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Index Theorems on Torsional Geometries

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Abstract

We study various topological invariants on a differential geometry in the presence of a totally anti-symmetric torsion H under the closed condition $\mathrm{d}H = 0$. By using the identification between the Clifford algebra on a geometry and the canonical quantization condition of fermion in the quantum mechanics, we construct the $\mathcal{N} = 1$ quantum mechanical sigma model in the Hamiltonian formalism and extend this model to $\mathcal{N} = 2$ system, equipped with the totally anti-symmetric tensor associated with the torsion on the target space geometry. Next we construct transition elements in the Lagrangian path integral formalism and apply them to the analyses of the Witten indices in supersymmetric systems. We improve the formulation of the Dirac index on the torsional geometry which has already been studied. We also formulate the Euler characteristic and the Hirzebruch signature on the torsional geometry.

1 Introduction

Flux compactification scenarios have become one of the most significant issues in the study of low energy effective theories from string theories (for instance, see [1, 2, 3] and references therein). Non-trivial fluxes induce a superpotential, which stabilizes moduli of a compactified geometry and decreases the number of "redundant" massless modes in the low energy effective theory in four dimensional spacetime. This mechanism, called the moduli stabilization, also gives a new insight into cosmology as well as string phenomenology ([4] and a huge number of related works).

Flux compactification provides another interesting issue to the compactified geometry itself: In a specific situation, for instance, the NS-NS three-form flux H_{mnp} behaves as a torsion on the compactified geometry and gives rise to a significant modification [5], i.e., the Kähler form is no longer closed. This phenomenon indicates that the fluxes modify the background geometry in supergravity in a crucial way. Of course, the Calabi-Yau condition [6] should be influenced by the back reactions from the fluxes onto the geometry.

If a certain n-dimensional manifold has a non-trivial structure group G on its tangent bundle, this manifold, called the G-structure manifold, admits the existence of nowhere vanishing tensors; for example, the metric ($G \subseteq O(n)$), the Levi-Civita anti-symmetric tensor ($G \subseteq SO(n)$), the almost complex structure ($G \subseteq U(m)$ where $n = 2m$), and the holomorphic m-form ($G \subseteq SU(m)$). This classification does not exclude the existence of torsion. (In this sense, a Calabi-Yau n-fold is one of the $SU(n)$-structure manifolds.) This classification is also studied in terms of Killing spinors on the manifold. In particular, the $SU(3)$-structure manifold has been investigated in terms of intrinsic torsion [7] and has been applied to the string theory compactification scenarios [8].

On the contrary, however, one has not understood a lot of mathematical properties of the G-structure manifold such as moduli and moduli spaces, topological invariants, and so forth. This is quite different from the case of Calabi-Yau geometries [9]. Because of the lack of this knowledge, one has not been able to discuss the massless modes on the ground state in the effective theory derived from string theory compactified on the G-structure manifold.

Similarly, various kinds of topological invariants on a torsional geometry have not been analyzed, although many topological invariants on a Riemannian geometry have been well investigated. Here let us briefly introduce some invariants: Suppose there exist Dirac fermions in an even dimensional geometry. We define chirality on the Dirac fermions and find the difference between the number of fermions with positive chirality and the number of fermions with negative chirality at the massless level. This difference is a topological invariant, which is called the index of the Dirac operator, or the Dirac index [10, 11, 12]. We also introduce the Euler characteristic as the difference between the number of harmonic even-forms and the number of odd-forms on the geometry, and the Hirzebruch signature as the difference between the number of self-dual forms and the number of anti-self-dual forms. These invariants are described in terms of polynomials of Riemann curvature two-form (see, for example, [13, 14, 15]). So far the index of the Dirac operator in the presence of torsion has been studied [16, 17, 18, 19]. Unfortunately, however, the other indices on a torsional geometry have not been analyzed so much. In particular, it is quite worth studying the Euler characteristic on a complex geometry in the presence of torsion, which will give a new insight on the number of generation in the flux compactification scenarios.

In this paper we analyze the above various topological indices on a geometry in the presence of a totally anti-symmetric torsion H_{mnp} with a closed condition $\mathrm{d}H = 0$ in terms of supersymmetric quantum mechanics [10]: First, for instance, we define the index of the Dirac operator on the geometry in the infinity limit of β:

$$\mathrm{index} \not{D} \ \equiv \ \lim_{\beta \to \infty} \mathrm{Tr} \big\{ \Gamma_{(5)} \mathrm{e}^{-\beta \mathscr{R}} \big\} \ = \ \lim_{\beta \to 0} \mathrm{Tr} \big\{ \Gamma_{(5)} \mathrm{e}^{-\beta \mathscr{R}} \big\} \ , \tag{1.1}$$

where $\mathscr{R}$ is an appropriate regulator, given by the square of the Dirac operator (or, equivalently, the Laplacian) in a usual case. Notice that since a topological value is definitely independent of the continuous parameter β, we can take the zero limit $\beta \to 0$. Next we identify the Clifford algebra on the geometry considered with the canonical quantization condition of fermionic operators in a supersymmetric quantum mechanical nonlinear sigma model, because we want to identify the Dirac operator on the geometry to the supercharge in the quantum mechanics. Next we define the Witten index in the quantum mechanics in such a way as

$$\lim_{\beta \to 0} \mathrm{Tr} \big\{ \Gamma_{(5)} \mathrm{e}^{-\beta \mathscr{R}} \big\} \ \to \ \lim_{\beta \to 0} \mathrm{Tr} \big\{ (-1)^F \mathrm{e}^{-\frac{\beta}{\hbar} \mathscr{H}} \big\} \ = \ \lim_{\beta \to 0} \int \mathrm{d}X \, \langle \, X \, | (-1)^F \mathrm{e}^{-\frac{\beta}{\hbar} \mathscr{H}} | \, X \, \rangle \ . \tag{1.2}$$

Here we identified the regulator and the chirality operator on the geometry with the Hamiltonian and the fermion number operator $(-1)^F$ in the quantum mechanics, respectively. The trace Tr denotes the sum of all transition elements whose final states $\langle \, X \, |$ correspond to the initial states $| \, X \, \rangle$. Third, we rewrite the Witten index from the Hamiltonian formalism, as described above, to the Lagrangian path integral formalism. During this process, we introduce discretized transition elements and adopt the Weyl-ordered form in order to avoid any ambiguous ordering of quantum operators. Then we integrate out momentum variables and obtain the transition elements described in the configuration space path integral. Fourth, we discuss the Feynman rule which defines free propagators and interaction terms in the supersymmetric systems. Finally, we evaluate the Witten indices in the quantum mechanical nonlinear sigma models in appropriate ways. This procedure is summarized in a clear way by de Boer, Peeters, Skenderis and van Nieuwenhuizen [20], and Bastianelli and van Nieuwenhuizen [21]. We will apply this technique to the analysis of index theorems on a torsional geometry.

This paper is organized as follows: In section 2 we construct $\mathcal{N} = 1$ and $\mathcal{N} = 2$ quantum supersymmetric Hamiltonians equipped with a non-vanishing totally anti-symmetric field H_{mnp}, which can be regarded as the torsion on the geometry considered. In section 3 we describe the transition elements in the Hamiltonian formalism and rewrite them to functional path integrals in the Lagrangian formalism. We also prepare bosonic and fermionic propagators in the quantum mechanics. This transition elements play significant roles in the evaluation of the Witten indices in next sections. In section 4 and 5 the Witten index in $\mathcal{N} = 1$ supersymmetric quantum mechanical nonlinear sigma model is analyzed. First we review the Witten index associated with the Dirac index on a usual Riemannian geometry without boundary. Next we generalize the index on a geometry in the presence of non-trivial torsion H under the condition $\mathrm{d}H = 0$. We obtain an explicit expression of the Pontrjagin class and of the Chern character on the torsional geometry. The Euler characteristic corresponding to the Witten index in $\mathcal{N} = 2$ supersymmetric system is discussed in section 6. This topological invariant is also generalized to the one on a torsional geometry under the $\mathrm{d}H = 0$ condition. In section 7 we also analyze the derivation of the Hirzebruch signature on a Riemannian geometry and on a torsional geometry from the $\mathcal{N} = 2$ supersymmetric quantum mechanics. We summarize this paper and discuss open problems and future works in section 8. We attach some appendices in the last few pages. In appendix A we list the convention of differential geometry which we adopt in this paper. In appendix B we briefly summarize the classification of torsional geometries which appear in heterotic string compactification in the presence of non-trivial fluxes. In appendix C a number of useful formulae, which play important roles in the computation of Feynman graphs, are listed.

2 Supersymmetric quantum Hamiltonians

First of all, we prepare a bosonic operator x^m and its canonical conjugate momentum p_m in quantum mechanics, whose canonical quantization condition is defined as a commutation relation between them in such a way as $[x^m, p_n] = i\hbar\, \delta^m_n$. Since we consider a quantum mechanical nonlinear sigma model, we regard x^m as a coordinate on the target space of the sigma model, where its index runs $m = 1, \ldots, D$. Since the target space is curved, the differential representation of the canonical momentum operator is given as $g^{\frac{1}{4}} p_m g^{-\frac{1}{4}} = -i\hbar\, \partial_m$ equipped with the determinant of the target space metric $g = \det g_{mn}$. We also introduce a real fermionic operator ψ^a in the quantum mechanics, equipped with the local Lorentz index $a = 1, \ldots, D$. In the quantum mechanics of real fermions, we define the canonical quantization condition as an anti-commutation relation $\{\psi^a, \psi^b\} = \hbar\, \delta^{ab}$. Since, under the identification $\psi^a \equiv \sqrt{\frac{\hbar}{2}}\Gamma^a$, the structure of this quantization condition can be interpreted as the $SO(D)$ Clifford algebra given by the anti-commutation relation between the Dirac gamma matrices $\{\Gamma^a, \Gamma^b\} = 2\delta^{ab}$ on the target geometry, we will investigate the Dirac index on this curved geometry in terms of the Witten index in the quantum mechanics. First let us discuss $\mathcal{N} = 1$ supersymmetry, and extend this to $\mathcal{N} = 2$ supersymmetry under a certain condition. We should choose $\mathcal{N} = 1$ or $\mathcal{N} = 2$ in the case when we want to study the index density for the Pontrjagin classes, or for the Euler characteristics, respectively [10][1].

2.1 $\mathcal{N} = 1$ real supersymmetry

Now let us introduce the $\mathcal{N} = 1$ supersymmetry algebra with respect to a real fermionic charge Q^1:

$$\{Q^1, Q^1\} \ = \ 2\hbar\, \mathscr{H}^1 \ . \tag{2.1}$$

Note that $\mathscr{H}^1$ is the quantum Hamiltonian in $\mathcal{N} = 1$ system, where the superscript "1" indicates $\mathcal{N} = 1$. We will realize this algebra in terms of quantum operators x^m, p_m and ψ^a. It is useful to introduce a covariant momentum $\pi_m^{(\alpha)}$ associated with a covariant derivative $D_m(\omega^{(\alpha)})$ with respect to spin connection $\omega_{mab}^{(\alpha)} = \omega_{mab} + \alpha H_{mab}$ equipped with the torsion, given by a totally anti-symmetric three-form flux H_{mab} on the target space, such as

$$\pi_m^{(\alpha)} \ \equiv \ p_m - \frac{\hbar}{2}\omega_{mab}^{(\alpha)}\, \Sigma^{ab} \ = \ p_m - \frac{\hbar}{2}\big(\omega_{mab} + \alpha H_{mab}\big)\Sigma^{ab} \ , \tag{2.2}$$

where α is a constant which tunes the magnitude of the torsion. In this paper, we adopt $\alpha = -1/3$ in the same way as in [22]. In particular, we use the description $\hat{\omega}_{mab} \equiv \omega_{mab} - \frac{1}{3}H_{mab}$. Since the Dirac operator acts on spinors on the geometry, the Lorentz generator Σ^{ab} is given in the spinor representation, which can be described in terms of the real fermions via the identification $\Gamma^a = \sqrt{\frac{2}{\hbar}}\psi^a$ such as

$$\Sigma^{ab} \ = \ \frac{i}{4}\big(\Gamma^a\Gamma^b - \Gamma^b\Gamma^a\big) \ = \ \frac{i}{2\hbar}\big(\psi^a\psi^b - \psi^b\psi^a\big) \ \equiv \ \frac{i}{\hbar}\psi^{ab} \ . \tag{2.3}$$

We should also define the action of the covariant momentum on the fermionic operator:

$$g^{\frac{1}{4}}[\pi_m^{(\alpha)}, \psi^a]g^{-\frac{1}{4}} \ = \ 0 \ , \qquad g^{\frac{1}{4}}[\pi_m^{(\alpha)}, \psi^n]g^{-\frac{1}{4}} \ = \ i\hbar\, \Gamma^n_{(\alpha)pm}\, \psi^p \ , \tag{2.4}$$

where $\Gamma^n_{(\alpha)pm}$ is the affine connection equipped with torsion $\alpha H^n{}_{pm}$. The Levi-Civita connection, which is given as the symmetric part of the affine connection, is expressed by Γ^n_{0pm} defined in appendix A. Actually, the above commutator is associated with the covariant derivative of the Dirac gamma matrix on the target geometry.

By using the covariant momentum $\pi_m^{(-1/3)}$, let us represent the supercharge Q_H^1 and the Hamiltonian $\mathscr{H}_H^1$ (where the subscript H denotes that the operator contains the torsion H) as follows:

$$Q_H^1 \ \equiv \ \psi^m g^{\frac{1}{4}}\pi_m^{(-1/3)} g^{-\frac{1}{4}} \ = \ \psi^m g^{\frac{1}{4}}\Big(p_m - \frac{i}{2}\big(\omega_{mab} - \tfrac{1}{3}H_{mab}\big)\psi^{ab}\Big)g^{-\frac{1}{4}} \ , \tag{2.5a}$$

[1] Alvarez-Gaumé [10] and Mavromatos [16] refer the $\mathcal{N} = 2$ ($\mathcal{N} = 1$) model to $\mathcal{N} = 1$ ($\mathcal{N} = 1/2$) supersymmetric quantum mechanics.

$$\mathscr{H}_H^1 \;=\; \frac{1}{2}g^{-\frac{1}{4}}\pi_m^{(-1)}g^{mn}\sqrt{g}\,\pi_n^{(-1)}\,g^{-\frac{1}{4}}+\frac{\hbar^2}{8}R(\omega)-\frac{\hbar^2}{24}H_{mnp}H^{mnp}+\frac{1}{24}(\mathrm{d}H)_{abcd}\,\psi^{abcd}\;. \tag{2.5b}$$

Note that since we used the complete square in $\mathscr{H}_H^1$, the magnitude of the torsion in the covariant momentum is changed to $\pi_m^{(-1)}$. This is consistent with the analysis of the Killing spinor equation in the heterotic theory [22]. We can also formulate the $\mathcal{N}=1$ supersymmetric charges with introducing a (non-abelian) gauge fields on the target space:

$$Q_H^1 \;=\; \psi^m g^{\frac{1}{4}}\widetilde{\pi}_m^{(-1/3)}g^{-\frac{1}{4}}\;, \quad \{Q_H^1,Q_H^1\}\;=\;2\hbar\,\mathscr{H}_H^1\;, \tag{2.6a}$$

$$\begin{aligned}\mathscr{H}_H^1 \;&=\; \frac{1}{2}g^{-\frac{1}{4}}\widetilde{\pi}_m^{(-1)}g^{mn}\sqrt{g}\,\widetilde{\pi}_n^{(-1)}g^{-\frac{1}{4}}+\frac{\hbar^2}{8}\Big[R(\omega)-\frac{1}{3}H_{mnp}H^{mnp}\Big]+\frac{1}{24}(\mathrm{d}H)_{mnpq}\psi^{mnpq}\\ &\quad -\frac{1}{2}F_{mn}^{\alpha}\psi^{mn}(\hat{c}^\dagger T_\alpha \hat{c})\;,\end{aligned} \tag{2.6b}$$

$$\widetilde{\pi}_m^{(\alpha)} \;=\; p_m-\frac{i}{2}\Big(\omega_{mab}+\alpha H_{mab}\Big)\psi^{ab}-iA_m^{\alpha}(\hat{c}^\dagger T_\alpha \hat{c})\;, \tag{2.6c}$$

where we used the anti-hermitian matrix T_α as a generator of the gauge symmetry group. We also introduced a complex ghost field $\hat{c}^i$ living in the quantum mechanics.

2.2 $\mathcal{N}=2$ complex supersymmetry

Now we introduce two sets of real fermionic operators ψ_α^a ($\alpha=1,2$) and perform the complexification of fermionic operators via linear combination

$$\varphi^a \;\equiv\; \frac{1}{\sqrt{2}}(\psi_1^a+i\psi_2^a)\;, \quad \overline{\varphi}^a \;\equiv\; \frac{1}{\sqrt{2}}(\psi_1^a-i\psi_2^a)\;. \tag{2.7a}$$

Note that we used the convention $\overline{\varphi}^a=(\varphi^a)^\dagger$. Then the canonical quantization condition is extended in such a way as

$$\{\varphi^a,\varphi^b\}\;=\;0\;, \quad \{\overline{\varphi}^a,\overline{\varphi}^b\}\;=\;0\;, \quad \{\varphi^a,\overline{\varphi}^b\}\;=\;\hbar\,\delta^{ab}\;. \tag{2.7b}$$

This is nothing but the $SO(D,D)$ Clifford algebra. This complex fermion φ^a plays a central role in $\mathcal{N}=2$ supersymmetry, while ψ^a consists of $\mathcal{N}=1$ supersymmetry. Now let us construct the $\mathcal{N}=2$ supersymmetric model. First we define the algebraic equation in terms of a covariant momentum and fermionic operators. Second we look for the field theoretical representation of the covariant momentum which satisfies the algebraic equation.

Let us define the commutation relations between the covariant momentum $\pi_m^{(\alpha)}$ and the complex fermions, which are given in terms of the affine connection $\Gamma^p_{(\alpha)mn}$ in the same analogy as in the $\mathcal{N}=1$ system:

$$g^{\frac{1}{4}}[\pi_m^{(\alpha)},\varphi^n]g^{-\frac{1}{4}}\;=\;i\hbar\,\Gamma^n_{(\alpha)pm}\varphi^p\;, \quad g^{\frac{1}{4}}[\pi_m^{(\alpha)},\varphi_n]g^{-\frac{1}{4}}\;=\;-i\hbar\,\Gamma^p_{(\alpha)nm}\varphi_p\;. \tag{2.8}$$

The Lorentz generator coupled to the spin connection and the curvature tensor are expressed as

$$\Sigma^{ab} = \frac{i}{\hbar}\Big(\varphi^a\overline{\varphi}^b - \varphi^b\overline{\varphi}^a\Big) \,, \tag{2.9a}$$

$$g^{\frac{1}{4}}[\pi_m, \pi_n]\, g^{-\frac{1}{4}} = \frac{i\hbar^2}{2}R_{abmn}(\omega)\Sigma^{ab} = -\hbar\, R_{abmn}(\omega)\varphi^a\overline{\varphi}^b \,. \tag{2.9b}$$

Next, let us express $\mathcal{N} = 2$ supercharges Q_H in terms of the operators and the torsion given by three-form flux H. In the same way as the $\mathcal{N} = 1$ supercharge, we want to identify the "Dirac operator" with the $\mathcal{N} = 2$ supercharge. In the case on a Riemannian geometry, we identify the exterior derivative d on the geometry with the $\mathcal{N} = 2$ supercharge $Q \equiv \varphi^m g^{\frac{1}{4}}\pi_m g^{-\frac{1}{4}}$, where π_m is the covariant momentum in the $\mathcal{N} = 2$ quantum mechanics defined as

$$\pi_m = p_m - \frac{\hbar}{2}\omega_{mab}\,\Sigma^{ab} = p_m - i\omega_{mab}\,\varphi^a\overline{\varphi}^b \,. \tag{2.10}$$

Let us introduce the torsion on the geometry. Following the discussions of the sigma model with torsion [23, 24, 16] and the generalized complex geometry [25, 26], we extend the exterior derivative d to d_H in such a way as

$$\mathrm{d}_H \equiv \mathrm{d} + H\wedge \,, \quad (\mathrm{d}_H)^2 = (\mathrm{d}H)\wedge \,. \tag{2.11}$$

This means that d_H is satisfied with the nilpotency up to the derivative $\mathrm{d}H$. In addition, by using the Darboux theorem (see, for instance, section *4.7* in [25]), we can identify the one-form with the holomorphic variable, while the adjoint of the one-form can be identified with the anti-holomorphic variable. Thus, we identify the exterior derivative d_H and its adjoint $\mathrm{d}_H^\dagger$ with appropriate operators in terms of complex fermions φ^m and $\overline{\varphi}^m$ in the quantum mechanics:

$$\mathrm{d}_H = \mathrm{d} + H\wedge \quad \leftrightarrow \quad Q_H \equiv \varphi^m g^{\frac{1}{4}}\pi_m g^{-\frac{1}{4}} + \alpha H_{mnp}\,\varphi^m\varphi^n\varphi^p \,, \tag{2.12a}$$

$$\mathrm{d}_H^\dagger \quad \leftrightarrow \quad \overline{Q}_H \equiv \overline{\varphi}^m g^{\frac{1}{4}}\pi_m g^{-\frac{1}{4}} + \overline{\alpha} H_{mnp}\,\overline{\varphi}^m\overline{\varphi}^n\overline{\varphi}^p \,. \tag{2.12b}$$

Due to this, we wish to interpret Q_H as the "supersymmetry charge", associated with the exterior derivative $\mathrm{d}_H \equiv \mathrm{d} + H\wedge$, while $\overline{Q}_H$ associated with $\mathrm{d}_H^\dagger$, i.e., the adjoint of the derivative d_H. Here we also introduced the scale factor α, which should be fixed compared with the $\mathcal{N} = 1$ supercharge. Notice that, as in the previous work of the $\mathcal{N} = 2$ complex supersymmetry in [23, 24], let us assume the closed condition $\mathrm{d}H = 0$, while we have not imposed this condition on the $\mathcal{N} = 1$ real supersymmetry in the previous discussion. This strong condition should be relaxed when we consider the heterotic string compactification in the presence of the non-trivial NS-fluxes [22]. Even though we could introduce such non-vanishing flux $\mathrm{d}H \neq 0$ in the $\mathcal{N} = 1$ supersymmetric system, so far we have not found a consistent $\mathcal{N} = 2$ supersymmetry with the condition $\mathrm{d}H \neq 0$. We keep this extension as a future problem, and we will always impose $\mathrm{d}H = 0$ on the $\mathcal{N} = 2$ supersymmetric theory in this paper.

