, \gamma^\alpha e_j \rangle, \quad \forall k \in \mathbb{Z}^n,\, \forall i, 1 \leq i \leq 2^m. \qquad (52)$$

We conjecture that $\operatorname{Ker} \mathcal{D} = \operatorname{Ker} \mathcal{D}_A$ *at least for generic* Θ*'s*:
the constraints (52) should imply $c_{l,j} = 0$ for all j and all $l \neq 0$ meaning $\psi \in \operatorname{Ker} \mathcal{D}$. When $\frac{1}{2\pi}\Theta$ has only integer coefficients, the sin part of these constraints disappears giving the result.

Lemma 5.3. *If* $\frac{1}{2\pi}\Theta$ *is badly approximable,* $Sp\big(C^\infty(\mathbb{T}^n_\Theta), \mathcal{H}, \mathcal{D}\big) = \mathbb{Z}$ *and all these poles are simple.*

Proof. Let $B \in \mathcal{D}(\mathcal{A})$ and $p \in \mathbb{N}_0$. Suppose that B is of the form

$$B = a_r b_r \mathcal{D}^{q_{r-1}} |\mathcal{D}|^{p_{r-1}} a_{r-1} b_{r-1} \cdots \mathcal{D}^{q_1} |\mathcal{D}|^{p_1} a_1 b_1$$

where $r \in \mathbb{N}$, $a_i \in \mathcal{A}$, $b_i \in J\mathcal{A}J^{-1}$, $q_i, p_i \in \mathbb{N}_0$. We note $a_i =: \sum_l a_{i,l} U_l$ and $b_i =: \sum_i b_{i,l} U_l$. With the shorthand $k_{\mu_1,\mu_{q_i}} := k_{\mu_1} \cdots k_{\mu_{q_i}}$ and $\gamma^{\mu_1,\mu_{q_i}} = \gamma^{\mu_1} \cdots \gamma^{\mu_{q_i}}$, we get

$$\mathcal{D}^{q_1} |\mathcal{D}|^{p_1} a_1 b_1 \, U_k \otimes e_j = \sum_{l_1, l_1'} a_{1,l_1} b_{1,l_1'} U_{l_1} U_k U_{l_1'} \, |k + l_1 + l_1'|^{p_1} \, (k + l_1 + l_1')_{\mu_1,\mu_{q_1}} \otimes \gamma^{\mu_1,\mu_{q_1}} e_j$$

which gives, after r iterations,

$$BU_k \otimes e_j = \sum_{l,l'} \widetilde{a}_l \widetilde{b}_l U_{l_r} \cdots U_{l_1} U_k U_{l_1'} \cdots U_{l_r'} \prod_{i=1}^{r-1} |k + \widehat{l}_i + \widehat{l}_i'|^{p_i} (k + \widehat{l}_i + \widehat{l}_i')_{\mu_1^i, \mu_{q_i}^i} \otimes \gamma^{\mu_1^{r-1}, \mu_{q_{r-1}}^{r-1}} \cdots \gamma^{\mu_1^1, \mu_{q_1}^1} e_j$$

where $\widetilde{a}_l := a_{1,l_1} \cdots a_{r,l_r}$ and $\widetilde{b}_{l'} := b_{1,l_1'} \cdots b_{r,l_r'}$.
Let us note $F_\mu(k,l,l') := \prod_{i=1}^{r-1} |k + \widehat{l}_i + \widehat{l}_i'|^{p_i} (k + \widehat{l}_i + \widehat{l}_i')_{\mu_1^i, \mu_{q_i}^i}$ and $\gamma^\mu := \gamma^{\mu_1^{r-1}, \mu_{q_{r-1}}^{r-1}} \cdots \gamma^{\mu_1^1, \mu_{q_1}^1}$. Thus, with the shortcut $\sim_c$ meaning modulo a constant function towards the variable s,

$$\operatorname{Tr}\big(B|D|^{-2p-s}\big) \sim_c \sum_k{}' \sum_{l,l'} \widetilde{a}_l \widetilde{b}_{l'} \, \tau\big(U_{-k} U_{l_r} \cdots U_{l_1} U_k U_{l_1'} \cdots U_{l_r'}\big) \tfrac{F_\mu(k,l,l')}{|k|^{s+2p}} \operatorname{Tr}(\gamma^\mu) \,.$$

Since $U_{l_r} \cdots U_{l_1} U_k = U_k U_{l_r} \cdots U_{l_1} e^{-i \sum_1^r l_i . \Theta k}$ we get

$$\tau\big(U_{-k} U_{l_r} \cdots U_{l_1} U_k U_{l_1'} \cdots U_{l_r'}\big) = \delta_{\sum_1^r l_i + l_i', 0} \, e^{i\phi(l,l')} \, e^{-i \sum_1^r l_i . \Theta k}$$

where ϕ is a real valued function. Thus,

$$\begin{aligned}\operatorname{Tr}\big(B|D|^{-2p-s}\big) &\sim_c \sum_k{}' \sum_{l,l'} e^{i\phi(l,l')} \, \delta_{\sum_1^r l_i + l_i', 0} \, \widetilde{a}_l \widetilde{b}_{l'} \, \tfrac{F_\mu(k,l,l') \, e^{-i \sum_1^r l_i . \Theta k}}{|k|^{s+2p}} \operatorname{Tr}(\gamma^\mu) \\ &\sim_c f_\mu(s) \operatorname{Tr}(\gamma^\mu).\end{aligned}$$

The function $f_\mu(s)$ can be decomposed has a linear combination of zeta function of type described in Theorem 2.17 (or, if $r = 1$ or all the p_i are zero, in Theorem 2.5). Thus, $s \mapsto \operatorname{Tr}\big(B|D|^{-2p-s}\big)$ has only poles in $\mathbb{Z}$ and each pole is simple. Finally, by linearity, we get the result. □

The dimension spectrum of the noncommutative torus is simple:

Proposition 5.4. *(i) If $\frac{1}{2\pi}\Theta$ is badly approximable, the spectrum dimension of $\big(C^\infty(\mathbb{T}^n_\Theta), \mathcal{H}, \mathcal{D}\big)$ is equal to the set $\{\, n - k \,:\, k \in \mathbb{N}_0 \,\}$ and all these poles are simple.*
(ii) $\zeta_D(0) = 0$.

Proof. (i) Lemma 5.3 and Remark 3.9.
(ii) $\zeta_D(s) = \sum_{k \in \mathbb{Z}^n} \sum_{1 \leq j \leq 2^m} \langle U_k \otimes e_j \,,\, |D|^{-s} U_k \otimes e_j \rangle = 2^m (\sum_{k \in \mathbb{Z}^n}' \frac{1}{|k|^s} + 1) = 2^m (\, Z_n(s) + 1)$. By (21), we get the result. □

We have computed $\zeta_D(0)$ relatively easily but the main difficulty of the present work is precisely to calculate $\zeta_{D_A}(0)$.

5.3 Noncommutative integral computations

We fix a self-adjoint 1-form A on the noncommutative torus of dimension n.

Proposition 5.5. *If $\frac{1}{2\pi}\Theta$ is badly approximable, then the first elements of the expansion (5) are given by*

$$\textstyle\int\!\!\!\!\!- |D_A|^{-n} = \int\!\!\!\!\!- |D|^{-n} = 2^{m+1}\pi^{n/2}\,\Gamma(\frac{n}{2})^{-1}. \tag{53}$$

$$\textstyle\int\!\!\!\!\!- |D_A|^{n-k} = 0 \text{ for } k \text{ odd.}$$

$$\textstyle\int\!\!\!\!\!- |D_A|^{n-2} = 0.$$

We need few technical lemmas:

Lemma 5.6. *On the noncommutative torus, for any $t \in \mathbb{R}$,*

$$\textstyle\int\!\!\!\!\!- \widetilde{A}\mathcal{D}|D|^{-t} = \int\!\!\!\!\!- \mathcal{D}\widetilde{A}|D|^{-t} = 0.$$

Proof. Using notations of (48), we have

$$\begin{aligned}\operatorname{Tr}(\widetilde{A}\mathcal{D}|D|^{-s}) &\sim_c \sum_j \sum_k{}' \langle U_k \otimes e_j, -ik_\mu |k|^{-s}[A_\alpha, U_k] \otimes \gamma^\alpha\gamma^\mu e_j\rangle \\ &\sim_c -i\operatorname{Tr}(\gamma^\alpha\gamma^\mu)\sum_k{}' k_\mu |k|^{-s}\langle U_k, [A_\alpha, U_k]\rangle = 0\end{aligned}$$

since $\langle U_k, [A_\alpha, U_k]\rangle = 0$. Similarly

$$\begin{aligned}\operatorname{Tr}(\mathcal{D}\widetilde{A}|D|^{-s}) &\sim_c \sum_j \sum_k{}' \langle U_k \otimes e_j, |k|^{-s}\sum_l a_{\alpha,l}\, 2\sin\tfrac{k.\Theta l}{2}(l+k)_\mu U_{l+k} \otimes \gamma^\mu\gamma^\alpha e_j\rangle \\ &\sim_c 2\operatorname{Tr}(\gamma^\mu\gamma^\alpha)\sum_k{}'\sum_l a_{\alpha,l}\sin\tfrac{k.\Theta l}{2}\,(l+k)_\mu\,|k|^{-s}\langle U_k, U_{l+k}\rangle = 0.\end{aligned}$$

∎

Any element h in the algebra generated by $\mathcal{A}$ and $[\mathcal{D},\mathcal{A}]$ can be written as a linear combination of terms of the form ${a_1}^{p_1}\cdots{a_n}^{p_r}$ where a_i are elements of $\mathcal{A}$ or $[\mathcal{D},\mathcal{A}]$. Such a term can be written as a series $b := \sum a_{1,\alpha_1,l_1}\cdots a_{q,\alpha_q,l_q}U_{l_1}\cdots U_{l_q}\otimes\gamma^{\alpha_1}\cdots\gamma^{\alpha_q}$ where a_{i,α_i} are Schwartz sequences and when $a_i =: \sum_l a_l U_l \in \mathcal{A}$, we set $a_{i,\alpha,l} = a_{i,l}$ with $\gamma^\alpha = 1$. We define

$$L(b) := \tau\big(\sum_l a_{1,\alpha_1,l_1}\cdots a_{q,\alpha_q,l_q}U_{l_1}\cdots U_{l_q}\big)\operatorname{Tr}(\gamma^{\alpha_1}\cdots\gamma^{\alpha_q}).$$

By linearity, L is defined as a linear form on the whole algebra generated by $\mathcal{A}$ and $[\mathcal{D},\mathcal{A}]$.

Lemma 5.7. *If h is an element of the algebra generated by $\mathcal{A}$ and $[\mathcal{D},\mathcal{A}]$,*

$$\operatorname{Tr}\big(h|D|^{-s}\big) \sim_c L(h)\,Z_n(s).$$

In particular, $\operatorname{Tr}\big(h|D|^{-s}\big)$ *has at most one pole at $s = n$.*

Proof. We get with b of the form $\sum a_{1,\alpha_1,l_1}\cdots a_{q,\alpha_q,l_q}U_{l_1}\cdots U_{l_q}\otimes\gamma^{\alpha_1}\cdots\gamma^{\alpha_q}$,

$$\begin{aligned}\operatorname{Tr}\big(b|D|^{-s}\big) &\sim_c \sum_{k\in\mathbb{Z}^n}{}' \langle U_k, \sum_l a_{1,\alpha_1,l_1}\cdots a_{q,\alpha_q,l_q}U_{l_1}\cdots U_{l_q}U_k\rangle\ \operatorname{Tr}(\gamma^{\alpha_1}\cdots\gamma^{\alpha_q})|k|^{-s} \\ &\sim_c \tau\big(\sum_l a_{1,\alpha_1,l_1}\cdots a_{q,\alpha_q,l_q}U_{l_1}\cdots U_{l_q}\big)\operatorname{Tr}(\gamma^{\alpha_1}\cdots\gamma^{\alpha_q})\,Z_n(s) = L(b)\,Z_n(s).\end{aligned}$$

The results follows now from linearity of the trace. ∎

Lemma 5.8. *If $\frac{1}{2\pi}\Theta$ is badly approximable, the function $s \mapsto \operatorname{Tr}\big(\varepsilon JAJ^{-1}A|D|^{-s}\big)$ extends meromorphically on the whole plane with only one possible pole at $s = n$. Moreover, this pole is simple and*

$$\operatorname*{Res}_{s=n} \operatorname{Tr}\big(\varepsilon JAJ^{-1}A|D|^{-s}\big) = a_{\alpha,0}\, a_0^\alpha\; 2^{m+1}\pi^{n/2}\,\Gamma(n/2)^{-1}.$$

Proof. With $A = L(-iA_\alpha) \otimes \gamma^\alpha$, we get $\epsilon JAJ^{-1} = R(iA_\alpha) \otimes \gamma^\alpha$, and by multiplication $\varepsilon JAJ^{-1}A = R(A_\beta)L(A_\alpha) \otimes \gamma^\beta\gamma^\alpha$. Thus,

$$\begin{aligned}\operatorname{Tr}\big(\varepsilon JAJ^{-1}A|D|^{-s}\big) &\sim_c \sum_{k\in\mathbb{Z}^n}{}' \langle U_k, A_\alpha U_k A_\beta\rangle\, |k|^{-s}\operatorname{Tr}(\gamma^\beta\gamma^\alpha) \\ &\sim_c \sum_{k\in\mathbb{Z}^n}{}' \sum_l a_{\alpha,l}\, a_{\beta,-l}\, e^{ik.\Theta l}\, |k|^{-s}\operatorname{Tr}(\gamma^\beta\gamma^\alpha) \\ &\sim_c 2^m \sum_{k\in\mathbb{Z}^n}{}' \sum_l a_{\alpha,l}\, a_{-l}^\alpha\, e^{ik.\Theta l}\, |k|^{-s}.\end{aligned}$$

Theorem 2.5 (ii) entails that $\sum'_{k\in\mathbb{Z}^n}\sum_l a_{\alpha,l}\, a_{-l}^\alpha\, e^{ik.\Theta l}\, |k|^{-s}$ extends meromorphically on the whole plane $\mathbb{C}$ with only one possible pole at $s = n$. Moreover, this pole is simple and we have

$$\operatorname*{Res}_{s=n} \sum_{k\in\mathbb{Z}^n}{}' \sum_l a_{\alpha,l}\, a_{-l}^\alpha\, e^{ik.\Theta l}\, |k|^{-s} = a_{\alpha,0}\, a_0^\alpha \operatorname*{Res}_{s=n} Z_n(s).$$

Equation (20) now gives the result. □

Lemma 5.9. *If $\frac{1}{2\pi}\Theta$ is badly approximable, then for any $t \in \mathbb{R}$,*

$$-\!\!\!\!\!\!\int X|D|^{-t} = \delta(t-n)\, 2^{m+1}\big(-\sum_l a_{\alpha,l}\, a_{-l}^\alpha + a_{\alpha,0}\, a_0^\alpha\big)\; 2\pi^{n/2}\,\Gamma(n/2)^{-1}.$$

where $X = \widetilde{A}\mathcal{D} + \mathcal{D}\widetilde{A} + \widetilde{A}^2$ and $A =: -i\sum_l a_{\alpha,l}\, U_l \otimes \gamma^\alpha$.

