On the Correlation Structure of Microstructure Noise in Theory and Practice*

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Abstract

We argue for incorporating the financial economics of market microstructure into the financial econometrics of asset return volatility estimation. In particular, we use market microstructure theory to derive the cross-correlation function between latent returns and market microstructure noise, which feature prominently in the recent volatility literature. The cross-correlation at zero displacement is typically negative, and cross-correlations at nonzero displacements are positive and decay geometrically. If market makers are sufficiently risk averse, however, the cross-correlation pattern is inverted. Our results are useful for assessing the validity of the frequently-assumed independence of latent price and microstructure noise, for explaining observed cross-correlation patterns, for predicting as-yet undiscovered patterns, and for making informed conjectures as to improved volatility estimation methods.

Keywords: Realized volatility, Market microstructure theory, High-frequency data, Financial econometrics

JEL classification: G14, G20, D82, D83, C51

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1 Introduction

Recent years have seen substantial progress in asset return volatility measurement, with important applications to asset pricing, portfolio allocation and risk management. In particular, so-called realized variances and covariances ("realized volatilities"), based on increasingly-available high-frequency data, have emerged as central for several reasons.[1] They are, for example, largely model-free (in contrast to traditional model-based approaches such as GARCH), they are computationally trivial, and they are in principle highly accurate.

A tension arises, however, linked to the last of the above desiderata. Econometric theory suggests the desirability of sampling as often as possible to obtain highly accurate volatility estimates, but financial market reality suggests otherwise. In particular, microstructure noise (MSN) such as bid/ask bounce associated with ultra-high-frequency sampling may contaminate the observed price, separating it from the latent ("true") price and potentially rendering naively-calculated realized volatilities unreliable.

Early work (e.g., Andersen, Bollerslev, Diebold and Labys (2001), Andersen, Bollerslev, Diebold and Ebens (2001), Barndorff-Nielsen and Shephard (2002a), Barndorff-Nielsen and Shephard (2002b), Andersen, Bollerslev, Diebold and Labys (2003)) addressed the sampling issue by attempting to sample often, but not "too often," implicitly or explicitly using the volatility signature plot of Andersen, Bollerslev, Diebold and Labys (2000) to guide sampling frequency, typically resulting in use of five- to thirty-minute returns.[2]

Much higher-frequency data are usually available, however, so reducing the sampling frequency to insure against MSN discards potentially valuable information. To use all information, more recent work has emphasized MSN-robust realized volatilities that use returns sampled at very high frequencies. Examples include Zhang, Mykland and Aït-Sahalia (2005), Bandi and Russell (2008), Aït-Sahalia, Mykland and Zhang (2005), Hansen and Lunde (2006), Barndorff-Nielsen, Hansen, Lunde and Shephard (2008a), and Barndorff-Nielsen, Hansen, Lunde and Shephard (2008c). That literature is almost entirely *statistical*, however, which is unfortunate because it makes important assumptions regarding the nature of the latent price, the MSN, and their interaction, and purely statistical thinking offers little guidance. A central example concerns the interaction (if any) between latent price and MSN. Some authors such as Bandi and Russell assume no correlation (perhaps erroneously),

[1]Several surveys are now available, ranging from the comparatively theoretical treatments of Barndorff-Nielsen and Shephard (2007) and Andersen, Bollerslev and Diebold (2009) to the applied perspective of Andersen, Bollerslev, Christoffersen and Diebold (2006).

[2]The volatility signature plot shows average daily realized volatility as a function of underlying sampling frequency.

whereas in contrast Barndorff-Nielsen et al. (2008a) and Barndorff-Nielsen, Hansen, Lunde and Shephard (2008b) allow for correlation (perhaps unnecessarily).

To improve this situation, we explicitly recognize that MSN results from the *strategic behavior of economic agents*, and we push toward integration of the financial economics of market microstructure with the financial econometrics of volatility estimation. In particular, we explore the implications of microstructure theory for the relationship between latent price and MSN, characterizing the cross-correlation structure between latent price and MSN, contemporaneously and dynamically, in a variety of leading environments, including those of Roll (1984), Glosten and Milgrom (1985), Kyle (1985), Easley and O'Hara (1992), and Hasbrouck (2002).[3]

We view this paper as both a general "call to action" for incorporation of microstructure theory into financial econometrics, and a detailed analysis of the fruits of doing so in the specific and important context of volatility estimation, where the payoff is several-fold. Among other things, attention to market microstructure theory enables us to assess the likely validity of the independence assumption, to offer explanations of observed cross-correlation patterns, to predict the existence of as-yet undiscovered patterns, and to make informed conjectures as to improved volatility estimation methods.

We proceed as follows. In section 2 we provide an overview of various market microstructure models and introduce our general framework, which nests a variety of such models, and we provide a generic (model-free) result on the nature of correlation between latent price and MSN. In sections 3 and 4 we provide a detailed analysis of models of private information, and we distinguish two types of latent prices based on the implied level of market efficiency, treating strong form efficiency in section 3 and semi-strong form efficiency in section 4. We draw some implications of our findings for empirical work in section 5, and we conclude in section 6.