In order to fix the coefficient α, let us truncate the supercharge Q_H to the supercharge Q_H^1 in the $\mathcal{N}=1$ supersymmetry (2.5) via the restriction $\psi_2^a = 0$ and $\psi_1^a \equiv \psi^a$:

$$Q_H \;\to\; \frac{1}{\sqrt{2}}\psi^m \Big\{ g^{\frac{1}{4}} p_m \, g^{-\frac{1}{4}} - \frac{i}{2}\big(\omega_{mab} - \alpha H_{mab}\big)\psi^a \psi^b \Big\} \;=\; \frac{1}{\sqrt{2}} Q_H^1 \; . \tag{2.13}$$

Since we have already known the $\mathcal{N}=1$ supercharge Q_H^1, we can fix the coefficient

$$\alpha \;=\; \frac{1}{3} \;=\; \overline{\alpha} \; . \tag{2.14}$$

Due to the first Bianchi identity $R_{[mnp]q}(\omega) = 0$ and $D_{[d}(\omega) H_{cab]} = \frac{1}{4}(\mathrm{d}H)_{dcab} = 0$, we find that the supersymmetry algebra is given by

$$\{Q_H, Q_H\} \;=\; \frac{\hbar}{6}\,(\mathrm{d}H)_{abcd}\,\varphi^{abcd} \;=\; 0 \; , \qquad \{\overline{Q}_H, \overline{Q}_H\} \;=\; \frac{\hbar}{6}\,(\mathrm{d}H)_{abcd}\,\overline{\varphi}^{abcd} \;=\; 0 \; , \tag{2.15a}$$

$$\{Q_H, \overline{Q}_H\} \;=\; 2\hbar\,\mathscr{H}_H \; . \tag{2.15b}$$

This supersymmetry algebra does not contain “central charge”. We can see the energy degeneration condition between bosonic state and fermionic state via the following commutator:

$$[Q_H, \mathscr{H}_H] \;=\; -\frac{1}{2\hbar}[\overline{Q}_H, (Q_H)^2] \;=\; 0 \; . \tag{2.16}$$

Now we explicitly express the Hamiltonian $\mathscr{H}_H$ in terms of the complex fermions under the condition $\mathrm{d}H = 0$:

$$\begin{aligned}
\mathscr{H}_H \;&=\; \frac{1}{2\hbar}\{Q_H, \overline{Q}_H\} \\
&=\; \frac{1}{2} g^{-\frac{1}{4}} \Big\{ \pi_m + \frac{i}{2} H_{mab}\big(\varphi^{ab} + \overline{\varphi}^{ab}\big) \Big\} g^{mn} \sqrt{g} \Big\{ \pi_n + \frac{i}{2} H_{ncd}\big(\varphi^{cd} + \overline{\varphi}^{cd}\big) \Big\} \, g^{-\frac{1}{4}} \\
&\quad - \frac{1}{2} R_{abmn}(\omega)\, \varphi^m \overline{\varphi}^n \varphi^a \overline{\varphi}^b + \frac{1}{6} D_a(\omega)[H_{bcd}] \Big\{ \overline{\varphi}^a \varphi^{bcd} + \varphi^a \overline{\varphi}^{bcd} - \frac{3}{2}\hbar\,\delta^{ab}\big(\varphi^{cd} + \overline{\varphi}^{cd}\big) \Big\} \\
&\quad + \frac{\hbar^2}{12} H_{mnp} H^{mnp} - \frac{\hbar}{2} H_{acd} H_b{}^{cd}\, \varphi^a \overline{\varphi}^b \\
&\quad - \frac{1}{4} H_{abe} H_{cd}{}^e\, \varphi^{ab} \overline{\varphi}^{cd} + \frac{1}{8} H_{abe} H_{cd}{}^e\, \varphi^{abcd} + \frac{1}{8} H_{abe} H_{cd}{}^e\, \overline{\varphi}^{abcd} \; .
\end{aligned} \tag{2.17}$$

There exists a comment on the Hamiltonians in the $\mathcal{N}=1$ and the $\mathcal{N}=2$ system. The $\mathcal{N}=1$ Hamiltonian cannot be obtained by truncation of the $\mathcal{N}=2$ Hamiltonian, because the truncation $\psi_2^a = 0$ is no longer consistent at the quantum level since the anti-commutation relation $\{\psi_1^a + i\psi_2^a, \psi_1^b + i\psi_2^b\}$ becomes non-zero via the truncation. On the other hand, we need not use such anti-commutation relation when we reduce the $\mathcal{N}=2$ supercharge to a charge in the $\mathcal{N}=1$ system.

3 Path integral formalism from Hamiltonian formalism

In this section first we will discuss a generic strategy to obtain the transition element $\langle\, x \,|\mathrm{e}^{-\frac{\beta}{\hbar}\mathscr{H}}|\, y \,\rangle$ which appears in (1.2). We will introduce a number of useful tools to investigate the quantum mechanical path integral such as the complete sets of eigenstates, and the Weyl-ordered form. Next we will move to the concrete constructions of the transition elements in the $\mathcal{N}=1$ and in the $\mathcal{N}=2$ systems. In this paper we omit many technical details which can be seen in the works [20, 21]. We mainly follow the convention defined in [21]. Before going to the main discussion, for later convenience, let us take a rescaling on the fermionic operators which we introduced in the previous section:

$$\mathcal{N}=2 \text{ system:} \quad \varphi^a \;\equiv\; \sqrt{\hbar}\,\varphi^a_{\diamond}\;, \tag{3.1a}$$

$$\mathcal{N}=1 \text{ system:} \quad \psi^a \;\equiv\; \sqrt{\hbar}\,\psi^a_{\diamond}\;, \qquad \text{ghost fields:} \quad \varphi^{\rm gh} \;\equiv\; \sqrt{\hbar}\,\varphi^{\rm gh}_{\diamond}\;. \tag{3.1b}$$

Notice that since we are now trying to find a configuration space path integral with respect to classical Lagrangians, the (Dirac) fermions in the system are now given by Grassmann odd c-numbers, then the (anti-)commutation relations between two fermions are trivial, i.e., $\{\varphi^a, \overline{\varphi}^b\} = 0 = \{\varphi^a, \varphi^b\} = \{\overline{\varphi}^a, \overline{\varphi}^b\}$. For simplicity we omit the symbol "$\diamond$" if there are no confusion.

3.1 General discussion

In order to formulate the transition elements we should prepare a number of tools. Let $\widehat{x}^m$ and $\widehat{p}_m$ be the operators of the coordinate and the momentum, respectively, while x^m and p_m denote their eigenvalues[2]. According to [20, 21], let us introduce the complete set of the $\widehat{x}$-eigenfunctions and the complete set of the $\widehat{p}$-eigenfunctions

$$\int \mathrm{d}^D x \,|\, x \,\rangle \sqrt{g(x)} \langle\, x \,| \;\equiv\; 1 \;\equiv\; \int \mathrm{d}^D p \,|\, p \,\rangle \langle\, p \,|\;, \tag{3.2}$$

where $g(x) = \det g_{mn}(x)$. We also define the inner products and the plane wave such as

$$\langle\, x \,|\, y \,\rangle \;\equiv\; \frac{1}{\sqrt{g(x)}} \delta^D(x-y)\;, \quad \langle\, p \,|\, p' \,\rangle \;\equiv\; \delta^D(p-p')\;, \tag{3.3a}$$

$$\langle\, x \,|\, p \,\rangle \;\equiv\; \frac{1}{(2\pi\hbar)^{D/2}} \exp\Big(\frac{i}{\hbar} p \cdot x\Big) g^{-\frac{1}{4}}\;, \tag{3.3b}$$

where the plane wave is normalized to

$$\int \mathrm{d}^D p\, \exp\Big(\frac{i}{\hbar} p \cdot (x-y)\Big) \;=\; (2\pi\hbar)^{D/2} \delta^D(x-y)\;, \tag{3.3c}$$

which appears when we evaluate the transition elements with infinitesimal short period. In order to discuss the path integrals for Dirac fermion operators, let us also introduce a set of coherent states for

[2] The symbol "$\widehat{\ }$" on an operator is omitted if there are no confusions.

fermionic operators in terms of the operator $\widehat{\varphi}^a$ satisfying $\{\widehat{\varphi}^a, \widehat{\overline{\varphi}}^b\} = \delta^{ab}$, and a complex Grassmann odd variable η:

$$
\begin{aligned}
|\,\eta\,\rangle \;&\equiv\; \mathrm{e}^{\widehat{\overline{\varphi}}^a \eta^a} |\,0\,\rangle \,, \quad \widehat{\varphi}^a|\,0\,\rangle \;=\; 0 \,, \quad \widehat{\varphi}^a|\,\eta\,\rangle \;=\; \eta^a|\,\eta\,\rangle \,, && (3.4a) \\
\langle\,\overline{\eta}\,| \;&\equiv\; \langle\,0\,|\mathrm{e}^{\overline{\eta}^a \widehat{\varphi}^a} \,, \quad \langle\,0\,|\widehat{\overline{\varphi}}^a \;=\; 0 \,, \quad \langle\,\overline{\eta}\,|\widehat{\overline{\varphi}}^a \;=\; \langle\,\overline{\eta}\,|\overline{\eta}^a \,. && (3.4b)
\end{aligned}
$$

The inner product of these coherent state is given by $\langle\,\overline{\eta}\,|\,\zeta\,\rangle = \mathrm{e}^{\overline{\eta}^a \zeta^a}$. In the same analogy as (3.2), we introduce a complete set of the Dirac fermion coherent states:

$$
1 \;=\; \int \prod_{a=1}^{D} \mathrm{d}\overline{\eta}^a \mathrm{d}\eta^a \,|\,\eta\,\rangle\, \mathrm{e}^{-\overline{\eta}^a \eta^a} \langle\,\eta\,| \,, \tag{3.5a}
$$

$$
\prod_{a=1}^{D} \mathrm{d}\overline{\eta}^a \;\equiv\; \mathrm{d}\overline{\eta}^D \mathrm{d}\overline{\eta}^{D-1} \cdots \mathrm{d}\overline{\eta}^1 \,, \quad \prod_{a=1}^{D} \mathrm{d}\eta^a \;\equiv\; \mathrm{d}\eta^1 \mathrm{d}\eta^2 \cdots \mathrm{d}\eta^D \,. \tag{3.5b}
$$

Generically we define a matrix element $M(z,y)$ of the quantum operators in the following way:

$$
M(z,y) \;=\; \langle\,z\,|\widehat{\mathcal{O}}(\widehat{x},\widehat{p})|\,y\,\rangle \,, \tag{3.6}
$$

where $|\,y\,\rangle$ and $\langle\,z\,|$ are the initial and final state, respectively. Now we are quite interested in the transition element with respect to the quantum Hamiltonian $\widehat{\mathscr{H}}$ and a parameter β:

$$
T(z,\overline{\eta};y,\zeta;\beta) \;\equiv\; \langle\,z,\overline{\eta}\,|\exp\Big(-\frac{\beta}{\hbar}\widehat{\mathscr{H}}\Big)|\,y,\zeta\,\rangle \,. \tag{3.7}
$$

Next we introduce $N-1$ complete sets of position eigenstates x_k and of the fermion coherent states λ_k into the above transition elements. At the same time let us also insert N complete sets of momentum eigenstates p_k and of another fermion coherent states ξ_k to yield

$$
\begin{aligned}
&\langle\,z,\overline{\eta}\,|\exp\Big(-\frac{\beta}{\hbar}\widehat{\mathscr{H}}\Big)|\,y,\zeta\,\rangle \\
&\quad= \int \prod_{i=1}^{N-1} \mathrm{d}^D x_i \prod_{i'=1}^{N-1} \mathrm{d}\overline{\lambda}_{i'}\lambda_{i'}\, \mathrm{e}^{-\overline{\lambda}_{i'}\lambda_{i'}} \prod_{j=1}^{N} \mathrm{d}^D p_j \prod_{j'=0}^{N-1} \mathrm{d}\overline{\xi}_{j'}\mathrm{d}\xi_{j'}\, \mathrm{e}^{-\overline{\xi}_{j'}\xi_{j'}} \\
&\qquad \times \prod_{k=0}^{N-1} \langle\,x_{k+1},\overline{\lambda}_{k+1}\,|\,p_k,\xi_k\,\rangle \exp\Big(-\frac{\epsilon}{\hbar}\mathscr{H}_{\mathrm{W}}(x_{k+\frac{1}{2}},p_{k+1};\overline{\xi}_k,\tfrac{1}{2}(\xi_k+\lambda_k))\Big)\langle\,p_k,\overline{\xi}_k\,|\,x_k,\lambda_k\,\rangle \\
&\quad= \big[g(z)g(y)\big]^{-\frac{1}{4}} \int \prod_{j=1}^{N} \frac{\mathrm{d}^D p_j}{(2\pi\hbar)^D} \prod_{i=1}^{N-1} \mathrm{d}^D x_i \prod_{j'=0}^{N-1} \mathrm{d}\overline{\xi}_{j'}\mathrm{d}\xi_{j'} \\
&\qquad \times \exp\Bigg(\overline{\eta}\cdot\xi_{N-1} + \frac{\epsilon}{\hbar}\sum_{k=0}^{N-1}\Big[ip_{k+1}\cdot\frac{x_{k+1}-x_k}{\epsilon} - \hbar\,\overline{\xi}_k\cdot\frac{\xi_k-\xi_{k-1}}{\epsilon} - \mathscr{H}_{\mathrm{W}}(x_{k+\frac{1}{2}},p_{k+1};\overline{\xi}_k,\xi_{k-\frac{1}{2}})\Big]\Bigg) \,.
\end{aligned} \tag{3.8}
$$

Notice that the subscript k denotes the k-th complete set of the coherent states, while the superscript a runs from 1 to D, the number of the dimensions of the target space. The contraction of the indices

a is given by $\overline{\eta}^a \eta^a = \delta_{ab}\overline{\eta}^a \eta^b = \overline{\eta} \cdot \eta$. We also note that $y = x_0$, $z = x_N$, $\overline{\eta} = \overline{\lambda}_N$, $\zeta = \lambda_0 = \xi_{-1}$. We adopt the midpoint rule $x_{k+\frac{1}{2}} = \frac{1}{2}(x_{k+1} + x_k)$ and $\xi_{k-\frac{1}{2}} = \frac{1}{2}(\xi_k + \xi_{k-1})$. We also used the inner products (3.3). The factors $\sqrt{g(x_k)}$ compensate exactly the $g^{\frac{1}{4}}$ factors from the plane waves in the inner products. Furthermore, we integrated the arguments λ_k and $\overline{\lambda}_k$ to yield a useful equation

$$\int \mathrm{d}\overline{\lambda}_k \mathrm{d}\lambda_k \, \mathrm{e}^{-\overline{\lambda}_k \cdot (\lambda_k - \xi_{k-1})} f(\lambda_k) \;=\; f(\xi_{k-1}) \, , \tag{3.9}$$

where $f(\lambda)$ is an arbitrary function of the fermionic variable λ. Notice that $\widehat{\mathscr{H}}$ is the quantum Hamiltonian in terms of quantum operators, while $\mathscr{H}_{\mathrm{W}}$ is its Weyl-ordered form. The translation from the operator to the Weyl-ordered form is given in terms of the symmetrized form $\widehat{\mathscr{H}}_{\mathrm{S}}$ by

$$\widehat{\mathscr{H}} \;=\; \widehat{\mathscr{H}}_{\mathrm{S}} + \text{further terms} \;=\; \mathscr{H}_{\mathrm{W}} \, . \tag{3.10}$$

Integrating out the (discretized) momenta and taking the continuum limit $N \to \infty$, $\epsilon/\beta \to \mathrm{d}\tau$ with $\sum_{k=0}^{N-1} \epsilon/\beta \to \int_{-1}^{0} \mathrm{d}\tau$, we obtain the continuum path integral description in a following form:

$$T(z, \overline{\eta}; y, \zeta; \beta) \;=\; \left(\frac{g(z)}{g(y)} \right)^{\frac{1}{4}} \frac{1}{(2\pi\beta\hbar)^{D/2}} \, \mathrm{e}^{\overline{\eta}_a \zeta^a} \Big\langle \exp \Big(- \frac{1}{\hbar} S^{(\mathrm{int})} - \frac{1}{\hbar} S^{(\mathrm{source})} \Big) \Big\rangle_0 \, . \tag{3.11}$$

Note the followings: The action $S^{(\mathrm{int})}$ is given in terms of the interaction terms in the Lagrangian derived from the Legendre transformation of Weyl-ordered Hamiltonian, which we will explicitly show later. We introduced the external source of fields contained in the action $S^{(\mathrm{source})}$ to define their propagators. The additional factor $\sqrt{g(z)}$ appears due to the expanding the metric in $S^{(\mathrm{int})}$ at the point z and due to the integrating out the free kinetic terms of fields (see, for detail, section *2.1* in [21]). The symbol $\langle \cdots \rangle_0$ denotes the contraction of interaction terms in terms of propagators and setting the external source to zero. From now on we simply abbreviate $\big\langle \mathrm{e}^{-\frac{1}{\hbar} S^{(\mathrm{int})} - \frac{1}{\hbar} S^{(\mathrm{source})}} \big\rangle_0$ as $\big\langle \exp(-\frac{1}{\hbar} S^{(\mathrm{int})}) \big\rangle$.

3.2 Weyl-ordered form of quantum Hamiltonians

The next task is to study the Weyl-ordered form of the Hamiltonians $\mathscr{H}_H^{\mathrm{W}}$ and obtain the actions $S^{(\mathrm{int})}$ in the $\mathcal{N} = 2$ and the $\mathcal{N} = 1$ systems, respectively. In the part of the bosonic operators, the symmetrized form is defined by

$$\prod_{m,n} N! \big\{ (\widehat{p}_m)^{k_m} (\widehat{x}^n)^{\ell_n} \big\}_{\mathrm{S}} \;\equiv\; \prod_{m,n} \Big(\frac{\partial}{\partial \alpha^m} \Big)^{k_m} \Big(\frac{\partial}{\partial \beta_n} \Big)^{\ell_n} \big(\alpha^m \widehat{p}_m + \beta_n \widehat{x}^n \big)^N \, , \tag{3.12a}$$

$$N \;\equiv\; \sum_m k_m + \sum_n \ell_n \, . \tag{3.12b}$$

In the $\mathcal{N} = 2$ complex fermions' case we define the following anti-symmetrized form:

$$\prod_{a,b} N! \big\{ (\widehat{\varphi}^a)^{m_a} (\widehat{\overline{\varphi}}_b)^{n_b} \big\}_{\mathrm{S}} \;\equiv\; \prod_{a,b} \Big(\frac{\partial}{\partial \alpha_a} \Big)^{m_a} \Big(\frac{\partial}{\partial \beta^b} \Big)^{n_b} \big(\alpha_a \widehat{\varphi}^a + \beta^b \widehat{\overline{\varphi}}_b \big)^N \, , \tag{3.13a}$$

$$N \equiv \sum_a m_a + \sum_b n_b \, , \tag{3.13b}$$

where we perform the left derivative with respect to the Grassmann odd variables α_a and β^b. In the $\mathcal{N}=1$ real fermions' case, the anti-symmetrized form is defined by

$$\big(\psi^{a_1} \cdots \psi^{a_N}\big)_{\rm S} \equiv \frac{1}{N!} \prod_i \Big(\frac{\partial}{\partial \alpha_{a_i}}\Big) \big(\alpha_a \psi^a\big)^N \, . \tag{3.14}$$

By using the above procedure, we obtain the Weyl-ordered form of the $\mathcal{N}=2$ Hamiltonian

$$\begin{aligned}
\mathscr{H}_H^{\rm W} &= \frac{1}{2}\Big(g^{mn} \pi_m^{(-1)} \pi_n^{(-1)}\Big)_{\rm S} - \frac{\hbar^2}{2} R_{cdab}(\omega) (\varphi^a \overline{\varphi}^b \varphi^c \overline{\varphi}^d)_{\rm S} + \frac{\hbar^2}{8} g^{mn} \Big\{ \Gamma^p_{0mq} \Gamma^q_{0np} + \omega_{mab} \omega_n{}^{ab} \Big\} \\
&\quad + \frac{\hbar^2}{12} H_{mnp} H^{mnp} + \frac{\hbar^2}{8} H_{abe} H_{cd}{}^e \Big\{ (\varphi^{abcd})_{\rm S} + (\overline{\varphi}^{abcd})_{\rm S} - 2 (\varphi^{ab} \overline{\varphi}^{cd})_{\rm S} \Big\} \, , \qquad (3.15a) \\
\pi_m^{(-1)} &\equiv \pi_m + \frac{i\hbar}{2} H_{mab} \big(\varphi^{ab} + \overline{\varphi}^{ab}\big) = p_m - i\hbar \, \omega_{mab} \, \varphi^a \overline{\varphi}^b + \frac{i\hbar}{2} H_{mab} \big(\varphi^{ab} + \overline{\varphi}^{ab}\big) \, , \qquad (3.15b)
\end{aligned}$$

and of the $\mathcal{N}=1$ Hamiltonian

$$\begin{aligned}
\mathscr{H}_H^{1;{\rm W}} &= \frac{1}{2}\Big(g^{mn} \widetilde{\pi}_m^{(-1)} \widetilde{\pi}_n^{(-1)}\Big)_{\rm S} + \frac{\hbar^2}{8} \Big\{ g^{mn} \Gamma^p_{0mq} \Gamma^q_{0np} + \frac{1}{2} g^{mn} \omega_{-mab} \, \omega_{-n}{}^{ab} \Big\} \\
&\quad - \frac{\hbar^2}{24} H_{mnp} H^{mnp} + \frac{\hbar^2}{24} ({\rm d}H)_{abcd} \, (\psi^{abcd})_{\rm S} - \frac{\hbar^2}{2} F^\alpha_{mn} (\psi^{mn})_{\rm S} (\hat{c}^\dagger T_\alpha \hat{c}) \, , \qquad (3.16a) \\
\widetilde{\pi}_m^{(-1)} &\equiv p_m - \frac{i\hbar}{2} \omega_{-mab} \, \psi^{ab} - i\hbar \, A^\alpha_m (\hat{c}^\dagger T_\alpha \hat{c}) \, . \qquad (3.16b)
\end{aligned}$$

To proceed computations in path integral formalism in the $\mathcal{N}=1$ system, we would like to add a second set of "free" Majorana fermions in order to simplify the path integral in the $\mathcal{N}=1$ system in the same way as the one in the $\mathcal{N}=2$ system. Denoting the original Majorana fermions ψ^a by ψ_1^a, and the new ones by ψ_2^a, and combining them, we again construct Dirac fermions χ^a and $\overline{\chi}^a$ as

$$\chi^a = \frac{1}{\sqrt{2}}\big(\psi_1^a + i\psi_2^a\big) \, , \quad \overline{\chi}^a = \frac{1}{\sqrt{2}}\big(\psi_1^a - i\psi_2^a\big) \, . \tag{3.17}$$

Notice that, in this context, ψ_2^a differs from the second component of the previously defined Dirac fermions φ^a because now ψ_2^a is introduced as a "free" fermion in the $\mathcal{N}=1$ Hamiltonian.