Proof. By Lemma 5.6, we get $-\!\!\!\!\!\int X|D|^{-t} = \operatorname{Res}_{s=0}\operatorname{Tr}(\widetilde{A}^2|D|^{-s-t})$. Since A and εJAJ^{-1} commute, we have $\widetilde{A}^2 = A^2 + JA^2J^{-1} + 2\varepsilon JAJ^{-1}A$. Thus,

$$\operatorname{Tr}(\widetilde{A}^2|D|^{-s-t}) = \operatorname{Tr}(A^2|D|^{-s-t}) + \operatorname{Tr}(JA^2J^{-1}|D|^{-s-t}) + 2\operatorname{Tr}(\varepsilon JAJ^{-1}A|D|^{-s-t}).$$

Since $|D|$ and J commute, we have with Lemma 5.7,

$$\operatorname{Tr}\big(\widetilde{A}^2|D|^{-s-t}\big) \sim_c 2L(A^2)\, Z_n(s+t) + 2\operatorname{Tr}\big(\varepsilon JAJ^{-1}A|D|^{-s-t}\big).$$

Thus Lemma 5.8 entails that $\operatorname{Tr}(\widetilde{A}^2|D|^{-s-t})$ is holomorphic at 0 if $t \neq n$. When $t = n$,

$$\operatorname*{Res}_{s=0} \operatorname{Tr}\big(\widetilde{A}^2|D|^{-s-t}\big) = 2^{m+1}\big(-\sum_l a_{\alpha,l}\, a_{-l}^\alpha + a_{\alpha,0}\, a_0^\alpha\ \big)\, 2\pi^{n/2}\,\Gamma(n/2)^{-1}, \tag{54}$$

which gives the result. □

Lemma 5.10. *If $\frac{1}{2\pi}\Theta$ is badly approximable, then*

$$-\!\!\!\!\!\!\int \widetilde{A}\mathcal{D}\widetilde{A}\mathcal{D}|D|^{-2-n} = -\tfrac{n-2}{n} -\!\!\!\!\!\!\int \widetilde{A}^2|D|^{-n}.$$

Proof. With $\mathcal{D}J = \varepsilon J\mathcal{D}$, we get

$$\int\!\!\!\!\!- \widetilde{A}\mathcal{D}\widetilde{A}\mathcal{D}|D|^{-2-n} = 2\int\!\!\!\!\!- A\mathcal{D}A\mathcal{D}|D|^{-2-n} + 2\int\!\!\!\!\!- \varepsilon JAJ^{-1}\mathcal{D}A\mathcal{D}|D|^{-2-n}.$$

Let us first compute $\int\!\!\!\!- A\mathcal{D}A\mathcal{D}|D|^{-2-n}$. We have, with $A =: -iL(A_\alpha)\otimes\gamma^\alpha =: -i\sum_l a_{\alpha,l}U_l\otimes\gamma^\alpha$,

$$\operatorname{Tr}\big(A\mathcal{D}A\mathcal{D}|D|^{-s-2-n}\big) \sim_c -{\sum_k}' \sum_{l_1,l_2} a_{\alpha_2,l_2}\, a_{\alpha_1,l_1}\, \tau(U_{-k}U_{l_2}U_{l_1}U_k)\, \tfrac{k_{\mu_1}(k+l_1)_{\mu_2}}{|k|^{s+2+n}} \operatorname{Tr}(\gamma^{\alpha,\mu})$$

where $\gamma^{\alpha,\mu} := \gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}$. Thus,

$$\int\!\!\!\!\!- A\mathcal{D}A\mathcal{D}|D|^{-2-n} = -\sum_l a_{\alpha_2,-l}\, a_{\alpha_1,l} \operatorname*{Res}_{s=0}\big({\sum_k}' \tfrac{k_{\mu_1}k_{\mu_2}}{|k|^{s+2+n}}\big) \operatorname{Tr}(\gamma^{\alpha,\mu}).$$

We have also, with $\varepsilon JAJ^{-1} = iR(A_\alpha)\otimes\gamma^a$,

$$\operatorname{Tr}\big(\varepsilon JAJ^{-1}\mathcal{D}A\mathcal{D}|D|^{-s-2-n}\big) \sim_c {\sum_k}' \sum_{l_1,l_2} a_{\alpha_2,l_2}a_{\alpha_1,l_1}\tau(U_{-k}U_{l_1}U_kU_{l_2})\tfrac{k_{\mu_1}(k+l_1)_{\mu_2}}{|k|^{s+2+n}} \operatorname{Tr}(\gamma^{\alpha,\mu}).$$

which gives

$$\int\!\!\!\!\!- \varepsilon JAJ^{-1}\mathcal{D}A\mathcal{D}|D|^{-2-n} = a_{\alpha_2,0}a_{\alpha_1,0} \operatorname*{Res}_{s=0}\big({\sum_k}' \tfrac{k_{\mu_1}k_{\mu_2}}{|k|^{s+2+n}}\big) \operatorname{Tr}(\gamma^{\alpha,\mu}).$$

Thus,

$$\tfrac{1}{2}\int\!\!\!\!\!- \widetilde{A}\mathcal{D}\widetilde{A}\mathcal{D}|D|^{-2-n} = \big(a_{\alpha_2,0}a_{\alpha_1,0} - \sum_l a_{\alpha_2,-l}a_{\alpha_1,l}\big) \operatorname{Res}_{s=0}\big({\sum_k}' \tfrac{k_{\mu_1}k_{\mu_2}}{|k|^{s+2+n}}\big) \operatorname{Tr}(\gamma^{\alpha,\mu}).$$

With ${\sum}'_k \frac{k_{\mu_1}k_{\mu_2}}{|k|^{s+2+n}} = \frac{\delta_{\mu_1\mu_2}}{n} Z_n(s+n)$ and $C_n := \operatorname{Res}_{s=0} Z_n(s+n) = 2\pi^{n/2}\Gamma(n/2)^{-1}$ we obtain

$$\tfrac{1}{2}\int\!\!\!\!\!- \widetilde{A}\mathcal{D}\widetilde{A}\mathcal{D}|D|^{-2-n} = \big(a_{\alpha_2,0}a_{\alpha_1,0} - \sum_l a_{\alpha_2,-l}a_{\alpha_1,l}\big)\tfrac{C_n}{n} \operatorname{Tr}(\gamma^{\alpha_2}\gamma^\mu\gamma^{\alpha_1}\gamma_\mu).$$

Since $\operatorname{Tr}(\gamma^{\alpha_2}\gamma^\mu\gamma^{\alpha_1}\gamma_\mu) = 2^m(2-n)\delta^{\alpha_2,\alpha_1}$, we get

$$\tfrac{1}{2}\int\!\!\!\!\!- \widetilde{A}\mathcal{D}\widetilde{A}\mathcal{D}|D|^{-2-n} = 2^m\big(-a_{\alpha,0}\, a_0^\alpha + \sum_l a_{\alpha,-l}\, a_l^\alpha\big)\tfrac{C_n(n-2)}{n}.$$

Equation (54) now proves the lemma. ☐

Lemma 5.11. *If $\frac{1}{2\pi}\Theta$ is badly approximable, then for any $P\in\Psi_1(\mathcal{A})$ and $q\in\mathbb{N}$, q odd,*

$$\int\!\!\!\!\!- P|D|^{-(n-q)} = 0.$$

Proof. There exist $B\in\mathcal{D}_1(\mathcal{A})$ and $p\in\mathbb{N}_0$ such that $P = BD^{-2p}+R$ where R is in OP^{-q-1}. As a consequence, $\int\!\!\!\!- P|D|^{-(n-q)} = \int\!\!\!\!- B|D|^{-n-2p+q}$. Assume $B = a_rb_r\mathcal{D}^{q_{r-1}}a_{r-1}b_{r-1}\cdots\mathcal{D}^{q_1}a_1b_1$ where $r\in\mathbb{N}$, $a_i\in\mathcal{A}$, $b_i\in J\mathcal{A}J^{-1}$, $q_i\in\mathbb{N}$. If we prove that $\int\!\!\!\!- B|D|^{-n-2p+q}=0$, then the general case will follow by linearity. We note $a_i =: \sum_l a_{i,l}\, U_l$ and $b_i =: \sum_l b_{i,l}\, U_l$. With the shorthand $k_{\mu_1,\mu_{q_i}} := k_{\mu_1} \cdots k_{\mu_{q_i}}$ and $\gamma^{\mu_1,\mu_{q_i}} = \gamma^{\mu_1} \cdots \gamma^{\mu_{q_i}}$, we get

$$\mathcal{D}^{q_1} a_1 b_1 U_k \otimes e_j = \sum_{l_1,l'_1} a_{1,l_1}\, b_{1,l'_1}\, U_{l_1} U_k U_{l'_1}\, (k + l_1 + l'_1)_{\mu_1,\mu_{q_1}} \otimes \gamma^{\mu_1,\mu_{q_1}} e_j$$

which gives, after iteration,

$$B\, U_k \otimes e_j = \sum_{l,l'} \widetilde{a}_l \widetilde{b}_l U_{l_r} \cdots U_{l_1} U_k U_{l'_1} \cdots U_{l'_r} \prod_{i=1}^{r-1} (k + \widehat{l}_i + \widehat{l'}_i)_{\mu_1^i,\mu_{q_i}^i} \otimes \gamma^{\mu_1^{r-1},\mu_{q_{r-1}}^{r-1}} \cdots \gamma^{\mu_1^1,\mu_{q_1}^1} e_j$$

where $\widetilde{a}_l := a_{1,l_1} \cdots a_{r,l_r}$ and $\widetilde{b}_{l'} := b_{1,l'_1} \cdots b_{r,l'_r}$. Let's note $Q_\mu(k,l,l') := \prod_{i=1}^{r-1} (k + \widehat{l}_i + \widehat{l'}_i)_{\mu_1^i,\mu_{q_i}^i}$ and $\gamma^\mu := \gamma^{\mu_1^{r-1},\mu_{q_{r-1}}^{r-1}} \cdots \gamma^{\mu_1^1,\mu_{q_1}^1}$. Thus,

$$⨍ B\, |D|^{-n-2p+q} = \operatorname*{Res}_{s=0} \sum_k{}' \sum_{l,l'} \widetilde{a}_l\, \widetilde{b}_{l'}\, \tau\big(U_{-k} U_{l_r} \cdots U_{l_1} U_k U_{l'_1} \cdots U_{l'_r}\big)\, \tfrac{Q_\mu(k,l,l')}{|k|^{s+2p+n-q}}\ \operatorname{Tr}(\gamma^\mu)\,.$$

Since $U_{l_r} \cdots U_{l_1} U_k = U_k U_{l_r} \cdots U_{l_1} e^{-i \sum_1^r l_i.\Theta k}$, we get

$$\tau\big(U_{-k} U_{l_r} \cdots U_{l_1} U_k U_{l'_1} \cdots U_{l'_r}\big) = \delta_{\sum_1^r l_i + l'_i, 0}\, e^{i\phi(l,l')}\, e^{-i \sum_1^r l_i.\Theta k}$$

where ϕ is a real valued function. Thus,

$$\begin{aligned} ⨍ B\, |D|^{-n-2p+q} &= \operatorname*{Res}_{s=0} \sum_k{}' \sum_{l,l'} e^{i\phi(l,l')}\, \delta_{\sum_1^r l_i + l'_i, 0}\, \widetilde{a}_l\, \widetilde{b}_{l'}\, \tfrac{Q_\mu(k,l,l') e^{-i \sum_1^r l_i.\Theta k}}{|k|^{s+2p+n-q}}\, \operatorname{Tr}(\gamma^\mu) \\ &=: \operatorname*{Res}_{s=0} f_\mu(s)\, \operatorname{Tr}(\gamma^\mu). \end{aligned}$$

We decompose $Q_\mu(k,l,l')$ as a sum $\sum_{h=0}^r M_{h,\mu}(l,l')\, Q_{h,\mu}(k)$ where $Q_{h,\mu}$ is a homogeneous polynomial in $(k_1, \cdots, k_n)$ and $M_{h,\mu}(l,l')$ is a polynomial in $\big((l_1)_1, \cdots, (l_r)_n, (l'_1)_1, \cdots, (l'_r)_n\big)$. Similarly, we decompose $f_\mu(s)$ as $\sum_{h=0}^r f_{h,\mu}(s)$. Theorem 2.5 (ii) entails that $f_{h,\mu}(s)$ extends meromorphically to the whole complex plane $\mathbb{C}$ with only one possible pole for $s+2p+n-q = n+d$ where $d := \deg Q_{h,\mu}$. In other words, if $d+q-2p \neq 0$, $f_{h,\mu}(s)$ is holomorphic at $s = 0$. Suppose now $d+q-2p = 0$ (note that this implies that d is odd, since q is odd by hypothesis), then, by Theorem 2.5 (ii)

$$\operatorname*{Res}_{s=0}\ f_{h,\mu}(s) = V \int_{u \in S^{n-1}} Q_{h,\mu}(u)\, dS(u)$$

where $V := \sum_{l,l' \in Z} M_{h,\mu}(l,l')\, e^{i\phi(l,l')}\, \delta_{\sum_1^r l_i + l'_i, 0}\, \widetilde{a}_l\, \widetilde{b}_{l'}$ and $Z := \{\, l,l' \ :\ \sum_{i=1}^r l_i = 0 \}$. Since d is odd, $Q_{h,\mu}(-u) = -Q_{h,\mu}(u)$ and $\int_{u \in S^{n-1}} Q_{h,\mu}(u)\, dS(u) = 0$. Thus, $\operatorname*{Res}_{s=0}\ f_{h,\mu}(s) = 0$ in any case, which gives the result. □

As we have seen, the crucial point of the preceding lemma is the decomposition of the numerator of the series $f_\mu(s)$ as polynomials in k. This has been possible because we restricted our pseudodifferential operators to $\Psi_1(\mathcal{A})$.

Proof of Proposition 5.5. The top element follows from Proposition 4.9 and according to (20),

$$⨍ |D|^{-n} = \operatorname*{Res}_{s=0}\ \operatorname{Tr}\big(|D|^{-s-n}\big) = 2^m \operatorname*{Res}_{s=0} Z_n(s+n) = \tfrac{2^{m+1}\pi^{n/2}}{\Gamma(n/2)}\,.$$

For the second equality, we get from Lemmas 5.7 and 4.6

$$\operatorname*{Res}_{s=n-k} \zeta_{D_A}(s) = \sum_{p=1}^{k} \sum_{r_1,\cdots,r_p=0}^{k-p} h(n-k,r,p) \,⨍\, \varepsilon^{r_1}(Y)\cdots\varepsilon^{r_p}(Y)|D|^{-(n-k)}.$$

Lemmas 4.4 and 5.11 imply that $⨍\, \varepsilon^{r_1}(Y)\cdots\varepsilon^{r_p}(Y)|D|^{-(n-k)} = 0$, which gives the result.
Last equality follows from Lemma 5.10 and Corollary 4.11. □

6 The spectral action

Here is the main result of this section.

Theorem 6.1. *Consider the n-NC-torus* $\big(C^\infty(\mathbb{T}^n_\Theta), \mathcal{H}, \mathcal{D}\big)$ *where* $n \in \mathbb{N}$ *and* $\frac{1}{2\pi}\Theta$ *is a real* $n\times n$ *skew-symmetric real badly approximable matrix, and a selfadjoint one-form* $A = L(-iA_\alpha)\otimes\gamma^\alpha$*. Then, the full spectral action of* $\mathcal{D}_A = \mathcal{D} + A + \epsilon JAJ^{-1}$ *is*
(i) for $n = 2$,

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = 4\pi\, \Phi_2\, \Lambda^2 + \mathcal{O}(\Lambda^{-2}),$$

(ii) for $n = 4$,

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = 8\pi^2\, \Phi_4\, \Lambda^4 - \tfrac{4\pi^2}{3}\, \Phi(0)\, \tau(F_{\mu\nu}F^{\mu\nu}) + \mathcal{O}(\Lambda^{-2}),$$

(iii) More generally, in

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) \;=\; \sum_{k=0}^{n} \Phi_{n-k}\, c_{n-k}(A)\, \Lambda^{n-k} + \mathcal{O}(\Lambda^{-1}),$$

$c_{n-2}(A) = 0$, $c_{n-k}(A) = 0$ *for* k *odd. In particular,* $c_0(A) = 0$ *when* n *is odd.*

This result (for $n = 4$) has also been obtained in [20] using the heat kernel method. It is however interesting to get the result via direct computations of (5) since it shows how this formula is efficient. As we will see, the computation of all the noncommutative integrals require a lot of technical steps. One of the main points, namely to isolate where the Diophantine condition on Θ is assumed, is outlined here.