2 General Framework and Results

Here we introduce a general price process, relate it to existing market microstructure models, and derive a generic result on the correlation between latent price and MSN.

[3]For insightful surveys of the key models, see O'Hara (1995) and Hasbrouck (2007).

2.1 Price and Noise Processes

Let p_t^* denote the (logarithm of the) *strong form efficient price* of some asset in period t. This price, strictly exogenous and at time t known only to the informed traders, follows the process

$$p_t^* = p_{t-1}^* + \mu_t^* + \sigma\varepsilon_t, \tag{1}$$

$$\varepsilon_t \overset{iid}{\sim} (0,1), \tag{2}$$

where μ_t^* denotes its drift.

Let q_t denote the direction of the trade in period t, where $q_t = +1$ denotes a buy and $q_t = -1$ a sell. Using this, the *semi-strong form efficient price*, which summarizes the current knowledge of the market maker, is

$$\tilde{p}_t = \tilde{\tilde{p}}_t + \lambda_t q_t. \tag{3}$$

$\lambda_t \geq 0$ captures the response to asymmetric information revealed by trade direction q_t, and $\tilde{\tilde{p}}_t$ is the expected efficient price before the trade occurs. This price evolves according to

$$\tilde{\tilde{p}}_t = \tilde{p}_{t-1} + \tilde{\mu}_t + c_t, \tag{4}$$

where $\tilde{\mu}_t$ is its drift, and c_t summarizes information about p_{t-1}^* revealed in period t. We use the term "*latent price*" as a general term comprising both types of efficient prices.

Assuming that the (logarithm of) price quotes are symmetric around the expected efficient price before the trade,[4] the (logarithm of the) observed *transaction price* can be written as

$$p_t = \tilde{\tilde{p}}_t + s_t q_t, \tag{5}$$

where s_t is one-half of the spread. In particular, the bid price is $p_t^{bid} = \tilde{\tilde{p}}_t - s_t$, the ask price is $p_t^{ask} = \tilde{\tilde{p}}_t + s_t$, and the mid price is $p_t^{mid} = \tilde{\tilde{p}}_t$.

We define returns as price changes net of drift. *Strong form efficient returns* are therefore

$$\Delta p_t^* \equiv p_t^* - p_{t-1}^* - \mu_t^* = \sigma\varepsilon_t, \tag{6}$$

[4] We use the approximation $ln(P+S) = ln\left(P\left(1+\frac{S}{P}\right)\right) = p + ln\left(1+\frac{S}{P}\right) \approx p + \frac{S}{P} \equiv p + s$, where P and S denote price and spread before taking the natural logarithm.

semi-strong form efficient returns are

$$\Delta\tilde{p}_t \equiv \tilde{p}_t - \tilde{p}_{t-1} - \tilde{\mu}_t = \lambda_t q_t + c_t, \tag{7}$$

and *market returns* are

$$\begin{aligned} \Delta p_t \equiv p_t - p_{t-1} - \tilde{\mu}_t &= \Delta\tilde{\tilde{p}}_t + s_t q_t - s_{t-1} q_{t-1} \\ &= \Delta\tilde{p}_t + (s_t - \lambda_t) q_t - (s_{t-1} - \lambda_{t-1}) q_{t-1}. \end{aligned} \tag{8}$$

In absence of persistent bubbles the drift of all three prices must be equal in the long run. We thus set $\mu_t^* = \tilde{\mu}_t \equiv \mu_t$.

Microstructure *noise* (MSN) is the difference between the observed market return and the latent return.[5] Depending on whether one considers the strong form efficient return or the semi-strong form efficient return, the noise is defined either as *strong form noise*

$$\Delta u_t \equiv \Delta p_t - \Delta p_t^*, \tag{9}$$

or as *semi-strong form noise*

$$\Delta u_t \equiv \Delta p_t - \Delta\tilde{p}_t. \tag{10}$$

As we show in this paper, these two types of noise differ fundamentally in their cross-correlation properties. It is therefore essential for a researcher to be clear in advance what type of latent price the object of interest is, because each type of efficiency requires different procedures to remove MSN appropriately.

A convenient estimator of the variance of the strong form efficient return, σ^2, is the realized variance (Andersen, Bollerslev, Diebold and Labys 2001). Realized variance during the time interval $[0, T]$ is defined as the sum of squared market returns over the interval, i.e. as

$$Var(\Delta p_t) = \sum_{t=1}^{T} \Delta p_t^2. \tag{11}$$

In the presence of MSN, the realized variance is generally a biased estimate of the variance of the efficient return, σ^2. To see this, decompose the noise into two components, i.e. $\Delta u_t = \Delta u_t^{ba} + \Delta u_t^{asy}$. The first component, Δu_t^{ba}, is assumed to be uncorrelated with the latent price of interest, reflecting for example the bid/ask bounce. The second component,

[5] We assume throughout that market prices p_t adjust sufficiently fast such that the noise process Δu_t is covariance stationary.