3.3 Explicit form of the transition element in $\mathcal{N}=2$ system

We are ready to discuss the explicit form of the transition element in the $\mathcal{N}=2$ system in the framework of the Lagrangian formalism. Let us first decompose the bosonic and fermionic variables into two parts, i.e., the background fields and quantum fluctuations in such a way as $x^m(\tau) = x^m_{\rm bg}(\tau) + q^m(\tau)$ and $\xi^a(\tau) = \xi^a_{\rm bg}(\tau) + \xi^a_{\rm qu}(\tau)$, respectively. These background fields follow the free equations of motion and their solutions are

$$x^m_{\rm bg}(\tau) = z^m + \tau(z^m - y^m) \, , \quad \xi^a_{\rm bg}(\tau) = \zeta^a \, , \quad \overline{\xi}^a_{\rm bg}(\tau) = \overline{\eta}^a \, , \tag{3.18}$$

with constraint (via the mean-value theorem)

$$q^m(-1) \;=\; q^m(0) \;=\; 0 \;, \qquad \int_{-1}^{0} \mathrm{d}\tau\, q^m(\tau) \;=\; 0 \;, \tag{3.19a}$$

$$\xi^a_{\mathrm{qu}}(-1) \;=\; \overline{\xi}^a_{\mathrm{qu}}(0) \;=\; 0 \;. \tag{3.19b}$$

Furthermore, we can rewrite the term $H_{e[ab}H_{cd]}{}^e$ in terms of the curvature:

$$R_{[abcd]}(\omega_+) \;=\; 2\partial_{[c}H_{abd]} + 2H^e{}_{[ca}H_{bd]e} \;=\; \frac{1}{2}(\mathrm{d}H)_{abcd} + 2H^e{}_{[ab}H_{cd]e} \;=\; 2H^e{}_{[ab}H_{cd]e} \;. \tag{3.20}$$

Then the description of the transition element in the configuration space path integral is given in the following form (see eq.(*2.81*) in [21]):

$$\langle\, z, \overline{\eta} \,|\, \exp\Big(- \frac{\beta}{\hbar} \widehat{\mathscr{H}}_H \Big) |\, y, \zeta \,\rangle \;=\; \left(\frac{g(z)}{g(y)} \right)^{\frac{1}{4}} \frac{1}{(2\pi\beta\hbar)^{D/2}} \, \mathrm{e}^{\overline{\eta}_a \zeta^a} \Big\langle \exp\Big(- \frac{1}{\hbar} S_H^{(\mathrm{int})} \Big) \Big\rangle \;, \tag{3.21a}$$

$$\overline{\eta}_a \zeta^a - \frac{1}{\hbar} S_H^{(\mathrm{int})} \;=\; - \frac{1}{\hbar} \big(S_H - S^{(0)} \big) \;, \tag{3.21b}$$

$$\begin{aligned}
-\frac{1}{\hbar} S_H \;=\;& -\frac{1}{\beta\hbar} \int_{-1}^{0} \mathrm{d}\tau\, \frac{1}{2} g_{mn}(x) \left(\frac{\mathrm{d}x^m}{\mathrm{d}\tau} \frac{\mathrm{d}x^n}{\mathrm{d}\tau} + b^m c^n + a^m a^n \right) + \overline{\eta}_a \zeta^a - \int_{-1}^{0} \mathrm{d}\tau\, \delta_{ab}\, \overline{\xi}^a_{\mathrm{qu}} \frac{\mathrm{d}}{\mathrm{d}\tau} \xi^b_{\mathrm{qu}} \\
& - \int_{-1}^{0} \mathrm{d}\tau\, \frac{\mathrm{d}x^m}{\mathrm{d}\tau} \left(\omega_{mab}(x) \overline{\xi}^a \xi^b - \frac{1}{2} H_{mab}(x) (\xi^{ab} + \overline{\xi}^{ab}) \right) \\
& - \frac{\beta\hbar}{16} \int_{-1}^{0} \mathrm{d}\tau\, R_{[abcd]}(\omega_+(x)) \big(\xi^{abcd} + \overline{\xi}^{abcd} - 2\xi^{ab}\overline{\xi}^{cd} \big) + \frac{\beta\hbar}{2} \int_{-1}^{0} \mathrm{d}\tau\, R_{cdab}(\omega(x))\, \xi^a \overline{\xi}^b \xi^c \overline{\xi}^d \\
& - \frac{\beta\hbar}{8} \int_{-1}^{0} \mathrm{d}\tau\, \mathcal{G}_2(x) \;,
\end{aligned} \tag{3.21c}$$

$$\mathcal{G}_2(x) \;\equiv\; g^{mn}(x) \Big\{ \Gamma^p_{0mq}(x) \Gamma^q_{0np}(x) + \omega_{mab}(x) \omega_n{}^{ab}(x) \Big\} + \frac{2}{3} H_{mnp}(x) H^{mnp}(x) \;, \tag{3.21d}$$

$$-\frac{1}{\hbar} S^{(0)} \;=\; -\frac{1}{\beta\hbar} \int_{-1}^{0} \mathrm{d}\tau\, \frac{1}{2} g_{mn}(z) \left(\frac{\mathrm{d}q^m}{\mathrm{d}\tau} \frac{\mathrm{d}q^n}{\mathrm{d}\tau} + b^m c^n + a^m a^n \right) - \int_{-1}^{0} \mathrm{d}\tau\, \delta_{ab}\, \overline{\xi}^a_{\mathrm{qu}} \frac{\mathrm{d}}{\mathrm{d}\tau} \xi^b_{\mathrm{qu}} \;. \tag{3.21e}$$

Note that we introduced anti-commuting ghost fields b^m, c^m and a commuting ghost field a^m associated with the integrating out of momentum variables. They also appear in the $\mathcal{N} = 1$ system. We should notice that the metric in $S^{(0)}$ is given at the point z, not at the intermediate point x, while the metric, spin connection, and the fluxes in S_H are given at the intermediate point x. We can also define the propagators in this system:

$$\big\langle q^m(\sigma) q^n(\tau) \big\rangle \;=\; -\beta\hbar\, g^{mn}(z)\, \Delta(\sigma, \tau) \;, \tag{3.22a}$$

$$\big\langle a^m(\sigma) a^n(\tau) \big\rangle \;=\; \beta\hbar\, g^{mn}(z)\, \delta(\sigma - \tau) \;, \tag{3.22b}$$

$$\big\langle b^m(\sigma) c^n(\tau) \big\rangle \;=\; -2\beta\hbar\, g^{mn}(z)\, \delta(\sigma - \tau) \;, \tag{3.22c}$$

$$\big\langle \xi^a_{\mathrm{qu}}(\sigma) \overline{\xi}^b_{\mathrm{qu}}(\tau) \big\rangle \;=\; \delta^{ab}\, \theta(\sigma - \tau) \;, \tag{3.22d}$$

$$\big\langle \xi^a_{\mathrm{qu}}(\sigma) \xi^b_{\mathrm{qu}}(\tau) \big\rangle \;=\; 0 \;=\; \big\langle \overline{\xi}^a_{\mathrm{qu}}(\sigma) \overline{\xi}^b_{\mathrm{qu}}(\tau) \big\rangle \;, \tag{3.22e}$$

where the $\delta(\sigma-\tau)$ is the "Kronecker delta", and $-1 \le \tau, \sigma \le 0$. The definitions of various functions are defined as $\Delta(\sigma,\tau) = \sigma(\tau+1)\theta(\sigma-\tau) + \tau(\sigma+1)\theta(\tau-\sigma) = \Delta(\tau,\sigma)$, $\theta(\tau-\tau) = \frac{1}{2}$, $\theta(\tau-\sigma) = -\theta(\sigma-\tau)+1$, and so forth, which we list in (C.1) (see also [21]).

3.4 Explicit form of the transition element in $\mathcal{N}=1$ system

We can also describe the transition element in the $\mathcal{N}=1$ supersymmetric quantum system in terms of the dynamical bosonic and fermionic fields and free Majorana fields (see eq.(*2.81*) in [21]):

$$\langle\, z, \overline{\eta}, \overline{\eta}_{\rm gh} \,|\exp\Big(-\frac{\beta}{\hbar}\widehat{\mathscr{H}}{}^1_H\Big)|\, y, \zeta, \zeta_{\rm gh}\,\rangle \ = \ \left(\frac{g(z)}{g(y)}\right)^{\frac{1}{4}} \frac{1}{(2\pi\beta\hbar)^{D/2}}\, {\rm e}^{\overline{\eta}_a \zeta^a}\, {\rm e}^{\overline{\eta}_{\rm gh}\cdot\zeta_{\rm gh}} \big\langle {\rm e}^{-\frac{1}{\hbar}S^{\rm (int)}_{1,H}} \big\rangle \,, \tag{3.23a}$$

$$\overline{\eta}_a \zeta^a + \overline{\eta}_{\rm gh}\cdot\zeta_{\rm gh} - \frac{1}{\hbar} S^{\rm (int)}_{1,H} \ = \ -\frac{1}{\hbar}\big(S_{1,H} - S^{(0)}_1\big)\,, \tag{3.23b}$$

$$\begin{aligned}
-\frac{1}{\hbar}S_{1,H} \ &= \ \overline{\eta}_a \zeta^a + \overline{\eta}_{\rm gh}\cdot\zeta_{\rm gh} \\
&\quad - \frac{1}{\beta\hbar}\int_{-1}^0 {\rm d}\tau\, \frac{1}{2} g_{mn}(x)\left(\frac{{\rm d}x^m}{{\rm d}\tau}\frac{{\rm d}x^n}{{\rm d}\tau} + b^m c^n + a^m a^n\right) - \int_{-1}^0 {\rm d}\tau\left(\delta_{ab}\,\overline{\xi}{}^a_{\rm qu}\frac{\rm d}{{\rm d}\tau}\xi^b_{\rm qu} + \hat{c}^\dagger_{i,{\rm qu}}\frac{\rm d}{{\rm d}\tau}\hat{c}^i_{\rm qu}\right) \\
&\quad - \frac{1}{2}\int_{-1}^0 {\rm d}\tau\,\frac{{\rm d}x^m}{{\rm d}\tau}\omega_{-mab}(x)\,\psi^a_1\psi^b_1 - \int_{-1}^0 {\rm d}\tau\,\frac{{\rm d}x^m}{{\rm d}\tau}A^\alpha_m(x)\,\big(\overline{\xi}_{\rm gh}\,T_\alpha\,\xi_{\rm gh}\big) \\
&\quad - \frac{\beta\hbar}{24}\int_{-1}^0 {\rm d}\tau\,({\rm d}H)_{abcd}(x)\,\psi^a_1\psi^b_1\psi^c_1\psi^d_1 + \frac{\beta\hbar}{2}\int_{-1}^0 {\rm d}\tau\,F^\alpha_{mn}(x)\,\psi^m_1\psi^n_1\,\big(\overline{\xi}_{\rm gh}\,T_\alpha\,\xi_{\rm gh}\big) \\
&\quad - \frac{\beta\hbar}{8}\int_{-1}^0 {\rm d}\tau\,\mathcal{G}_1(x)\,,
\end{aligned} \tag{3.23c}$$

$$\mathcal{G}_1(x) \ \equiv \ g^{mn}(x)\Big\{\Gamma^p_{0mq}(x)\Gamma^q_{0np}(x) + \frac{1}{2}\omega_{-mab}(x)\omega_{-n}{}^{ab}(x)\Big\} - \frac{1}{3}H_{mnp}(x)H^{mnp}(x)\,, \tag{3.23d}$$

$$-\frac{1}{\hbar}S^{(0)}_1 \ = \ -\int_{-1}^0 {\rm d}\tau\left(\frac{1}{2\beta\hbar}g_{mn}(z)\Big\{\frac{{\rm d}q^m}{{\rm d}\tau}\frac{{\rm d}q^n}{{\rm d}\tau} + b^m c^n + a^m a^n\Big\} + \Big\{\delta_{ab}\,\overline{\xi}{}^a_{\rm qu}\frac{\rm d}{{\rm d}\tau}\xi^b_{\rm qu} + \hat{c}^\dagger_{i,{\rm qu}}\frac{\rm d}{{\rm d}\tau}\hat{c}^i_{\rm qu}\Big\}\right). \tag{3.23e}$$

In the same way as (3.18), the dynamical fields are decomposed into the background fields and the quantum fields

$$\psi^a_1(\tau) \ = \ \psi^a_{1,{\rm bg}}(\tau) + \psi^a_{1,{\rm qu}}(\tau)\,, \qquad \psi^a_{1,{\rm bg}}(\tau) \ = \ \frac{1}{\sqrt{2}}\big(\zeta^a + \overline{\eta}^a\big)\,, \tag{3.24a}$$

$$\xi^i_{\rm gh}(\tau) \ = \ \zeta^i_{\rm gh} + \hat{c}^i_{\rm qu}(\tau)\,, \qquad \overline{\xi}_{i,{\rm gh}}(\tau) \ = \ \overline{\eta}_{i,{\rm gh}} + \hat{c}^\dagger_{i,{\rm qu}}(\tau)\,. \tag{3.24b}$$

Notice that the metric in $S^{(0)}_1$ is given at the point z, not at the intermediate point x, while the metric, spin connection, and the fluxes in $S_{1,H}$ are given at the intermediate point x. In the same analogy as the $\mathcal{N}=2$ system, we introduce the bosonic and fermionic propagators. The propagators with respect to the bosonic quantum fields q^m and the ghost fields b^m, c^m and a^m are same as the ones (3.22) in the $\mathcal{N}=2$ system. Here we newly introduce the propagators with respect to the real fermion $\psi^a_{1,{\rm qu}}$ given by the combination with two Dirac fermions (3.24a). Since we have already introduced the propagators with respect to the Dirac (complex) fermions $\xi^a_{\rm qu}$, we can derive the propagators of $\psi^a_{1,{\rm qu}}$ in such a way as

$$\big\langle \psi^a_{1,{\rm qu}}(\sigma) \psi^b_{1,{\rm qu}}(\tau) \big\rangle \ = \ \frac{1}{2} \delta^{ab} \Big(\theta(\sigma - \tau) - \theta(\tau - \sigma) \Big) \ . \tag{3.25}$$

The propagator of ghost field $\hat{c}^i_{\rm gh}$ is also given as

$$\big\langle \hat{c}^i_{\rm qu}(\sigma) \hat{c}^{\dagger}_{j,{\rm qu}}(\tau) \big\rangle \ = \ \delta^i_j \, \theta(\sigma - \tau) \ . \tag{3.26}$$

4 Witten index in $\mathcal{N} = 1$ quantum mechanics

In this section we will discuss the Witten index in the $\mathcal{N} = 1$ quantum mechanical system derived from the path integral formalism. To obtain this, we will analyze Feynman path integral in terms of Feynman (dis)connected graphs. Since the form of the Witten index (or equivalently, the Dirac index) is same as the one of the chiral anomaly, we refer to the derivation of the chiral anomaly given in section *6.1* and *6.2* of [21].

4.1 Formulation

As mentioned before, by using the identification between the Clifford algebra on the target geometry and the anti-commutation relations of fermions in the quantum mechanics, we can describe the Dirac index equipped with the regulator $\mathscr{R}$ in terms of the transition element of $\mathcal{N} = 1$ quantum mechanics

$$\begin{aligned} \text{index} \not{D}(\hat{\omega}) \ &\equiv \ \lim_{\beta \to 0} {\rm Tr} \big\{ \Gamma_{(5)} {\rm e}^{-\beta \mathscr{R}} \big\} \ = \ \lim_{\beta \to 0} {\rm Tr} \big\{ (-1)^F {\rm e}^{-\frac{\beta}{\hbar} \widehat{\mathscr{H}}{}^1_H} \big\} \\ &= \ \lim_{\beta \to 0} \frac{(-i)^{D/2}}{2^{D/2}} \, {\rm Tr} \prod_{a=1}^D \big(\widehat{\varphi}^a + \widehat{\overline{\varphi}}{}^a \big) \, {\rm e}^{-\frac{\beta}{\hbar} \widehat{\mathscr{H}}{}^1_H} \ . \end{aligned} \tag{4.1}$$

Note that the chirality operator $\Gamma_{(5)}$ on the target geometry can be identified with the fermion number operator $(-1)^F$ in the $\mathcal{N} = 1$ quantum mechanics. Since the chirality operator is defined as $\Gamma_{(5)} = (-i)^{D/2} \Gamma^1 \Gamma^2 \cdots \Gamma^D$, the number operator $(-1)^F$ is replaced in terms of the fermion operators

$$\Gamma^a \ \equiv \ \sqrt{2} \psi^a_{1,\diamond} \ = \ \big(\widehat{\varphi}^a_{\diamond} + \widehat{\overline{\varphi}}{}^a_{\diamond} \big) \ , \quad \Gamma_{(5)} \ \equiv \ (-i)^{D/2} \prod_{a=1}^D \big(\widehat{\varphi}^a_{\diamond} + \widehat{\overline{\varphi}}{}^a_{\diamond} \big) \ . \tag{4.2}$$

Notice that the fermion $\psi^a_{2,\diamond}$, which is now included in the path integral measure while does not appear in the Hamiltonian, has dimension $2^{D/2}$. Then we should divide by $2^{D/2}$ from the formulation $(-i)^{D/2} \prod_{a=1}^D (\widehat{\varphi}^a_{\diamond} + \widehat{\overline{\varphi}}{}^a_{\diamond})$ by hand. (See the explanation in section *6.1* in [21] and we will find that this factor is canceled out via the fermionic measure computation.) From now on we omit the symbol

"$\diamond$". The symbol Tr in the above expression of the index is defined as

$$\mathrm{Tr}\,\mathcal{O} \;\equiv\; \int \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D} \big(\mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}_a\big)\, \mathrm{e}^{\overline{\zeta}\zeta} \,\langle\, x_0, \overline{\zeta}\,|\mathcal{O}|\, x_0, \zeta\,\rangle \;. \tag{4.3}$$

Then, inserting the complete set of the fermion coherent states (3.5), we obtain the explicit form of the Dirac index, i.e., the Witten index with respect to the $\mathcal{N}=1$ quantum mechanical path integral:

$$\begin{aligned}\mathrm{index}\not{D}(\hat{\omega}) \;=\; \lim_{\beta\to 0} \frac{(-i)^{D/2}}{2^{D/2}} \int \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D} \Big(\mathrm{d}\overline{\eta}_a \mathrm{d}\eta^a \, \mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}_a\Big)\, \mathrm{e}^{\overline{\zeta}\zeta} \,\langle\, \overline{\zeta}\,|\prod_{b=1}^{D} \big(\widehat{\varphi}^b + \widehat{\overline{\varphi}}{}^b\big)|\,\eta\,\rangle\, \mathrm{e}^{-\overline{\eta}\eta} \\ \times \langle\, x_0, \overline{\eta}\,|\exp\Big(-\frac{\beta}{\hbar}\widehat{\mathscr{H}}{}^1_H\Big)|\, x_0, \zeta\,\rangle \;. \end{aligned} \tag{4.4a}$$

Here the appearing transition element has already described in the previous section such as

$$\langle\, x_0, \overline{\eta}\,|\exp\Big(-\frac{\beta}{\hbar}\widehat{\mathscr{H}}{}^1_H\Big)|\, x_0, \zeta\,\rangle \;=\; \frac{1}{(2\pi\beta\hbar)^{D/2}}\, \mathrm{e}^{\overline{\eta}_a \zeta^a} \Big\langle \exp\Big(-\frac{1}{\hbar} S^{(\mathrm{int})}_{1,H}\Big)\Big\rangle \;, \tag{4.4b}$$

$$\begin{aligned} -\frac{1}{\hbar} S^{(\mathrm{int})}_{1,H} \;=&\; -\frac{1}{2\beta\hbar} \int_{-1}^{0} \mathrm{d}\tau \, \Big\{ g_{mn}(x) - g_{mn}(x_0) \Big\} \Big(\dot{q}^m \dot{q}^n + b^m c^n + a^m a^n \Big) \\ & - \frac{1}{2} \int_{-1}^{0} \mathrm{d}\tau \, \dot{q}^m \omega_{-mab}(x)\, \psi_1^{ab} - \frac{\beta\hbar}{24} \int_{-1}^{0} \mathrm{d}\tau \, (\mathrm{d}H)_{abcd}(x)\, \psi_1^{abcd} - \frac{\beta\hbar}{8} \int_{-1}^{0} \mathrm{d}\tau \, \mathcal{G}_1(x) \;, \end{aligned} \tag{4.4c}$$

where $x = x_0 + q$, $\omega_{-mab}(x) = \omega_{mab}(x) - H_{mab}(x)$ and $\psi_1^a = \psi_{1,\mathrm{bg}}^a + \psi_{1,\mathrm{qu}}^a(\tau)$. The functional $\mathcal{G}_1(x)$ is defined in (3.23d).

Now let us simplify the fermionic measure in the above form in order to approach to the useful form to compute the Witten index more explicitly. The fermionic terms are summarized as

$$\begin{aligned} \int \prod_{a=1}^{D} \Big(\mathrm{d}\overline{\eta}_a \mathrm{d}\eta^a \mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}_a\Big)\, \mathrm{e}^{\overline{\zeta}\zeta - \overline{\eta}\eta} \langle\, \overline{\zeta}\,|\prod_{b=1}^{D} \big(\widehat{\varphi}^b + \widehat{\overline{\varphi}}{}^b\big)|\,\eta\,\rangle \;=&\; \int \prod_{a=1}^{D} \Big(\mathrm{d}\overline{\eta}_a \mathrm{d}\eta^a \mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}_a\Big)\, \mathrm{e}^{\overline{\zeta}\zeta - \overline{\eta}\eta + \overline{\zeta}\eta} \prod_{b=1}^{D} \big(\eta^b + \overline{\zeta}{}^b\big) \\ =&\; \int \prod_{a=1}^{D} \Big(\mathrm{d}\overline{\eta}_a \mathrm{d}\eta^a \mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}_a\Big)\, \mathrm{e}^{\overline{\zeta}\zeta - \overline{\eta}\eta} \prod_{b=1}^{D} \big(\eta^b + \overline{\zeta}{}^b\big) \;. \end{aligned} \tag{4.5a}$$

The last factor becomes a fermionic delta function $\delta(\eta + \overline{\zeta})$, hence $\langle\, \overline{\zeta}\,|\,\eta\,\rangle = \mathrm{e}^{\overline{\zeta}\eta}$ can be replaced by unity. For the same reason, we rewrite other exponential factor in such a way as $\overline{\zeta}\zeta - \overline{\eta}\eta = -\frac{1}{2}(\eta - \overline{\zeta})(\zeta - \overline{\eta})$. Let us see the measure:

$$\prod_{a=1}^{D} \mathrm{d}\overline{\eta}_a \mathrm{d}\eta^a \mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}_a \;=\; \prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \cdot 2^D \mathrm{d}(\overline{\zeta} + \eta)^D \cdots \mathrm{d}(\overline{\zeta} + \eta)^1 \mathrm{d}(\eta - \overline{\zeta})^1 \cdots \mathrm{d}(\eta - \overline{\zeta})^D \;. \tag{4.5b}$$

Thus, combining the above two equations, we show

$$\begin{aligned} \int \prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \Big[2^D \mathrm{d}(\overline{\zeta} + \eta)^D \cdots \mathrm{d}(\overline{\zeta} + \eta)^1 \mathrm{d}(\eta - \overline{\zeta})^1 \cdots \mathrm{d}(\eta - \overline{\zeta})^D \Big]\, \mathrm{e}^{-\frac{1}{2}(\eta - \overline{\zeta})(\zeta - \overline{\eta})} \prod_b \big(\eta^b + \overline{\zeta}{}^b\big) \\ = \int \prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \prod_b \big(\zeta^b - \overline{\eta}{}^b\big) \;. \end{aligned} \tag{4.5c}$$

This is again the fermionic delta function, which annihilates the exponential factor $\mathrm{e}^{\overline{\eta}\zeta}$ from the Weyl-ordered Hamiltonian. We perform this fermionic delta function to the transition element. Generically we consider the following equation in the $\mathcal{N}=1$ system:

$$\int \prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \prod_b \big(\zeta^b - \overline{\eta}^b\big)\, \mathrm{e}^{\overline{\eta}\zeta} F\Big(\frac{\zeta+\overline{\eta}}{\sqrt{2}}\Big) \ = \ 2^{D/2} \int \prod_a \mathrm{d}\psi^a_{1,\mathrm{bg}}\, F(\psi^a_{1,\mathrm{bg}}) \, . \tag{4.5d}$$

The factor $2^{D/2}$ cancels the factor $2^{-D/2}$ in (4.4), which we introduced caused by the free fermion ψ^a_2. Next, rescaling the fermions ψ^a_1 by a factor $(\beta\hbar)^{-\frac{1}{2}}$ as $\psi^a_1 \to (\beta\hbar)^{-\frac{1}{2}}\psi^a_1$, we remove the $\beta\hbar$ dependence of the path integral measure. Here we show the Witten index (the Dirac index) in the path integral formalism:

$$\begin{aligned}
\mathrm{index}\not{D}(\hat{\omega}) \ &= \ \lim_{\beta\to 0} \frac{(-i)^{D/2}}{(2\pi)^{D/2}} \int \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D} \mathrm{d}\psi^a_{1,\mathrm{bg}} \Big\langle \exp\Big(- \frac{1}{\hbar} S^{(\mathrm{int})}_{1,H} \Big) \Big\rangle \, , && (4.6a) \\
-\frac{1}{\hbar} S^{(\mathrm{int})}_{1,H} \ &= \ -\frac{1}{\beta\hbar} \int_{-1}^{0} \mathrm{d}\tau\, \frac{1}{2} \Big\{ g_{mn}(x) - g_{mn}(x_0) \Big\} \Big(\dot{q}^m \dot{q}^n + b^m c^n + a^m a^n \Big) \\
&\quad - \frac{1}{2\beta\hbar} \int_{-1}^{0} \mathrm{d}\tau\, \dot{q}^m \omega_{-mab}(x) \big(\psi_{1,\mathrm{bg}} + \psi_{1,\mathrm{qu}} \big)^{ab} \\
&\quad - \frac{1}{24\beta\hbar} \int_{-1}^{0} \mathrm{d}\tau\, (\mathrm{d}H)_{abcd}(x)\, \big(\psi^a_{1,\mathrm{bg}} + \psi^a_{1,\mathrm{qu}} \big)^{abcd} - \frac{\beta\hbar}{8} \int_{-1}^{0} \mathrm{d}\tau\, \mathcal{G}_1(x) \, , && (4.6b)
\end{aligned}$$

where $x = x_0 + q$. In addition, all the bosonic and fermionic propagators are proportional to $\beta\hbar$:

$$\begin{aligned}
\big\langle q^m(\sigma) q^n(\tau) \big\rangle \ &= \ -\beta\hbar\, g^{mn}(x_0)\, \Delta(\sigma,\tau) \, , && (4.7a) \\
\big\langle q^m(\sigma) \dot{q}^n(\tau) \big\rangle \ &= \ -\beta\hbar\, g^{mn}(x_0) \Big(\sigma + \theta(\tau-\sigma) \Big) \, , && (4.7b) \\
\big\langle \dot{q}^m(\sigma) \dot{q}^n(\tau) \big\rangle \ &= \ -\beta\hbar\, g^{mn}(x_0) \Big(1 - \delta(\tau-\sigma) \Big) \, , && (4.7c) \\
\big\langle a^m(\sigma) a^n(\tau) \big\rangle \ &= \ \beta\hbar\, g^{mn}(x_0)\, \delta(\sigma-\tau) \, , && (4.7d) \\
\big\langle b^m(\sigma) c^n(\tau) \big\rangle \ &= \ -2\beta\hbar\, g^{mn}(x_0)\, \delta(\sigma-\tau) \, , && (4.7e) \\
\big\langle \psi^a_{1,\mathrm{qu}}(\sigma) \psi^b_{1,\mathrm{qu}}(\tau) \big\rangle \ &= \ \frac{1}{2}\beta\hbar\, \delta^{ab} \Big(\theta(\sigma-\tau) - \theta(\tau-\sigma) \Big) \, . && (4.7f)
\end{aligned}$$

The properties of these functions are seen in (C.1). In the end of the evaluation of the path integral, we should take a limit $\beta \to 0$. There are a number of comments to verify the path integral:

- Disconnected graphs should contribute to the functional integrals, called the Feynman amplitudes [18, 21].
- Graphs of higher order in $\beta\hbar$ do not contribute to Feynman amplitudes in the vanishing limit $\beta \to 0$.
- All the ghost fields might contribute to the Feynman graphs from two-loops level, which we should investigate if $(\mathrm{d}H)_{abcd}\psi^{abcd}_{1,\mathrm{bg}}$ appear in the graphs. (In this paper we do not analyze this non-vanishing case.)