Remark 6.2. *Note that all terms must be gauge invariants, namely, according to (47), invariant by* $A_\alpha \longrightarrow \gamma_u(A_\alpha) = uA_\alpha u^* + u\delta_\alpha(u^*)$*. A particular case is* $u = U_k$ *where* $U_k\delta_\alpha(U_k^*) = -ik_\alpha U_0$*. In the same way, note that there is no contradiction with the commutative case where, for any selfadjoint one-form A,* $\mathcal{D}_A = \mathcal{D}$ *(so A is equivalent to 0!), since we assume in Theorem 6.1 that* Θ *is badly approximable, so* $\mathcal{A}$ *cannot be commutative.*

Conjecture 6.3. *The constant term of the spectral action of* $\mathcal{D}_A$ *on the noncommutative n-torus is proportional to the constant term of the spectral action of* $\mathcal{D} + A$ *on the commutative n-torus.*

Remark 6.4. *The appearance of a Diophantine condition for* Θ *has been characterized in dimension 2 by Connes [7, Prop. 49] where in this case,* $\Theta = \theta\big(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\big)$ *with* $\theta \in \mathbb{R}$*. In fact, the Hochschild cohomology* $H(\mathcal{A}_\Theta, \mathcal{A}_\Theta{}^*)$ *satisfies dim* $H^j(\mathcal{A}_\Theta, \mathcal{A}_\Theta{}^*) = 2$ *(or 1) for* $j = 1$ *(or* $j = 2$*) if and only if the irrational number* θ *satisfies a Diophantine condition like* $|1-e^{i2\pi n\theta}|^{-1} = \mathcal{O}(n^k)$ *for some k.*
Recall that when the matrix Θ *is quite irrational (see [23, Cor. 2.12]), then the* C^**-algebra generated by* $\mathcal{A}_\Theta$ *is simple.*

6.1 Computations of $⨍$

In order to get this theorem, let us prove a few technical lemmas.
We suppose from now on that Θ is a skew-symmetric matrix in $\mathcal{M}_n(\mathbb{R})$. No other hypothesis is assumed for Θ, except when it is explicitly stated.
When A is a selfadjoint one-form, we define for $n \in N$, $q \in \mathbb{N}$, $2 \leq q \leq n$ and $\sigma \in \{-,+\}^q$

$$\begin{aligned}\mathbb{A}^+ &:= A\mathcal{D}D^{-2},\\ \mathbb{A}^- &:= \epsilon JAJ^{-1}\mathcal{D}D^{-2},\\ \mathbb{A}^\sigma &:= \mathbb{A}^{\sigma_q}\cdots\mathbb{A}^{\sigma_1}.\end{aligned}$$

Lemma 6.5. *We have for any $q \in \mathbb{N}$,*

$$⨍(\widetilde{A}D^{-1})^q = ⨍(\widetilde{A}\mathcal{D}D^{-2})^q = \sum_{\sigma\in\{+,-\}^q} ⨍\mathbb{A}^\sigma.$$

Proof. Since $P_0 \in OP^{-\infty}$, $D^{-1} = \mathcal{D}D^{-2} \mod OP^{-\infty}$ and $⨍(\widetilde{A}D^{-1})^q = ⨍(\widetilde{A}\mathcal{D}D^{-2})^q$. □

Lemma 6.6. *Let A be a selfadjoint one-form, $n \in \mathbb{N}$ and $q \in \mathbb{N}$ with $2 \leq q \leq n$ and $\sigma \in \{-,+\}^q$. Then*

$$⨍\mathbb{A}^\sigma = ⨍\mathbb{A}^{-\sigma}.$$

Proof. Let us first check that $JP_0 = P_0J$. Since $\mathcal{D}J = \varepsilon J\mathcal{D}$, we get $\mathcal{D}JP_0 = 0$ so $JP_0 = P_0JP_0$. Since J is an antiunitary operator, we get $P_0J = P_0JP_0$ and finally, $P_0J = JP_0$. As a consequence, we get $JD^2 = D^2J$, $J\mathcal{D}D^{-2} = \varepsilon\mathcal{D}D^{-2}J$, $J\mathbb{A}^+J^{-1} = \mathbb{A}^-$ and $J\mathbb{A}^-J^{-1} = \mathbb{A}^+$.
In summary, $J\mathbb{A}^{\sigma_i}J^{-1} = \mathbb{A}^{-\sigma_i}$.
The trace property of $⨍$ now gives

$$⨍\mathbb{A}^\sigma = ⨍\mathbb{A}^{\sigma_q}\cdots\mathbb{A}^{\sigma_1} = ⨍J\mathbb{A}^{\sigma_q}J^{-1}\cdots J\mathbb{A}^{\sigma_1}J^{-1}⨍\mathbb{A}^{-\sigma_q}\cdots\mathbb{A}^{-\sigma_1} = ⨍\mathbb{A}^{-\sigma}.$$ □

Definition 6.7. *In [4] has been introduced the vanishing tadpole hypothesis:*

$$⨍AD^{-1} = 0, \text{ for all } A \in \Omega^1_{\mathcal{D}}(\mathcal{A}). \tag{55}$$

By the following lemma, this condition is satisfied for the noncommutative torus, a fact more or less already known within the noncommutative community [34].

Lemma 6.8. *Let $n \in \mathbb{N}$, $A = L(-iA_\alpha)\otimes\gamma^\alpha = -i\sum_{l\in\mathbb{Z}^n} a_{\alpha,l}\,U_l\otimes\gamma^\alpha$, $A_\alpha \in \mathcal{A}_\Theta$, $\{a_{\alpha,l}\}_l \in \mathcal{S}(\mathbb{Z}^n)$, be a hermitian one-form. Then,*
(i) $⨍A^qD^{-q} = ⨍(\epsilon JAJ^{-1})^qD^{-q} = 0$ for $1 \leq q < n$ (case $q = 1$ is tadpole hypothesis.)
(ii) If $\frac{1}{2\pi}\Theta$ is badly-approximable, then $⨍(A + \epsilon JAJ^{-1})^qD^{-q} = 0$ for $1 \leq q < n$.

Proof. (i) Let's compute

$$⨍A^p(\epsilon JAJ^{-1})^{p'}D^{-(p+p')}$$

where $q := p + p' \in \mathbb{N}$ with $q \leq n$. With $A = L(-iA_\alpha)\otimes\gamma^\alpha$ and $\epsilon JAJ^{-1} = R(iA_\alpha)\otimes\gamma^\alpha$, we get

$$A^p = L(-iA_{\alpha_1})\cdots L(-iA_{\alpha_p})\otimes\gamma^{\alpha_1}\cdots\gamma^{\alpha_p}$$

and

$$(\epsilon JAJ^{-1})^{p'} = R(iA_{\alpha'_1})\cdots R(iA_{\alpha'_{p'}}) \otimes \gamma^{\alpha'_1}\cdots\gamma^{\alpha'_{p'}}.$$

We note $\widetilde{a}_{\alpha,l} := a_{\alpha_1,l_1}\cdots a_{\alpha_p,l_p}$. Since

$$L(-iA_{\alpha_1})\cdots L(-iA_{\alpha_p})R(iA_{\alpha'_1})\cdots R(iA_{\alpha'_{p'}})U_k = (-i)^p\, i^{p'} \sum_{l,l'} \widetilde{a}_{\alpha,l}\,\widetilde{a}_{\alpha',l'}\, U_{l_1}\cdots U_{l_p}U_kU_{l'_{p'}}\cdots U_{l'_1},$$

and

$$U_{l_1}\cdots U_{l_p}U_k = U_kU_{l_1}\cdots U_{l_p}\, e^{-i(\sum_i l_i).\Theta k},$$

we get, with

$$\begin{aligned}
&q := p+p',\\
&U_{l,l'} := U_{l_1}\cdots U_{l_p}U_{l'_{p'}}\cdots U_{l'_1},\\
&g_{\mu,\alpha,\alpha'}(s,k,l,l') := e^{ik.\Theta\sum_j l_j}\,\tfrac{k_{\mu_1}\ldots k_{\mu_q}}{|k|^{s+2q}}\,\widetilde{a}_{\alpha,l}\,\widetilde{a}_{\alpha',l'},\\
&\gamma^{\alpha,\alpha',\mu} := \gamma^{\alpha_1}\cdots\gamma^{\alpha_p}\gamma^{\alpha'_1}\cdots\gamma^{\alpha'_{p'}}\gamma^{\mu_1}\cdots\gamma^{\mu_q},
\end{aligned}$$

$$A^p(\epsilon JAJ^{-1})^{p'}D^{-q}|D|^{-s}U_k\otimes e_i \sim_c (-i)^p\, i^{p'}\sum_{l,l'} g_{\mu,\alpha,\alpha'}(s,k,l,l')\,U_kU_{l,l'}\otimes\gamma^{\alpha,\alpha',\mu}e_i.$$

Thus, $⨍ A^p(\epsilon JAJ^{-1})^{p'}D^{-(p+p')} = \operatorname{Res}_{s=0} f(s)$ where

$$\begin{aligned}
f(s) :&= \operatorname{Tr}\big(A^p(\epsilon JAJ^{-1})^{p'}D^{-(p+p')}|D|^{-s}\big)\\
&\sim_c (-i)^p\, i^{p'}\sum_{k\in\mathbb{Z}^n}{}' \langle U_k\otimes e_i, \sum_{l,l'} g_{\mu,\alpha,\alpha'}(s,k,l,l')U_kU_{l,l'}\otimes\gamma^{\alpha,\alpha',\mu}e_i\rangle\\
&\sim_c (-i)^p\, i^{p'}\sum_{k\in\mathbb{Z}^n}{}' \,\tau\big(\sum_{l,l'} g_{\mu,\alpha,\alpha'}(s,k,l,l')U_{l,l'}\big)\operatorname{Tr}(\gamma^{\mu,\alpha,\alpha'})\\
&\sim_c (-i)^p\, i^{p'}\sum_{k\in\mathbb{Z}^n}{}' \sum_{l,l'} g_{\mu,\alpha,\alpha'}(s,k,l,l')\,\tau\big(U_{l,l'}\big)\operatorname{Tr}(\gamma^{\mu,\alpha,\alpha'}).
\end{aligned}$$

It is straightforward to check that the series $\sum'_{k,l,l'} g_{\mu,\alpha,\alpha'}(s,k,l,l')\,\tau\big(U_{l,l'}\big)$ is absolutely summable if $\Re(s) > R$ for a $R > 0$. Thus, we can exchange the summation on k and l,l', which gives

$$f(s) \sim_c (-i)^p\, i^{p'}\sum_{l,l'}\sum_{k\in\mathbb{Z}^n}{}' g_{\mu,\alpha,\alpha'}(s,k,l,l')\,\tau\big(U_{l,l'}\big)\operatorname{Tr}(\gamma^{\mu,\alpha,\alpha'}).$$

If we suppose now that $p' = 0$ and $p = q < n$, we see that,

$$f(s) \sim_c (-i)^q \sum_{l}\sum_{k\in\mathbb{Z}^n}{}' \tfrac{k_{\mu_1}\ldots k_{\mu_q}}{|k|^{s+2q}}\,\widetilde{a}_{\alpha,l}\,\delta_{\sum_{i=1}^q l_i}\operatorname{Tr}(\gamma^{\mu,\alpha,\alpha'})$$

which is, by Proposition 2.16, analytic at 0. In particular, for $(p,p') = (1,0)$, we see that $⨍ AD^{-1} = 0$, i.e. the vanishing tadpole hypothesis is satisfied. Similarly, if we suppose $p = 0$ and $p' = q < n$, we get

$$f(s) \sim_c (-i)^q \sum_{l'}\sum_{k\in\mathbb{Z}^n}{}' \tfrac{k_{\mu_1}\ldots k_{\mu_q}}{|k|^{s+2q}}\,\widetilde{a}_{\alpha,l'}\,\delta_{\sum_{i=1}^q {l'}_i}\operatorname{Tr}(\gamma^{\mu,\alpha,\alpha'})$$

which is holomorphic at 0.
(ii) By linearity and since A and ϵJAJ^{-1} commute, it is sufficient to check that for any $p,p'\geq 0$ such that $1\leq p+p'<n$,

$$\int\!\!\!\!\!- A^p(\epsilon JAJ^{-1})^{p'}D^{-(p+p')}=0.$$

By Theorem 2.5 (ii), since we suppose here that $\frac{1}{2\pi}\Theta$ is badly approximable, the functions $\sum_{l,l'} g(s,l,l')$ where $g(s,l,l'):=\sum'_k g_{\mu,\alpha,\alpha'}(s,k,l,l')\,\tau\big(U_{l,l'}\big)$ are holomorphic at 0. This entails the result. ☐

6.1.1 Even dimensional case

Lemma 6.9. *Same hypothesis as in Lemma 6.8.*
(i) Case $n=2$:

$$\int\!\!\!\!\!- A^qD^{-q}=-\delta_{q,2}\,4\pi\,\tau\big(A_\alpha A^\alpha\big)\,.$$

(ii) Case $n=4$: with the shorthand $\delta_{\mu_1,\dots,\mu_4}:=\delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4}+\delta_{\mu_1\mu_3}\delta_{\mu_2\mu_4}+\delta_{\mu_1\mu_4}\delta_{\mu_2\mu_3}$,

$$\int\!\!\!\!\!- A^qD^{-q}=\delta_{q,4}\,\tfrac{\pi^2}{12}\tau\big(A_{\alpha_1}\cdots A_{\alpha_4}\big)\,\mathrm{Tr}(\gamma^{\alpha_1}\cdots\gamma^{\alpha_4}\gamma^{\mu_1}\cdots\gamma^{\mu_4})\delta_{\mu_1,\dots,\mu_4}\,.$$

Proof. (i,ii) The same computation as in Lemma 6.8 (i) (with $p'=0$, $p=q=n$) gives

$$\int\!\!\!\!\!- A^nD^{-n}=\operatorname*{Res}_{s=0}(-i)^n\big(\sum_{k\in\mathbb{Z}^n}{}'\tfrac{k_{\mu_1}\dots k_{\mu_n}}{|k|^{s+2n}}\big)\,\tau\big(\sum_{l\in(\mathbb{Z}^n)^n}\widetilde{a}_{\alpha,l}U_{l_1}\cdots U_{l_n}\big)\ \mathrm{Tr}(\gamma^{\alpha_1}\cdots\gamma^{\alpha_n}\gamma^{\mu_1}\cdots\gamma^{\mu_n})$$

and the result follows from Proposition 2.16. ☐

We will use few notations:
If $n\in\mathbb{N}$, $q\geq 2$, $l:=(l_1,\cdots,l_{q-1})\in(\mathbb{Z}^n)^{q-1}$, $\alpha:=(\alpha_1,\cdots,\alpha_q)\in\{1,\cdots,n\}^q$, $k\in\mathbb{Z}^n\backslash\{0\}$, $\sigma\in\{-,+\}^q$, $(a_i)_{1\leq i\leq n}\in(\mathcal{S}(\mathbb{Z}^n))^n$,

$$\begin{aligned}
&l_q:=-\sum_{1\leq j\leq q-1}l_j\,,\quad \lambda_\sigma:=(-i)^q\prod_{j=1\dots q}\sigma_j\,,\quad \widetilde{a}_{\alpha,l}:=a_{\alpha_1,l_1}\dots a_{\alpha_q,l_q}\,,\\
&\phi_\sigma(k,l):=\sum_{1\leq j\leq q-1}(\sigma_j-\sigma_q)\,k.\Theta l_j+\sum_{2\leq j\leq q-1}\sigma_j\,(l_1+\dots+l_{j-1}).\Theta l_j\,,\\
&g_\mu(s,k,l):=\tfrac{k_{\mu_1}(k+l_1)_{\mu_2}\dots(k+l_1+\dots+l_{q-1})_{\mu_q}}{|k|^{s+2}|k+l_1|^2\dots|k+l_1+\dots+l_{q-1}|^2}\,,
\end{aligned}$$

with the convention $\sum_{2\leq j\leq q-1}=0$ when $q=2$, and $g_\mu(s,k,l)=0$ whenever $\widehat{l_i}=-k$ for a $1\leq i\leq q-1$.