Δu_t^{asy}, is correlated with the efficient price, and reflects for example the effect of asymmetric information. Realized variance can now be decomposed – here shown for the strong form efficient price – as

$$\begin{aligned} Var(\Delta p_t) &= Var(\Delta p_t^* + \Delta u_t^{ba} + \Delta u_t^{asy}) \\ &= \sigma^2 + Var(\Delta u_t^{ba}) + Var(\Delta u_t^{asy}) + 2Cov(\Delta p_t^*, \Delta u_t^{asy}). \end{aligned} \tag{12}$$

The bias of the realized variance can stem from any of the last three terms, which are all nonzero in general. Realized variance estimation under the independent noise assumption accounts for the second and third positive terms, but ignores the last term, which is typically negative (Hansen and Lunde 2006). Correcting the estimates for independent noise only always reduces the volatility estimate. But because such a correction ignores the last term, which is the second channel through which asymmetric information affects the realized variance estimate, the overall reduction might be too much. Further, serial correlation of noise, or equivalently a cross-correlation between noise and latent returns at nonzero displacement, requires the use of robust estimators for both the variance and the covariance terms. In this paper we determine what correlation and serial correlation market microstructure theory predicts, and how market microstructure theory can be useful for obtaining improved estimates of integrated variance.

2.2 Institutional Setting

Price and noise processes as defined in the previous subsection suffice to mechanically derive expressions for their cross-correlation. However, this reduced form setup does not give much guidance about sign and time pattern of these cross-correlations. Without any microstructure foundation a purely statistical MSN correction blindly removes any kind of correlation. It may unintentionally remove part of the information component of the price, thereby introducing a new type of bias into the "corrected" price series. A more careful noise correction removes *only* noise patterns that can be traced back to market microstructure phenomena. For this reason we provide in this section a general setup, which contains many market microstructure models as special cases. These models allow us to determine the sign and describe the pattern of cross-correlations *due to MSN*. These are the patterns that in our view any serious MSN correction must remove, not more, and not less.

As we will see, the key determinants of the shape of the cross-correlation function between latent returns and MSN are market structure and the market maker's loss function.

2.2.1 Market Microstructure

Whereas the strong form efficient price is exogenous, the semi-strong form efficient price and the market price are the outcome of the market participants' optimizing behavior. Generally speaking, the market price depends on the information available about the strong form efficient price and the market participants' response to this information. The *information* process matters in two ways: First, via its information content, and second, via the time span in which it is not publicly known but valid. The *price updating* rule determines how, and how quickly, market prices respond to new information. Of particular importance is whether the market maker can quote prices dependent on the direction of trade, i.e. whether he can charge a spread, because direction-dependent quotes allow prices to react instantaneously.

Let Ω_t denote all public information available at time t. In particular the market maker has no information beyond Ω_t.[6] For convenience of exposition we use

Assumption 1 *The probability density function of* ε_t *is symmetric around its zero mean, monotonically increasing on* $]-\infty;0]$ *and monotonically decreasing on* $[0;\infty[$.

We analyze limit-order markets, populated by informed and uninformed traders. There are many market makers[7] which are in perfect competition with each other, and which serve as counterparty to all trades. The timing of information and actions in any given period, t, which is infinitely often repeated, is as follows:

1. p^*_{t-1} becomes public information, thus $\left\{p^*_{t-1}\right\} \in \Omega_t$.

2. p^*_t changes randomly.

3. The market maker observes Ω_t which contains at least all transaction prices and trades up to the previous period, i.e. $\{p_i, q_i\} \in \Omega_t \quad \forall i < t$. Ω_t may contain additional information about the *current* strong form efficient price, p^*_t, for example the direction of the price innovation, $\{\text{sgn}(\varepsilon_t)\}$.

4. The market maker quotes a pricing scheme for period t, i.e. a mid price $p^{mid}_t > 0$ and a spread $2s_t \geq 0$. The market maker is bound to transact one unit at this price.

[6] Drift μ_t, variance σ^2 and probability density function of ε_t are public knowledge. We assume perfect memory, $\Omega_i \subset \Omega_t \ \forall i \leq t$, and that given the information set Ω_t the market participants' optimizing behavior determines a unique market price $p(\Omega_t)$, with corresponding market return $\Delta p(\Omega_t, \Omega_{t-1})$.

[7] At least there needs to be one market maker and many *potential* competitors.

5. If informed traders are present (or "active"), they observe p_t^* and the market maker pricing scheme $\{p_t^{mid}, s_t\}$. If based on their private knowledge $p_t^* > p_t^{ask} \equiv p_t^{mid} + s_t$, then they try to buy an infinite amount, whereas if $p_t^* < p_t^{bid} \equiv p_t^{mid} - s_t$, they try to (short-) sell an infinite amount. However, the market maker fills the demand only up to his commitment limit, one unit. If a transaction takes place, the transaction price is $p_t = p_t^{ask}$ or $p_t = p_t^{bid}$, respectively. If neither buy nor sell is profitable, or if informed traders are not active in this period, then no informed trade occurs.

6. If there was no informed trade in step 5, uninformed traders trade randomly for exogenous reasons. For these traders buying at p_t^{ask} and selling at p_t^{bid} has equal probability, which allows market makers to earn the spread without risk. Denote this constant income per period by $\pi = \pi(s_t)$. Uninformed traders are the only source of revenue of the market maker.