- Terms linear in the quantum fields $\dot{q}^m$ do not contribute because of the periodic boundary condition $q^m(-1) = q^m(0) = 0$.

- Terms linear in the quantum fields q^m do not contribute because of the periodic boundary condition and the mean-value theorem (3.19a).

- Terms linear in $\psi^a_{1,\text{qu}}$ contribute because there are no restrictions on the quantum fermion fields except for $\xi^a_{\text{qu}}(-1) = \overline{\xi}^a_{\text{qu}}(0) = 0$. Notice that such terms only come from the expansion of $(\mathrm{d}H)_{abcd}(x)\,\psi_1^{abcd}$ (which does not appear in this paper).

- We could, for convenience, choose a frame with $\partial_m g_{pq}(x_0) = 0$, called the Riemann normal coordinate frame. Due to this we find $\partial_m e_n{}^a = \partial_m E_a{}^n = 0$, $\Gamma^p_{0nq}(x_0) = 0$ and $\omega_{mab}(x_0) = 0$ (notice that $\omega_{-mab}(x_0) = \omega_{mab}(x_0) - H_{mab}(x_0) \neq 0$). Notice, however that $\partial_p \partial_q e_m{}^a(x_0) \neq 0$, $\partial_m \omega_{nab}(x_0) \neq 0$ and so forth.

- The Feynman amplitudes should be independent of the target space metric, at least invariant under the rescale of the metric.

Note that we rewrite the derivative of the spin connection in such a way as

$$\begin{aligned}
\partial_n \omega_{-mab}(x_0) \int_{-1}^0 \mathrm{d}\tau\, \dot{q}^m q^n \;&=\; -\frac{1}{2}\Big(\partial_m \omega_{-nab}(x_0) - \partial_n \omega_{-mab}(x_0)\Big) \int_{-1}^0 \mathrm{d}\tau\, \dot{q}^m q^n \\
&=\; \frac{1}{2}\Big[R_{abmn}(\omega(x_0)) - 2\partial_{[m} H_{n]ab}(x_0)\Big] \int_{-1}^0 \mathrm{d}\tau\, q^m \dot{q}^n \\
&=\; \frac{1}{2}\Big[R_{mnab}(\omega(x_0)) - 2D_{[m}(\Gamma_0,\omega) H_{n]ab}(x_0)\Big] \int_{-1}^0 \mathrm{d}\tau\, q^m \dot{q}^n \\
&\equiv\; \frac{1}{2}\mathcal{R}_{mnab} \int_{-1}^0 \mathrm{d}\tau\, q^m \dot{q}^n \;,
\end{aligned} \tag{4.8}$$

where we used the symmetricity on the Riemann tensor without torsion $R_{abmn}(\omega) = R_{mnab}(\omega)$ and the periodicity of the bosonic quantum fields $q^m(0) = q^m(-1)$. Furthermore we also generalized the derivative to the covariant derivative because now we analyze on a point x_0 on which the torsion free connections vanish: $\Gamma^p_{0mn}(x_0) = \omega_{mab}(x_0) = 0$. For simplicity we express the torsion free covariant derivative $D_m(\Gamma_0,\omega)$ as $\widetilde{D}_m$.

Let us evaluate the functional integral in terms of the bosonic propagators (4.7) at the point x_0. The exponent $\langle \exp(-\frac{1}{\hbar} S^{(\text{int})}_{1,H}) \rangle$ contains both connected and disconnected Feynman graphs. First we analyze connected graphs, then we summarize them to obtain the products of connected graphs. Let us introduce the effective action W_H by $\mathrm{e}^{-\frac{1}{\hbar}W_H} = \langle \exp(-\frac{1}{\hbar} S^{(\text{int})}_{1,H}) \rangle$, which is expanded as

$$-\frac{1}{\hbar} W_H \;=\; \sum_{k=1}^{\infty} \frac{1}{k!} \Big\langle\!\!\Big\langle \Big(-\frac{1}{\hbar} S^{(\text{int})}_{1,H} \Big)^k \Big\rangle\!\!\Big\rangle \;, \tag{4.9}$$

where $\langle\!\langle \cdots \rangle\!\rangle$ indicates the value given only by the connected Feynman graphs.

For later discussions, it is also worth mentioning that the volume form and the Riemann curvature two-form are given in terms of the vielbein one-form $e^a = e_m{}^a \mathrm{d}x^m$ in the following way:

$$\mathrm{d}^{2n} x_0 \sqrt{g(x_0)}\, \mathcal{E}^{b_1 \cdots b_{2n}} \ = \ e^{b_1} \wedge \cdots \wedge e^{b_{2n}} \ . \tag{4.10}$$

Furthermore, we also find the following formula:

$$\int \prod_{a=1}^{D} \mathrm{d}\psi^a_{1,\mathrm{bg}}\, \psi^{a_1 \cdots a_D}_{1,\mathrm{bg}} \ = \ (-)^{D/2}\, \mathcal{E}^{a_1 a_2 \cdots a_D} \ . \tag{4.11}$$

The trace of the odd number of the curvature two-form vanishes because the permutation of the two-form is symmetric but the flip of the indices is anti-symmetric $\mathrm{tr}(R^{2k-1}) = 0$.

This is the generic form without any constraints on the flux H. In this paper we will impose the closed condition $\mathrm{d}H = 0$ for simplicity, although we should analyze this most generic form to obtain deeper information on a generic torsional geometry. Peeters and Waldron [18] have already analyzed a couple of four-dimensional torsional geometries with boundaries in terms of the above path integral formalism. An extension of their work to an analysis on higher dimensional torsional geometry with(out) boundary is an interesting, and important future work, which will give rise to a new insight on the string theory compactification in the presence of non-trivial fluxes.

4.2 Pontrjagin classes

4.2.1 Riemannian geometry: $H = \mathrm{d}H = 0$ case

In this case $S_1^{(\mathrm{int})}$ becomes much simpler than (4.6a) because there are no terms from H-flux. The spin connection ω_- is also reduced to ω. We also easily find that the terms equipped with higher derivatives carrying more than three bosonic quantum fields q^m always generate higher-loops Feynman graphs because of the absence of the tadpole graphs. Furthermore, the terms of order in $\beta\hbar$ do not contribute to the final result. Then we truncate $S_1^{(\mathrm{int})}$ in the following way:

$$-\frac{1}{\hbar} S_1^{(\mathrm{int})} \ = \ -\frac{1}{2\beta\hbar} R_{mn}(\omega(x_0)) \int_{-1}^{0} \mathrm{d}\tau\, q^m \dot{q}^n \ , \qquad R_{mn} \ \equiv \ \frac{1}{2} R_{mnab}(\omega(x_0))\, \psi^a_{1,\mathrm{bg}} \psi^b_{1,\mathrm{bg}} \ , \tag{4.12}$$

where we used (4.8) with $H = \mathrm{d}H = 0$. Then, the path integral form of the Witten index without H-flux is reduced to

$$\mathrm{index} \not{D}(\omega) \ = \ \lim_{\beta \to 0} \frac{(-i)^{D/2}}{(2\pi)^{D/2}} \int \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D} \mathrm{d}\psi^a_{1,\mathrm{bg}} \Big\langle \exp\Big(- \frac{1}{2\beta\hbar} R_{mn} \int_{-1}^{0} \mathrm{d}\tau\, q^m \dot{q}^n \Big) \Big\rangle \ . \tag{4.13}$$

Let us first evaluate the sum of connected graphs:

$$-\frac{1}{\hbar} W \ = \ \log \Big\langle \exp\Big(- \frac{1}{2\beta\hbar} R_{mn} \int_{-1}^{0} \mathrm{d}\tau\, q^m \dot{q}^n \Big) \Big\rangle$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!} \Big(- \frac{1}{2\beta\hbar} \Big)^k R_{m_1 n_1} \cdots R_{m_k n_k} \int_{-1}^0 \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_k \Big\langle\!\Big\langle \big(q^{m_1} \dot{q}^{n_1} \big)(\tau_1) \cdots \big(q^{m_k} \dot{q}^{n_k} \big)(\tau_k) \Big\rangle\!\Big\rangle \ . \quad (4.14)$$

Since the two indices in the Riemann tensors are anti-symmetric whereas the propagators are symmetric with respect to the exchanging of bosonic quantum fields, we easily find that the contraction at the same "time" τ_i yields a vanishing amplitude. We also know that the partial integration is allowed since $q^m(\tau_i) = 0$ at the end points. Then, there are $(k-1)!$ ways to contract k vertices and the symmetry of each vertex in both q yields a factor 2^{k-1}. Then we find that the effective action (4.14) is described as

$$\begin{aligned} -\frac{1}{\hbar} W \ &= \ \sum_{k=1}^{\infty} \frac{1}{k!} \Big(- \frac{1}{2\beta\hbar} \Big)^k (k-1)! \, 2^{k-1} \big(- \beta\hbar \big)^k \cdot R_{m_1 n_1} R_{m_2 n_2} \cdots R_{m_k n_k} \, g^{n_1 m_2} g^{n_2 m_3} \cdots g^{n_k m_1} \\ & \qquad \times \int_{-1}^0 \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_k \, \partial_{\tau_1} \Delta(\tau_1, \tau_2) \partial_{\tau_2} \Delta(\tau_2, \tau_3) \cdots \partial_{\tau_{k-1}} \Delta(\tau_{k-1}, \tau_k) \partial_{\tau_k} \Delta(\tau_k, \tau_1) \\ &\equiv \ \frac{1}{2} \sum_{k=2}^{\infty} \frac{1}{k} \mathrm{tr}(R^k) I_k \ , \qquad (4.15a) \\ I_k \ &\equiv \ \int_{-1}^0 \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_k \big[\tau_2 + \theta(\tau_1 - \tau_2) \big] \big[\tau_3 + \theta(\tau_2 - \tau_3) \big] \cdots \big[\tau_1 + \theta(\tau_k - \tau_1) \big] \ , \qquad (4.15b) \end{aligned}$$

where we used $\mathrm{tr} R^1 = 0$. By using the formula (see appendix *A.4* in [20])

$$\sum_{k=2}^{\infty} \frac{y^k}{k} I_k \ = \ \log \frac{y/2}{\sinh(y/2)} \ = \ -\frac{1}{3!} \Big(\frac{y}{2} \Big)^2 + \cdots \ , \quad (4.16)$$

we summarize the form of the effective action

$$-\frac{1}{\hbar} W \ = \ \frac{1}{2} \mathrm{tr} \log \Big(\frac{R/2}{\sinh(R/2)} \Big) \ . \quad (4.17)$$

Furthermore, in order to remove the overall factor in front of the path integral (4.4), we rescale the background fermions $\psi^a_{1,\mathrm{bg}} \to \sqrt{\frac{-i}{2\pi}} \psi^a_{1,\mathrm{bg}}$. Then we obtain the path integral form of the Witten index in such a way as

$$\begin{aligned} \mathrm{index} \not{D}(\omega) \ &= \ \int \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D} \mathrm{d}\psi^a_{1,\mathrm{bg}} \ \exp \left[\frac{1}{2} \mathrm{tr} \log \left(\frac{-iR/4\pi}{\sinh(-iR/4\pi)} \right) \right] \ , \qquad (4.18a) \\ \mathrm{tr}(R^k) \ &= \ R_{m_1 n_1} R_{m_2 n_2} \cdots R_{m_k n_k} \, g^{n_1 m_2} g^{n_2 m_3} \cdots g^{n_k m_1} \ . \qquad (4.18b) \end{aligned}$$

Due to the property of $\mathrm{tr}(R^k)$, this value becomes zero when $D = 4k+2$. Let us simplify the formula (4.18) by integrating the background fermion $\psi^a_{1,\mathrm{bg}}$ of (4.18) with noticing the formulae (4.10) (in particular, eq.(4.11)):

$$\mathrm{index} \not{D}(\omega) \ = \ \int_{\mathcal{M}} \exp \left[\frac{1}{2} \mathrm{tr} \log \left(\frac{iR/4\pi}{\sinh(iR/4\pi)} \right) \right] \ , \qquad R_{mn} \ = \ \frac{1}{2} R_{mnab}(\omega) \, e^a \wedge e^b \ . \quad (4.19)$$

This is the well-known form of Dirac index on a Riemannian geometry $\mathcal{M}$. The integrand is called the (Dirac) $\hat{\mathcal{A}}$-genus.

4.2.2 Torsional geometry: $H \neq 0$, $\mathrm{d}H = 0$ case

This case is still simple. Since there does not exist an interaction term with single quantum fermion, all the Feynman amplitudes are of order in $(\beta\hbar)^k$, where k is a non-negative integer. Thus, since we are interested only in the amplitudes of order in $(\beta\hbar)^0$ which remain in the vanishing limit $\beta \to 0$, we can neglect the last term in (4.6b) which yields graphs of higher order in $\beta\hbar$. We can also neglect the interaction terms including more than three quantum fields which yield more than two-loops graphs. Thus we truncate $S_{1,H}^{(\text{int})}$ carrying only two bosonic and fermionic quantum fields to

$$-\frac{1}{\hbar}S_{1,H}^{(\text{int})} \;=\; -\frac{1}{2\beta\hbar}\mathcal{R}_{mn}\int_{-1}^{0}\mathrm{d}\tau\, q^m \dot{q}^n + \frac{1}{\beta\hbar}H_{mab}\,\psi_{1,\text{bg}}^a \int_{-1}^{0}\mathrm{d}\tau\, \dot{q}^m \psi_{1,\text{qu}}^b \;, \tag{4.20a}$$

where we used (4.8) with $\mathrm{d}H = 0$ and

$$\mathcal{R}_{mn} \;\equiv\; \frac{1}{2}\Big\{ R_{mnab}(\omega(x_0)) - 2\widetilde{D}_{[m}H_{n]ab} \Big\} \psi_{1,\text{bg}}^a \psi_{1,\text{bg}}^b \;. \tag{4.20b}$$

We can easily find that $\mathrm{index}\not{D} = 0$ when $D = 2n+1$ because odd number of $\mathrm{d}\psi_{1,\text{bg}}^a$ gives rise to odd number of $\psi_{1,\text{qu}}^a$ in Feynman graph, which always yields zero amplitude. Furthermore we can also claim that $\mathrm{index}\not{D}\big|_{D=2n} = 0$ in which the anti-symmetrized indices in R_{abcd} and H_{abc} are contracted with which carries symmetric tensors via quantum propagators.

The effective action, or the functional integral of the connected graphs are given in terms of (4.9):

$$\begin{aligned}
-\frac{1}{\hbar}W_H \;&=\; \sum_{N=1}^{\infty}\frac{1}{N!}\Big\langle\!\!\Big\langle \Big(-\frac{1}{\hbar}S_1^{(\text{int})}\Big)^N \Big\rangle\!\!\Big\rangle \\
&=\; \sum_{N=1}^{\infty}\frac{1}{N!}\sum_{k=0}^{N}\frac{N!}{k!(N-k)!}\Big\langle\!\!\Big\langle \Big(-\frac{1}{2\beta\hbar}\mathcal{R}_{mn}\int_{-1}^{0}\mathrm{d}\tau\, q^m\dot{q}^n\Big)^k \Big(\frac{1}{\beta\hbar}H_{pab}\,\psi_{1,\text{bg}}^a\int_{-1}^{0}\mathrm{d}\sigma\,\dot{q}^p\psi_{1,\text{qu}}^b\Big)^{N-k} \Big\rangle\!\!\Big\rangle \\
&\sim\; \sum_{N=1}^{\infty}\sum_{k=0}^{N}\delta_{2\ell,N-k}\Big(-\frac{1}{2\beta\hbar}\Big)^N \frac{(-2)^{2\ell}}{k!(2\ell)!}\,\mathcal{R}_{m_1n_1}\cdots\mathcal{R}_{m_kn_k}H_{p_1a_1b_1}\cdots H_{p_{2\ell}a_{2\ell}b_{2\ell}}\,\psi_{1,\text{bg}}^{a_1}\cdots\psi_{1,\text{bg}}^{a_{2\ell}} \\
&\qquad \times \int_{-1}^{0}\mathrm{d}\tau_1\cdots\mathrm{d}\tau_k\int_{-1}^{0}\mathrm{d}\sigma_1\cdots\mathrm{d}\sigma_{2\ell}\,\Big\langle\!\!\Big\langle \prod_{i=1}^{k}(q^{m_i}\dot{q}^{n_i})(\tau_i)\prod_{j=1}^{2\ell}(\dot{q}^{p_j}\psi^{b_j})(\sigma_j)\Big\rangle\!\!\Big\rangle \;,
\end{aligned} \tag{4.21}$$

where we abbreviated $\psi_{1,\text{qu}}^a \equiv \psi^a$. Notice that the vertex $H_{pab}\psi_{1,\text{bg}}^a\int\dot{q}^p\psi^b$ should be paired and be contracted with each other in the amplitude because an amplitude including an isolated, uncontracted quantum fermionic field is forbidden. Then we should introduce the even number $N-k = 2\ell$ in the above equation, and we extracted only the terms which contribute to the Feynman graphs with non-trivial values in the limit $\beta \to 0$. In addition, the contraction between a pair of second terms in (4.20a) always vanishes

$$\begin{aligned}
&H_{p_ia_ib_i}H_{p_ja_jb_j}\,\psi_{1,\text{bg}}^{a_i}\psi_{1,\text{bg}}^{a_j}\int\mathrm{d}\sigma_i\int\mathrm{d}\sigma_j\,\dot{q}^{p_i}(\sigma_i)\dot{q}^{p_j}(\sigma_j)\cdot\big(\underbrace{\psi^{b_i}(\sigma_i)\psi^{b_j}}(\sigma_j)\big) \\
&\qquad\qquad \sim\; H_{p_ia_ib_i}H_{p_ja_j}{}^{b_i}\,\psi_{1,\text{bg}}^{a_i}\psi_{1,\text{bg}}^{a_j}\int\mathrm{d}\sigma_i\,\dot{q}^{p_i}\dot{q}^{p_j}(\sigma_i) \;=\; 0 \;,
\end{aligned} \tag{4.22}$$

because the two indices p_i and p_j are symmetric, while the indices a_i and a_j are anti-symmetric under the contraction of two quantum fermionic fields $\underbrace{\psi^{b_i}(\sigma_i)\psi^{b_j}}(\sigma_j) = \frac{1}{2}\beta\hbar\delta^{b_i b_j}(\theta(\sigma_i - \sigma_j) - \theta(\sigma_j - \sigma_i))$. Thus, we can remove the second term in (4.20a) and reduce the path integral. Then the effective action (4.21) is extremely simplified:

$$-\frac{1}{\hbar}W_H \;=\; \sum_{k=1}^{\infty}\frac{1}{k!}\Big(-\frac{1}{2\beta\hbar}\Big)^k \mathcal{R}_{m_1 n_1}\cdots\mathcal{R}_{m_k n_k}\int_{-1}^{0}\mathrm{d}\tau_1\cdots\mathrm{d}\tau_k\Big\langle\!\!\Big\langle (q^{m_1}\dot{q}^{n_1})(\tau_1)\cdots(q^{m_k}\dot{q}^{n_k})(\tau_k)\Big\rangle\!\!\Big\rangle\;. \quad (4.23)$$

This is exactly same equation as (4.14) except for the Riemann curvature tensors. Then, after the rescaling of the background fermion fields, the result is given in the following way:

$$\mathrm{index}\not{D}(\hat{\omega}) \;=\; \int \mathrm{d}^D x_0\sqrt{g(x_0)}\prod_{a=1}^{D}\mathrm{d}\psi^a_{1,\mathrm{bg}}\;\exp\left[\frac{1}{2}\mathrm{tr}\log\left(\frac{-i\mathcal{R}/4\pi}{\sinh(-i\mathcal{R}/4\pi)}\right)\right]\,, \qquad (4.24\mathrm{a})$$

$$\mathcal{R}_{mn} \;=\; \frac{1}{2}\mathcal{R}_{mnab}\,\psi^a_{1,\mathrm{bg}}\psi^b_{1,\mathrm{bg}} \;\equiv\; \frac{1}{2}\Big\{R_{mnab}(\omega(x_0)) - 2\widetilde{D}_{[m}H_{n]ab}(x_0)\Big\}\psi^a_{1,\mathrm{bg}}\psi^b_{1,\mathrm{bg}}\;. \qquad (4.24\mathrm{b})$$

The most significant point is that we obtained the additional second term $\widetilde{D}_{[m}H_{n]ab}$ in the "curvature two-form" $\mathcal{R}$, which did not appear in the Mavromatos' work [16]. The reason is that *the torsional spin connection was fixed $\omega_{-mab}(x_0) = 0$ with $g_{mn}(x_0) = \delta_{mn}$ and $\partial_p g_{mn}(x_0) = 0$ simultaneously, and the Lagrangian was also expanded in a similar way as the Riemann normal coordinate expansion*. However, this simultaneous fixing becomes **inconsistent** because the torsion (in this case, the H-flux) also vanishes at the same point x_0. In addition, Peeters and Waldron's work [18] should also be improved. The third term in the equation (*2.17*) in [18] should not appear in the correct expansion.