Lemma 6.10. *Let $A=L(-iA_\alpha)\otimes\gamma^\alpha=-i\sum_{l\in\mathbb{Z}^n}a_{\alpha,l}\,U_l\otimes\gamma^\alpha$ where $A_\alpha=-A_\alpha^*\in\mathcal{A}_\Theta$ and $\{\,a_{\alpha,l}\,\}_l\in\mathcal{S}(\mathbb{Z}^n)$, with $n\in\mathbb{N}$, be a hermitian one-form, and let $2\leq q\leq n$, $\sigma\in\{-,+\}^q$.*
Then, $\int\!\!\!\!- \mathbb{A}^\sigma=\operatorname*{Res}_{s=0}\,f(s)$ where

$$f(s):=\sum_{l\in(\mathbb{Z}^n)^{q-1}}\sum_{k\in\mathbb{Z}^n}{}'\lambda_\sigma\ e^{\frac{i}{2}\phi_\sigma(k,l)}\ g_\mu(s,k,l)\ \widetilde{a}_{\alpha,l}\ \ \mathrm{Tr}(\gamma^{\alpha_q}\gamma^{\mu_q}\cdots\gamma^{\alpha_1}\gamma^{\mu_1}).$$

Proof. By definition, $⨍ \mathbb{A}^\sigma = \operatorname{Res}_{s=0} f(s)$ where

$$\operatorname{Tr}(\mathbb{A}^{\sigma_q}\cdots\mathbb{A}^{\sigma_1}|D|^{-s}) \sim_c \sum_{k\in\mathbb{Z}^n}{}' \langle U_k\otimes e^i, |k|^{-s}\,\mathbb{A}^{\sigma_q}\cdots\mathbb{A}^{\sigma_1}U_k\otimes e_i\rangle =: f(s)$$

Let $r\in\mathbb{Z}^n$ and $v\in\mathbb{C}^{2^m}$. Since $A = L(-iA_\alpha)\otimes\gamma^\alpha$, and $\epsilon JAJ^{-1} = R(iA_\alpha)\otimes\gamma^\alpha$, we get

$$\mathbb{A}^+U_r\otimes v = A\mathcal{D}D^{-2}U_r\otimes v = A\tfrac{r_\mu}{|r|^2+\delta_{r,0}}U_r\otimes\gamma^\mu v = -i\tfrac{r_\mu}{|r|^2+\delta_{r,0}}A_\alpha U_r\otimes\gamma^\alpha\gamma^\mu v\,,$$
$$\mathbb{A}^-U_r\otimes v = \epsilon JAJ^{-1}\mathcal{D}D^{-2}U_r\otimes v = \epsilon JAJ^{-1}\tfrac{r_\mu}{|r|^2+\delta_{r,0}}U_r\otimes\gamma^\mu v = i\tfrac{r_\mu}{|r|^2+\delta_{r,0}}U_rA_\alpha\otimes\gamma^\alpha\gamma^\mu v.$$

With $U_lU_r = e^{\frac{i}{2}r.\Theta l}U_{r+l}$ and $U_rU_l = e^{-\frac{i}{2}r.\Theta l}U_{r+l}$, we obtain, for any $1\le j\le q$,

$$\mathbb{A}^{\sigma_j}U_r\otimes v = \sum_{l\in\mathbb{Z}^n}(-\sigma_j)\,i\,e^{\sigma_j\,\frac{i}{2}r.\Theta l}\,\tfrac{r_\mu}{|r|^2+\delta_{r,0}}\,a_{\alpha,l}\,U_{r+l}\otimes\gamma^\alpha\gamma^\mu v.$$

We now apply q times this formula to get

$$|k|^{-s}\mathbb{A}^{\sigma_q}\cdots\mathbb{A}^{\sigma_1}U_k\otimes e_i = \lambda_\sigma\sum_{l\in(\mathbb{Z}^n)^q} e^{\frac{i}{2}\phi_\sigma(k,l)}\;g_\mu(s,k,l)\;\widetilde{a}_{\alpha,l}\;U_{k+\sum_j l_j}\otimes\gamma^{\alpha_q}\gamma^{\mu_q}\cdots\gamma^{\alpha_1}\gamma^{\mu_1}e_i$$

with

$$\phi_\sigma(k,l) := \sigma_1\,k.\Theta l_1 + \sigma_2\,(k+l_1).\Theta l_2 + \ldots + \sigma_q\,(k+l_1+\ldots+l_{q-1}).\Theta l_q.$$

Thus,

$$\begin{aligned} f(s) &= \sum_{k\in\mathbb{Z}^n}{}'\tau\big(\lambda_\sigma\sum_{l\in(\mathbb{Z}^n)^q}e^{\frac{i}{2}\phi_\sigma(k,l)}\;g_\mu(s,k,l)\;\widetilde{a}_{\alpha,l}\;U_{\sum_j l_j}e^{\frac{i}{2}k.\Theta\sum_j l_j}\big)\operatorname{Tr}(\gamma^{\alpha_q}\gamma^{\mu_q}\cdots\gamma^{\alpha_1}\gamma^{\mu_1})\\ &= \sum_{k\in\mathbb{Z}^n}{}'\lambda_\sigma\sum_{l\in(\mathbb{Z}^n)^q}e^{\frac{i}{2}\phi_\sigma(k,l)}\;g_\mu(s,k,l)\;\widetilde{a}_{\alpha,l}\;\delta(\sum_j l_j)\operatorname{Tr}(\gamma^{\alpha_q}\gamma^{\mu_q}\cdots\gamma^{\alpha_1}\gamma^{\mu_1})\\ &= \sum_{k\in\mathbb{Z}^n}{}'\lambda_\sigma\sum_{l\in(\mathbb{Z}^n)^{q-1}}e^{\frac{i}{2}\phi_\sigma(k,l)}\;g_\mu(s,k,l)\;\widetilde{a}_{\alpha,l}\;\operatorname{Tr}(\gamma^{\alpha_q}\gamma^{\mu_q}\cdots\gamma^{\alpha_1}\gamma^{\mu_1}) \end{aligned}$$

where in the last sum l_q is fixed to $-\sum_{1\le j\le q-1}l_j$ and thus,

$$\phi_\sigma(k,l) = \sum_{1\le j\le q-1}(\sigma_j-\sigma_q)\,k.\Theta l_j + \sum_{2\le j\le q-1}\sigma_j\,(l_1+\ldots+l_{j-1}).\Theta l_j.$$

By Lemma 2.10, there exists a $R>0$ such that for any $s\in\mathbb{C}$ with $\Re(s)>R$, the family

$$\big(e^{\frac{i}{2}\phi_\sigma(k,l)}\;g_\mu(s,k,l)\;\widetilde{a}_{\alpha,l}\big)_{(k,l)\in(\mathbb{Z}^n\setminus\{\,0\,\})\times(\mathbb{Z}^n)^{q-1}}$$

is absolutely summable as a linear combination of families of the type considered in that lemma. As a consequence, we can exchange the summations on k and l, which gives the result. □

In the following, we will use the shorthand

$$c := \tfrac{4\pi^2}{3}.$$

Lemma 6.11. *Suppose $n=4$. Then, with the same hypothesis of Lemma 6.10,*

(i) $\quad \tfrac{1}{2}\fint (\mathbb{A}^+)^2 = \tfrac{1}{2}\fint (\mathbb{A}^-)^2 = c \sum_{l\in\mathbb{Z}^4} a_{\alpha_1,l}\, a_{\alpha_2,-l}\,\big(l^{\alpha_1} l^{\alpha_2} - \delta^{\alpha_1\alpha_2}|l|^2\big).$

(ii) $\quad -\tfrac{1}{3}\fint (\mathbb{A}^+)^3 = -\tfrac{1}{3}\fint (\mathbb{A}^-)^3 = 4c \sum_{l_i\in\mathbb{Z}^4} a_{\alpha_3,-l_1-l_2}\, a_{l_2}^{\alpha_1}\, a_{\alpha_1,l_1}\ \sin \tfrac{l_1.\Theta l_2}{2}\, l_1^{\alpha_3}.$

(iii) $\quad \tfrac{1}{4}\fint (\mathbb{A}^+)^4 = \tfrac{1}{4}\fint (\mathbb{A}^-)^4 = 2c \sum_{l_i\in\mathbb{Z}^4} a_{\alpha_1,-l_1-l_2-l_3}\, a_{\alpha_2,l_3}\, a_{l_2}^{\alpha_1}\, a_{l_1}^{\alpha_2} \sin \tfrac{l_1.\Theta(l_2+l_3)}{2}\ \sin \tfrac{l_2.\Theta l_3}{2}.$

(iv) Suppose $\frac{1}{2\pi}\Theta$ badly approximable. Then the crossed terms in $\fint(\mathbb{A}^+ + \mathbb{A}^-)^q$ vanish: if C is the set of all $\sigma\in\{-,+\}^q$ with $2\le q\le 4$, such that there exist i,j satisfying $\sigma_i\neq\sigma_j$, we have $\sum_{\sigma\in C}\fint \mathbb{A}^\sigma = 0$.

Proof. (i) Lemma 6.10 entails that $\fint \mathbb{A}^{++} = \underset{s=0}{\operatorname{Res}} \sum_{l\in\mathbb{Z}^n} -f(s,l)$ where

$$f(s,l) := {\sum_{k\in\mathbb{Z}^n}}' \tfrac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+2}|k+l|^2}\, \widetilde{a}_{\alpha,l}\ \operatorname{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}) \ \text{ and } \ \widetilde{a}_{\alpha,l} := a_{\alpha_1,l}\, a_{\alpha_2,-l}\,.$$

We will now reduce the computation of the residue of an expression involving terms like $|k+l|^2$ in the denominator to the computation of residues of zeta functions. To proceed, we use (16) into an expression like the one appearing in $f(s,l)$. We see that the last term on the righthandside yields a $Z_n(s)$ while the first one is less divergent by one power of k. If this is not enough, we repeat this operation for the new factor of $|k+l|^2$ in the denominator. For $f(s,l)$, which is quadratically divergent at $s=0$, we have to repeat this operation three times before ending with a convergent result. All the remaining terms are expressible in terms of Z_n functions. We get, using three times (16),

$$\tfrac{1}{|k+l|^2} = \tfrac{1}{|k|^2} - \tfrac{2k.l+|l|^2}{|k|^4} + \tfrac{(2k.l+|l|^2)^2}{|k|^6} - \tfrac{(2k.l+|l|^2)^3}{|k|^6|k+l|^2}\,. \tag{56}$$

Let us define

$$f_{\alpha,\mu}(s,l) := {\sum_{k\in\mathbb{Z}^n}}' \tfrac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+2}|k+l|^2}\, \widetilde{a}_{\alpha,l}$$

so that $f(s,l) = f_{\alpha,\mu}(s,l)\operatorname{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1})$. Equation (56) gives

$$f_{\alpha,\mu}(s,l) = f_1(s,l) - f_2(s,l) + f_3(s,l) - r(s,l)$$

with obvious identifications. Note that the function

$$r(s,l) = {\sum_{k\in\mathbb{Z}^n}}' \tfrac{k_{\mu_1}(k+l)_{\mu_2}(2kl+|l|^2)^3}{|k|^{s+8}|k+l|^2}\, \widetilde{a}_{\alpha,l}$$

is a linear combination of functions of the type $H(s,l)$ satisfying the hypothesis of Corollary 2.13. Thus, $r(s,l)$ satisfies (H1) and with the previously seen equivalence relation modulo functions satisfying this hypothesis we get $f_{\alpha,\mu}(s,l) \sim f_1(s,l) - f_2(s,l) + f_3(s,l)$.
Let's now compute $f_1(s,l)$.

$$f_1(s,l) = {\sum_{k\in\mathbb{Z}^n}}' \tfrac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+4}}\, \widetilde{a}_{\alpha,l} = \widetilde{a}_{\alpha,l} {\sum_{k\in\mathbb{Z}^n}}' \tfrac{k_{\mu_1}k_{\mu_2}}{|k|^{s+4}} + 0.$$

Proposition 2.1 entails that $s \mapsto \sum_{k\in\mathbb{Z}^n}{}' \frac{k_{\mu_1}k_{\mu_2}}{|k|^{s+4}}$ is holomorphic at 0. Thus, $f_1(s,l)$ satisfies (H1), and $f_{\alpha,\mu}(s,l) \sim -f_2(s,l) + f_3(s,l)$.
Let's now compute $f_2(s,l)$ modulo (H1). We get, using several times Proposition 2.1,

$$\begin{aligned} f_2(s,l) &= \sum_{k\in\mathbb{Z}^n}{}' \tfrac{k_{\mu_1}(k+l)_{\mu_2}(2kl+|l|^2)}{|k|^{s+6}}\ \widetilde{a}_{\alpha,l} = \sum_{k\in\mathbb{Z}^n}{}' \tfrac{(2kl)k_{\mu_1}k_{\mu_2}+(2kl)k_{\mu_1}l_{\mu_2}+|l|^2k_{\mu_1}k_{\mu_2}+l_{\mu_2}|l|^2k_{\mu_1}}{|k|^{s+6}}\ \widetilde{a}_{\alpha,l} \\ &\sim 0 + \sum_{k\in\mathbb{Z}^n}{}' \tfrac{(2kl)k_{\mu_1}l_{\mu_2}}{|k|^{s+6}}\ \widetilde{a}_{\alpha,l} + \sum_{k\in\mathbb{Z}^n}{}' \tfrac{|l|^2k_{\mu_1}k_{\mu_2}}{|k|^{s+6}}\ \widetilde{a}_{\alpha,l} + 0 \end{aligned}$$

Recall that $\sum_{k\in\mathbb{Z}^n}' \frac{k_ik_j}{|k|^{s+6}} = \frac{\delta_{ij}}{n} Z_n(s+4)$. Thus,

$$f_2(s,l) \sim 2l^i l_{\mu_2}\widetilde{a}_{\alpha,l}\tfrac{\delta_{i\mu_1}}{n} Z_n(s+4) + |l|^2\, \widetilde{a}_{\alpha,l}\, \tfrac{\delta_{\mu_1\mu_2}}{n} Z_n(s+4).$$

Finally, let us compute $f_3(s,l)$ modulo (H1) following the same principles:

$$\begin{aligned} f_3(s,l) &= \sum_{k\in\mathbb{Z}^n}{}' \tfrac{k_{\mu_1}(k+l)_{\mu_2}(2kl+|l|^2)^2}{|k|^{s+8}}\ \widetilde{a}_{\alpha,l} \\ &= \sum_{k\in\mathbb{Z}^n}{}' \tfrac{(2kl)^2k_{\mu_1}k_{\mu_2}+(2kl)^2k_{\mu_1}l_{\mu_2}+|l|^4k_{\mu_1}k_{\mu_2}+|l|^4k_{\mu_1}l_{\mu_2}+(4kl)|l|^2k_{\mu_1}k_{\mu_2}+(4kl)|l|^2k_{\mu_1}l_{\mu_2}}{|k|^{s+8}}\ \widetilde{a}_{\alpha,l} \\ &\sim 4l^il^j \sum_{k\in\mathbb{Z}^n}{}' \tfrac{k_ik_jk_{\mu_1}k_{\mu_2}}{|k|^{s+8}}\widetilde{a}_{\alpha,l} + 0. \end{aligned}$$

In conclusion,

$$f_{\alpha,\mu}(s,l) \sim -\tfrac{1}{4}(2l_{\mu_1}l_{\mu_2} + |l|^2\ \delta_{\mu_1\mu_2})\widetilde{a}_{\alpha,l} Z_n(s+4) + 4l^il^j\, \widetilde{a}_{\alpha,l} \sum_{k\in\mathbb{Z}^n}{}' \tfrac{k_ik_jk_{\mu_1}k_{\mu_2}}{|k|^{s+8}} =: g_{\alpha,\mu}(s,l).$$

Proposition (2.1) entails that $Z_n(s+4)$ and $s \mapsto \sum_{k\in\mathbb{Z}^n}{}' \frac{k_ik_jk_{\mu_1}k_{\mu_2}}{|k|^{s+8}}$ extend holomorphically in a punctured open disk centered at 0. Thus, $g_{\alpha,\mu}(s,l)$ satisfies (H2) and we can apply Lemma 2.14 to get

$$-\!\!\!\!\!\!\int (\mathbb{A}^+)^2 = \operatorname*{Res}_{s=0} \sum_{l\in\mathbb{Z}^n} f(s,l) = \sum_{l\in\mathbb{Z}^n} \operatorname*{Res}_{s=0}\, g_{\alpha,\mu}(s,l)\, \mathrm{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}) =: \sum_{l\in\mathbb{Z}^n} \operatorname*{Res}_{s=0}\, g(s,l).$$

The problem is now reduced to the computation of $\operatorname*{Res}_{s=0} g(s,l)$. Recall that $\mathrm{Res}_{s=0}\, Z_4(s+4) = 2\pi^2$ by (20) or (17), and

$$\mathrm{Res}_{s=0} \sum_{k\in\mathbb{Z}^n}{}' \tfrac{k_ik_jk_lk_m}{|k|^{s+8}} = (\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})\, \tfrac{\pi^2}{12}.$$

Thus,

$$\operatorname*{Res}_{s=0}\, g_{\alpha,\mu}(s,l) = -\tfrac{\pi^2}{3}\, \widetilde{a}_{\alpha,l}\, (l_{\mu_1}l_{\mu_2} + \tfrac{1}{2}|l|^2\delta_{\mu_1\mu_2}).$$

We will use

$$\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_{2j}}) = \mathrm{Tr}(1) \sum_{\text{all pairings of } \{1\cdots 2j\}} s(P)\, \delta_{\mu_{P_1}\mu_{P_2}}\delta_{\mu_{P_3}\mu_{P_4}}\cdots\delta_{\mu_{P_{2j-1}}\mu_{P_{2j}}} \qquad (57)$$

where $s(P)$ is the signature of the permutation P when $P_{2m-1} < P_{2m}$ for $1 \le m \le n$. This gives

$$\mathrm{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}) = 2^m(\delta^{\alpha_2\mu_2}\delta^{\alpha_1\mu_1} - \delta^{\alpha_1\alpha_2}\delta^{\mu_2\mu_1} + \delta^{\alpha_2\mu_1}\delta^{\mu_2\alpha_1}). \tag{58}$$