7. If private information is valid for only one period, then the market continues with step 1. Otherwise, if information remains private for $T > 1$ periods, no further information is revealed at this moment and the market continues with step 3. Eventually after T loops p_t^* becomes public information and the market continues with step 1.

A second assumption helps us in greatly simplifying the model without affecting its basic behavior.

Assumption 2 *Ex ante ($t = 0$) buys ($q_t = +1$) and sells ($q_t = -1$) are equally likely, so that $E(q_t) = 0$. There is no "momentum" in uninformed trading, and thus trades are not serially correlated beyond the time of a strong form efficient price change, that is, $E(q_{\kappa T+\tau_1}|q_{\kappa T-\tau_2}) = 0 \ \ \forall \kappa \in Z, \forall \tau_1 \geq 0, \forall \tau_2 \geq 1$.*

This setup has an immediate implication. If no informed traders are present in the market, then $E\left(q_t \varepsilon_{t-\tau}\right) = 0 \quad \forall t, \ \forall \tau$, because uninformed trades are unrelated to p_t^*. In contrast, for informed trades $E(q_t \varepsilon_{t-\tau}) \geq 0 \ \ \forall t, \forall \tau \geq 0$, because informed traders buy only if the strong form price increased, and sell if it fell. Taken together, and using Jensen's inequality, it holds that

$$0 \leq E\left(q_t \varepsilon_{t-\tau}\right) \leq E\left(|\varepsilon_t|\right) \leq 1, \ \forall t, \forall \tau. \tag{13}$$

Having detailed the market microstructure, we now describe the behavior of the market maker.

2.2.2 The Market Maker

The loss function of the market maker pins down the optimal spread size and the response to a trade, and is thus a key determinant of the sign of the cross-correlations. Before trading occurs, the market maker has a belief about p_t^*, summarized by the prior probability density function $f(p_t^*)$. We require $f(p_t^*) = f(p_{t-1}^* + \mu_t + \varepsilon_t)$ to be consistent with Assumption 1 and denote the corresponding cumulative distribution function with $F(\cdot)$. Let $\underline{p}$ and $\overline{p}$ denote the lower and upper end of the support of p_t^* that the market maker has determined by previous experimentation.[8] We define the loss function of a market maker with risk aversion parameter $n \geq 1$ as

$$\breve{l}_n(x) = -|x|^n . \tag{14}$$

The market maker's per period loss is a function of disadvantageous differences between the strong form efficient price and the transaction price in periods of informed trading. In periods without any informed trading the market maker's loss is zero.[9] The expected loss in period t when the market price is set at p_t is[10]

$$\begin{aligned} L_n\left(p_t, \underline{p}, \overline{p}, F(\cdot)\right) &= E\left(l_n(p_t - p_t^*)\right) \\ &= -\int_{\underline{p}}^{\overline{p}} |p_t - p_t^*|^n \, f\left(p_t^*\right) \mu\left(p_t, p_t^*\right) dp_t^*, \end{aligned} \tag{15}$$

where $\mu\left(p_t, p_t^*\right) = Prob\,(\text{informed trade}|p_t^*, p_t)$.

The higher the risk aversion n, the more sensitive is the expected loss, $E\left(l_n(p_t - p_t^*)\right)$, to the support of p_t^*, that is, to $\underline{p}$ and $\overline{p}$. A well-known result is that the optimal choice for a risk neutral market maker ($n = 1$) is to set p_t equal to the median of $f(\cdot)$, and for a modestly risk averse market maker ($n = 2$) to the mean. An extremely risk averse ($n \to \infty$) market maker minimizes his expected loss at the price in the middle of the support of $f(\cdot)$, i.e. $p_t = \frac{\underline{p}+\overline{p}}{2}$.[11] If $f(\cdot)$ is unbounded on one side, this p_t is infinite.

[8] In the first period, either $\underline{p}$ and $\overline{p}$ are known, or are set to $\underline{p} = -\infty$ and $\overline{p} = \infty$.

[9] More comprehensive and realistic loss functions are possible, of course. For example, the loss function may be defined over *all* market maker income per period, not just over *deviations* from the income from uninformed trading. However, this would add extra complication without changing the effect of risk aversion on market maker behavior.

[10] $L_n(p_t, \underline{p}, \overline{p}, F(\cdot))^{1/n}$ is related to the ℓ_n metric. However, it differs in that it is reweighted, and sums over infinitely many elements. In particular, for $n \to \infty$ we have

$$L_{n\to\infty}(p_t, \underline{p}, \overline{p}, F(\cdot))^{1/n} = -sup\left\{|p_t - p_t^*|, p_t^* \in \left\{p|\underline{p} \leq p \leq \overline{p}, f(p) > 0\right\}\right\} = -sup\left\{|p_t - \overline{p}|, |p_t - \underline{p}|\right\}.$$

[11] See section 3.3.3.