Let us further integrate the background fermion $\psi^a_{1,\mathrm{bg}}$ of (4.24) in the same analogy as (4.19):

$$\mathrm{index}\not{D}(\hat{\omega}) \;=\; \int_{\mathcal{M}}\exp\left[\frac{1}{2}\mathrm{tr}\log\left(\frac{i\mathcal{R}/4\pi}{\sinh(i\mathcal{R}/4\pi)}\right)\right]\,, \qquad (4.25\mathrm{a})$$

$$\mathcal{R}_{mn} \;=\; \frac{1}{2}\mathcal{R}_{mnab}(\omega)\,e^a\wedge e^b \;\equiv\; \frac{1}{2}\Big\{R_{mnab}(\omega) - 2\widetilde{D}_{[m}H_{n]ab}\Big\}e^a\wedge e^b\;. \qquad (4.25\mathrm{b})$$

As an typical example, let us describe the modified first Pontrjagin class (cf. Mavromatos [16], Yajima [17], and Peeters and Waldron [18]):

$$\begin{aligned}\mathrm{index}\not{D}(\hat{\omega}) \;=\;& \frac{1}{192\pi^2}\int_{\mathcal{M}_4}\mathrm{tr}\big\{R(\omega)\wedge R(\omega)\big\} \\ &+\frac{1}{192\pi^2}\int_{\mathcal{M}_4}(\mathrm{vol.})\,\mathcal{E}^{abcd}\left(R^{a_2b_2}{}_{ab}(\omega)\,\widetilde{D}_{[a_2}H_{b_2]cd} + \delta^{a_1b_2}\delta^{a_2b_1}\,\widetilde{D}_{[a_1}H_{b_1]ab}\widetilde{D}_{[a_2}H_{b_2]cd}\right)\,,\end{aligned} \qquad (4.26)$$

where we used $e^a\wedge e^b\wedge e^c\wedge e^d = (\mathrm{vol.})\,\mathcal{E}^{abcd}$. As in [18], let us introduce a dual field of H_{abc} in four-dimensional Euclidean space:

$$\widetilde{H}_a \;\equiv\; \frac{1}{k}\mathcal{E}_{abcd}\,H^{bcd}\,, \qquad H_{abc} \;=\; -\frac{k}{3!}\mathcal{E}_{abcd}\widetilde{H}^d\;. \qquad (4.27)$$

Then the covariant derivatives are rewritten in a way as

$$\widetilde{D}_{[a_2} H_{b_2]cd} \ = \ -\frac{k}{6} \mathcal{E}_{cde[a_2} \, \widetilde{D}_{b_2]} \widetilde{H}^e \ . \tag{4.28}$$

We substitute this into the above equation:

$$\begin{aligned} \text{index} \not{D}(\hat{\omega}) \ &= \ \frac{1}{192\pi^2} \int_{\mathcal{M}_4} \text{tr}\big\{ R(\omega) \wedge R(\omega) \big\} + \frac{1}{192\pi^2} \int_{\mathcal{M}_4} (\text{vol.}) \bigg(\frac{k}{6} \mathcal{E}_{a_2 cde} \, \mathcal{E}^{abcd} \, R^{a_2 b_2}{}_{ab}(\omega) \, \widetilde{D}_{b_2} \widetilde{H}^e \bigg) \\ & \quad + \frac{1}{192\pi^2} \int_{\mathcal{M}_4} (\text{vol.}) \bigg(\frac{k^2}{36} \delta^{a_1 b_2} \delta^{a_2 b_1} \, \mathcal{E}^{abcd} \, \mathcal{E}_{abe[a_1} \, \widetilde{D}_{b_1]} \widetilde{H}^e \, \mathcal{E}_{cdf[a_2} \, \widetilde{D}_{b_2]} \widetilde{H}^f \bigg) \ . \end{aligned} \tag{4.29}$$

Let us simplify more:

$$\mathcal{E}_{a_2 cde} \, \mathcal{E}^{abcd} \, R^{a_2 b_2}{}_{ab}(\omega) \, \widetilde{D}_{b_2} \widetilde{H}^e \ = \ 2 \widetilde{D}_b \Big(R^{ab}{}_{ae}(\omega) \, \widetilde{H}^e \Big) - 2 \widetilde{H}^e \, \widetilde{D}_b R^{ab}{}_{ae}(\omega) \ , \tag{4.30a}$$

$$\delta^{a_1 b_2} \delta^{a_2 b_1} \, \mathcal{E}^{abcd} \, \mathcal{E}_{abe[a_1} \, \widetilde{D}_{b_1]} \widetilde{H}^e \, \mathcal{E}_{cdf[a_2} \, \widetilde{D}_{b_2]} \widetilde{H}^f \ = \ -\mathcal{E}^{abcd} \, \widetilde{D}_a \widetilde{H}_b \, \widetilde{D}_c \widetilde{H}_d \ . \tag{4.30b}$$

Furthermore we introduce the "field strength of the dual field $\widetilde{H}_a$" in such a way as

$$\widetilde{F}_{ab}(\widetilde{H}) \ \equiv \ \widetilde{D}_a \widetilde{H}_b - \widetilde{D}_b \widetilde{H}_a \ = \ 2 \widetilde{D}_{[a} \widetilde{H}_{b]} \ , \qquad \widetilde{F}(\widetilde{H}) \ \equiv \ \frac{1}{2} \widetilde{F}_{ab}(\widetilde{H}) \, e^a \wedge e^b \ , \tag{4.30c}$$

$$\widetilde{D}_b \widetilde{H}^b \ = \ \frac{1}{k} \mathcal{E}^{bcde} \widetilde{D}_b H_{cde} \ = \ \frac{1}{4k} \mathcal{E}^{bcde} (\mathrm{d}H)_{bcde} \ = \ 0 \ . \tag{4.30d}$$

In addition we derive the following equation from the second Bianchi identity $0 = \nabla_{[m} R^{np}{}_{qr]}(\Gamma_0)$ as in (A.4b):

$$\begin{aligned} 0 \ &= \ \delta^c_a \Big(\widetilde{D}_b R^{ab}{}_{ce}(\omega) + \widetilde{D}_c R^{ab}{}_{eb}(\omega) + \widetilde{D}_e R^{ab}{}_{bc}(\omega) \Big) \ = \ \widetilde{D}_b R^{ab}{}_{ae}(\omega) + \widetilde{D}_a R^{ab}{}_{eb}(\omega) + \widetilde{D}_e R^{ab}{}_{ba}(\omega) \\ &= \ 2 \widetilde{D}_b R^{ab}{}_{ae}(\omega) - \widetilde{D}_e R(\omega) \ . \end{aligned} \tag{4.30e}$$

Substituting them into (4.29), we obtain

$$\begin{aligned} \text{index} \not{D}(\hat{\omega}) \ &= \ \frac{1}{192\pi^2} \int_{\mathcal{M}_4} \text{tr}\big\{ R(\omega) \wedge R(\omega) \big\} - \frac{k^2}{432\pi^2} \int_{\mathcal{M}_4} \widetilde{F}(\widetilde{H}) \wedge \widetilde{F}(\widetilde{H}) \\ & \quad + \frac{k}{576\pi^2} \int_{\mathcal{M}_4} (\text{vol.}) \, \widetilde{D}_b \Big\{ R^{ab}{}_{ae}(\omega) \, \widetilde{H}^e - \frac{1}{2} R(\omega) \, \widetilde{H}^b \Big\} \ . \end{aligned} \tag{4.31}$$

5 $\mathcal{N} = 1$ quantum mechanics for internal gauge symmetry

In this section we will focus on the gauge field and the invariant polynomial derived from the path integral. The transition element is described in terms of the quantum Hamiltonian in (2.6). Since the $\hat{c}$-ghost field in (2.6) are independent of the other fields, the path integral of this $\hat{c}$-ghost can be evaluated on a flat geometry and can be applied to an arbitrary curved geometry. Thus let us first formulate the path integral of this ghost field on a flat geometry, and we apply this result on the computation on a generic curved geometry. Here we again follow the convention in [21].

5.1 Formulation

The Dirac index is given by the Witten index in a following way:

$$\mathrm{index}\not{D}(\hat{\omega},A) \equiv \lim_{\beta\to 0}\mathrm{Tr}'\big\{(-1)^F \mathrm{e}^{-\frac{\beta}{\hbar}\widehat{\mathscr{H}}{}^1_H}\big\} = \lim_{\beta\to 0}\frac{(-i)^{D/2}}{2^{D/2}}\mathrm{Tr}\prod_{a=1}^{D}\big(\widehat{\varphi}^a + \widehat{\overline{\varphi}}{}^a\big)P_{\mathrm{gh}}\,\mathrm{e}^{-\frac{\beta}{\hbar}\widehat{\mathscr{H}}{}^1_H}\ , \tag{5.1a}$$

$$P_{\mathrm{gh}} \equiv : x\,\mathrm{e}^{-x} : \ , \qquad x \equiv \hat{c}_i^{\dagger}\hat{c}^i\ , \tag{5.1b}$$

where we expressed the trace with prime in order to evaluate the trace only over the one-particle ghost sector. We also introduce the one-particle ghost "projection operator" P_{gh} instead of the trace with prime. We should also define the completeness relation of the fermionic states as

$$I_{\mathrm{gh}} \equiv \int\prod_{i=1}^{\dim R}\mathrm{d}\overline{\eta}_{i,\mathrm{gh}}\mathrm{d}\eta^i_{\mathrm{gh}}\,|\,\eta_{\mathrm{gh}}\,\rangle\,\mathrm{e}^{-\overline{\eta}_{\mathrm{gh}}\cdot\eta_{\mathrm{gh}}}\langle\,\overline{\eta}_{\mathrm{gh}}\,|\ , \qquad I_{\mathrm{f}} \equiv \int\prod_{a=1}^{D}\mathrm{d}\overline{\eta}_{a,\mathrm{f}}\mathrm{d}\eta^a_{\mathrm{f}}\,|\,\eta_{\mathrm{f}}\,\rangle\,\mathrm{e}^{-\overline{\eta}_{\mathrm{f}}\cdot\eta_{\mathrm{f}}}\langle\,\overline{\eta}_{\mathrm{f}}\,|\ . \tag{5.2}$$

The trace formulae for the ghost and physical fermionic states are also independently defined by

$$\mathrm{tr}_{\mathrm{gh}}\mathcal{O} \equiv \int\prod_{i=1}^{\dim R}\mathrm{d}\chi^i_{\mathrm{gh}}\mathrm{d}\overline{\chi}_{i,\mathrm{gh}}\,\mathrm{e}^{\overline{\chi}_{\mathrm{gh}}\cdot\chi_{\mathrm{gh}}}\langle\,\overline{\chi}_{\mathrm{gh}}\,|\mathcal{O}|\,\chi_{\mathrm{gh}}\,\rangle\ , \qquad \mathrm{tr}_{\mathrm{f}}\mathcal{O} \equiv \int\prod_{a=1}^{D}\mathrm{d}\chi^a_{\mathrm{f}}\mathrm{d}\overline{\chi}_{a,\mathrm{f}}\,\mathrm{e}^{\overline{\chi}_{\mathrm{f}}\cdot\chi_{\mathrm{f}}}\langle\,\overline{\chi}_{\mathrm{f}}\,|\mathcal{O}|\,\chi_{\mathrm{f}}\,\rangle\ . \tag{5.3}$$

In a usual case this trace formula gives the anti-periodic boundary condition on the fermion. The fermion number operator $(-1)^F$, which acts on the physical fermion states, flips the condition to the periodic boundary condition (see section *2.4* in [21]). By using these formulae, we rewrite the Dirac index given by (5.1):

$$\begin{aligned}\mathrm{index}\not{D}(\hat{\omega},A) = \lim_{\beta\to 0}\frac{(-i)^{D/2}}{2^{D/2}}&\int\mathrm{d}^D x_0\sqrt{g(x_0)}\\ &\times\mathrm{tr}_{\mathrm{f}}\,\mathrm{tr}_{\mathrm{gh}}\big\langle\,x_0,\overline{\chi}_{\mathrm{gh}},\overline{\chi}_{\mathrm{f}}\,\big|\prod_{a=1}^{D}\big(\widehat{\varphi}^a+\widehat{\overline{\varphi}}{}^a\big)P_{\mathrm{gh}}I_{\mathrm{gh}}I_{\mathrm{f}}\,\mathrm{e}^{-\frac{\beta}{\hbar}\widehat{\mathscr{H}}{}^1_H}\big|\,x_0,\chi_{\mathrm{f}},\chi_{\mathrm{gh}}\,\big\rangle\ .\end{aligned} \tag{5.4}$$

Of course the ghost Hilbert space and the physical fermion Hilbert space are independent of each other. Then these completeness relation act on the individual spaces without any interruption. Now let us evaluate the trace in the ghost sector:

$$\begin{aligned}&\mathrm{tr}_{\mathrm{gh}}\langle\,\overline{\chi}_{\mathrm{gh}}\,|P_{\mathrm{gh}}I_{\mathrm{gh}}\mathrm{e}^{-\frac{\beta}{\hbar}\widehat{\mathscr{H}}{}^1_H}|\,\chi_{\mathrm{gh}}\,\rangle\\ &\quad= \int\prod_i\mathrm{d}\chi^i_{\mathrm{gh}}\mathrm{d}\overline{\chi}_{i,\mathrm{gh}}\,\mathrm{e}^{\overline{\chi}_{\mathrm{gh}}\cdot\chi_{\mathrm{gh}}}\prod_j\mathrm{d}\overline{\eta}_{j,\mathrm{gh}}\mathrm{d}\eta^j_{\mathrm{gh}}\,\mathrm{e}^{-\overline{\eta}_{\mathrm{gh}}\cdot\eta_{\mathrm{gh}}}\,\langle\,\overline{\chi}_{\mathrm{gh}}\,|P_{\mathrm{gh}}|\,\eta_{\mathrm{gh}}\,\rangle\langle\,\overline{\eta}_{\mathrm{gh}}\,|\mathrm{e}^{-\frac{\beta}{\hbar}\widehat{\mathscr{H}}{}^1_H}|\,\chi_{\mathrm{gh}}\,\rangle\ .\end{aligned} \tag{5.5}$$

Since $P_{\mathrm{gh}} =: x\mathrm{e}^{-x}:$ projects the ghost coherent state $|\,\eta_{\mathrm{gh}}\,\rangle$ onto its one-particle part $P_{\mathrm{gh}}|\,\eta_{\mathrm{gh}}\,\rangle = c_i^{\dagger}\eta^i_{\mathrm{gh}}|\,0\,\rangle$, the matrix element of the ghost projection operator P_{gh} is easily computed and yields

$$\langle\,\overline{\chi}_{\mathrm{gh}}\,|P_{\mathrm{gh}}|\,\eta_{\mathrm{gh}}\,\rangle = \sum_{i=1}^{\dim R}\overline{\chi}_{i,\mathrm{gh}}\,\eta^i_{\mathrm{gh}} = \overline{\chi}_{\mathrm{gh}}\cdot\eta_{\mathrm{gh}}\ . \tag{5.6}$$

Then we can integrate out the ghost variables η^i_{gh} and $\overline{\chi}_{i,\text{gh}}$ and define a new kind of projection operator in the following way:

$$\int \prod_i \text{d}\eta^i_{\text{gh}} \text{d}\overline{\chi}_{i,\text{gh}} \, \text{e}^{\overline{\chi}_{\text{gh}}\cdot\chi_{\text{gh}} - \overline{\eta}_{\text{gh}}\cdot\eta_{\text{gh}}} \, \langle\, \overline{\chi}_{\text{gh}} \,|P_{\text{gh}}|\, \eta_{\text{gh}} \,\rangle \;=\; \sum_{i=1}^{\dim R} \prod_{\ell\neq i} \Big(\overline{\eta}_{\ell,\text{gh}} \chi^{\ell}_{\text{gh}} \Big) \;\equiv\; P^{\text{gh}}_{\overline{\eta},\chi} \, . \tag{5.7}$$

This operator annihilates all terms containing more than two ghost fields $\overline{\eta}_{\text{gh}}$ and χ_{gh}. Because of this we interpret this operator as a kind of "projection operator" onto terms which are linear in $\overline{\eta}_{\text{gh}}$ and χ_{gh}, and onto terms independent of any ghost fields.

By using (4.5), (5.7) and (3.23), and rescaling physical fermions as $\psi^a_1 \to (\beta\hbar)^{-\frac{1}{2}}\psi^a_1$, while keeping the scale of the ghost fields unchanged, we can evaluate the Dirac index (5.1):

$$\text{index}\not{D}(\hat{\omega}, A) \;=\; \lim_{\beta\to 0} \frac{(-i)^{D/2}}{(2\pi)^{D/2}} \int \text{d}^D x_0 \sqrt{g(x_0)} \prod_{i=1}^{\dim R} \text{d}\chi^i_{\text{gh}} \text{d}\overline{\eta}_{i,\text{gh}} \, P^{\text{gh}}_{\overline{\eta},\chi} \, \text{e}^{\overline{\eta}_{\text{gh}}\cdot\chi_{\text{gh}}} \prod_{a=1}^{D} \text{d}\psi^a_{1,\text{bg}} \big\langle \text{e}^{-\frac{1}{\hbar} S^{(\text{int})}_{1,H}} \big\rangle \, , \tag{5.8a}$$

$$\begin{aligned} -\frac{1}{\hbar} S^{(\text{int})}_{1,H} \;=&\; -\frac{1}{2\beta\hbar} \int_{-1}^{0} \text{d}\tau \, \Big\{ g_{mn}(x) - g_{mn}(x_0) \Big\} \Big(\dot{q}^m \dot{q}^n + b^m c^n + a^m a^n \Big) \\ & - \frac{1}{2\beta\hbar} \int_{-1}^{0} \text{d}\tau \, \dot{q}^m \omega_{-mab}(x) \, \psi^{ab}_1 - \frac{1}{24\beta\hbar} \int_{-1}^{0} \text{d}\tau \, (\text{d}H)_{abcd}(x) \, \psi^{abcd}_1 - \frac{\beta\hbar}{8} \int_{-1}^{0} \text{d}\tau \, \mathcal{G}_1(x) \\ & - \int_{-1}^{0} \text{d}\tau \, \dot{q}^m A^{\alpha}_m(x) \, \big(\overline{\xi}_{\text{gh}} \, T_{\alpha} \, \xi_{\text{gh}} \big) + \frac{1}{2} \int_{-1}^{0} \text{d}\tau \, F^{\alpha}_{ab}(x) \; \psi^a_1 \psi^b_1 \, \big(\overline{\xi}_{\text{gh}} \, T_{\alpha} \, \xi_{\text{gh}} \big) \, , \end{aligned} \tag{5.8b}$$

with $x = x_0 + q$ and the boundary conditions $q^m(-1) = q^m(0) = 0$, $\int_{-1}^{0} \text{d}\tau \, q^m(\tau) = 0$, and

$$\psi^a_{1,\text{qu}}(-1) \;=\; \frac{1}{\sqrt{2}} \overline{\xi}^a_{\text{qu}}(-1) \, , \quad \psi^a_{1,\text{qu}}(0) \;=\; \frac{1}{\sqrt{2}} \xi^a_{\text{qu}}(0) \, , \quad \hat{c}^i_{\text{qu}}(-1) \;=\; \hat{c}^{\dagger}_{i,\text{qu}}(0) \;=\; 0 \, . \tag{5.9}$$

In addition we can rewrite the expansion of gauge field in such a way as

$$\begin{aligned} \partial_n A^{\alpha}_m(x_0) \int_{-1}^{0} \text{d}\tau \, \dot{q}^m q^n \;=&\; -\frac{1}{2} \Big(\partial_m A^{\alpha}_n(x_0) - \partial_n A^{\alpha}_m(x_0) \Big) \int_{-1}^{0} \text{d}\tau \, \dot{q}^m q^n \\ =&\; \frac{1}{2} \Big[F^{\alpha}_{mn}(x_0) - f^{\alpha}{}_{\beta\gamma} \, A^{\beta}_m(x_0) A^{\gamma}_n(x_0) \Big] \int_{-1}^{0} \text{d}\tau \, q^m \dot{q}^n \\ \equiv&\; \frac{1}{2} \mathcal{F}^{\alpha}_{mn}(x_0) \int_{-1}^{0} \text{d}\tau \, q^m \dot{q}^n \, , \end{aligned} \tag{5.10a}$$

$$F^{\alpha}_{mn}(x_0) \;=\; \partial_m A^{\alpha}_n(x_0) - \partial_n A^{\alpha}_m(x_0) + f^{\alpha}{}_{\beta\gamma} \, A^{\beta}_m(x_0) A^{\gamma}_n(x_0) \, , \tag{5.10b}$$

where F^{α}_{mn} is the field strength of the gauge field and $f^{\alpha}{}_{\beta\gamma}$ is the structure constant of the gauge group. Notice that the ghost fermions ξ_{gh} and $\overline{\xi}_{\text{gh}}$ obey the anti-periodic boundary condition, while the physical fermions ξ_{f} and $\overline{\xi}_{\text{f}}$ follow the periodic boundary condition because of the insertion of $(-1)^F$. This indicates that any closed-loop graphs of the ghost fields yield zero amplitudes and that only tree graphs contribute to non-vanishing amplitudes. Because of this, disconnected graphs with respect to the $\hat{c}$-ghost amplitudes does not appear in this path integral transition element. This statement is quite strong.

5.2 Chern character

5.2.1 Chern character on flat geometry without H-flux

Let us first consider the simplest system on a flat geometry with vanishing flux $H = \mathrm{d}H = 0$. In this case there are no (background) interaction terms which carries negative powers of $\beta\hbar$, contractions of any physical fields q^m and $\psi^a_{1,\mathrm{qu}}$ become irrelevant under the vanishing limit $\beta \to 0$. Then we can neglect the term linear in $A^\alpha_m(x_0 + q)$ and the path integral (5.8) is reduced to

$$\mathrm{index}\not{D}(A) \;=\; \lim_{\beta\to 0} \frac{(-i)^{D/2}}{(2\pi)^{D/2}} \int \mathrm{d}^D x_0 \prod_{i=1}^{\dim R} \mathrm{d}\chi^i_{\mathrm{gh}} \mathrm{d}\overline{\eta}_{i,\mathrm{gh}}\, P^{\mathrm{gh}}_{\overline{\eta},\chi}\, \mathrm{e}^{\overline{\eta}_{\mathrm{gh}}\cdot\chi_{\mathrm{gh}}} \prod_{a=1}^{D} \mathrm{d}\psi^a_{1,\mathrm{bg}} \big\langle \mathrm{e}^{-\frac{1}{\hbar}S^{(\mathrm{int})}_{1,H}} \big\rangle \;, \tag{5.11a}$$

$$-\frac{1}{\hbar} S^{(\mathrm{int})}_{1,H} \;=\; (F(x_0))^i{}_j \int_{-1}^{0} \mathrm{d}\tau\, \big(\overline{\xi}_{\mathrm{gh}} + \hat{c}^\dagger_{\mathrm{qu}}(\tau)\big)_i \big(\xi_{\mathrm{gh}} + \hat{c}_{\mathrm{qu}}(\tau)\big)^j \;, \tag{5.11b}$$

where $(F(x_0))^i{}_j = \frac{1}{2} F^\alpha_{ab}(x_0) \psi^{ab}_{1,\mathrm{bg}} (T_\alpha)^i{}_j$. As we mentioned before, we only analyze the ghost tree graphs via the expansion of the above form:

$$\begin{aligned}
&\Big\langle \exp\Big(F^i{}_j \int_{-1}^{0} \mathrm{d}\tau\, \big(\overline{\xi}_{\mathrm{gh}} + \hat{c}^\dagger_{\mathrm{qu}}(\tau)\big)_i \big(\xi_{\mathrm{gh}} + \hat{c}_{\mathrm{qu}}(\tau)\big)^j \Big) \Big\rangle \\
&\quad= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \overline{\eta}_{j,\mathrm{gh}} \big(F^k\big)^j{}_l\, \chi^l_{\mathrm{gh}} \Big[k! \int_{-1}^{0} \mathrm{d}\sigma_1 \cdots \mathrm{d}\sigma_k\, \theta(\sigma_1 - \sigma_2)\, \theta(\sigma_2 - \sigma_3) \cdots \theta(\sigma_{k-1} - \sigma_k) \Big] \\
&\quad= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \overline{\eta}_{j,\mathrm{gh}} \big(F^k\big)^j{}_l\, \chi^l_{\mathrm{gh}} \;.
\end{aligned} \tag{5.12}$$

Note that the factor $k!$ in the square bracket in the second line is due to the fact that we can order the k vertices into a tree in $k!$ ways. We also used the following integral:

$$\int_{-1}^{0} \mathrm{d}\sigma_1 \cdots \mathrm{d}\sigma_k\, \theta(\sigma_1 - \sigma_2)\, \theta(\sigma_2 - \sigma_3) \cdots \theta(\sigma_{k-1} - \sigma_k) \;=\; \frac{1}{k!} \;. \tag{5.13}$$

Integral of the ghost fields of (5.12) gives the following simple result:

$$\begin{aligned}
&\int \prod_{i=1}^{\dim R} \mathrm{d}\chi^i_{\mathrm{gh}} \mathrm{d}\overline{\eta}_{i,\mathrm{gh}}\, P^{\mathrm{gh}}_{\overline{\eta},\chi}\, \mathrm{e}^{\overline{\eta}_{\mathrm{gh}}\cdot\chi_{\mathrm{gh}}} \Big\langle \exp\Big(F^i{}_j \int_{-1}^{0} \mathrm{d}\tau\, \big(\overline{\xi}_{\mathrm{gh}} + \hat{c}^\dagger_{\mathrm{qu}}(\tau)\big)_i \big(\xi_{\mathrm{gh}} + \hat{c}_{\mathrm{qu}}(\tau)\big)^j \Big) \Big\rangle \\
&\qquad= \int \prod_{i=1}^{\dim R} \mathrm{d}\chi^i_{\mathrm{gh}} \mathrm{d}\overline{\eta}_{i,\mathrm{gh}}\, P^{\mathrm{gh}}_{\overline{\eta},\chi}\, \mathrm{e}^{\overline{\eta}_{\mathrm{gh}}\cdot\chi_{\mathrm{gh}}} \bigg(1 + \sum_{k=1}^{\infty} \frac{1}{k!} \overline{\eta}_{j,\mathrm{gh}} \big(F^k\big)^j{}_l\, \chi^l_{\mathrm{gh}} \bigg) \\
&\qquad= \sum_{j=1}^{\dim R} \bigg[\delta^j{}_j + \sum_{k=1}^{\infty} \frac{1}{k!} \big(F^k\big)^j{}_j \bigg] \;=\; \sum_{i=1}^{\dim R} \exp\big(F\big)^i{}_i \;\equiv\; \mathrm{Tr}_R \exp\big(F\big) \;,
\end{aligned} \tag{5.14}$$

where the symbol Tr_R denotes the trace in the R representation of the gauge group. Summarizing the integral and rescaling the background fermion in such a way as $\psi^a_{1,\mathrm{bg}} \to \sqrt{\frac{-i}{2\pi}}\psi^a_{1,\mathrm{bg}}$, we obtain

$$\mathrm{index}\not{D}(A) \;=\; \int \mathrm{d}^D x_0 \prod_{a=1}^{D} \mathrm{d}\psi^a_{1,\mathrm{bg}}\, \mathrm{Tr}_R \exp\Big(-\frac{i}{2\pi} F \Big) \;, \qquad F \;=\; \frac{1}{2} F^\alpha_{ab}(x_0)\, \psi^a_{1,\mathrm{bg}} \psi^b_{1,\mathrm{bg}}\, T_\alpha \;. \tag{5.15}$$

This is nothing but the Chern character of the gauge fields A_m^α. When we explicitly calculate, we should use the formulae (4.10). In the same way as (4.19), let us integrate the background fermions with respect to (4.10) and obtain

$$\text{index}\not{D}(A) \ = \ \int_{\mathcal{M}} \text{Tr}_R \exp\Big(\frac{i}{2\pi} F\Big) \, , \quad F \ = \ \frac{1}{2} F_{ab} \, e^a \wedge e^b \ = \ \mathrm{d}A + A \wedge A \, . \tag{5.16}$$

5.2.2 Torsional geometry: $H \neq 0$, $\mathrm{d}H = 0$ case

Let us easily generalize the equation (5.15) to the one on a curved geometry $\mathcal{M}$ (in the presence of torsion H). Since the Hilbert spaces of the physical states and the $\hat{c}$-ghost states are independent of each other, the functional integrals of the Dirac index are also performed independently. Then, combining the functional integral of the physical field sector (4.18) and the functional integral of the $\hat{c}$-ghost sector (5.15), we obtain the Dirac index in the following representation:

$$\text{index}\not{D}(\hat{\omega}, A) \ = \ \int_{\mathcal{M}} \exp\left[\frac{1}{2}\text{tr}\log\left(\frac{i\mathcal{R}/4\pi}{\sinh(i\mathcal{R}/4\pi)}\right)\right] \text{Tr}_R \exp\Big(\frac{i}{2\pi} F\Big) \, , \tag{5.17a}$$

$$\mathcal{R}_{mn} \ = \ \frac{1}{2}\Big\{R_{mnab}(\omega) - 2\widetilde{D}_{[m} H_{n]ab}\Big\} e^a \wedge e^b \, , \quad F \ = \ \frac{1}{2} F_{ab} \, e^a \wedge e^b \, . \tag{5.17b}$$

The index on a Riemannian geometry without torsion can be easily obtained when we choose $H = 0$ in this form.