Thus,

$$\begin{aligned}\operatorname*{Res}_{s=0} g(s,l) &= -c\,\widetilde{a}_{\alpha,l}\,(l_{\mu_1}l_{\mu_2} + \tfrac{1}{2}|l|^2\delta_{\mu_1\mu_2})(\delta^{\alpha_2\mu_2}\delta^{\alpha_1\mu_1} - \delta^{\alpha_1\alpha_2}\delta^{\mu_2\mu_1} + \delta^{\alpha_2\mu_1}\delta^{\mu_2\alpha_1})\\ &= -2c\,\widetilde{a}_{\alpha,l}\,(l^{\alpha_1}l^{\alpha_2} - \delta^{\alpha_1\alpha_2}|l|^2).\end{aligned}$$

Finally,

$$\tfrac{1}{2}\fint(\mathbb{A}^+)^2 = \tfrac{1}{2}\fint(\mathbb{A}^-)^2 = c\sum_{l\in\mathbb{Z}^n} a_{\alpha_1,l}\,a_{\alpha_2,-l}\,\big(l^{\alpha_1}l^{\alpha_2} - \delta^{\alpha_1\alpha_2}|l|^2\big).$$

(ii) Lemma 6.10 entails that $\fint \mathbb{A}^{+++} = \operatorname*{Res}_{s=0}\sum_{(l_1,l_2)\in(\mathbb{Z}^n)^2} f(s,l)$ where

$$\begin{aligned}f(s,l) &:= \sum_{k\in\mathbb{Z}^n}{}' i\,e^{\frac{i}{2}l_1\Theta l_2}\,\tfrac{k_{\mu_1}(k+l_1)_{\mu_2}(k+\widehat{l}_2)_{\mu_3}}{|k|^{s+2}|k+l_1|^2|k+\widehat{l}_2|^2}\,\widetilde{a}_{\alpha,l}\,\mathrm{Tr}(\gamma^{\alpha_3}\gamma^{\mu_3}\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1})\\ &=: f_{\alpha,\mu}(s,l)\,\mathrm{Tr}(\gamma^{\alpha_3}\gamma^{\mu_3}\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}),\end{aligned}$$

and $\widetilde{a}_{\alpha,l} := a_{\alpha_1,l_1}\,a_{\alpha_2,l_2}\,a_{\alpha_3,-\widehat{l}_2}$ with $\widehat{l}_2 := l_1 + l_2$.
We use the same technique as in (i):

$$\begin{aligned}\tfrac{1}{|k+l_1|^2} &= \tfrac{1}{|k|^2} - \tfrac{2k.l_1+|l_1|^2}{|k|^4} + \tfrac{(2k.l_1+|l_1|^2)^2}{|k|^4|k+l_1|^2}\,,\\ \tfrac{1}{|k+\widehat{l}_2|^2} &= \frac{1}{|k|^2} - \tfrac{2k.\widehat{l}_2+|\widehat{l}_2|^2}{|k|^4} + \tfrac{(2k.\widehat{l}_2+|\widehat{l}_2|^2)^2}{|k|^4|k+\widehat{l}_2|^2}\end{aligned}$$

and thus,

$$\tfrac{1}{|k+l_1|^2|k+\widehat{l}_2|^2} = \tfrac{1}{|k|^4} - \tfrac{2k.l_1}{|k|^6} - \tfrac{2k.\widehat{l}_2}{|k|^6} + R(k,l) \tag{59}$$

where the remain $R(k,l)$ is a term of order at most -6 in k. Equation (59) gives

$$f_{\alpha,\mu}(s,l) = f_1(s,l) + r(s,l)$$

where $r(s,l)$ corresponds to $R(k,l)$. Note that the function

$$r(s,l) = \sum_{k\in\mathbb{Z}^n}{}' i\,e^{\frac{i}{2}l_1\Theta l_2}\,\tfrac{k_{\mu_1}(k+l)_{\mu_2}(k+\widehat{l}_2)_{\mu_3}R(k,l)}{|k|^{s+2}}\,\widetilde{a}_{\alpha,l}$$

is a linear combination of functions of the type $H(s,l)$ satisfying the hypothesis of Corollary (2.13). Thus, $r(s,l)$ satisfies (H1) and $f_{\alpha,\mu}(s,l) \sim f_1(s,l)$.
Let us compute $f_1(s,l)$ modulo (H1)

$$\begin{aligned}f_1(s,l) &= \sum_{k\in\mathbb{Z}^n}{}' i\,e^{\frac{i}{2}l_1\Theta l_2}\,\tfrac{k_{\mu_1}(k+l_1)_{\mu_2}(k+\widehat{l}_2)_{\mu_3}}{|k|^{s+6}}\,\widetilde{a}_{\alpha,l} - \sum_{k\in\mathbb{Z}^n}{}' i\,e^{\frac{i}{2}l_1\Theta l_2}\,\tfrac{k_{\mu_1}(k+l_1)_{\mu_2}(k+\widehat{l}_2)_{\mu_3}(2k.l_1+2k.\widehat{l}_2)}{|k|^{s+8}}\,\widetilde{a}_{\alpha,l}\\ &\sim \sum_{k\in\mathbb{Z}^n}{}' i\,e^{\frac{i}{2}l_1\Theta l_2}\,\tfrac{k_{\mu_1}k_{\mu_2}\widehat{l}_{2\mu_3}+k_{\mu_1}k_{\mu_3}l_{1\mu_2}}{|k|^{s+6}}\,\widetilde{a}_{\alpha,l} - \sum_{k\in\mathbb{Z}^n}{}' i\,e^{\frac{i}{2}l_1\Theta l_2}\,\tfrac{k_{\mu_1}k_{\mu_2}k_{\mu_3}(2k.l_1+2k.\widehat{l}_2)}{|k|^{s+8}}\,\widetilde{a}_{\alpha,l}\\ &= i\,e^{\frac{i}{2}l_1\Theta l_2}\,\widetilde{a}_{\alpha,l}\big((l_{1\mu_2}\delta_{\mu_1\mu_3} + \widehat{l}_{2\mu_3}\delta_{\mu_1\mu_2})\,\tfrac{1}{4}Z_4(s+4) - 2(l_1^i + \widehat{l}_2^i)\sum_{k\in\mathbb{Z}^n}{}'\tfrac{k_{\mu_1}k_{\mu_2}k_{\mu_3}k_i}{|k|^{s+8}}\big)\\ &=: g_{\alpha,\mu}(s,l).\end{aligned}$$

Since $g_{\alpha,\mu}(s,l)$ satisfies (H2), we can apply Lemma 2.14 to get

$$\begin{aligned}
-\!\!\!\!\!\!\int (\mathbb{A}^+)^3 &= \operatorname*{Res}_{s=0} \sum_{(l_1,l_2)\in(\mathbb{Z}^n)^2} f(s,l) \\
&= \sum_{(l_1,l_2)\in(\mathbb{Z}^n)^2} \operatorname*{Res}_{s=0} g_{\alpha,\mu}(s,l)\, \mathrm{Tr}(\gamma^{\alpha_3}\gamma^{\mu_3}\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}) =: \sum_l X_l.
\end{aligned}$$

Recall that $l_3 := -l_1 - l_2 = -\widehat{l_2}$. By (17) and (19),

$$\begin{aligned}
&\operatorname*{Res}_{s=0} g_{\alpha,\mu}(s,l) i\, e^{\frac{i}{2} l_1 \Theta l_2}\, \widetilde{a}_{\alpha,l} \big(2(-l_1^i + l_3^i)\tfrac{\pi^2}{12}(\delta_{\mu_1\mu_2}\delta_{\mu_3 i} + \delta_{\mu_1\mu_3}\delta_{\mu_2 i} + \delta_{\mu_1 i}\delta_{\mu_2\mu_3}) \\
&\qquad + (l_{1\mu_2}\delta_{\mu_1\mu_3} - l_{3\mu_3}\delta_{\mu_1\mu_2})\tfrac{\pi^2}{2}\big).
\end{aligned}$$

We decompose X_l in five terms: $X_l = 2^m\ \frac{\pi^2}{2}\ i\, e^{\frac{i}{2} l_1 \Theta l_2}\, \widetilde{a}_{\alpha,l}\, (T_1 + T_2 + T_3 + T_4 + T_5)$ where

$$\begin{aligned}
T_0 &:= \tfrac{1}{3}(-l_1^i + l_3^i)(\delta_{\mu\nu}\delta_{\rho i} + \delta_{\mu\rho}\delta_{\nu i} + \delta_{\mu i}\delta_{\nu\rho}) + l_{1\nu}\delta_{\mu\rho} - l_{3\rho}\delta_{\mu\nu}, \\
T_1 &:= (\delta^{\alpha_3\rho}\delta^{\alpha_2\nu}\delta^{\alpha_1\mu} - \delta^{\alpha_3\rho}\delta^{\alpha_2\alpha_1}\delta^{\mu\nu} + \delta^{\alpha_3\rho}\delta^{\alpha_2\mu}\delta^{\alpha_1\nu})T_0, \\
T_2 &:= (-\delta^{\alpha_2\alpha_3}\delta^{\rho\nu}\delta^{\alpha_1\mu} + \delta^{\alpha_2\alpha_3}\delta^{\alpha_1\rho}\delta^{\mu\nu} - \delta^{\alpha_2\alpha_3}\delta^{\rho\mu}\delta^{\alpha_1\nu})T_0, \\
T_3 &:= (\delta^{\alpha_3\nu}\delta^{\alpha_2\rho}\delta^{\alpha_1\mu} - \delta^{\alpha_3\nu}\delta^{\alpha_1\rho}\delta^{\alpha_2\mu} + \delta^{\alpha_3\nu}\delta^{\rho\mu}\delta^{\alpha_1\alpha_2})T_0, \\
T_4 &:= (-\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\rho}\delta^{\mu\nu} + \delta^{\alpha_1\alpha_3}\delta^{\rho\nu}\delta^{\alpha_2\mu} - \delta^{\alpha_1\alpha_3}\delta^{\rho\mu}\delta^{\alpha_2\nu})T_0, \\
T_5 &:= (\delta^{\alpha_3\mu}\delta^{\alpha_2\rho}\delta^{\alpha_1\nu} - \delta^{\alpha_3\mu}\delta^{\rho\nu}\delta^{\alpha_1\alpha_2} + \delta^{\alpha_3\mu}\delta^{\alpha_1\rho}\delta^{\alpha_2\nu})T_0.
\end{aligned}$$

With the shorthand $p := -l_1 - 2l_3$, $q := 2l_1 + l_3$, $r := -p - q = -l_1 + l_3$, we compute each T_i, and find

$$\begin{aligned}
3T_1 &= \delta^{\alpha_1\alpha_2}(2-2^m)p^{\alpha_3} + \delta^{\alpha_3\alpha_1}q^{\alpha_2} - \delta^{\alpha_2\alpha_1}q^{\alpha_3} + \delta^{\alpha_3\alpha_2}q^{\alpha_1} + \delta^{\alpha_3\alpha_2}r^{\alpha_1} - \delta^{\alpha_2\alpha_1}r^{\alpha_3} + \delta^{\alpha_3\alpha_1}r^{\alpha_2}, \\
3T_2 &= (2^m - 2)\delta^{\alpha_2\alpha_3}p^{\alpha_1} - 2^m\delta^{\alpha_2\alpha_3}q^{\alpha_1} - 2^m\delta^{\alpha_2\alpha_3}r^{\alpha_1}, \\
3T_3 &= \delta^{\alpha_1\alpha_3}p^{\alpha_2} - \delta^{\alpha_2\alpha_3}p^{\alpha_1} + \delta^{\alpha_1\alpha_2}p^{\alpha_3} + 2^m\delta^{\alpha_2\alpha_1}q^{\alpha_3} + \delta^{\alpha_3\alpha_2}r^{\alpha_1} - \delta^{\alpha_3\alpha_1}r^{\alpha_2} + \delta^{\alpha_1\alpha_2}r^{\alpha_3}, \\
3T_4 &= -\delta^{\alpha_1\alpha_3}2^m p^{\alpha_2} - \delta^{\alpha_1\alpha_3}2^m q^{\alpha_2} + \delta^{\alpha_1\alpha_3}(2^m - 2)r^{\alpha_2}, \\
3T_5 &= \delta^{\alpha_1\alpha_3}p^{\alpha_2} - \delta^{\alpha_1\alpha_2}p^{\alpha_3} + \delta^{\alpha_3\alpha_2}p^{\alpha_1} + \delta^{\alpha_3\alpha_2}q^{\alpha_1} - \delta^{\alpha_1\alpha_2}q^{\alpha_3} + \delta^{\alpha_3\alpha_1}q^{\alpha_2} + (2-2^m)\delta^{\alpha_1\alpha_2}r^{\alpha_3}.
\end{aligned}$$

Thus,

$$X_l = 2^m\ \tfrac{2\pi^2}{3} i\, e^{\frac{i}{2} l_1.\Theta l_2}\, \widetilde{a}_{\alpha,l}\, (q^{\alpha_3}\delta^{\alpha_1\alpha_2} + r^{\alpha_2}\delta^{\alpha_1\alpha_3} + p^{\alpha_1}\delta^{\alpha_2\alpha_3}) \tag{60}$$

and

$$-\!\!\!\!\!\!\int (\mathbb{A}^+)^3 = i\, 2c\, (S_1 + S_2 + S_3),$$

where S_1, S_2 and S_3 correspond to respectively $q^{\alpha_3}\delta^{\alpha_1\alpha_2}$, $r^{\alpha_2}\delta^{\alpha_1\alpha_3}$ and $p^{\alpha_1}\delta^{\alpha_2\alpha_3}$. In S_1, we permute the l_i variables the following way: $l_1 \mapsto l_3$, $l_2 \mapsto l_1$, $l_3 \mapsto l_2$. Therefore, $l_3.\Theta\, l_1 \mapsto l_3.\Theta\, l_1$ and $q \mapsto r$. With a similar permutation of the α_i, we see that $S_1 = S_2$. We apply the same principles to prove that $S_1 = S_3$ (using permutation $l_1 \mapsto l_2$, $l_2 \mapsto l_3$, $l_3 \mapsto l_1$). Thus,

$$\tfrac{1}{3}\ -\!\!\!\!\!\!\int (\mathbb{A}^+)^3 = i\ 2c\ \sum_{l_i} \widetilde{a}_{\alpha,l}\, e^{\frac{i}{2} l_1.\Theta l_2}\ (l_1 - l_2)^{\alpha_3}\delta^{\alpha_1\alpha_2} = S_4 - S_5,$$

where S_4 correspond to l_1 and S_5 to l_2. We permute the l_i variables in S_5 the following way: $l_1 \mapsto l_2$, $l_2 \mapsto l_1$, $l_3 \mapsto l_3$, with a similar permutation on the α_i. Since $l_1.\Theta\, l_2 \mapsto -l_1.\Theta\, l_2$, we finally get

$$\tfrac{1}{3} \;⨍(\mathbb{A}^+)^3 = -4c \sum_{l_i} a_{\alpha_1,l_1}\, a_{\alpha_2,l_2}\, a_{\alpha_3,-l_1-l_2}\ \sin\tfrac{l_1.\Theta l_2}{2}\ l_1^{\alpha_3}\, \delta^{\alpha_1\alpha_2}.$$

(iii) Lemma 6.10 entails that $⨍\, \mathbb{A}^{++++} = \operatorname{Res}_{s=0} \sum_{(l_1,l_2,l_3)\in(\mathbb{Z}^n)^3} f_{\mu,\alpha}(s,l)\, \operatorname{Tr}\gamma^{\mu,\alpha}$ where

$$\begin{aligned}
&\theta := l_1.\Theta l_2 + l_1.\Theta l_3 + l_2.\Theta l_3,\\
&\operatorname{Tr}\gamma^{\mu,\alpha} := \operatorname{Tr}(\gamma^{\alpha_4}\gamma^{\mu_4}\gamma^{\alpha_3}\gamma^{\mu_3}\gamma^{\alpha_2}\gamma^{\mu 2}\gamma^{\alpha_1}\gamma^{\mu_1}),\\
&f_{\mu,\alpha}(s,l) := {\sum_{k\in\mathbb{Z}^n}}' e^{\frac{i}{2}\theta}\, \tfrac{k_{\mu_1}(k+l_1)_{\mu_2}(k+\widehat{l}_2)_{\mu_3}(k+\widehat{l}_3)_{\mu_4}}{|k|^{s+2}|k+l_1|^2|k+\widehat{l}_2|^2|k+\widehat{l}_3|^2}\ \widetilde{a}_{\alpha,l},\\
&\widetilde{a}_{\alpha,l} := a_{\alpha_1,l_1}\, a_{\alpha_2,l_2}\, a_{\alpha_3,l_3}\, a_{\alpha_4,-l_1-l_2-l_3}.
\end{aligned}$$