Between any two changes in the strong form efficient price the market maker chooses the pricing scheme $\{p_t^{mid}, s_t\}$ such that his discounted loss-adjusted profit

$$\Pi_t(\underline{p}, \overline{p}, F) = E\left[(1-\delta)\sum_{i=0}^{\infty}\delta^i \pi(s_{t+i})\right] + V_t(\underline{p}_t, \overline{p}_t, F_t) \tag{16}$$

is maximized. δ denotes the market maker's discount factor, and $V_t(\cdot)$ is the total expected loss from trades with informed traders from period t onwards. The following assumption pins down the market marker behavior further.

Assumption 3 *Perfect competition among market makers implies that the market maker earns zero expected profit on each transaction, which pins down the spread* $2s_t$*, and the market makers' revenue* $\pi(s_t)$ *from transactions with uninformed traders. Individual market makers take the spread as given.*

Note that in general the spread must exceed the expected adverse selection effect, i.e. $s_t > \lambda_t$, because the market maker must cover his processing cost on top of the adverse selection cost. Under Assumption 3 the market maker's profit maximization problem (16) reduces to minimizing his expected loss from trades with informed traders, which can be written in recursive form as

$$\begin{aligned} V_t(\underline{p}_t, \overline{p}_t, F_t) \;=\; & \sum_{\tilde{\Omega}_{t+1}} P(\tilde{\Omega}_{t+1}) \max_{p^{bid}, p^{ask}} \Big[L_n(p^{bid}, \underline{p}_{t+1}, p^{bid}, \tilde{F}_{t+1}) \\ & + \delta V_{t+1}(\underline{p}_{t+1}, p^{bid}, \tilde{F}_{t+1}|_{\underline{p}_{t+1}}^{p^{bid}}) \left(\tilde{F}_{t+1}(p^{bid}) - \tilde{F}_{t+1}(\underline{p}_{t+1})\right) \\ & + \delta V_{t+1}(p^{bid}, p^{ask}, \tilde{F}_{t+1}|_{p^{bid}}^{p^{ask}}) \left(\tilde{F}_{t+1}(p^{ask}) - \tilde{F}_{t+1}(p^{bid})\right) \\ & + \delta V_{t+1}(p^{ask}, \overline{p}_{t+1}, \tilde{F}_{t+1}|_{p^{ask}}^{\overline{p}_{t+1}}) \left(\tilde{F}_{t+1}(\overline{p}_{t+1}) - \tilde{F}_{t+1}(p^{ask})\right) \\ & + L_n(p^{ask}, p^{ask}, \overline{p}_{t+1}, \tilde{F}_{t+1}) \Big], \end{aligned} \tag{17}$$

where $\tilde{F}_{t+1}$ is the (Bayesian) update of F_t using information $\tilde{\Omega}_t = \Omega_t \backslash \Omega_{t-1}$, $\tilde{F}(x)|_{x_1}^{x_2}$ is the cumulative distribution $\tilde{F}$ of x conditional on $x \in [x_1, x_2]$, $\overline{p}_{t+1}$ and $\underline{p}_{t+1}$ are the updated upper and lower bound of this distribution, $P(\tilde{\Omega})$ is the probability that the market maker observes the signal $\tilde{\Omega}$, and $p^{bid} \equiv p_{t+1}^{bid}(\underline{p}_{t+1}, \overline{p}_{t+1}, \tilde{F}_{t+1})$ and $p^{ask} \equiv p_{t+1}^{ask}(\underline{p}_{t+1}, \overline{p}_{t+1}, \tilde{F}_{t+1})$.

If $\tilde{\Omega}_{t+1}$ contains only information about period t and earlier, but no signal about $t+1$ values, and if the spread is fixed at a constant, then the market maker's problem becomes independent of time and his only choice variable is the location of the spread interval, p^{mid}.

(17) simplifies to

$$
\begin{aligned}
V(\underline{p},\overline{p},F) = \max_{p^{mid}} \quad & \left[L_n\left(p^{mid}-s,\underline{p},p^{mid}-s,F\right)\right. \\
+ \quad & \delta V\left(\underline{p},p^{mid}-s,F|_{\underline{p}}^{p^{mid}-s}\right)\left(F\left(p^{mid}-s\right)-F\left(\underline{p}\right)\right) \\
+ \quad & \delta V(p^{mid}-s,p^{mid}+s,F|_{p^{mid}-s}^{p^{mid}+s})\left(F(p^{mid}+s)-F(p^{mid}-s)\right) \\
+ \quad & \delta V\left(p^{mid}+s,\overline{p},F|_{p^{mid}+s}^{\overline{p}}\right)\left(F\left(\overline{p}\right)-F\left(p^{mid}+s\right)\right) \\
+ \quad & \left.L_n\left(p^{mid}+s,p^{mid}+s,\overline{p},F\right)\right], \qquad (18)
\end{aligned}
$$

where $p^{mid} \equiv p^{mid}(\underline{p},\overline{p},F)$.