6 Witten index in $\mathcal{N} = 2$ quantum mechanics

In this section let us analyze the Euler characteristics with non-vanishing H-flux under the $\mathrm{d}H = 0$ condition. In the case of vanishing H-flux, we will find a form of the Gauss-Bonnet theorem.

6.1 Formulation

The Euler characteristics χ on the target space geometry can also be expressed in terms of the $\mathcal{N} = 2$ supersymmetric quantum mechanics (see section *14.3* in [15])

$$\chi \ \equiv \ \lim_{\beta \to 0} \text{Tr}\big\{\Gamma_{(5)} \widetilde{\Gamma}_{(5)} \mathrm{e}^{-\beta \mathscr{R}}\big\} \ = \ \lim_{\beta \to 0} \text{Tr} \prod_{a=1}^{D} \big(\widehat{\varphi}^a + \widehat{\overline{\varphi}}^a\big) \prod_{b=1}^{D} \big(\widehat{\varphi}^b - \widehat{\overline{\varphi}}^b\big) \, \mathrm{e}^{-\frac{\beta}{\hbar}\widehat{\mathscr{H}}} \, . \tag{6.1}$$

The chirality operators $\Gamma_{(5)}$ and $\widetilde{\Gamma}_{(5)}$ are given in terms of $\Gamma^a = \sqrt{2}\widehat{\psi}{}^a_{1,\diamond}$ and $\widetilde{\Gamma}^a = \sqrt{2}\widehat{\psi}{}^a_{2,\diamond}$, respectively:

$$\Gamma_{(5)} \ \equiv \ (-i)^{D/2} \Gamma^1 \cdots \Gamma^D \ = \ (-i)^{D/2} 2^{D/2} \widehat{\psi}{}^a_{1,\diamond} \cdots \widehat{\psi}{}^D_{1,\diamond} \ = \ (-i)^{D/2} \prod_{a=1}^{D} \big(\widehat{\varphi}^a_\diamond + \widehat{\overline{\varphi}}{}^a_\diamond\big) \, , \tag{6.2a}$$

$$\widetilde{\Gamma}_{(5)} \equiv (-i)^{D/2}\widetilde{\Gamma}^1\cdots\widetilde{\Gamma}^D = (-i)^{D/2}2^{D/2}\widehat{\psi}^a_{2,\diamond}\cdots\widehat{\psi}^D_{2,\diamond} = (-i)^{D/2}(-i)^D\prod_{a=1}^D\big(\widehat{\varphi}^a_\diamond - \widehat{\overline{\varphi}}{}^a_\diamond\big)\,. \qquad (6.2b)$$

Notice that since the non-trivial values are given when D is even number, we find $(-i)^{2D} = 1$. From now on we omit the symbol "$\diamond$" of the fermions. Then we formulate the Euler characteristic in terms of the transition element and effective action (where $x = x_0 + q$):

$$\chi = \lim_{\beta\to 0}\int \mathrm{d}^D x_0 \sqrt{g(x_0)}\prod_{a=1}^D\Big(\mathrm{d}\overline{\eta}_a\mathrm{d}\eta^a\mathrm{d}\zeta^a\mathrm{d}\overline{\zeta}^a\mathrm{d}\overline{\lambda}_a\mathrm{d}\lambda^a\Big)\,\mathrm{e}^{\overline{\zeta}\zeta}\,\mathrm{e}^{-\overline{\lambda}\lambda}\,\mathrm{e}^{-\overline{\eta}\eta}$$
$$\times\langle\,\overline{\zeta}\,|\prod_{b=1}^D\big(\widehat{\varphi}^b + \widehat{\overline{\varphi}}{}^b\big)|\,\lambda\,\rangle\langle\,\overline{\lambda}\,|\prod_{c=1}^D\big(\widehat{\varphi}^c - \widehat{\overline{\varphi}}{}^c\big)|\,\eta\,\rangle\langle\,x_0,\overline{\eta}\,|\exp\Big(-\frac{\beta}{\hbar}\widehat{\mathscr{H}}_H\Big)|\,x_0,\zeta\,\rangle\,, \qquad (6.3a)$$

$$\langle\,x_0,\overline{\eta}\,|\exp\Big(-\frac{\beta}{\hbar}\widehat{\mathscr{H}}_H\Big)|\,x_0,\zeta\,\rangle = \frac{1}{(2\pi\beta\hbar)^{D/2}}\mathrm{e}^{\overline{\eta}\zeta}\Big\langle\exp\Big(-\frac{1}{\hbar}S_H^{(\mathrm{int})}\Big)\Big\rangle\,, \qquad (6.3b)$$

$$\begin{aligned}
-\frac{1}{\hbar}S_H^{(\mathrm{int})} =& -\frac{1}{\beta\hbar}\int_{-1}^0\mathrm{d}\tau\,\frac{1}{2}\Big[g_{mn}(x) - g_{mn}(x_0)\Big]\Big(\dot{q}^m\dot{q}^n + b^m c^n + a^m a^n\Big) \\
& - \int_{-1}^0\mathrm{d}\tau\,\dot{q}^m\bigg(\omega_{mab}(x)\,(\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^a(\zeta+\xi_{\mathrm{qu}})^b - \frac{1}{2}H_{mab}(x)\Big\{(\zeta+\xi_{\mathrm{qu}})^{ab} + (\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^{ab}\Big\}\bigg) \\
& - \frac{\beta\hbar}{16}\int_{-1}^0\mathrm{d}\tau\,R_{[abcd]}(\omega_+(x))\Big\{(\zeta+\xi_{\mathrm{qu}})^{abcd} + (\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^{abcd} - 2(\zeta+\xi_{\mathrm{qu}})^{ab}(\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^{cd}\Big\} \\
& + \frac{\beta\hbar}{2}\int_{-1}^0\mathrm{d}\tau\,R_{cdab}(\omega(x))\Big\{(\zeta+\xi_{\mathrm{qu}})^a(\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^b(\zeta+\xi_{\mathrm{qu}})^c(\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^d\Big\} \\
& - \frac{\beta\hbar}{8}\int_{-1}^0\mathrm{d}\tau\,\mathcal{G}_2(x)\,, \qquad (6.3c)
\end{aligned}$$

where the functional $\mathcal{G}_2(x)$ is given in (3.21d). Now let us analyze fermionic measure in the form (6.3). The effective action $S^{(\mathrm{int})}$ contains ξ^a and $\overline{\xi}^a$ whose boundaries are ζ and $\overline{\eta}$, respectively, and η, $\overline{\zeta}$ and λ, $\overline{\lambda}$ do not appear in $S^{(\mathrm{int})}$. Then let us rewrite the path integral measure with fermions:

$$\begin{aligned}
\prod_a\mathrm{d}\overline{\eta}_a\mathrm{d}\eta^a\,\mathrm{d}\zeta^a\mathrm{d}\overline{\zeta}_a\,\mathrm{d}\overline{\lambda}_a\mathrm{d}\lambda^a =& \prod_a\big(\mathrm{d}\overline{\eta}_a\mathrm{d}\zeta^a\big)\big(\mathrm{d}\overline{\lambda}_a\mathrm{d}\eta^a\big)\big(\mathrm{d}\overline{\zeta}_a\mathrm{d}\lambda^a\big) \\
=& \prod_a\big(\mathrm{d}\overline{\eta}_a\mathrm{d}\zeta^a\big)\Big(2^D\mathrm{d}(\overline{\lambda}+\eta)_a\mathrm{d}(\eta-\overline{\lambda})^a\Big)\Big(2^D\mathrm{d}(\overline{\zeta}+\lambda)_a\mathrm{d}(\lambda-\overline{\zeta})^a\Big)\,, \qquad (6.4a)
\end{aligned}$$

where we implicitly used the orderings of $\mathrm{d}\overline{\eta}$ and $\mathrm{d}\zeta$ (3.5). Under the integral with $\prod_b(\lambda^b+\overline{\zeta}^b)\prod_c(\eta^b-\overline{\lambda}^b)$ which can be regarded as the fermionic delta functions, we can see $\overline{\zeta} = -\lambda$ and $\eta = \overline{\lambda}$. Then, after a tedious computation, we obtain

$$\int\prod_a\mathrm{d}\overline{\eta}_a\mathrm{d}\eta^a\,\mathrm{d}\zeta^a\mathrm{d}\overline{\zeta}_a\,\mathrm{d}\overline{\lambda}_a\mathrm{d}\lambda^a\,\mathrm{e}^{\overline{\zeta}\zeta-\overline{\lambda}\lambda-\overline{\eta}\eta+\overline{\zeta}\lambda+\overline{\lambda}\eta+\overline{\eta}\zeta}\prod_b\big(\lambda^b+\overline{\zeta}^b\big)\prod_c\big(\eta^b-\overline{\lambda}^b\big) = \int\prod_a\mathrm{d}\overline{\eta}_a\mathrm{d}\zeta^a\,, \qquad (6.4b)$$

where we used the fermionic delta functions:

$$\int\prod_a\mathrm{d}(\overline{\zeta}+\eta)_a\prod_b(\eta^b+\overline{\zeta}^b) = 1\,, \quad (-1)^D\int\prod_{a=1}^D\mathrm{d}\overline{\zeta}\,\mathrm{e}^{-\overline{\zeta}(\eta-\zeta)} = \prod_{a=1}^D\big(\eta^a-\zeta^a\big)\,. \qquad (6.4c)$$

Then we rescale the fermion to remove the $\beta\hbar$ dependence on the measure in such a way as

$$\frac{1}{(\beta\hbar)^{D/2}} \prod_{a=1}^{D} \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \ \equiv \ \prod_{a=1}^{D} \mathrm{d}\overline{\eta}'_a \mathrm{d}\zeta'^a \ , \quad \xi^a \ \equiv \ (\beta\hbar)^{-\frac{1}{4}} \xi'^a \ . \tag{6.5}$$

Then the rescaled $S^{(\mathrm{int})}$ is given by (where we omit the prime symbol)

$$\begin{aligned}
-\frac{1}{\hbar} S_H^{(\mathrm{int})} \ &= \ -\frac{1}{\beta\hbar} \int_{-1}^{0} \mathrm{d}\tau \, \frac{1}{2} \Big[g_{mn}(x) - g_{mn}(x_0) \Big] \Big(\dot{q}^m \dot{q}^n + b^m c^n + a^m a^n \Big) \\
&\quad - \frac{1}{\sqrt{\beta\hbar}} \int_{-1}^{0} \mathrm{d}\tau \, \dot{q}^m \bigg(\omega_{mab}(x) (\overline{\eta} + \overline{\xi}_{\mathrm{qu}})^a (\zeta + \xi_{\mathrm{qu}})^b - \frac{1}{2} H_{mab}(x) \Big\{ (\zeta + \xi_{\mathrm{qu}})^{ab} + (\overline{\eta} + \overline{\xi}_{\mathrm{qu}})^{ab} \Big\} \bigg) \\
&\quad - \frac{1}{16} \int_{-1}^{0} \mathrm{d}\tau \, R_{[abcd]}(\omega_+(x)) \Big\{ (\zeta + \xi_{\mathrm{qu}})^{abcd} + (\overline{\eta} + \overline{\xi}_{\mathrm{qu}})^{abcd} - 2 (\zeta + \xi_{\mathrm{qu}})^{ab} (\overline{\eta} + \overline{\xi}_{\mathrm{qu}})^{cd} \Big\} \\
&\quad + \frac{1}{2} \int_{-1}^{0} \mathrm{d}\tau \, R_{cdab}(\omega(x)) \Big\{ (\zeta + \xi_{\mathrm{qu}})^a (\overline{\eta} + \overline{\xi}_{\mathrm{qu}})^b (\zeta + \xi_{\mathrm{qu}})^c (\overline{\eta} + \overline{\xi}_{\mathrm{qu}})^d \Big\} \\
&\quad - \frac{\beta\hbar}{8} \int_{-1}^{0} \mathrm{d}\tau \, \mathcal{G}_2(x) \ .
\end{aligned} \tag{6.6}$$

Since the bosonic and fermionic propagators are now proportional to $\beta\hbar$ and $\sqrt{\beta\hbar}$, respectively (we have also rescaled the fermion propagator), we easily find that each contraction among quantum fields yields Feynman graphs of higher order in $\beta\hbar$, which goes to zero in the limit $\beta\hbar \to 0$. Only the interaction terms given by background fields x_0^m, ζ^a and $\overline{\eta}^a$ are independent of β and they give rise to the relevant Feynman graphs. Then, we can truncate $S^{(\mathrm{int})}$ in order to obtain the Euler characteristics on the D-dimensional geometry $\mathcal{M}$ in the path integral formalism:

$$\chi \ = \ \frac{1}{(2\pi)^{D/2}} \int \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D} \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \left\langle \exp\Big(- \frac{1}{\hbar} S_H^{(\mathrm{int})} \Big) \right\rangle \ , \tag{6.7a}$$

$$-\frac{1}{\hbar} S_H^{(\mathrm{int})} \ = \ -\frac{1}{4} R_{abcd}(\omega(x_0)) \, \zeta^{ab} \overline{\eta}^{cd} - \frac{1}{16} R_{[abcd]}(\omega_+(x_0)) \big(\zeta^{abcd} + \overline{\eta}^{abcd} - 2 \zeta^{ab} \overline{\eta}^{cd} \big) \ , \tag{6.7b}$$

where we used $R_{cdab}(\omega) = R_{abcd}(\omega)$ and the second Bianchi identity $R_{abcd}(\omega) + R_{acdb}(\omega) + R_{adbc}(\omega) = 0$ without torsion: $R_{abcd}(\omega) \, \zeta^a \overline{\eta}^b \zeta^c \overline{\eta}^d = -\frac{1}{2} R_{abcd}(\omega) \, \zeta^{ab} \overline{\eta}^{cd}$. Since there exist only background fields, we do not have to introduce quantum propagators to contract interaction terms. The Feynman amplitude of the path integral is given only by the expansion of $\exp(-\frac{1}{\hbar} S^{(\mathrm{int})})$ with noticing that the number of ζ should be equal to the number of $\overline{\eta}$ to saturate the fermionic path integral measure. Since each term in (6.7b) carries even number of background fermions ζ and $\overline{\eta}$, the path integral with $D = 2n+1$ becomes trivial.

6.2 Euler characteristics

Next let us investigate the path integral forms in various geometries in diverse dimensions. We can easily find that the second and the third terms in (6.7b) do not contribute to the Feynman graphs in the case of $D = 2$. This is consistent with the fact there does not exist a totally antisymmetric torsion in two-dimensional geometry.

6.2.1 Riemannian geometry: $H = \mathrm{d}H = 0$ case

This is the simplest case in this paper because, as mentioned before, we need not introduce any propagators to contract interaction terms. All interaction terms in the effective action are given in terms of background fields:

$$-\frac{1}{\hbar}S^{(\mathrm{int})} \ = \ -\frac{1}{4}R_{abcd}(\omega)\,\zeta^{ab}\overline{\eta}^{cd} \ , \tag{6.8}$$

Then the path integral form is described in the following way:

$$\begin{aligned}\chi \ &= \ \frac{1}{(2\pi)^{D/2}}\int \mathrm{d}^D x_0\sqrt{g(x_0)}\prod_{a=1}^{D}\mathrm{d}\overline{\eta}^a\mathrm{d}\zeta^a\ \exp\Big(-\frac{1}{4}R_{abcd}(\omega)\,\zeta^{ab}\overline{\eta}^{cd}\Big) \\ &= \ \frac{1}{(8\pi)^n n!}\,\mathcal{E}_{a_1\cdots a_{2n}}\mathcal{E}^{b_1\cdots b_{2n}}\int \mathrm{d}^{2n}x_0\sqrt{g(x_0)}\,\Big(R^{a_1a_2}{}_{b_1b_2}(\omega)\cdots R^{a_{2n-1}a_{2n}}{}_{b_{2n-1}b_{2n}}(\omega)\Big) \\ &= \ \frac{1}{(4\pi)^n n!}\,\mathcal{E}_{a_1\cdots a_{2n}}\int_{\mathcal{M}} R^{a_1a_2}(\omega)\wedge\cdots\wedge R^{a_{2n-1}a_{2n}}(\omega)\ ,\end{aligned} \tag{6.9}$$

where we used the formulae in Euclidean space:

$$\mathrm{d}^{2n}x_0\sqrt{g(x_0)}\,\mathcal{E}^{b_1\cdots b_{2n}} \ = \ e^{b_1}\wedge\cdots\wedge e^{b_{2n}}\ , \qquad R_{ab}(\omega) \ = \ \frac{1}{2}R_{abcd}(\omega)\,e^c\wedge e^d\ . \tag{6.10}$$

Non-trivial value of χ is given only when $D = 2n$ and all indices of totally antisymmetric tensor $\mathcal{E}_{abcd\cdots}$ are the frame (local Lorentz) indices with Euclidean signature. We also used the following formulae in the same way as (4.11):

$$\int \mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{2n}\,\zeta_{1\cdots 2n} \ = \ (-1)^n\ , \qquad \int \mathrm{d}\overline{\eta}^{2n}\cdots\mathrm{d}\overline{\eta}^1\,\overline{\eta}^{12\cdots 2n} \ = \ 1\ . \tag{6.11}$$

6.2.2 Torsional geometry: $H \neq 0$, $\mathrm{d}H = 0$ case

This case is still simple even though four non-trivial interaction terms appear in the effective action:

$$-\frac{1}{\hbar}S_H^{(\mathrm{int})} \ = \ -\frac{1}{4}R_{abcd}(\omega)\,\zeta^{ab}\overline{\eta}^{cd} - \frac{1}{16}R_{[abcd]}(\omega_+)\big(\zeta^{abcd}+\overline{\eta}^{abcd}-2\zeta^{ab}\overline{\eta}^{cd}\big)\ . \tag{6.12}$$

On a D-dimensional geometry with $D = 2n$, the exponent $\big\langle\exp(-\frac{1}{\hbar}S^{(\mathrm{int})})\big\rangle$ can be reduced to

$$\begin{aligned}\Big\langle\exp\Big(-\frac{1}{\hbar}S_H^{(\mathrm{int})}\Big)\Big\rangle \ &= \ \exp\Big(-\frac{1}{4}R_{abcd}(\omega)\,\zeta^{ab}\overline{\eta}^{cd} - \frac{1}{16}R_{[abcd]}(\omega_+)\big(\zeta^{abcd}+\overline{\eta}^{abcd}-2\zeta^{ab}\overline{\eta}^{cd}\big)\Big) \\ &\sim \sum_{k+2\ell=n}\frac{1}{4^{2\ell}k!\ell!\ell!}\Big(-\frac{1}{4}\Big)^{k+2\ell}\Big(\mathfrak{R}_{abcd}\,\zeta^{ab}\overline{\eta}^{cd}\Big)^k\Big(R_{[efgh]}(\omega_+)\,\zeta^{efgh}\Big)^\ell\Big(R_{[ijkl]}(\omega_+)\,\overline{\eta}^{ijkl}\Big)^\ell\ ,\end{aligned} \tag{6.13}$$

where $0 \le k, \ell \le n$. To simplify the expression we introduced

$$\mathfrak{R}_{abcd} \ \equiv \ R_{abcd}(\omega) - \frac{1}{2} R_{[abcd]}(\omega_{+}) \ . \tag{6.14}$$

Thus, by using (6.11), we obtain the explicit formula

$$\begin{aligned}
\chi \ &= \ \frac{1}{(8\pi)^n} \sum_{k+2\ell=n} \frac{1}{4^{2\ell} k! \, \ell! \, \ell!} \, \mathcal{E}^{a_1 \cdots a_{2k} c_1 \cdots c_{4\ell}} \, \mathcal{E}^{b_1 \cdots b_{2k} d_1 \cdots d_{4\ell}} \\
&\quad \times \int \mathrm{d}^{2n} x_0 \sqrt{g(x_0)} \Big(\mathfrak{R}_{a_1 a_2 b_1 b_2} \cdots \mathfrak{R}_{a_{2k-1} a_{2k} b_{2k-1} b_{2k}} \Big) \\
&\quad\quad \times \Big(R_{c_1 c_2 c_3 c_4}(\omega_{+}) \cdots R_{c_{4\ell-3} c_{4\ell-2} c_{4\ell-1} c_{4\ell}}(\omega_{+}) \Big) \Big(R_{d_1 d_2 d_3 d_4}(\omega_{+}) \cdots R_{d_{4\ell-3} d_{4\ell-2} d_{4\ell-1} d_{4\ell}}(\omega_{+}) \Big) \ .
\end{aligned} \tag{6.15}$$

As mentioned before, the two-dimensional Euler characteristic with H-flux should be reduced to the ordinary one. Here let us consider the Euler characteristics on $D = 4, 6, 8$ dimensional geometries. We can easily compute them and obtain the following explicit expressions in terms of $\mathfrak{R}_{ab} = \frac{1}{2} \mathfrak{R}_{abcd} \, e^c \wedge e^d$ and $R_4(\omega_{+}) = \frac{1}{4!} R_{abcd}(\omega_{+}) \, e^a \wedge e^b \wedge e^c \wedge e^d$:

$$\chi(\mathcal{M}_4) \ = \ \frac{1}{(4\pi)^2 2!} \, \mathcal{E}^{a_1 \cdots a_4} \int_{\mathcal{M}} \bigg(\mathfrak{R}_{a_1 a_2} \wedge \mathfrak{R}_{a_3 a_4} + \frac{3}{4} R_{a_1 a_2 a_3 a_4}(\omega_{+}) \, R_4(\omega_{+}) \bigg) \ , \tag{6.16a}$$

$$\chi(\mathcal{M}_6) \ = \ \frac{1}{(4\pi)^3 3!} \, \mathcal{E}^{a_1 \cdots a_6} \int_{\mathcal{M}} \bigg(\mathfrak{R}_{a_1 a_2} \wedge \mathfrak{R}_{a_3 a_4} \wedge \mathfrak{R}_{a_5 a_6} + \frac{9}{4} R_{a_1 a_2 a_3 a_4}(\omega_{+}) \, \mathfrak{R}_{a_5 a_6} \wedge R_4(\omega_{+}) \bigg) \ , \tag{6.16b}$$

$$\begin{aligned}
\chi(\mathcal{M}_8) \ = \ \frac{1}{(4\pi)^4 4!} \, \mathcal{E}^{a_1 \cdots a_8} \int_{\mathcal{M}} \bigg(& \mathfrak{R}_{a_1 a_2} \wedge \mathfrak{R}_{a_3 a_4} \wedge \mathfrak{R}_{a_5 a_6} \wedge \mathfrak{R}_{a_7 a_8} \\
& + 9 R_{a_1 a_2 a_3 a_4}(\omega_{+}) \, \mathfrak{R}_{a_5 a_6} \wedge \mathfrak{R}_{a_7 a_8} \wedge R_4(\omega_{+}) \\
& + \frac{27}{8} R_{a_1 a_2 a_3 a_4}(\omega_{+}) \, R_{a_5 a_6 a_7 a_8}(\omega_{+}) \, R_4(\omega_{+}) \wedge R_4(\omega_{+}) \bigg) \ .
\end{aligned} \tag{6.16c}$$

Each first term represents the formula of the Gauss-Bonnet theorem in each dimensional Riemannian geometry.