Using (16) and Corollary 2.13 successively, we find

$$f_{\mu,\alpha}(s,l) \sim {\sum_{k\in\mathbb{Z}^n}}' e^{\frac{i}{2}\theta}\, \tfrac{k_{\mu_1}k_{\mu_2}k_{\mu_3}k_{\mu_4}}{|k|^{s+2}|k+l_1|^2|k+l_1+l_2|^2|k+l_1+l_2+l_3|^2}\, \widetilde{a}_{\alpha,l} \sim {\sum_{k\in\mathbb{Z}^n}}' e^{\frac{i}{2}\theta}\, \tfrac{k_{\mu_1}k_{\mu_2}k_{\mu_3}k_{\mu_4}}{|k|^{s+8}}\, \widetilde{a}_{\alpha,l}.$$

Since the function ${\sum_{k\in\mathbb{Z}^n}}' e^{\frac{i}{2}\theta}\, \tfrac{k_{\mu_1}k_{\mu_2}k_{\mu_3}k_{\mu_4}}{|k|^{s+8}}\, \widetilde{a}_{\alpha,l}$ satisfies (H2), Lemma 2.14 entails that

$$⨍(\mathbb{A}^+)^4 = \sum_{(l_1,l_2,l_3)\in(\mathbb{Z}^n)^3} e^{\frac{i}{2}\theta}\, \widetilde{a}_{\alpha,l}\, \operatorname{Res}_{s=0} {\sum_{k\in\mathbb{Z}^n}}' \tfrac{k_{\mu_1}k_{\mu_2}k_{\mu_3}k_{\mu_4}}{|k|^{s+8}}\, \operatorname{Tr}\gamma^{\mu,\alpha} =: \sum_l X_l.$$

Therefore, with (19), we get $X_l = \frac{\pi^2}{12}\, \widetilde{a}_{\alpha,l}\, e^{\frac{i}{2}\theta}\, (A+B+C)$, where

$$\begin{aligned}
A &:= \operatorname{Tr}(\gamma^{\alpha_4}\gamma^{\mu_4}\gamma^{\alpha_3}\gamma_{\mu_4}\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma_{\mu_2}),\\
B &:= \operatorname{Tr}(\gamma^{\alpha_4}\gamma^{\mu_4}\gamma^{\alpha_3}\gamma^{\mu_2}\gamma^{\alpha_2}\gamma_{\mu_4}\gamma^{\alpha_1}\gamma_{\mu_2}),\\
C &:= \operatorname{Tr}(\gamma^{\alpha_4}\gamma^{\mu_4}\gamma^{\alpha_3}\gamma_{\mu_2}\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma_{\mu_4}).
\end{aligned}$$

Using successively $\{\gamma^\mu,\gamma^\nu\} = 2\delta^{\mu\nu}$ and $\gamma^\mu\gamma_\mu = 2^m\, 1_{2^m}$, we see that

$$\begin{aligned}
&A = C = 4\ \operatorname{Tr}(\gamma^{\alpha_4}\gamma^{\alpha_3}\gamma^{\alpha_2}\gamma^{\alpha_1}),\\
&B = -4\ \big(\operatorname{Tr}(\gamma^{\alpha_4}\gamma^{\alpha_3}\gamma^{\alpha_1}\gamma^{\alpha_2}) + \operatorname{Tr}(\gamma^{\alpha_4}\gamma^{\alpha_2}\gamma^{\alpha_3}\gamma^{\alpha_1})\big).
\end{aligned}$$

Thus, $A+B+C = 8\ 2^m\big(\delta^{\alpha_4\alpha_3}\delta^{\alpha_2\alpha_1} + \delta^{\alpha_4\alpha_1}\delta^{\alpha_3\alpha_2} - 2\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}\big)$, and

$$X_l = \tfrac{2\pi^2}{3}\ 2^m\ e^{\frac{i}{2}\theta}\, \widetilde{a}_{\alpha,l}\, \big(\delta^{\alpha_4\alpha_3}\delta^{\alpha_2\alpha_1} + \delta^{\alpha_4\alpha_1}\delta^{\alpha_3\alpha_2} - 2\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}\big). \tag{61}$$

By (61), we get

$$⨍(\mathbb{A}^+)^4 = 2c\ (-2T_1 + T_2 + T_3),$$

where

$$\begin{aligned}
T_1 &:= \sum_{l_1,\dots,l_4} a_{\alpha_4,l_4}\, a_{\alpha_3,l_3}\, a_{\alpha_2,l_2}\, a_{\alpha_1,l_1}\, e^{\frac{i}{2}\theta}\ \delta_{0,\sum_i l_i}\ \delta^{\alpha_4\alpha_2}\, \delta^{\alpha_3\alpha_1},\\
T_2 &:= \sum_{l_1,\dots,l_4} a_{\alpha_4,l_4}\, a_{\alpha_3,l_3}\, a_{\alpha_2,l_2}\, a_{\alpha_1,l_1}\, e^{\frac{i}{2}\theta}\ \delta_{0,\sum_i l_i}\ \delta^{\alpha_4\alpha_3}\, \delta^{\alpha_2\alpha_1},\\
T_3 &:= \sum_{l_1,\dots,l_4} a_{\alpha_4,l_4}\, a_{\alpha_3,l_3}\, a_{\alpha_2,l_2}\, a_{\alpha_1,l_1}\, e^{\frac{i}{2}\theta}\ \delta_{0,\sum_i l_i}\ \delta^{\alpha_4\alpha_1}\, \delta^{\alpha_3\alpha_2}.
\end{aligned}$$

We now proceed to the following permutations of the l_i variables in the T_1 term : $l_1 \mapsto l_2$, $l_2 \mapsto l_1$, $l_3 \mapsto l_4$, $l_4 \mapsto l_3$. While $\sum_i l_i$ is invariant, θ is modified : $\theta \mapsto l_2.\Theta l_1 + l_2.\Theta l_4 + l_1.\Theta l_4$. With $\delta_{0,\sum_i l_i}$ in factor, we can let l_4 be $-l_1 - l_2 - l_3$, so that $\theta \mapsto -\theta$. We also permute the α_i in the same way. Thus,

$$T_1 = \sum_{l_1,\dots,l_4} a_{\alpha_3,l_3}\, a_{\alpha_4,l_4}\, a_{\alpha_1,l_1}\, a_{\alpha_2,l_2}\, e^{-\frac{i}{2}\theta}\ \delta_{0,\sum_i l_i}\ \delta^{\alpha_3\alpha_1}\, \delta^{\alpha_4\alpha_2}.$$

Therefore,

$$2T_1 = 2\sum_{l_1,\dots,l_4} a_{\alpha_4,l_4}\, a_{\alpha_3,l_3}\, a_{\alpha_2,l_2}\, a_{\alpha_1,l_1}\ \cos\tfrac{\theta}{2}\ \delta_{0,\sum_i l_i}\ \delta^{\alpha_4\alpha_2}\, \delta^{\alpha_3\alpha_1}. \tag{62}$$

The same principles are applied to T_2 and T_3. Namely, the permutation $l_1 \mapsto l_1$, $l_2 \mapsto l_3$, $l_3 \mapsto l_2$, $l_4 \mapsto l_4$ in T_2 and the permutation $l_1 \mapsto l_2$, $l_2 \mapsto l_3$, $l_3 \mapsto l_1$, $l_4 \mapsto l_4$ in T_3 (the α_i variables are permuted the same way) give

$$T_2 = \sum_{l_1,\dots,l_4} a_{\alpha_4,l_4} a_{\alpha_3,l_3} a_{\alpha_2,l_2}\, a_{\alpha_1,l_1}\, e^{\frac{i}{2}\phi}\ \delta_{0,\sum_i l_i}\ \delta^{\alpha_4\alpha_2}\, \delta^{\alpha_3\alpha_1},$$

$$T_3 = \sum_{l_1,\dots,l_4} a_{\alpha_4,l_4}\, a_{\alpha_3,l_3} a_{\alpha_2,l_2}\, a_{\alpha_1,l_1}\, e^{-\frac{i}{2}\phi}\ \delta_{0,\sum_i l_i}\ \delta^{\alpha_4\alpha_2}\, \delta^{\alpha_3\alpha_1}$$

where $\phi := l_1.\Theta\ l_2 + l_1.\Theta\ l_3 - l_2.\Theta\ l_3$. Finally, we get

$$\begin{aligned} \textstyle\int\!\!\!\!-(\mathbb{A}^+)^4 &= 4c \sum_{l_1,\dots,l_4} a_{\alpha_1,l_4}\, a_{\alpha_2,l_3}\, a^{\alpha_1}_{l_2}\, a^{\alpha_2}_{l_1}\, \delta_{0,\sum_i l_i}(\cos\tfrac{\phi}{2} - \cos\tfrac{\theta}{2}) \\ &= 8c \sum_{l_1,\dots,l_3} a_{\alpha_1,-l_1-l_2-l_3}\, a_{\alpha_2,l_3}\, a^{\alpha_1}_{l_2}\, a^{\alpha_2}_{l_1} \sin\tfrac{l_1.\Theta(l_2+l_3)}{2}\ \sin\tfrac{l_2.\Theta l_3}{2}. \end{aligned} \tag{63}$$

(iv) Suppose $q = 2$. By Lemma 6.10, we get

$$\textstyle\int\!\!\!\!- \mathbb{A}^\sigma = \operatorname*{Res}_{s=0} \sum_{l\in\mathbb{Z}^n} \lambda_\sigma f_{\alpha,\mu}(s,l)\, \mathrm{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1})$$

where

$$f_{\alpha,\mu}(s,l) := {\sum_{k\in\mathbb{Z}^n}}' \tfrac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+2}|k+l|^2}\, e^{i\eta\, k.\Theta l}\, \widetilde{a}_{\alpha,l}$$

and $\eta := \frac{1}{2}(\sigma_1 - \sigma_2) \in \{-1, 1\}$. As in the proof of (i), since the presence of the phase does not change the fact that $r(s,l)$ satisfies (H1), we get

$$f_{\alpha,\mu}(s,l) \sim f_1(s,l) - f_2(s,l) + f_3(s,l).$$

where

$$f_1(s,l) = {\sum_{k\in\mathbb{Z}^n}}' \tfrac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+4}}\, e^{i\eta\, k.\Theta l}\, \widetilde{a}_{\alpha,l},$$

$$f_2(s,l) = {\sum_{k\in\mathbb{Z}^n}}' \tfrac{k_{\mu_1}(k+l)_{\mu_2}(2k.l+|l|^2)}{|k|^{s+6}}\, e^{i\eta\, k.\Theta l}\, \widetilde{a}_{\alpha,l},$$

$$f_3(s,l) = {\sum_{k\in\mathbb{Z}^n}}' \tfrac{k_{\mu_1}(k+l)_{\mu_2}(2k.l+|l|^2)^2}{|k|^{s+8}}\, e^{i\eta\, k.\Theta l}\, \widetilde{a}_{\alpha,l}.$$

Suppose that $l=0$. Then $f_2(s,0)=f_3(s,0)=0$ and Proposition 2.1 entails that

$$f_1(s,0)=\sum\nolimits'_{k\in\mathbb{Z}^n}\tfrac{k_{\mu_1}k_{\mu_2}}{|k|^{s+4}}\,\widetilde{a}_{\alpha,0}$$

is holomorphic at 0 and so is $f_{\alpha,\mu}(s,0)$.
Since $\frac{1}{2\pi}\Theta$ is badly approximable, Theorem 2.5 3 gives us the result.
Suppose $q=3$. Then Lemma 6.10 implies that

$$⨍\,\mathbb{A}^\sigma=\operatorname*{Res}_{s=0}\sum\nolimits_{l\in(\mathbb{Z}^n)^2}f_{\mu,\alpha}(s,l)\ \operatorname{Tr}(\gamma^{\mu_3}\gamma^{\alpha_3}\cdots\gamma^{\mu_1}\gamma^{\alpha_1})$$

where

$$f_{\mu,\alpha}(s,l):=\sum\nolimits'_{k\in\mathbb{Z}^n}\lambda_\sigma e^{ik.\Theta(\varepsilon_1l_1+\varepsilon_2l_2)}e^{\frac{i}{2}\sigma_2l_1.\Theta l_2}\tfrac{k_{\mu_1}(k+l_1)_{\mu_2}(k+l_1+l_2)_{\mu_3}}{|k|^{s+2}|k+l_1|^2|k+l_1+l_2|^2}\,\widetilde{a}_{\alpha,l},$$

and $\varepsilon_i:=\frac{1}{2}(\sigma_i-\sigma_3)\in\{-1,0,1\}$. By hypothesis $(\varepsilon_1,\varepsilon_2)\neq(0,0)$. There are six possibilities for the values of $(\varepsilon_1,\varepsilon_2)$, corresponding to the six possibilities for the values of σ: $(-,-,+)$, $(-,+,+)$, $(+,-,+)$, $(+,+,-)$, $(-,+,-)$, and $(+,-,-)$. As in (ii), we see that

$$\begin{aligned}f_{\mu,\alpha}(s,l)\sim\Big(&\sum_{k\in\mathbb{Z}^n}{}'\frac{e^{ik.\Theta(\varepsilon_1l_1+\varepsilon_2l_2)}k_{\mu_1}(k+l_1)_{\mu_2}(k+\widehat{l}_2)_{\mu_3}}{|k|^{s+6}}\\&-\sum_{k\in\mathbb{Z}^n}{}'\frac{e^{ik.\Theta(\varepsilon_1l_1+\varepsilon_2l_2)}k_{\mu_1}(k+l_1)_{\mu_2}(k+\widehat{l}_2)_{\mu_3}(2k.l_1+2k.\widehat{l}_2)}{|k|^{s+8}}\,\lambda_\sigma\,\widetilde{a}_{\alpha,l}\,e^{\frac{i}{2}\sigma_2l_1.\Theta l_2}.\end{aligned}$$

With $Z:=\{(l_1,l_2)\ :\ \varepsilon_1l_1+\varepsilon_2l_2=0\}$, Theorem 2.5 (iii) entails that $\sum_{l\in(\mathbb{Z}^n)^2\setminus Z}f_{\mu,\alpha}(s,l)$ is holomorphic at 0. To conclude we need to prove that

$$\sum_\sigma g(\sigma):=\sum_\sigma\sum_{l\in Z}f_{\mu,\alpha}(s,l)\ \operatorname{Tr}(\gamma^{\mu_3}\gamma^{\alpha_3}\cdots\gamma^{\mu_1}\gamma^{\alpha_1})$$

is holomorphic at 0. By definition, $\lambda_\sigma=i\sigma_1\sigma_2\sigma_3$ and as a consequence, we check that

$$g(-,-,+)=-g(+,+,-),\quad g(+,-,+)=-g(+,-,-),\quad g(-,+,+)=-g(-,+,-),$$

which implies that $\sum_\sigma g(\sigma)=0$. The result follows.
Suppose finally that $q=4$. Again, Lemma 6.10 implies that

$$⨍\,\mathbb{A}^\sigma=\operatorname*{Res}_{s=0}\sum_{l\in(\mathbb{Z}^n)^3}f_{\mu,\alpha}(s,l)\ \operatorname{Tr}(\gamma^{\mu_4}\gamma^{\alpha_4}\cdots\gamma^{\mu_1}\gamma^{\alpha_1})$$

where

$$f_{\mu,\alpha}(s,l):=\sum_{k\in\mathbb{Z}^n}{}'\lambda_\sigma\,e^{ik.\Theta\sum_{i=1}^3\varepsilon_il_i}\,e^{\frac{i}{2}(\sigma_2l_1.\Theta l_2+\sigma_3(l_1+l_2).\Theta l_3)}\,\tfrac{k_{\mu_1}(k+l_1)_{\mu_2}(k+l_1+l_2)_{\mu_3}(k+l_1+l_2+l_3)_{\mu_4}}{|k|^{s+2}|k+l_1|^2|k+l_1+l_2|^2|k+l_1+l_2+l_3|^2}\,\widetilde{a}_{\alpha,l}$$

and $\varepsilon_i:=\frac{1}{2}(\sigma_i-\sigma_4)\in\{-1,0,1\}$. By hypothesis $(\varepsilon_1,\varepsilon_2,\varepsilon_3)\neq(0,0,0)$. There are fourteen possibilities for the values of $(\varepsilon_1,\varepsilon_2,\varepsilon_3)$, corresponding to the fourteen possibilities for the values of σ: $(-,-,-,+)$, $(-,-,+,+)$, $(-,+,-,+)$, $(+,-,-,+)$, $(-,+,+,+)$, $(+,-,+,+)$, $(+,+,-,+)$, $(+,+,+,-)$, $(-,-,+,-)$, $(-,+,-,-)$, $(+,-,-,-)$, $(-,+,+,-)$, $(+,-,+,-)$ and $(+,+,-,-)$. As in (ii), we see that, with the shorthand $\theta_\sigma:=\sigma_2l_1.\Theta l_2+\sigma_3(l_1+l_2).\Theta l_3$,

$$f_{\mu,\alpha}(s,l)\sim\sum\nolimits'_{k\in\mathbb{Z}^n}\lambda_\sigma\,e^{ik.\Theta\sum_{i=1}^3\varepsilon_il_i}\,e^{\frac{i}{2}\theta_\sigma}\,\tfrac{k_{\mu_1}k_{\mu_2}k_{\mu_3}k_{\mu_4}}{|k|^{s+8}}\widetilde{a}_{\alpha,l}=:g_{\mu,\alpha}(s,l)$$