The recursive problem (17) encompasses most cases that we discuss in this paper. Unfortunately, (17) and even (18) are hard to solve – in general the policy functions $p^{bid}(\cdot)$ and $p^{ask}(\cdot)$ are not available in closed form.[12]

In the following sections we look at specializations of the general market maker problem (17) and examine the effect of various model setups on the cross-correlation function. For both strong form and semi-strong form efficient returns we first examine the multiperiod case ($\delta \neq 0$), where private information is not revealed until after many periods. We then specialize to the one-period case ($\delta = 0$), a case where private information becomes public, and worthless, after only one period.

2.3 Cross-Correlations Between Latent Price and MSN

We focus in this paper on the *cross-correlation* between latent returns and noise contemporaneously and at all displacements. Throughout, we refer to this quantity simply as the "cross-correlation".

Proposition 1 ***(General cross-correlations)*** *Under the price processes given by (1)–(5) the contemporaneous cross-correlation ρ_0 is positive only if the market return, $\Delta p_t(\Omega_t,\Omega_{t-1})$, is more volatile than the latent return, that is, for strong form efficient returns*

$$
Corr(\Delta p_t^*,\Delta u_t) > 0 \Leftrightarrow E(\Delta p_t \Delta p_t^*) > Var(\Delta p_t^*) \Leftrightarrow Corr(\Delta p_t \Delta p_t^*) > \sqrt{\frac{Var(\Delta p_t^*)}{Var(\Delta p_t)}}, \qquad (19)
$$

[12] For characterizations of the general solution see Aghion, Bolton, Harris and Jullien (1991) and Aghion, Espinosa and Jullien (1993). Their solution shows that in general optimal learning requires λ_t in (3) to vary over time.

and for semi-strong form efficient returns

$$Corr(\Delta\tilde{p}_t, \Delta u_t) > 0 \Leftrightarrow E(\Delta p_t \Delta\tilde{p}_t) > Var(\Delta\tilde{p}_t) \Leftrightarrow Corr(\Delta p_t \Delta\tilde{p}_t) > \sqrt{\frac{Var(\Delta\tilde{p}_t)}{Var(\Delta p_t)}}. \quad (20)$$

The cross-correlation ρ_τ at displacement $\tau \geq 1$ is positive if and only if the current market price responds stronger in the direction of a previous latent price change than the current latent price itself, that is, for strong form efficient returns

$$Corr(\Delta p^*_{t-\tau}, \Delta u_t) > 0 \Leftrightarrow E(\Delta p_t \Delta p^*_{t-\tau}) > 0, \quad (21)$$

and for semi-strong form efficient returns

$$Corr(\Delta\tilde{p}_{t-\tau}, \Delta u_t) > 0 \Leftrightarrow E(\Delta p_t \Delta\tilde{p}_{t-\tau}) > E(\Delta\tilde{p}_t \Delta\tilde{p}_{t-\tau}). \quad (22)$$

Proof: See appendix A.

The importance of Proposition 1 stems from its generality. Without referring to any specific model of market participants' behavior, it nevertheless isolates the conditions on $\Delta p_t(\Omega_t, \Omega_{t-1})$ that determine the cross-correlation pattern. The next step, of course, is to characterize the properties of $\Delta p_t(\Omega_t, \Omega_{t-1})$ in the leading models of market microstructure. We now do so, treating in turn strong form and semi-strong form efficient prices.

3 Strong form Correlation

Here we characterize cross-correlations in an environment of strong form efficient prices. Accordingly, in this section "efficient price" means "strong form efficient price".

Suppose there is a single change in the strong form efficient price at a known time from a publicly known level, for example at the beginning of the day,[13] which lasts T periods. This allows studying the effect of one strong form efficient price change in isolation.

We first calculate the correlations between strong form efficient returns

$$\Delta p^*_t = \Delta p^*_{\kappa T} = \begin{cases} \sigma\varepsilon_{\kappa T} & \forall \kappa \in Z \\ 0 & \forall \kappa \notin Z \end{cases} \quad (23)$$

[13]With "day" we mean the average time between two changes in the strong form efficient price, which could be several days, or, more likely, just a few hours. Engle and Patton (2004) and Owens and Steigerwald (2005), for example, find evidence of multiple information arrivals during a calendar day.

and the corresponding noise[14]

$$\Delta u_t = \Delta p_t - \Delta p_t^* = \Delta\tilde{\tilde{p}}_t + s_t q_t - s_{t-1}q_{t-1} - \Delta p_t^*. \tag{24}$$

3.1 The General Multi-period Case

In the period of a change in the strong form efficient price the expectation about this price changes by[15]

$$\begin{aligned} \Delta\tilde{\tilde{p}}_0 &= \tilde{\tilde{p}}_0 - \tilde{\tilde{p}}_{-1} - \mu_0 \\ &= \sigma\varepsilon_{-T} - \sum_{t=2}^{T} \lambda_{-t} q_{-t}, \end{aligned} \tag{25}$$

and in all other periods by

$$\Delta\tilde{\tilde{p}}_t = \lambda_{t-1} q_{t-1}. \tag{26}$$

From (24) we get for $t = \kappa T$

$$\Delta u_0 = \sigma(\varepsilon_{-T} - \varepsilon_0) + s_0 q_0 - s_{-1} q_{-1} - \sum_{t=2}^{T} \lambda_{-t} q_{-t} \tag{27}$$

and $\forall t \neq \kappa T$

$$\Delta u_t = \lambda_{t-1} q_{t-1} + s_t q_t - s_{t-1} q_{t-1}, \tag{28}$$

where the first term reflects information-revealing trades, and the second and third term reflect the bid/ask bounce.