7 Witten index in $\mathcal{N} = 2$ quantum mechanics II

Finally we will discuss the derivation of the Hirzebruch signature on a torsional geometry in the path integral formalism. We also use $\mathcal{N} = 2$ supersymmetric quantum mechanical path integral, while we only insert $\Gamma_{(5)}$ into the transition element instead of the insertion $\Gamma_{(5)} \widetilde{\Gamma}_{(5)}$ in the case of the Euler characteristics. We review the derivation of the signature on a Riemannian geometry. Next we discuss the analysis of the signature on a torsional geometry in the same strategy.

7.1 Formulation

As mentioned in the introduction, the Hirzebruch signature is a topological invariant which gives the difference between the number of self-dual forms and the number of anti-self-dual forms on a geometry. Since we analyze the difference of the forms, we analyze another Witten index defined in the $\mathcal{N}=2$ supersymmetric quantum mechanics in the following form (see section *14.3* in [15]):

$$\sigma \ \equiv \ \lim_{\beta\to 0} \mathrm{Tr}\big\{\Gamma_{(5)} \mathrm{e}^{-\beta\mathscr{R}}\big\} \ = \ \lim_{\beta\to 0} (-i)^{D/2} \mathrm{Tr} \prod_{a=1}^{D} \big(\widehat{\varphi}^a + \widehat{\overline{\varphi}}{}^a\big) \, \mathrm{e}^{-\frac{\beta}{\hbar}\widehat{\mathscr{H}}} \ . \tag{7.1}$$

Here we did not insert $2^{-D/2}$ because in this system ψ_2^a is also dynamical. The chirality operators $\Gamma_{(5)}$ is again given in terms of the operators $\widehat{\psi}{}^a_{1,\diamond}$:

$$\Gamma_{(5)} \ \equiv \ (-i)^{D/2} \Gamma^1 \cdots \Gamma^D \ = \ (-i)^{D/2} 2^{D/2} \widehat{\psi}{}^a_{1,\diamond} \cdots \widehat{\psi}{}^D_{1,\diamond} \ = \ (-i)^{D/2} \prod_{a=1}^{D} \big(\widehat{\varphi}{}^a_{\diamond} + \widehat{\overline{\varphi}}{}^a_{\diamond}\big) \ . \tag{7.2}$$

Notice that since the non-trivial values are given when D is even number, we find $(-i)^{2D} = 1$. From now on we omit the symbol "$\diamond$" of the fermions. In addition, we prepare the trace formula and the complete set of the fermion coherent states (3.5). We obtain the explicit expression of the topological invariants with respect to the $\mathcal{N}=2$ quantum mechanical path integral in the same way as (6.3):

$$\begin{aligned}\sigma \ &= \ \lim_{\beta\to 0} \frac{(-i)^{D/2}}{(2\pi\beta\hbar)^{D/2}} \int \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D} \Big(\mathrm{d}\overline{\eta}_a \mathrm{d}\eta^a \, \mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}_a\Big) \\ &\qquad \times \mathrm{e}^{\overline{\zeta}\zeta + \overline{\zeta}\eta - \overline{\eta}\eta + \overline{\eta}\zeta} \prod_b \big(\eta^b + \overline{\zeta}{}^b\big) \Big\langle \exp\Big(-\frac{1}{\hbar} S_H^{(\mathrm{int})}\Big)\Big\rangle \ ,\end{aligned} \tag{7.3}$$

where $S^{(\mathrm{int})}$ in (7.3) is also given by (6.3c) which appeared in the previous subsection. Now let us consider the fermionic measure in this path integral form. In the same way as the Dirac index, we obtain

$$\begin{aligned}&\int \prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\eta^a \mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}_a \, \mathrm{e}^{\overline{\zeta}\zeta + \overline{\zeta}\eta - \overline{\eta}\eta + \overline{\eta}\zeta} \prod_b (\eta^b + \overline{\zeta}{}^b) \\ &\quad = \ (-2)^D \int \prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \, \mathrm{d}(\overline{\zeta} + \eta)_a \mathrm{d}(\eta - \overline{\zeta})^a \, \mathrm{e}^{-\frac{1}{2}(\eta - \overline{\zeta})(\zeta - \overline{\eta})} \prod_b (\eta^b + \overline{\zeta}{}^b) \\ &\quad = \ \int \prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \prod_b (\zeta^b - \overline{\eta}{}^b) \ .\end{aligned} \tag{7.4}$$

This measure gives the fermionic delta function which indicates the coincidence of the background fermions $\zeta^a = \overline{\eta}{}^a$:

$$\int \prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \prod_b (\zeta^b - \overline{\eta}{}^b) \, f(\overline{\eta}) \ = \ f(\zeta) \ . \tag{7.5}$$

To remove the β dependence in the path integral measure, we rescale the fermion

$$\frac{1}{(\beta\hbar)^{D/2}} \int \prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \prod_b (\zeta^b - \overline{\eta}^b) \ \equiv \ \int \prod_a \mathrm{d}\overline{\eta}'_a \mathrm{d}\zeta'^a \prod_b (\zeta'^b - \overline{\eta}'^b) \ , \tag{7.6a}$$

$$\overline{\eta}^a \ \equiv \ \Big(\frac{1}{\beta\hbar}\Big)^{1/2} \overline{\eta}'^a \ , \quad \zeta^a \ \equiv \ \Big(\frac{1}{\beta\hbar}\Big)^{1/2} \zeta'^a \ . \tag{7.6b}$$

Then the rescaled $S^{(\mathrm{int})}$ (3.21c) in the path integral with $\mathrm{d}H = 0$ is given by (where we omit the prime symbol)

$$\sigma \ = \ \lim_{\beta \to 0} \frac{(-i)^{D/2}}{(2\pi)^{D/2}} \int \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D} \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \prod_{b=1}^{D} (\zeta^b - \overline{\eta}^b) \Big\langle \exp\Big(- \frac{1}{\hbar} S_H^{(\mathrm{int})} \Big) \Big\rangle \ , \tag{7.7a}$$

$$\begin{aligned}
-\frac{1}{\hbar} S_H^{(\mathrm{int})} \ &= \ - \frac{1}{\beta\hbar} \int_{-1}^{0} \mathrm{d}\tau \, \frac{1}{2} \Big[g_{mn}(x) - g_{mn}(x_0) \Big] \Big(\dot{q}^m \dot{q}^n + b^m c^n + a^m a^n \Big) \\
& \quad - \frac{1}{\beta\hbar} \int_{-1}^{0} \mathrm{d}\tau \, \dot{q}^m \bigg(\omega_{mab}(x) \, (\overline{\eta} + \overline{\xi}_{\mathrm{qu}})^a (\zeta + \xi_{\mathrm{qu}})^b - \frac{1}{2} H_{mab}(x) \Big\{ (\zeta + \xi_{\mathrm{qu}})^{ab} + (\overline{\eta} + \overline{\xi}_{\mathrm{qu}})^{ab} \Big\} \bigg) \\
& \quad - \frac{1}{16\beta\hbar} \int_{-1}^{0} \mathrm{d}\tau \, R_{[abcd]}(\omega_+(x)) \Big\{ (\zeta + \xi_{\mathrm{qu}})^{abcd} + (\overline{\eta} + \overline{\xi}_{\mathrm{qu}})^{abcd} - 2 (\zeta + \xi_{\mathrm{qu}})^{ab} (\overline{\eta} + \overline{\xi}_{\mathrm{qu}})^{cd} \Big\} \\
& \quad + \frac{1}{2\beta\hbar} \int_{-1}^{0} \mathrm{d}\tau \, R_{cdab}(\omega(x)) \Big\{ (\zeta + \xi_{\mathrm{qu}})^a (\overline{\eta} + \overline{\xi}_{\mathrm{qu}})^b (\zeta + \xi_{\mathrm{qu}})^c (\overline{\eta} + \overline{\xi}_{\mathrm{qu}})^d \Big\} \\
& \quad - \frac{\beta\hbar}{8} \int_{-1}^{0} \mathrm{d}\tau \, \mathcal{G}_2(x) \ .
\end{aligned} \tag{7.7b}$$

The bosonic and fermionic propagators are of order in $\beta\hbar$. Let us truncate this action. In the same analogy to the Dirac index, disconnected Feynman graphs might contribute to the amplitude. In the same way as previous case, the fermion propagator is given by

$$\big\langle \xi_{\mathrm{qu}}^a(\sigma) \overline{\xi}_{\mathrm{qu}}^b(\tau) \big\rangle \ = \ \beta\hbar \, \delta^{ab} \theta(\sigma - \tau) \ . \tag{7.8}$$

7.2 Hirzebruch signature

7.2.1 Riemannian geometry: $H = \mathrm{d}H = 0$ case

This case is quite simple. Since there are no background interaction terms of order in $(\beta\hbar)^{-1}$ which contribute to the disconnected graphs, we only consider one-loop Feynman graphs. Then, we neglect interaction terms carrying more than three quantum fields. We can also neglect the last line in (7.7b) which yields the graphs of higher order in $\beta\hbar$. We also use the condition by Riemann normal coordinate frame $\partial_p g_{mn}(x_0) = \omega_{mab}(x_0) = 0$ at the point x_0. We can further neglect interaction terms which are irrelevant in the vanishing limit $\beta \to 0$. By using the Riemann normal coordinates on the second line in (7.7b), the fermionic delta function (7.4) and the first Bianchi identity (A.4a) acting on the fourth line in (7.7b), we obtain a much simpler expression of the Hirzebruch signature:

$$\sigma \ = \ \lim_{\beta \to 0} \frac{(-i)^{D/2}}{(2\pi)^{D/2}} \int \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D} \mathrm{d}\zeta^a \Big\langle \exp\Big(- \frac{1}{\hbar} S^{(\mathrm{int})} \Big) \Big\rangle \ , \tag{7.9a}$$

$$-\frac{1}{\hbar}S^{(\text{int})} = -\frac{1}{2\beta\hbar}R_{mnab}(\omega(x_0))\,\zeta^{ab}\int_{-1}^{0}\mathrm{d}\tau\,q^m\dot{q}^n$$
$$+\frac{1}{2\beta\hbar}R_{abcd}(\omega(x_0))\,\zeta^{ab}\Big(-\frac{1}{2}\int_{-1}^{0}\mathrm{d}\tau\,\xi^c_{\text{qu}}\xi^d_{\text{qu}} - \frac{1}{2}\int_{-1}^{0}\mathrm{d}\tau\,\overline{\xi}^c_{\text{qu}}\overline{\xi}^d_{\text{qu}} + \int_{-1}^{0}\mathrm{d}\tau\,\xi^c_{\text{qu}}\overline{\xi}^d_{\text{qu}}\Big) . \qquad (7.9\text{b})$$

We should notice that the fermionic fields in the above path integral have **anti-periodic** boundary condition. Originally the fermionic fields are introduced as the fields with anti-periodic boundary condition (see the discussion in section *2.4* of [21]), which is changed by the insertion of operators. Now, in the form (7.9) there are no additional operator insertions in the path integral measure. Thus the fermions in (7.9) keep the anti-periodic boundary condition.

We can easily find that the Feynman graphs will be described as the trace of Riemann curvature two-form in the same way as the Pontrjagin classes. Here let us remember a property that the trace of odd number of Riemann curvature two-form vanishes $\mathrm{tr}(R^{2k-1}) = 0$. On the other hand, the Feynman one-loop graph which contains all of three interaction terms in the second line in (7.9b) always has odd number of the interaction vertices. This indicates that the third interaction term in the second line $\xi^c_{\text{qu}}\overline{\xi}^d_{\text{qu}}$ should not connected to the other two interactions ($\xi^c_{\text{qu}}\xi^d_{\text{qu}}$ and $\overline{\xi}^c_{\text{qu}}\overline{\xi}^d_{\text{qu}}$) in the graphs. These other two terms should be connected with each other. Furthermore, because of the anti-periodicity of the fermions, we also find that the closed loop graphs which contain only the third interaction $\xi^c_{\text{qu}}\overline{\xi}^d_{\text{qu}}$ vanish in the same reason as the vanishing closed loop graphs of $\hat{c}$-ghost, which also has the anti-periodic boundary condition. The term in the first line exactly gives a same Feynman graphs as the Pontrjagin classes (4.12). Summarizing these comments, here let us again describe effective action in (7.9):

$$-\frac{1}{\hbar}S^{(\text{int})} = -\frac{1}{\beta\hbar}R_{mn}\int_{-1}^{0}\mathrm{d}\tau\,q^m\dot{q}^n - \frac{1}{2\beta\hbar}R_{cd}\int_{-1}^{0}\mathrm{d}\tau\,\xi^c_{\text{qu}}\xi^d_{\text{qu}} - \frac{1}{2\beta\hbar}R_{cd}\int_{-1}^{0}\mathrm{d}\tau\,\overline{\xi}^c_{\text{qu}}\overline{\xi}^d_{\text{qu}}$$
$$\equiv -\frac{1}{\hbar}\mathcal{S}_{\text{p}} - \frac{1}{\hbar}\mathcal{S} - \frac{1}{\hbar}\overline{\mathcal{S}} , \qquad (7.10\text{a})$$
$$R_{cd} \equiv \frac{1}{2}R_{cdab}(\omega(x_0))\,\zeta^{ab} = \frac{1}{2}R_{abcd}(\omega(x_0))\,\zeta^{ab} . \qquad (7.10\text{b})$$

Let us rewrite the exponent $\langle\exp(-\frac{1}{\hbar}S^{(\text{int})})\rangle$ in terms of the effective action W in such a way as

$$-\frac{1}{\hbar}W = \log\Big\langle\exp\Big(-\frac{1}{\hbar}S^{(\text{int})}\Big)\Big\rangle = \sum_{k=1}^{\infty}\frac{1}{k!}\Big\langle\!\Big\langle\Big(-\frac{1}{\hbar}S^{(\text{int})}\Big)^k\Big\rangle\!\Big\rangle$$
$$\sim \sum_{k=1}^{\infty}\frac{1}{k!}\Big\langle\!\Big\langle\Big(-\frac{1}{\hbar}\mathcal{S}_{\text{p}}\Big)^k\Big\rangle\!\Big\rangle + \sum_{k=0}^{\infty}\frac{1}{k!}\frac{k!}{(k/2)!(k/2)!}\Big\langle\!\Big\langle\Big(-\frac{1}{\hbar}\mathcal{S}\Big)^{k/2}\Big(-\frac{1}{\hbar}\overline{\mathcal{S}}\Big)^{k/2}\Big\rangle\!\Big\rangle$$
$$= \sum_{k=1}^{\infty}\frac{1}{k!}\Big(-\frac{1}{\beta\hbar}\Big)^k R_{m_1n_1}\cdots R_{m_kn_k}\int_{-1}^{0}\mathrm{d}\tau_1\cdots\mathrm{d}\tau_k\Big\langle\!\Big\langle(q^{m_1}\dot{q}^{n_1})(\tau_1)\cdots(q^{m_k}\dot{q}^{n_k})(\tau_k)\Big\rangle\!\Big\rangle$$
$$+\sum_{\ell=1}^{\infty}\frac{1}{\ell!\ell!}\Big(-\frac{1}{2\beta\hbar}\Big)^{2\ell}R_{a_1b_1}\cdots R_{a_\ell b_\ell}R_{c_1d_1}\cdots R_{c_\ell d_\ell}\int_{-1}^{0}\mathrm{d}\tau_1\cdots\mathrm{d}\tau_\ell\int_{-1}^{0}\mathrm{d}\sigma_1\cdots\mathrm{d}\sigma_\ell$$

$$
\times \left\langle\!\!\left\langle \big(\xi^{a_1}_{\rm qu}\xi^{b_1}_{\rm qu}\big)(\tau_1)\cdots\big(\xi^{a_\ell}_{\rm qu}\xi^{b_\ell}_{\rm qu}\big)(\tau_\ell)\big(\overline{\xi}^{c_1}_{\rm qu}\overline{\xi}^{d_1}_{\rm qu}\big)(\sigma_1)\cdots\big(\overline{\xi}^{c_\ell}_{\rm qu}\overline{\xi}^{d_\ell}_{\rm qu}\big)(\sigma_\ell)\right\rangle\!\!\right\rangle , \tag{7.11}
$$

where we extracted terms which contribute to the Feynman graphs in the vanishing limit $\beta \to 0$. The bracket $\langle\!\langle\cdots\rangle\!\rangle$ gives connected Feynman graphs. The number of the vertices $\mathcal{S}$ should be equal to the number of the vertices $\overline{\mathcal{S}}$ in order to obtain non-trivial graphs. Because of this, we find that k should be even: $k = 2\ell$.

Since we have already analyzed the first connected graphs in the Pontrjagin classes (4.15), it is easy to analyze the first term in (7.11):

$$
\begin{aligned}
\text{(1st term)} \ &= \ \sum_{k=1}^{\infty}\frac{1}{k!}\Big(-\frac{1}{\beta\hbar}\Big)^k (k-1)!\,2^{k-1}\big(-\beta\hbar\big)^k \cdot R_{m_1n_1}R_{m_2n_2}\cdots R_{m_kn_k}\,g^{n_1m_2}g^{n_2m_3}\cdots g^{n_km_1} \\
&\qquad \times \int_{-1}^{0}\mathrm{d}\tau_1\cdots\mathrm{d}\tau_k\,\partial_{\tau_1}\Delta(\tau_1,\tau_2)\partial_{\tau_2}\Delta(\tau_2,\tau_3)\cdots\partial_{\tau_{k-1}}\Delta(\tau_{k-1},\tau_k)\partial_{\tau_k}\Delta(\tau_k,\tau_1) \\
&\equiv \ \frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{k}\mathrm{tr}\big\{(2R)^k\big\}I_k \ = \ \frac{1}{2}\mathrm{tr}\log\Big(\frac{R}{\sinh R}\Big) .
\end{aligned} \tag{7.12a}
$$

Next let us here evaluate the second connected graphs in (7.11). In order to make one-loop graphs, ℓ vertices $\mathcal{S}$ and ℓ vertices $\overline{\mathcal{S}}$ should be alternatively located on the one-loop graph in $(\ell-1)!\ell!$ ways. Furthermore, there are $2^{2\ell-1}$ ways to contract these vertices in terms of fermion propagator (7.8) to yield the trace of curvature two-forms $\mathrm{tr}(R_2^{2\ell})$ with sign $(-1)^{\ell+1}$, which comes from permutation of indices. Then, the effective action (7.11) is evaluated in the following way:

$$
\begin{aligned}
\text{(2nd term)} \ &= \ \sum_{\ell=1}^{\infty}\frac{1}{\ell!\ell!}\Big(-\frac{1}{2\beta\hbar}\Big)^{2\ell}(\ell-1)!\ell!\,2^{2\ell-1}\,(-1)^{\ell+1}\,\big(\beta\hbar\big)^{2\ell}\cdot\mathrm{tr}\big(R^{2\ell}\big) \\
&\qquad \times \int_{-1}^{0}\prod_{i=1}^{\ell}\mathrm{d}\tau_i\,\mathrm{d}\sigma_i\,\theta(\tau_1-\sigma_1)\theta(\tau_1-\sigma_\ell)\,\theta(\tau_2-\sigma_2)\theta(\tau_2-\sigma_1)\cdots\theta(\tau_\ell-\sigma_\ell)\theta(\tau_\ell-\sigma_{\ell-1}) \\
&\equiv \ \frac{1}{2}\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell+1}}{\ell}\,\mathrm{tr}\big(R^{2\ell}\big)J_{2\ell} \ = \ \frac{1}{2}\mathrm{tr}\log\Big(\cosh R\Big) .
\end{aligned} \tag{7.13}
$$

The term $\ell = 0$ does not contribute to connected graphs because this term does not carry any background fermions. The function $J_{2\ell}$ is defined in such a way as

$$
J_{2\ell} \ \equiv \ \int_{-1}^{0}\prod_{i=1}^{\ell}\mathrm{d}\tau_i\,\mathrm{d}\sigma_i\,\theta(\tau_1-\sigma_1)\theta(\tau_1-\sigma_\ell)\,\theta(\tau_2-\sigma_2)\theta(\tau_2-\sigma_1)\cdots\theta(\tau_\ell-\sigma_\ell)\theta(\tau_\ell-\sigma_{\ell-1}) . \tag{7.14}
$$

Thus, substituting (7.12a) and (7.13) into (7.11), we obtain

$$
-\frac{1}{\hbar}W \ = \ \frac{1}{2}\mathrm{tr}\log\Big(\frac{R}{\sinh R}\Big) + \frac{1}{2}\mathrm{tr}\log\Big(\cosh R\Big) \ = \ \frac{1}{2}\mathrm{tr}\log\Big(\frac{R}{\tanh R}\Big) . \tag{7.15}
$$

Rescaling $\zeta^a \to \sqrt{\frac{-i}{2\pi}}\zeta^a$, we finally obtain the Hirzebruch signature on a Riemannian geometry

$$
\sigma \ = \ \int \mathrm{d}^D x_0\sqrt{g(x_0)}\prod_{a=1}^{D}\mathrm{d}\zeta^a\ \exp\Big[\frac{1}{2}\mathrm{tr}\log\Big(\frac{-iR/2\pi}{\tanh(-iR/2\pi)}\Big)\Big] , \tag{7.16}
$$

or, if we integrate out the fermionic fields and using the following formula (in the same way as (6.11)), we simplify (7.16) and obtain

$$\sigma \ = \ \int_{\mathcal{M}} \exp \Big[\frac{1}{2} \mathrm{tr} \log \Big(\frac{iR/2\pi}{\tanh(iR/2\pi)} \Big) \Big] \, , \qquad R_{mn} \ = \ \frac{1}{2} R_{mnab}(\omega) \, e^a \wedge e^b \, . \tag{7.17}$$

7.2.2 Torsional geometry: $H \neq 0$, $\mathrm{d}H = 0$ case

Now let us analyze the signature on the torsional geometry. It seems that the action (7.7b) carries the background interaction terms of order in $(\beta\hbar)^{-1}$, which cause the divergence of the amplitude in the vanishing limit $\beta \to 0$. Fortunately, however, the fermionic delta function (7.4) removes this difficulty:

$$- \frac{1}{16\beta\hbar} \int_{-1}^{0} \mathrm{d}\tau \, R_{[abcd]}(\omega_+(x)) \Big\{ \zeta^{abcd} + \overline{\eta}^{abcd} - 2 \zeta^{ab} \overline{\eta}^{cd} \Big\} \Big|_{(7.4)} \ = \ 0 \, . \tag{7.18}$$