With $Z_\sigma := \{(l_1,l_2,l_3)\ :\ \sum_{i=1}^3 \varepsilon_i l_i = 0\}$, Theorem 2.5 (iii), the series $\sum_{l\in(\mathbb{Z}^n)^3\setminus Z_\sigma} f_{\mu,\alpha}(s,l)$ is holomorphic at 0. To conclude, we need to prove that

$$\sum_\sigma g(\sigma) := \sum_\sigma \operatorname*{Res}_{s=0} \sum_{l\in Z_\sigma} g_{\mu,\alpha}(s,l)\ \operatorname{Tr}(\gamma^{\mu_4}\gamma^{\alpha_4}\cdots\gamma^{\mu_1}\gamma^{\alpha_1}) = 0.$$

Let C be the set of the fourteen values of σ and C_7 be the set of the seven first values of σ given above. Lemma 6.6 implies

$$\sum_{\sigma\in C} g(\sigma) = 2\sum_{\sigma\in C_7} g(\sigma).$$

Thus, in the following, we restrict to these seven values. Let us note $F_\mu(s) := \sum'_{k\in\mathbb{Z}^n} \frac{k_{\mu_1}k_{\mu_2}k_{\mu_3}k_{\mu_4}}{|k|^{s+8}}$ so that

$$g(\sigma) = \operatorname*{Res}_{s=0} F_\mu(s)\,\lambda_\sigma \sum_{l\in Z_\sigma} e^{\frac{i}{2}\theta_\sigma}\,\widetilde{a}_{\alpha,l}\ \operatorname{Tr}(\gamma^{\mu_4}\gamma^{\alpha_4}\cdots\gamma^{\mu_1}\gamma^{\alpha_1}).$$

Recall from (61) that

$$\operatorname*{Res}_{s=0} F_\mu(s)\operatorname{Tr}(\gamma^{\mu_4}\gamma^{\alpha_4}\cdots\gamma^{\mu_1}\gamma^{\alpha_1}) = 2c\big(\delta^{\alpha_4\alpha_3}\delta^{\alpha_2\alpha_1} + \delta^{\alpha_4\alpha_1}\delta^{\alpha_3\alpha_2} - 2\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}\big).$$

As a consequence, we get, with $\widetilde{a}_{\alpha,l} := a_{\alpha_1,l_1}\cdots a_{\alpha_4,l_4}$,

$$\begin{aligned} g(\sigma) &= 2c\lambda_\sigma \sum_{l\in(\mathbb{Z}^n)^4} e^{\frac{i}{2}\theta_\sigma}\,\widetilde{a}_{\alpha,l}\,\delta_{\sum_{i=1}^4 l_i,0}\,\delta_{\sum_{i=1}^3 \varepsilon_i l_i,0}\big(\delta^{\alpha_4\alpha_3}\delta^{\alpha_2\alpha_1} + \delta^{\alpha_4\alpha_1}\delta^{\alpha_3\alpha_2} - 2\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}\big) \\ &=: 2c\lambda_\sigma(T_1+T_2-2T_3). \end{aligned}$$

We proceed to the following change of variable in T_1: $l_1\mapsto l_1$, $l_2\mapsto l_3$, $l_3\mapsto l_2$, $l_4\mapsto l_4$. Thus, we get $\theta_\sigma\mapsto\psi_\sigma := \sigma_2 l_1.\Theta l_3 + \sigma_3(l_1+l_3).\Theta l_2$, and $\sum_{i=1}^3 \varepsilon_i l_i \mapsto \varepsilon_1 l_1+\varepsilon_3 l_2+\varepsilon_2 l_3 =: u_\sigma(l)$. With a similar permutation on the α_i, we get

$$T_1 = \sum_{l\in(\mathbb{Z}^n)^4} e^{\frac{i}{2}\psi_\sigma}\,\widetilde{a}_{\alpha,l}\,\delta_{\sum_{i=1}^4 l_i,0}\,\delta_{\varepsilon_1 l_1+\varepsilon_3 l_2+\varepsilon_2 l_3,0}\,\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}.$$

We proceed to the following change of variable in T_2: $l_1\mapsto l_2$, $l_2\mapsto l_3$, $l_3\mapsto l_1$, $l_4\mapsto l_4$. Thus, we get $\theta_\sigma\mapsto\phi_\sigma := \sigma_2 l_2.\Theta l_3 + \sigma_3(l_2+l_3).\Theta l_1$, and $\sum_{i=1}^3 \varepsilon_i l_i \mapsto \varepsilon_3 l_1+\varepsilon_1 l_2+\varepsilon_2 l_3 =: v_\sigma(l)$. After a similar permutation on the α_i, we get

$$T_2 = \sum\nolimits_{l\in(\mathbb{Z}^n)^4} e^{\frac{i}{2}\phi_\sigma}\,\widetilde{a}_{\alpha,l}\,\delta_{\sum_{i=1}^4 l_i,0}\,\delta_{\varepsilon_3 l_1+\varepsilon_1 l_2+\varepsilon_2 l_3,0}\,\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}.$$

Finally, we proceed to the following change of variable in T_3: $l_1\mapsto l_2$, $l_2\mapsto l_1$, $l_3\mapsto l_4$, $l_4\mapsto l_3$. Thus, we get $\theta_\sigma\mapsto-\theta_\sigma$, and $\sum_{i=1}^3 \varepsilon_i l_i \mapsto (\varepsilon_2-\varepsilon_3)l_1+(\varepsilon_1-\varepsilon_3)l_2-\varepsilon_3 l_3 =: w_\sigma(l)$. With a similar permutation on the α_i, we get

$$T_3 = \sum\nolimits_{l\in(\mathbb{Z}^n)^4} e^{-\frac{i}{2}\theta_\sigma}\,\widetilde{a}_{\alpha,l}\,\delta_{\sum_{i=1}^4 l_i,0}\,\delta_{(\varepsilon_2-\varepsilon_3)l_1+(\varepsilon_1-\varepsilon_3)l_2-\varepsilon_3 l_3,0}\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}.$$

As a consequence, we get

$$g(\sigma) = 2c\sum\nolimits_{l\in(\mathbb{Z}^n)^4} K_\sigma(l_1,l_2,l_3)\,\widetilde{a}_{\alpha,l}\,\delta_{\sum_{i=1}^4 l_i,0}\,\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1},$$

where $K_\sigma(l_1,l_2,l_3)=\lambda_\sigma\big(e^{\frac{i}{2}\psi_\sigma}\,\delta_{u_\sigma(l),0}+e^{\frac{i}{2}\phi_\sigma}\,\delta_{v_\sigma(l),0}-e^{\frac{i}{2}\theta_\sigma}\,\delta_{\sum_{i=1}^3\varepsilon_i l_i,0}-e^{-\frac{i}{2}\theta_\sigma}\,\delta_{w_\sigma(l),0}\big)$.
The computation of $K_\sigma(l_1,l_2,l_3)$ for the seven values of σ yields

$$\begin{aligned}
K_{--++}(l_1,l_2,l_3)&=\delta_{l_1+l_3,0}+\delta_{l_2+l_3,0}-\delta_{l_1+l_2,0}-\delta_{l_1+l_2,0},\\
K_{-+-+}(l_1,l_2,l_3)&=\delta_{l_1+l_2,0}+\delta_{l_1+l_2,0}-\delta_{l_1+l_3,0}-\delta_{l_1+l_3,0},\\
K_{--++}(l_1,l_2,l_3)&=\delta_{l_2+l_3,0}+\delta_{l_1+l_3,0}-\delta_{l_2+l_3,0}-\delta_{l_2+l_3,0},\\
K_{---+}(l_1,l_2,l_3)&=-\big(e^{\frac{i}{2}l_1.\Theta l_2}\delta_{\sum_{i=1}^3 l_i,0}+e^{\frac{i}{2}l_2.\Theta l_1}\delta_{\sum_{i=1}^3 l_i,0}-e^{\frac{i}{2}l_2.\Theta l_1}\delta_{\sum_{i=1}^3 l_i,0}-e^{\frac{i}{2}l_1.\Theta l_2}\delta_{l_3,0}\big),\\
K_{-+++}(l_1,l_2,l_3)&=-\big(e^{\frac{i}{2}l_3.\Theta l_2}\delta_{l_1,0}+e^{\frac{i}{2}l_3.\Theta l_1}\delta_{l_2,0}-e^{\frac{i}{2}l_2.\Theta l_3}\delta_{l_1,0}-e^{\frac{i}{2}l_3.\Theta l_1}\delta_{l_2,0}\big),\\
K_{+-++}(l_1,l_2,l_3)&=-\big(e^{\frac{i}{2}l_1.\Theta l_2}\delta_{l_3,0}+e^{\frac{i}{2}l_2.\Theta l_1}\delta_{l_3,0}-e^{\frac{i}{2}l_1.\Theta l_3}\delta_{l_2,0}-e^{\frac{i}{2}l_3.\Theta l_2}\delta_{l_1,0}\big),\\
K_{++-+}(l_1,l_2,l_3)&=-\big(e^{\frac{i}{2}l_1.\Theta l_3}\delta_{l_2,0}+e^{\frac{i}{2}l_2.\Theta l_3}\delta_{l_1,0}-e^{\frac{i}{2}l_1.\Theta l_2}\delta_{l_3,0}-e^{\frac{i}{2}l_2.\Theta l_1}\delta_{\sum_{i=1}^3 l_i,0}\big).
\end{aligned}$$

Thus,

$$\sum_{\sigma\in C_7}K_\sigma(l_1,l_2,l_3)=2i(\delta_{l_3,0}-\delta_{\sum_{i=1}^3 l_i,0})\sin\tfrac{l_1.\Theta l_2}{2}$$

and

$$\sum_{\sigma\in C_7}g(\sigma)=i4c\sum_{l\in(\mathbb{Z}^n)^4}(\delta_{l_3,0}-\delta_{\sum_{i=1}^3 l_i,0})\sin\tfrac{l_1.\Theta l_2}{2}\,\widetilde{a}_{\alpha,l}\,\delta_{\sum_{i=1}^4 l_i,0}\,\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}.$$

The following change of variables: $l_1\mapsto l_2$, $l_1\mapsto l_2$, $l_3\mapsto l_4$, $l_4\mapsto l_3$ gives

$$\sum_{l\in(\mathbb{Z}^n)^4}\delta_{\sum_1^3 l_i,0}\sin\tfrac{l_1.\Theta l_2}{2}\,\widetilde{a}_{\alpha,l}\,\delta_{\sum_1^4 l_i,0}\,\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}=-\sum_{l\in(\mathbb{Z}^n)^4}\delta_{l_3,0}\sin\tfrac{l_1.\Theta l_2}{2}\,\widetilde{a}_{\alpha,l}\,\delta_{\sum_1^4 l_i,0}\,\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}$$

so

$$\sum_{\sigma\in C_7}g(\sigma)=i8c\sum_{l\in(\mathbb{Z}^n)^4}\delta_{l_3,0}\,\sin\tfrac{l_1.\Theta l_2}{2}\,\widetilde{a}_{\alpha,l}\,\delta_{\sum_1^4 l_i,0}\,\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}.$$

Finally, the change of variables: $l_2\mapsto l_4, l_4\mapsto l_2$ gives

$$\sum_{l\in(\mathbb{Z}^n)^4}\delta_{l_3,0}\,\sin\tfrac{l_1.\Theta l_2}{2}\,\widetilde{a}_{\alpha,l}\,\delta_{\sum_1^4 l_i,0}\,\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}=-\sum_{l\in(\mathbb{Z}^n)^4}\delta_{l_3,0}\,\sin\tfrac{l_1.\Theta l_2}{2}\,\widetilde{a}_{\alpha,l}\,\delta_{\sum_1^4 l_i,0}\,\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}$$

which entails that $\sum_{\sigma\in C_7}g(\sigma)=0$. □

Lemma 6.12. *Suppose $n=4$ and $\frac{1}{2\pi}\Theta$ badly approximable. For any self-adjoint one-form A,*

$$\zeta_{D_A}(0)-\zeta_D(0)=-c\,\tau(F_{\alpha_1,\alpha_2}F^{\alpha_1\alpha_2}).$$

Proof. By (34) and Lemma 6.5 we get

$$\zeta_{D_A}(0)-\zeta_D(0)=\sum_{q=1}^n\tfrac{(-1)^q}{q}\sum_{\sigma\in\{+,-\}^q}⨍\mathbb{A}^\sigma.$$

By Lemma 6.11 (iv), we see that the crossed terms all vanish. Thus, with Lemma 6.6, we get

$$\zeta_{D_A}(0)-\zeta_D(0)=2\sum_{q=1}^n\tfrac{(-1)^q}{q}⨍(\mathbb{A}^+)^q. \tag{64}$$

By definition,

$$F_{\alpha_1\alpha_2} = i\sum_k \big(a_{\alpha_2,k}\, k_{\alpha_1} - a_{\alpha_1,k}\, k_{\alpha_2}\big)U_k + \sum_{k,\,l} a_{\alpha_1,k}\, a_{\alpha_2,l}\,[U_k,U_l]$$
$$= i\sum_k \big[(a_{\alpha_2,k}\, k_{\alpha_1} - a_{\alpha_1,k}\, k_{\alpha_2}) - 2\sum_l a_{\alpha_1,k-l}\, a_{\alpha_2,l}\, \sin(\tfrac{k.\Theta l}{2})\big]\, U_k.$$

Thus

$$\tau(F_{\alpha_1\alpha_2}F^{\alpha_1\alpha_2}) = \sum_{\alpha_1,\,\alpha_2=1}^{2^m}\ \sum_{k\in\mathbb{Z}^4} \big[(a_{\alpha_2,k}\, k_{\alpha_1} - a_{\alpha_1,k}\, k_{\alpha_2}) - 2\sum_{l'\in\mathbb{Z}^4} a_{\alpha_1,k-l'}\, a_{\alpha_2,l'}\, \sin(\tfrac{k.\Theta l'}{2})\big]$$
$$\big[(a_{\alpha_2,-k}\, k_{\alpha_1} - a_{\alpha_1,-k}\, k_{\alpha_2}) - 2\sum_{l"\in\mathbb{Z}^4} a_{\alpha_1,-k-l"}\, a_{\alpha_2,l"}\, \sin(\tfrac{k.\Theta l"}{2})\big].$$

One checks that the term in a^q of $\tau(F_{\alpha_1\alpha_2}F^{\alpha_1\alpha_2})$ corresponds to the term $⨍(\mathbb{A}^+)^q$ given by Lemma 6.11. For $q=2$, this is

$$-2\sum_{l\in\mathbb{Z}^4,\,\alpha_1,\,\alpha_2} a_{\alpha_1,l}\, a_{\alpha_2,-l}\, \big(l_{\alpha_1}l_{\alpha_2} - \delta_{\alpha_1\alpha_2}|l|^2\big).$$

For $q=3$, we compute the crossed terms:

$$i\sum_{k,k',l} (a_{\alpha_2,k}\, k_{\alpha_1} - a_{\alpha_1,k}\, k_{\alpha_2})\ a_{k'}^{\alpha_1}\ a_l^{\alpha_2}\big(U_k[U_{k'},l] + [U_{k'},U_l]U_k\big),$$

which gives the following a^3-term in $\tau(F_{\alpha_1\alpha_2}F^{\alpha_1\alpha_2})$

$$-8\sum_{l_i} a_{\alpha_3,-l_1-l_2}\, a_{l_2}^{\alpha_1}\, a_{\alpha_1,l_1}\ \sin\tfrac{l_1.\Theta l_2}{2}\ l_1^{\alpha_3}.$$

For $q=4$, this is

$$-4\sum_{l_i} a_{\alpha_1,-l_1-l_2-l_3}\, a_{\alpha_2,l_3}\, a_{l_2}^{\alpha_1}\, a_{l_1}^{\alpha_2} \sin\tfrac{l_1.\Theta(l_2+l_3)}{2}\ \sin\tfrac{l_2.\Theta l_3}{2}$$

which corresponds to the term $⨍(\mathbb{A}^+)^4$. We get finally,

$$\sum_{q=1}^{n} \tfrac{(-1)^q}{q} ⨍(\mathbb{A}^+)^q = -\tfrac{c}{2}\tau(F_{\alpha_1,\alpha_2}F^{\alpha_1\alpha_2}). \qquad (65)$$