This immediately leads to the contemporaneous cross-covariance

$$Cov(\Delta p_t^*, \Delta u_t) = \frac{\sigma}{T} \left(s_0 E(q_0 \varepsilon_0) - \sigma \right). \tag{29}$$

For cross-covariance at higher displacements $\tau \in [1; T-1]$ we get

$$Cov(\Delta p_{t-\tau}^*, \Delta u_t) = \frac{\sigma}{T} \left((\lambda_{\tau-1} - s_{\tau-1}) E(q_{\tau-1} \varepsilon_0) + s_\tau E(q_\tau \varepsilon_0) \right), \tag{30}$$

[14] The drift, μ_t, is time-varying. Because it is publicly available information, it plays no role in our cross-correlation analysis. In contrast, q_t is driven by unobserved private information and is a key determinant of the cross-correlation patterns.

[15] As a shorthand notation we use $p_x \equiv p_{\kappa T + x}$ $\forall \kappa, x \in Z$.

for cross-covariance at displacement T

$$Cov(\Delta p^*_{t-T}, \Delta u_t) = \frac{\sigma}{T}\left(\sigma - s_{T-1}E(q_{T-1}\varepsilon_0) - \sum_{i=0}^{T-2}\lambda_i E(q_i\varepsilon_0)\right), \tag{31}$$

and for all higher order displacements $\tau > T$

$$Cov(\Delta p^*_{t-\tau}, \Delta u_t) = 0. \tag{32}$$

Expressions for the variance terms $Var(\Delta p^*_t)$ and $Var(\Delta u_t)$ are given in appendix B. The general expressions for the cross-correlations are complicated enough to make their discussion here unattractive, but we will use them on numerous occasions throughout this paper.

As indicated earlier, for any displacement τ ceteris paribus the term $E(q_\tau\varepsilon_0)$ is the smaller, the more uninformed trades take place. This term enters the expression for the contemporaneous cross-covariance (29) linearly and enters the denominator of the cross-correlation under a square root. Therefore, the contemporaneous cross-correlation is the smaller, the less informed traders are active. In absence of any informed traders, the market microstructure is reduced to a bid/ask bounce, as in Roll (1984). In this case, shown in the first row of Table 1, the contemporaneous cross-correlation (29) is negative, the cross-correlations at displacement T is positive and all other cross-correlations are zero.

If the spread is zero,[16] the contemporaneous cross-correlation is negative as well, but the cross-correlations at displacements up to $T-1$ are positive.

In general, however, the sign of the cross-correlations depends on the behavior of the market maker and traders. We now turn to models that allow us to introduce these explicitly.

3.2 Special Multi-period Cases

Because the market maker loses in every trade with an informed trader, he has an incentive to find out the strong form efficient price. He learns about the informed traders' private information by setting prices and observing the resulting trades. As he "learns by experimentation"[17] over time, the value of private information of the informed trader slowly vanishes. Although there are many possible interactions of strategic actions by market participants, we will see that rational behavior ensures that they all share the same cross-correlation sign

[16] A sequence of only bid prices (or only ask prices) is equivalent to $s_t = 0 \;\; \forall t$.

[17] Aghion et al. (1991), Aghion et al. (1993)

pattern and differ only in the absolute value of the cross-correlation.

The market maker does not observe p_t^* directly, but only signals which allow him to narrow down the range of the current p_t^* level. He observes in particular the response of traders to his previous price quote and uses this signal to revise his quote. Because the strong form efficient price, p_t^*, by assumption does not change after the initial jump for T periods, the market maker can use the entire sequence of signals to learn p_t^* over time. His optimization task is to quote prices that minimize his losses by learning about p_t^* as quickly as possible.

With $\delta \neq 0$ the recursive problem (17) is hard to solve, and in particular there are in general no closed form policy functions p_t^{bid} and p_t^{ask}. Therefore we follow the market microstructure literature by discussing interesting polar cases, which can be solved because $f(p_t^*)$ is degenerate. We assume in this section that market makers are risk neutral ($n = 1$) and limit our discussion to the mid price in order to study the learning effect in isolation.