There are no other background interaction terms of order $(\beta\hbar)^{-1}$. Then we again find that only the interaction terms carrying two quantum fields contribute to the non-trivial Feynman amplitudes. We also find that the terms such as

$$- \frac{1}{16\beta\hbar} \partial_m R_{[abcd]}(\omega_+(x_0)) \, \zeta^{abc} \int_{-1}^{0} \mathrm{d}\tau \, q^m \xi^d_{\mathrm{qu}} \tag{7.19}$$

yield only vanishing amplitudes because of the same reason in (4.22). By using the equation (4.8), we reduce the effective action (7.7b) to

$$\begin{aligned} - \frac{1}{\hbar} S^{(\mathrm{int})}_H \ &= \ - \frac{1}{2\beta\hbar} \mathcal{R}_{mnab} \, \zeta^{ab} \int_{-1}^{0} \mathrm{d}\tau \, \dot{q}^m q^n \\ &\quad + \frac{1}{2\beta\hbar} \mathfrak{R}_{cdab} \, \zeta^{ab} \int_{-1}^{0} \mathrm{d}\tau \Big(- \frac{1}{2} \int_{-1}^{0} \mathrm{d}\tau \, \xi^c_{\mathrm{qu}} \xi^d_{\mathrm{qu}} - \frac{1}{2} \int_{-1}^{0} \mathrm{d}\tau \, \overline{\xi}^c_{\mathrm{qu}} \overline{\xi}^d_{\mathrm{qu}} + \int_{-1}^{0} \mathrm{d}\tau \, \xi^c_{\mathrm{qu}} \overline{\xi}^d_{\mathrm{qu}} \Big) \, , \end{aligned} \tag{7.20}$$

where the tensors $\mathcal{R}_{mnab}$ and $\mathfrak{R}_{abcd}$ have already appeared in the previous sections as (4.25b) and (6.14), respectively. Since the form (7.20) is exactly same as (7.9b), we can use the same analysis and obtain the explicit form the effective action W_H as

$$\begin{aligned} - \frac{1}{\hbar} W_H \ &= \ \log \Big\langle \exp \Big(- \frac{1}{\hbar} S^{(\mathrm{int})}_H \Big) \Big\rangle \\ &= \ \frac{1}{2} \mathrm{tr} \log \Big(\frac{\mathcal{R}}{\sinh \mathcal{R}} \Big) + \mathrm{tr} \log \Big(\cosh \mathfrak{R} \Big) \ = \ \frac{1}{2} \mathrm{tr} \log \Big(\frac{\mathcal{R} \cosh \mathfrak{R}}{\sinh \mathcal{R}} \Big) \, . \end{aligned} \tag{7.21}$$

Rescaling $\zeta^a \to \sqrt{\frac{-i}{2\pi}} \zeta^a$, we finally obtain the Hirzebruch signature on a torsional geometry

$$\sigma \ = \ \int \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D} \mathrm{d}\zeta^a \, \exp \Big[\frac{1}{2} \mathrm{tr} \log \Big(\frac{(-i\mathcal{R}/2\pi) \cosh(-i\mathfrak{R}/2\pi)}{\sinh(-i\mathcal{R}/2\pi)} \Big) \Big] \, , \tag{7.22}$$

or, if we integrate out the fermionic fields and using the following formula (in the same way as (6.11)), we simplify (7.22) and obtain

$$\sigma = \int_{\mathcal{M}} \exp\left[\frac{1}{2}\mathrm{tr}\log\left(\frac{(i\mathcal{R}/2\pi)\cosh(i\mathfrak{R}/2\pi)}{\sinh(i\mathcal{R}/2\pi)}\right)\right] , \tag{7.23a}$$

$$\mathcal{R}_{mn} = \frac{1}{2}\Big\{R_{mnab}(\omega) - 2\widetilde{D}_{[m}H_{n]ab}\Big\}e^a \wedge e^b , \quad \mathfrak{R}_{ab} = \frac{1}{2}\Big\{R_{abcd}(\omega) - \frac{1}{2}R_{[abcd]}(\omega_+)\Big\}e^c \wedge e^d . \tag{7.23b}$$

8 Summary and discussions

In this paper we have studied various topological invariants on a torsional geometry in the framework of supersymmetric quantum mechanical path integral formalism. First we constructed the $\mathcal{N} = 1$ supersymmetric quantum mechanics (2.6) whose target space corresponds to the torsional geometry. We extended this to the $\mathcal{N} = 2$ quantum mechanics (2.17) with introducing a closed condition of the torsion. Next we described the transition elements which appear in the calculation of the Witten index. Following the work [21], we rewrote the transition elements from the Hamiltonian formalism to the Lagrangian formalism (3.21) in the $\mathcal{N} = 2$ case, and (3.23) in the $\mathcal{N} = 1$ case. Since we have already known these topological invariants on a Riemannian geometry in the framework of the quantum mechanical path integral, we applied the same formalism to the analyses of the Witten indices which should be interpreted as the Dirac index (5.17), and as the Euler characteristic (6.15) on a torsional geometry. We also analyzed the Hirzebruch signature on it (7.23). These modified values should also be topological invariants because we started from the well-defined supersymmetric algebras (2.1) in the $\mathcal{N} = 1$ case and (2.15) in the $\mathcal{N} = 2$ case, respectively. In these systems we can define the bosonic and fermionic states whose energy levels are degenerated. We should also find the zero energy eigenstates, which gives the Witten index as the topological value. We evalueted these Witten indices in various supersymmetric systems.

We improved the formulation of the Dirac index on a torsional geometry which have already been investigated by Mavromatos [16], Yajima [17], Peeters and Waldron [18], and so forth. The point is that we should carefully use the Riemann normal coordinate frame on the spin connection (and the affine connection) equipped with torsion. In addition, the extension of the Euler characteristic should play a crucial role in the analysis of the number of generation in the effective theory derived from string compactification with fluxes.

In this paper we imposed the closed condition on the totally anti-symmetric torsion $\mathrm{d}H = 0$. Peeters and Waldron [18] investigated the Dirac index on a four-dimensional geometry with boundary in the presence of a totally anti-symmetric torsion H, and discussed the role of $\mathrm{d}H$ in the Feynman graphs. The four-form $\mathrm{d}H$ can be described as the Nieh-Yan four-form $\mathcal{N}(e, H) = \mathrm{d}(e^A \wedge H_A)$, which appears in [27] and is applied to the analysis of the chiral anomaly [28], and the Dirac index [18]. To complete the analysis of the index theorems on a torsional geometry in the presence of non-vanishing $\mathrm{d}H$ is of particular importance when we study the string theory compactified on a G-structure manifold [29, 8, 22].

This four-form $\mathrm{d}H$ also appears and plays a crucial role in the anomaly cancellation mechanism in heterotic string theory (for instructive references, see [30, 15, 21]). In the usual anomaly cancellation in heterotic string, the Bianchi identity of the NS-NS three-form H is given in terms of the Riemann curvature two-form and the field strength of the gauge field [31]: $\mathrm{d}H = -\alpha'[\mathrm{tr}\{R(\omega) \wedge R(\omega)\} - \mathrm{tr}(F \wedge F)]$. In the presence of non-vanishing H-flux in the spacetime, the spin connection ω in the Bianchi identity is modified to $\omega_{+MAB} = \omega_{MAB} + H_{MAB}$ and the Bianchi identity is rewritten such as

$$\mathrm{d}H \ = \ -\alpha' \Big[\mathrm{tr}\big\{ R(\omega_+) \wedge R(\omega_+) \big\} - \mathrm{tr}\big(F \wedge F \big) \Big] \ , \tag{8.1}$$

when the supersymmetry variation of the gravitino is given by

$$\delta \psi_M \ = \ \Big(\partial_M + \frac{1}{4} \omega_{-MAB} \Gamma^{AB} \Big) \epsilon + \cdots \ , \tag{8.2}$$

where ϵ is the supersymmetry parameter and $\omega_{-MAB} = \omega_{MAB} - H_{MAB}$ [5]. The modification of the Bianchi identity (8.1) was, for instance, investigated by Hull [32] in the framework of the worldsheet sigma model. Bergshoeff and de Roo applied (8.1) to the supergravity Lagrangian with higher-order α' corrections [33]. Recent papers follow this modification and analyze the structures in the effective theories from the heterotic string (see, for instance, [34, 35, 36, 37, 38, 22, 9] and references therein). Since there is a deep relation between the anomalies and the index theorems, and the index theorems should be modified due to the existence of torsion, the formulations in the anomaly cancellation and the Bianchi identity of heterotic string theory might be modified further than (8.1). In order to re-investigate this insight, we should complete the analyses of the index theorems on a generic torsional geometry.

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Appendix

A Convention

We introduce vielbeins $e_M{}^A$ and their inverses $E_A{}^M$, which come from the spacetime metric g_{MN} and the metric η_{AB} on orthogonal frame via $g_{MN} = \eta_{AB}\, e_M{}^A\, e_N{}^B$ and $\eta_{AB} = g_{MN}\, E_A{}^M\, E_B{}^N$. By using these geometrical variables, let us define the covariant derivatives $D_M(\Gamma)$ in such a way as

$$D_M(\Gamma) A_N \;=\; \partial_M A_N - \Gamma^P{}_{NM} A_P \;, \tag{A.1a}$$

$$D_M(\Gamma) g_{NP} \;\equiv\; 0 \;=\; \partial_M g_{NP} - \Gamma^Q{}_{NM} g_{QP} - \Gamma^Q{}_{PM} g_{NQ} \;, \tag{A.1b}$$

$$\Gamma^P_{0MN} \;=\; \frac{1}{2} g^{PQ} \big(\partial_M g_{QN} + \partial_N g_{MQ} - \partial_Q g_{MN} \big) \;, \quad \Gamma^P{}_{[NM]} \;=\; T^P{}_{NM} \;, \tag{A.1c}$$

$$[D_M(\Gamma), D_N(\Gamma)] A_Q \;=\; -R^P{}_{QMN}(\Gamma) A_P + 2T^P{}_{MN}\, D_Q(\Gamma) A_P \;, \tag{A.1d}$$

$$R^P{}_{QMN}(\Gamma) \;=\; \partial_M \Gamma^P{}_{QN} - \partial_N \Gamma^P{}_{QM} + \Gamma^P{}_{RM} \Gamma^R{}_{QN} - \Gamma^R{}_{RN} \Gamma^R{}_{QM} \;. \tag{A.1e}$$

Note that A_M in the above equations are vector. $\Gamma^P{}_{MN}$ is the affine connection whose two lower indices are not symmetric in general case. The symmetric part of the affine connection is given in terms of the Levi-Civita connection $\Gamma^P_{0MN} = \Gamma^P{}_{(MN)}$, while the anti-symmetric part is defined as a torsion $T^P{}_{MN} = \Gamma^P{}_{[MN]}$. We also introduce the covariant derivative induced by the local Lorentz transformation acting on a generic field ϕ^i as

$$D_M(\omega) \phi^i \;=\; \Big\{ \delta^i_j\, \partial_M - \frac{i}{2} \omega_M{}^{AB} \cdot (\Sigma_{AB})^i{}_j \Big\} \phi^j \;, \tag{A.2}$$

where Σ_{AB} is the Lorentz generator whose explicit form depends on the representation of the field ϕ^i. The curvature tensor associated with this covariant derivative is given in terms of the spin connection

$$[D_M(\omega), D_N(\omega)] \phi \;=\; -\frac{i}{2} R^{AB}{}_{MN}(\omega)\, \Sigma_{AB} \phi \;, \tag{A.3a}$$

$$R^{AB}{}_{MN}(\omega) \;=\; \partial_M \omega_N{}^{AB} - \partial_N \omega_M{}^{AB} + \omega_M{}^A{}_C\, \omega_N{}^{CB} - \omega_N{}^A{}_C\, \omega_M{}^{CB} \;. \tag{A.3b}$$

We also describe the first and second Bianchi identity on Riemann tensor:

$$\text{1st:} \quad 0 \;=\; R^M{}_{NPQ}(\Gamma_0) + R^M{}_{PQN}(\Gamma_0) + R^M{}_{QNP}(\Gamma_0) \;, \tag{A.4a}$$

$$\text{2nd:} \quad 0 \;=\; \nabla_M R^N{}_{PQR}(\Gamma_0) + \nabla_Q R^N{}_{PRM}(\Gamma_0) + \nabla_R R^N{}_{PMQ}(\Gamma_0) \;. \tag{A.4b}$$

It is worth mentioning the definition of differential forms on a D-dimensional Riemannian manifold with Euclidean signature. because an analysis of Killing spinor equation and algebraic equations derived from the supersymmetry variations of fermions becomes much simpler. Here let us enumerate a set of definitions of p-form, the Hodge star, volume form, and so forth:

$$\omega_p \;=\; \frac{1}{p!} \omega_{M_1 \cdots M_p}\, \mathrm{d}x^{M_1} \wedge \cdots \wedge \mathrm{d}x^{M_p} \;, \tag{A.5a}$$

$$*\omega_p \; = \; \frac{\sqrt{g_D}}{p!(D-p)!}\, \varepsilon_{N_{p+1}\cdots N_D}{}^{M_1\cdots M_p}\, \omega_{M_1\cdots M_p}\, \mathrm{d}x^{N_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{N_D} \; , \tag{A.5b}$$

$$* * \omega_p \; = \; (-1)^{p(D-p)} \omega_p \; , \tag{A.5c}$$

$$T_{M_1\cdots M_D} \varepsilon^{M_1\cdots M_D} \; = \; T_{M_1\cdots M_D} g^{M_1 N_1} \cdots g^{M_D N_D} \varepsilon_{N_1\cdots N_D} \; = \; T^{N_1\cdots N_D} \Big(\frac{1}{g_D} \varepsilon_{N_1\cdots N_D} \Big) \; , \tag{A.5d}$$

$$\mathrm{d}x^{M_1} \wedge \cdots \wedge \mathrm{d}x^{M_D} \; = \; g_D\, \varepsilon^{M_1\cdots M_D}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^D \; , \tag{A.5e}$$

$$\mathrm{d}^D x \; \equiv \; \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^D \; = \; \frac{1}{D!} \varepsilon_{M_1\cdots M_D}\, \mathrm{d}x^{M_1} \wedge \cdots \wedge \mathrm{d}x^{M_D} \; , \tag{A.5f}$$

$$g_D\, \varepsilon^{M_1\cdots M_p}{}_{N_{p+1}\cdots N_D} \cdot \varepsilon_{M_1\cdots M_p}{}^{L_{p+1}\cdots L_D} \; = \; p!(D-p)! \cdot \delta^{L_{p+1}}_{[N_{p+1}} \cdots \delta^{L_D}_{N_D]} \; , \tag{A.5g}$$

where g_D is the determinant of the metric g_{MN} in D-dimensional geometry. The variables with anti-symmetrized indices $\varepsilon_{M_1\cdots M_D}$ and $\varepsilon^{M_1\cdots M_D}$ are tensor density of weight $+1$ and -1, respectively. We also introduce the antisymmetric tensor $\mathcal{E}_{A_1\cdots A_D}$ in the frame coordinate system with normalization $\mathcal{E}_{12\cdots D} = 1$. The relation between $\varepsilon_{M_1\cdots M_D}$ and $\mathcal{E}_{A_1\cdots A_D}$ is given by the contraction with other anti-symmetric tensors:

$$T_{M_1\cdots M_D}\, \varepsilon^{M_1\cdots M_D} \; = \; \Big(\frac{1}{\sqrt{g_D}} T_{A_1\cdots A_D} \Big) \mathcal{E}^{A_1\cdots A_D} \; = \; T^{M_1\cdots M_D} \Big(\frac{1}{g_D} \varepsilon_{M_1\cdots M_D} \Big) \; , \tag{A.6a}$$

$$T^{M_1\cdots M_D}\, \varepsilon_{M_1\cdots M_D} \; = \; \Big(\sqrt{g_D}\, T^{A_1\cdots A_D} \Big) \mathcal{E}_{A_1\cdots A_D} \; , \tag{A.6b}$$

$$T_{A_1\cdots A_D}\, \mathcal{E}^{A_1\cdots A_D} \; = \; T^{A_1\cdots A_D}\, \mathcal{E}_{A_1\cdots A_D} \; . \tag{A.6c}$$

B $SU(3)$-structure manifolds in heterotic string

In this appendix we discuss classification of six-dimensional compactified torsional geometries in (heterotic) string theory. Since we mainly study supergravity theories as low energy effective theories of string theories, we always assume the existence of the metric g_{mn} on the geometry. We also assume that the existence of almost complex structure $J_m{}^n$ associated with the metric, and dilaton field Φ. In generic case of the string compactification, we can also introduce non-trivial NS-NS flux H_{mnp} with its Bianchi identity on the geometry. In type II theories appropriate R-R fluxes are also incorporated. All of these are strongly related via the preserved condition of supersymmetry. In heterotic case, supersymmetry variations of gravitino, gaugino and dilatino give rise to Killing spinor equation and some algebraic equations among the above bosonic variables (see, for instance, [22]). These analysis becomes much clear when we introduce a set of mathematical definitions such as

$$\text{Lee-form :} \qquad \theta \; \equiv \; J \lrcorner \mathrm{d}J \; = \; \frac{3}{2} J^{mn} \partial_{[m} J_{np]}\, \mathrm{d}y^p \; , \tag{B.1a}$$

$$\text{Nijenhuis tensor :} \qquad N_{mn}{}^p \; \equiv \; J_m{}^q \partial_{[q} J_{n]}{}^p - J_n{}^q \partial_{[q} J_{m]}{}^p \; = \; J_m{}^q \nabla_{[q} J_{n]}{}^p - J_n{}^q \nabla_{[q} J_{m]}{}^p \; , \tag{B.1b}$$

$$\text{Bismut torsion :} \qquad T^{(\mathrm{B})}_{mnp} \; \equiv \; \frac{3}{2} J_m{}^q J_n{}^r J_p{}^s \nabla_{[s} J_{qr]} \; = \; -\frac{3}{2} J_{[m}{}^q \nabla_{|q|} J_{np]} \; . \tag{B.1c}$$

When we discuss the number of generation in four-dimensional effective theories from the Euler characteristic χ on compactified geometry, we should impose that the compactified geometry is complex, i.e., the Nijenhuis tensor vanishes $\mathcal{N}_{mn}{}^{p} = 0$. For instance, if there are no fermion condensations and H-flux condensation in heterotic string compactified on a manifold with $SU(3)$-structure satisfying $D_m(\omega_-)J_{np} = 0$, the compactified geometry is always complex and becomes so-called a conformally balanced manifold, on which the dilaton field is related to the Lee-form $\theta = 2\mathrm{d}\Phi$ and $\mathrm{d}(\mathrm{e}^{-2\Phi} J \wedge J) = 0$. The H-flux corresponds to the Bismut torsion $T^{(\mathrm{B})}$ [39] and is also given in terms of the complex structure and the dilaton field [5]

$$H \;=\; T^{(\mathrm{B})} \;=\; \frac{1}{2} * \mathrm{e}^{2\Phi} \mathrm{d}\big(\mathrm{e}^{-2\Phi} J\big) \;=\; \frac{i}{2}(\partial - \overline{\partial}) J \; . \tag{B.2}$$

Furthermore, we can classify the appearing compactified geometry under a specific condition in the following way (see also the discussions in [40, 41, 37]):

$$\theta \;=\; 2\mathrm{d}\Phi \, , \quad \mathrm{d}\big(\mathrm{e}^{-2\Phi} J \wedge J\big) \;=\; 0 \qquad \rightarrow \text{conformally balanced} \tag{B.3a}$$

$$\text{if } \theta \;=\; 0 \qquad \rightarrow \text{balanced} \tag{B.3b}$$

$$\text{if } \mathrm{d}\big(\mathrm{e}^{-\Phi} J\big) \;=\; 0 \qquad \rightarrow \text{conformally Kähler} \tag{B.3c}$$

$$\text{if } \mathrm{d}T^{(\mathrm{B})} = 0 \qquad \rightarrow \text{strong Kähler with torsion} \tag{B.3d}$$

C Formulae

In the formulation of discretized and continuum path integral in quantum mechanics, we define a number of functions without ambiguities [21]. Here let us summarize functions which appear in propagators and their derivatives in the quantum mechanics.

$$\Delta(\sigma,\tau) \;=\; \sigma(\tau+1)\theta(\sigma-\tau) + \tau(\sigma+1)\theta(\tau-\sigma) \;=\; \Delta(\tau,\sigma) \, , \tag{C.1a}$$

$$\theta(\sigma-\tau)\big|_{\tau=\sigma} \;=\; \frac{1}{2} \, , \quad \theta(\tau-\sigma) \;=\; -\theta(\sigma-\tau) + 1 \, , \tag{C.1b}$$

$$\partial_\sigma \theta(\sigma-\tau) \;=\; \delta(\sigma-\tau) \, , \quad \partial_\sigma^2 \Delta(\sigma,\tau) \;=\; \delta(\sigma-\tau) \, , \tag{C.1c}$$

$$\int_{-1}^{0} \mathrm{d}\sigma \int_{-1}^{0} \mathrm{d}\tau \, \Delta(\sigma,\tau) \;=\; -\frac{1}{12} \, , \quad \int_{-1}^{0} \mathrm{d}\sigma \int_{-1}^{0} \mathrm{d}\tau \, \delta(\sigma-\tau) \, \theta(\sigma-\tau) \, \theta(\tau-\sigma) \;=\; \frac{1}{4} \, . \tag{C.1d}$$

Notice that $\delta(\sigma-\tau)$ should be regarded as the "Kronecker delta" instead of the delta function because this function appears in the discretized form of the path integral and we should take the continuum limit carefully.

By using the above basic functions, we should compute various kinds of integral when we analyze loop diagrams in the path integral formalism. In this paper we mainly use a set of useful formulae which appear in the derivation of invariant polynomials such as the Dirac genus, the Chern characters, the Hirzebruch signature, and so forth. Here we only list the formula for these invariant polynomials. When we derive the Dirac genus, we use the integral I_k defined as

$$I_k \equiv \int_{-1}^{0} \mathrm{d}\tau_1 \cdots \int_{-1}^{0} \mathrm{d}\tau_k \, \partial_{\tau_1}\Delta(\tau_1,\tau_2)\partial_{\tau_2}\Delta(\tau_2,\tau_3)\cdots\partial_{\tau_{k-1}}\Delta(\tau_{k-1},\tau_k)\partial_{\tau_k}\Delta(\tau_k,\tau_1) \, , \tag{C.2a}$$

$$\partial_{\tau_i}\Delta(\tau_i,\tau_{i+1}) = \tau_i + \theta(\tau_i - \tau_{i+1}) \, , \quad \sum_{k=2}^{\infty} \frac{y^k}{k} I_k = \log \frac{y/2}{\sinh(y/2)} \, . \tag{C.2b}$$

The following two integrals play important roles in the derivations of the Chern classes and the Hirzebruch signature:

$$\int_{-1}^{0} \mathrm{d}\sigma_1 \int_{-1}^{0} \mathrm{d}\sigma_2 \cdots \int_{-1}^{0} \mathrm{d}\sigma_k \, \theta(\sigma_1 - \sigma_2)\theta(\sigma_2 - \sigma_3)\cdots\theta(\sigma_{k-1} - \sigma_k)\theta(\sigma_k - \sigma_1) = 0 \, , \tag{C.3a}$$

$$\int_{-1}^{0} \mathrm{d}\sigma_1 \int_{-1}^{0} \mathrm{d}\sigma_2 \cdots \int_{-1}^{0} \mathrm{d}\sigma_k \, \theta(\sigma_1 - \sigma_2)\theta(\sigma_2 - \sigma_3)\cdots\theta(\sigma_{k-1} - \sigma_k) = \frac{1}{k!} \, , \tag{C.3b}$$

for $k \geq 2$. We also use the following integral when we derive the Hirzebruch signature:

$$J_{2\ell} = \int_{-1}^{0} \prod_{i=1}^{\ell} \mathrm{d}\tau_i \, \mathrm{d}\sigma_i \, \theta(\tau_1 - \sigma_\ell)\theta(\tau_1 - \sigma_1)\,\theta(\tau_2 - \sigma_1)\theta(\tau_2 - \sigma_2)\cdots\theta(\tau_\ell - \sigma_{\ell-1})\theta(\tau_\ell - \sigma_\ell) \, , \tag{C.4a}$$

$$\sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} y^{2\ell} J_{2\ell} = \log\big(\cosh y\big) \, . \tag{C.4b}$$

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