Equations (64) and (65) yield the result. □

Lemma 6.13. *Suppose $n=2$. Then, with the same hypothesis as in Lemma 6.10,*

(i) $\quad ⨍(\mathbb{A}^+)^2 = ⨍(\mathbb{A}^-)^2 = 0.$

(ii) Suppose $\frac{1}{2\pi}\Theta$ badly approximable. Then

$$⨍\,\mathbb{A}^+\mathbb{A}^- = ⨍\,\mathbb{A}^-\mathbb{A}^+ = 0.$$

Proof. (*i*) Lemma 6.10 entails that $\int\!\!\!\!\!- \mathbb{A}^{++} = \underset{s=0}{\text{Res}} \sum_{l\in\mathbb{Z}^2} -f(s,l)$ where

$$f(s,l) := {\sum}'_{k\in\mathbb{Z}^2} \tfrac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+2}|k+l|^2}\, \widetilde{a}_{\alpha,l}\ \ \text{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}) =: f_{\mu,\alpha}(s,l)\,\text{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1})$$

and $\widetilde{a}_{\alpha,l} := a_{\alpha_1,l}\, a_{\alpha_2,-l}$. This time, since $n = 2$, it is enough to apply just once (16) to obtain an absolutely convergent series. Indeed, we get with (16)

$$f_{\mu,\alpha}(s,l) = \sum_{k\in\mathbb{Z}^2}{}' \tfrac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+4}}\, \widetilde{a}_{\alpha,l} - \sum_{k\in\mathbb{Z}^2}{}' \tfrac{k_{\mu_1}(k+l)_{\mu_2}(2k.l+|l|^2)}{|k|^{s+4}|k+l|^2}\, \widetilde{a}_{\alpha,l}.$$

and the function $r(s,l) := \sum'_{k\in\mathbb{Z}^2} \frac{k_{\mu_1}(k+l)_{\mu_2}(2k.l+|l|^2)}{|k|^{s+4}|k+l|^2}\, \widetilde{a}_{\alpha,l}$ is a linear combination of functions of the type $H(s,l)$ satisfying the hypothesis of Corollary 2.13. As a consequence, $r(s,l)$ satisfies (H1) and

$$f_{\mu,\alpha}(s,l) \sim \sum_{k\in\mathbb{Z}^2}{}' \tfrac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+4}}\, \widetilde{a}_{\alpha,l} \sim \sum_{k\in\mathbb{Z}^2}{}' \tfrac{k_{\mu_1}k_{\mu_2}}{|k|^{s+4}}\, \widetilde{a}_{\alpha,l}$$

Note that the function $(s,l) \mapsto h_{\mu,\alpha}(s,l) := \sum'_{k\in\mathbb{Z}^2} \frac{k_{\mu_1}k_{\mu_2}}{|k|^{s+4}}\, \widetilde{a}_{\alpha,l}$ satisfies (H2). Thus, Lemma 2.14 yields

$$\underset{s=0}{\text{Res}}\ f(s,l) = \sum_{l\in\mathbb{Z}^2} \underset{s=0}{\text{Res}}\ h_{\mu,\alpha}(s,l)\,\text{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}).$$

By Proposition 2.16, we get $\underset{s=0}{\text{Res}}\ h_{\mu,\alpha}(s,l) = \delta_{\mu_1\mu_2}\,\pi\,\widetilde{a}_{\alpha,l}$. Therefore,

$$\int\!\!\!\!\!\!- \mathbb{A}^{++} = -\pi \sum_{l\in\mathbb{Z}^2} \widetilde{a}_{\alpha,l}\,\text{Tr}(\gamma^{\alpha_2}\gamma^{\mu}\gamma^{\alpha_1}\gamma_{\mu}) = 0$$

according to (58).

(*ii*) By Lemma 6.10, we obtain that $\int\!\!\!\!\!- \mathbb{A}^{-+} = \underset{s=0}{\text{Res}} \sum_{l\in\mathbb{Z}^2} \lambda_\sigma f_{\alpha,\mu}(s,l)\,\text{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1})$ where $\lambda_\sigma = -(-i)^2 = 1$ and

$$f_{\alpha,\mu}(s,l) := \sum_{k\in\mathbb{Z}^2}{}' \tfrac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+2}|k+l|^2}\, e^{i\eta\, k.\Theta l}\, \widetilde{a}_{\alpha,l}$$

and $\eta := \frac{1}{2}(\sigma_1 - \sigma_2) = -1$. As in the proof of (*i*), since the presence of the phase does not change the fact that $r(s,l)$ satisfies (H1), we get

$$f_{\alpha,\mu}(s,l) \sim \sum_{k\in\mathbb{Z}^2}{}' \tfrac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+4}}\, e^{i\eta\, k.\Theta l}\, \widetilde{a}_{\alpha,l} := g_{\alpha,\mu}(s,l)$$

Since $\frac{1}{2\pi}\Theta$ is badly approximable, the functions $s \mapsto \sum_{l\in\mathbb{Z}^2\setminus\{0\}} g_{\alpha,\mu}(s,l)$ are holomorphic at $s = 0$ by Theorem 2.5 3. As a consequence,

$$\int\!\!\!\!\!\!- \mathbb{A}^{-+} = \underset{s=0}{\text{Res}}\ g_{\alpha,\mu}(s,0)\,\text{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}) = \underset{s=0}{\text{Res}} \sum_{k\in\mathbb{Z}^2}{}' \tfrac{k_{\mu_1}k_{\mu_2}}{|k|^{s+4}}\, \widetilde{a}_{\alpha,0}\ \text{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}).$$

Recall from Proposition 2.1 that $\text{Res}_{s=0} \sum'_{k\in\mathbb{Z}^2} \frac{k_i k_j}{|k|^{s+4}} = \delta_{ij}\,\pi$. Thus, again with (58),

$$\int\!\!\!\!\!\!- \mathbb{A}^{-+} = \widetilde{a}_{\alpha,0}\,\pi\ \text{Tr}(\gamma^{\alpha_2}\gamma^{\mu}\gamma^{\alpha_1}\gamma_{\mu}) = 0.$$ ☐

Lemma 6.14. *Suppose $n = 2$ and $\frac{1}{2\pi}\Theta$ badly approximable. For any self-adjoint one-form A,*

$$\zeta_{D_A}(0) - \zeta_D(0) = 0.$$

Proof. As in Lemma 6.12, we use (34) and Lemma 6.5 so the result follows from Lemma 6.13. ☐

6.1.2 Odd dimensional case

Lemma 6.15. *Suppose n odd and $\frac{1}{2\pi}\Theta$ badly approximable. Then for any self-adjoint 1-form A and $\sigma \in \{-,+\}^q$ with $2 \leq q \leq n$,*

$$⨍ \mathbb{A}^\sigma = 0\,.$$

Proof. Since $\mathbb{A}^\sigma \in \Psi_1(\mathcal{A})$, Lemma 5.11 with $k = n$ gives the result. □

Corollary 6.16. *With the same hypothesis of Lemma 6.15, for any self-adjoint one-form A, $\zeta_{D_A}(0) - \zeta_D(0) = 0$.*

Proof. As in Lemma 6.12, we use (34) and Lemma 6.5 so the result follows from Lemma 6.15. □

6.2 Proof of the main result

Proof of Theorem 6.1. (i) By (5) and Proposition 5.5, we get

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) \;=\; 4\pi\Phi_2\,\Lambda^2 + \Phi(0)\,\zeta_{D_A}(0) + \mathcal{O}(\Lambda^{-2}),$$

where $\Phi_2 = \frac{1}{2}\int_0^\infty \Phi(t)\,dt$. By Lemma 6.14, $\zeta_{D_A}(0) - \zeta_D(0) = 0$ and from Proposition 5.4, $\zeta_D(0) = 0$, so we get the result.
(ii) Similarly, $\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = 8\pi^2\,\Phi_4\,\Lambda^4 + \Phi(0)\,\zeta_{D_A}(0) + \mathcal{O}(\Lambda^{-2})$ with $\Phi_4 = \frac{1}{2}\int_0^\infty \Phi(t)\,t\,dt$. Lemma 6.12 implies that $\zeta_{D_A}(0) - \zeta_D(0) = -c\,\tau(F_{\mu\nu}F^{\mu\nu})$ and by Proposition 5.4, $\zeta_{D_A}(0) = -c\,\tau(F_{\mu\nu}F^{\mu\nu})$ leading to the result.
(iii) is a direct consequence of (5), Propositions 5.4, 5.5, and Corollary 6.16. □

A Appendix

A.1 Proof of Lemma 3.3

(i) We have $|D|T|D|^{-1} = T + \delta(T)|D|^{-1}$ and $|D|^{-1}T|D| = T - |D|^{-1}\delta(T)$. A recurrence proves that for any $k \in \mathbb{N}$, $|D|^kT|D|^{-k} = \sum_{q=0}^{k}\binom{q}{k}\,\delta^q(T)|D|^{-q}$ and we get $|D|^{-k}T|D|^k = \sum_{q=0}^{k}(-1)^q\binom{q}{k}\,|D|^{-q}\delta^q(T)$.
As a consequence, since T, $|D|^{-q}$ and $\delta^q(T)$ are in OP^0 for any $q \in \mathbb{N}$, for any $k \in \mathbb{Z}$, $|D|^kT|D|^{-k} \in OP^0$. Let us fix $p \in \mathbb{N}_0$ and define $F_p(s) := \delta^p(|D|^sT|D|^{-s})$ for $s \in \mathbb{C}$. Since for $k \in \mathbb{Z}$, $F_p(k)$ is bounded, a complex interpolation proves that $F_p(s)$ is bounded, which gives $|D|^sT|D|^{-s} \in OP^0$.
(ii) Let $T \in OP^\alpha$ and $T' \in OP^\beta$. Thus, $T|D|^{-\alpha}$, $T'|D|^{-\beta}$ are in OP^0. By (i) we get $|D|^\beta T|D|^{-\alpha}|D|^{-\beta} \in OP^0$, so $T'|D|^{-\beta}|D|^\beta T|D|^{-\beta-\alpha} \in OP^0$. Thus, $T'T|D|^{-(\alpha+\beta)} \in OP^0$.
(iii) For $T \in OP^\alpha$, $|D|^{\alpha-\beta}$ and $T|D|^{-\alpha}$ are in OP^0, thus $T|D|^{-\beta} = T|D|^{-\alpha}|D|^{\alpha-\beta} \in OP^0$.
(iv) follows from $\delta(OP^0) \subseteq OP^0$.
(v) Since $\nabla(T) = \delta(T)|D| + |D|\delta(T) - [P_0, T]$, the result follows from (ii), (iv) and the fact that P_0 is in $OP^{-\infty}$.

A.2 Proof of Lemma 3.6

The non-trivial part of the proof is the stability under the product of operators. Let $T, T' \in \Psi(\mathcal{A})$. There exist $d, d' \in \mathbb{Z}$ such that for any $N \in \mathbb{N}$, $N > |d| + |d'|$, there exist P, P' in $\mathcal{D}(\mathcal{A})$,

$p, p' \in \mathbb{N}_0$, $R \in OP^{-N-d'}$, $R' \in OP^{-N-d}$ such that $T = PD^{-2p} + R$, $T' = P'D^{-2p'} + R'$, $PD^{-2p} \in OP^d$ and $P'D^{-2p'} \in OP^{d'}$.
Thus, $TT' = PD^{-2p}P'D^{-2p'} + RP'D^{-2p'} + PD^{-2p}R' + RR'$.
We also have $RP'D^{-2p'} \in OP^{-N-d'+d'} = OP^{-N}$ and similarly, $PD^{-2p}R' \in OP^{-N}$. Since $RR' \in OP^{-2N}$, we get

$$TT' \sim PD^{-2p}P'D^{-2p'} \mod OP^{-N}.$$

If $p = 0$, then $TT' \sim QD^{-2p'} \mod OP^{-N}$ where $Q = PP' \in \mathcal{D}(\mathcal{A})$ and $QD^{-2p'} \in OP^{d+d'}$.
Suppose $p \neq 0$. A recurrence proves that for any $q \in \mathbb{N}_0$,

$$D^{-2}P' \sim \sum_{k=0}^{q} (-1)^k \nabla^k(P') D^{-2k-2} + (-1)^{q+1} D^{-2} \nabla^{q+1}(P') D^{-2q-2} \mod OP^{-\infty}$$

By Lemma 3.3 (v), the remainder is in $OP^{d'+2p'-q-3}$, since $P' \in OP^{d'+2p'}$. Another recurrence gives for any $q \in \mathbb{N}_0$,

$$D^{-2p}P' \sim \sum_{k_1,\cdots,k_p=0}^{q} (-1)^{|k|_1} \nabla^{|k|_1}(P') D^{-2|k|_1-2p} \mod OP^{d'+2p'-q-1-2p}.$$

Thus, with $q_N = N + d + d' - 1$,

$$TT' \sim \sum_{k_1,\cdots,k_p=0}^{q_N} (-1)^{|k|_1} P \nabla^{|k|_1}(P') D^{-2|k|_1-2(p+p')} \mod OP^{-N}.$$

The last sum can be written $Q_N D^{-2r_N}$ where $r_N := p\, q_N + (p+p')$. Since $Q_N \in \mathcal{D}(\mathcal{A})$ and $Q_N D^{-2r_N} \in OP^{d+d'}$, the result follows.

A.3 Proof of Proposition 3.11

Let $P \in OP^{k_1}$, $Q \in OP^{k_2} \in \Psi(\mathcal{A})$. With $[Q, |D|^{-s}] = \big(Q - \sigma_{-s}(Q)\big)\, |D|^{-s}$ and the equivalence $Q - \sigma_{-s}(Q) \sim -\sum_{r=1}^{N} g(-s,r)\, \varepsilon^r(Q) \mod OP^{-N-1+k_2}$, we get

$$P[Q, |D|^{-s}] \sim -\sum_{r=1}^{N} g(-s,r)\, P\varepsilon^r(Q)|D|^{-s} \mod OP^{-N-1+k_1+k_2-\Re(s)}$$

which gives, if we choose $N = n + k_1 + k_2$,

$$\operatorname*{Res}_{s=0}\ \mathrm{Tr}\left(P[Q,|D|^{-s}]\right) = -\sum_{r=1}^{n+k_1+k_2} \operatorname*{Res}_{s=0}\, g(-s,r)\, \mathrm{Tr}\left(P\varepsilon^r(Q)|D|^{-s}\right).$$

By hypothesis $s \mapsto \mathrm{Tr}\left(P\varepsilon^r(Q)|D|^{-s}\right)$ has only simple poles. Thus, since $s=0$ is a zero of the analytic function $s \mapsto g(-s,r)$ for any $r \geq 1$, we have $\operatorname*{Res}_{s=0} g(-s,r)\ \mathrm{Tr}\left(P\varepsilon^r(Q)|D|^{-s}\right) = 0$, which entails that $\operatorname*{Res}_{s=0}\ \mathrm{Tr}\left(P[Q,|D|^{-s}]\right) = 0$ and thus

$$⨍ PQ = \operatorname*{Res}_{s=0}\ \mathrm{Tr}\left(P|D|^{-s}Q\right).$$

When $s \in \mathbb{C}$ with $\Re(s) > 2\max(k_1 + n + 1, k_2)$, the operator $P|D|^{-s/2}$ is trace-class while $|D|^{-s/2}Q$ is bounded, so $\operatorname{Tr}\big(P|D|^{-s}Q\big) = \operatorname{Tr}\big(|D|^{-s/2}QP|D|^{-s/2}\big) = \operatorname{Tr}\big(\sigma_{-s/2}(QP)|D|^{-s}\big)$. Thus, using (29) again,

$$\operatorname*{Res}_{s=0}\ \operatorname{Tr}\big(P|D|^{-s}Q\big) = -\!\!\!\!\!\!\int QP + \sum_{r=1}^{n+k_1+k_2} \operatorname*{Res}_{s=0}\ g(-s/2,r)\operatorname{Tr}\big(\varepsilon^r(QP)|D|^{-s}\big).$$

As before, for any $r \geq 1$, $\operatorname*{Res}_{s=0} g(-s/2,r)\operatorname{Tr}\big(\varepsilon^r(QP)|D|^{-s}\big) = 0$ since $g(0,r) = 0$ and the spectral triple is simple. Finally,

$$\operatorname*{Res}_{s=0}\ \operatorname{Tr}\big(P|D|^{-s}Q\big) = -\!\!\!\!\!\!\int QP.$$

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