3.2.1 Perfect Signal, No Strategic Traders

The market maker's learning speed depends on the reliability of the signal. Let us start with a situation where the signal is known to be free of noise and strategic manipulation by market participants. To learn as much as possible the market maker minimizes the length of the interval in which p_t^* may be located. In the special case of a constant spread during the interval between two latent price changes he solves (18) with $n = 1$

$$\begin{aligned} V(\underline{p}, \overline{p}, F) = \max_{p^{mid}} \quad & \Bigg[- \int_{\underline{p}}^{p^{mid}-s} (p^{mid} - s - p^*) f(p^*) dp^* \\ + \quad & \delta V \left(\underline{p}, p^{mid} - s, F|_{\underline{p}}^{p^{mid}-s} \right) \left(F\left(p^{mid} - s\right) - F\left(\underline{p}\right) \right) \\ + \quad & \delta V(p^{mid} - s, p^{mid} + s, F|_{p^{mid}-s}^{p^{mid}+s}) \left(F(p^{mid} + s) - F(p^{mid} - s) \right) \\ + \quad & \delta V \left(p^{mid} + s, \overline{p}, F|_{p^{mid}+s}^{\overline{p}} \right) \left(F\left(\overline{p}\right) - F\left(p^{mid} + s\right) \right) \\ - \quad & \int_{p^{mid}+s}^{\overline{p}} (p^* - p^{mid} - s) f(p^*) dp^* \Bigg]. \end{aligned} \tag{33}$$

Assuming that the spread s is sufficiently small, then from (29) the contemporaneous cross-correlation is negative, because in this case p_t shows barely any instantaneous reaction to Δp_t^*.[18] Because further by assumption p_t^* does not change for several periods ($\Delta p_{\kappa t}^* = 0$

[18] ρ_0 is negative, but strictly larger than negative one. This obtains, because p_t responds every period to noisy signals about p_0^*, which increases the noise variance.

$\forall \kappa \notin Z$) and learning takes several periods, the contemporaneous cross-correlation is larger in absolute value than cross-correlations at nonzero displacements. Likewise, by (30) the sign of cross-correlation at displacement one and higher is positive, because the more the market maker learns, the closer p_t gets to p_t^*, and the more noise shrinks to zero. Aghion et al. (1991) provide a thorough discussion of the market maker's learning problem.[19] If, further, the adverse selection coefficient λ in all periods is sufficiently small as well, by (31) the cross-correlation at displacement T is positive. We summarize these qualitative results in the second row of Table 1.

[Table 1 about here.]

3.2.2 Noisy Signal

The models so far did not account for signal uncertainty and strategic behavior. Here we do so. Consider first a market in which the market maker observes only a noisy signal of whether p_t^* has changed, but in which traders do not behave strategically yet. The market maker then has to learn both about the quality of the signal and about the latent price. Glosten and Milgrom (1985) describe a market maker who does not know whether he is trading with an informed or an uninformed trader and thus cannot tell whether his signal, the direction of trade q_t, contains any valuable information. For example, the market maker cannot tell whether a "buy" originates from an informed trader, in which case it would indicate an increase in the strong form efficient price, or whether it is just a random trade of an uninformed trader. Thus, this noisy "buy" increases the likelihood of an increase in the strong form efficient price less than a "buy" in the "perfect signal" environment of the previous paragraph.

Glosten and Milgrom (1985) show that if learning is costless, the expectations of market makers and traders converge as the number of trades increases.[20] Because of the uncertainty of whether a trade reflects the private information of the informed traders or not, the market maker adjusts only *partially* to the price indicated by any signal. Therefore, whereas the cross-correlations have the same sign as under signal certainty summarized in the second row of Table 1, all absolute values are dampened toward zero.

Easley and O'Hara (1992) additionally consider the information conveyed by periods of no trading in a model where the strong form efficient price is not a martingale. Their model

[19] See also appendix C for a simple example.

[20] But see Aghion et al. (1991) for situations in which learning stops before reaching p^*. In that case the cross-correlations cut off at some $\tau < T$.

is more abstract, but has the advantage that this pattern can be derived explicitly. Suppose at the beginning of the trading day watchful traders observe with probability α the new strong form efficient price, p_t^*, thereby becoming informed traders. This price is "low" ($\underline{p}$) with probability δ and "high" ($\overline{p}$) otherwise. The two possible latent price levels, $\underline{p}$ and $\overline{p}$, and their probability δ are publicly known, but the actual realization of p_t^* is not.

The direction-of-trade signal, q_t, is thereby uncertain in two ways in this model. Not only does the market maker not know if a specific trade originates from informed traders, thereby being informative; the market maker does not even know if there *are* any informed traders. He learns by updating in a Bayesian manner his belief about the probabilities that nobody observed a signal, that some traders observed a high p_t^*, and that some observed a low p_t^*, using his information set of all previous quotes and trades, Ω_t. Even non-trading intervals contain information about p_t^*, because they lower the probability that watchful traders have observed the strong form efficient price at the beginning of the trading day and therefore lower the probability of informed trading, too.[21]

The case of signal certainty discussed at the beginning of this subsection is trivial in this model: Signal certainty implies the absence of any uninformed traders. Because p_t^* can assume only one of two price levels, the first trade reveals the true strong form efficient price. Until the first trade occurs, the expected efficient price is $\delta\underline{p} + (1-\delta)\overline{p}$.

Turning to signal uncertainty, suppose first that informed traders trade at every profitable situation.[22] The contemporaneous cross-correlation in this case is for large T[23]

$$Corr(\Delta p_t^*, \Delta u_t) = -\left(K_1 + O\left(2^{-T}\right)\right)\frac{(\overline{p}-\underline{p})^2}{T} < 0, \tag{34}$$

where $K_1 = K_1(\alpha, \delta)$ and O is the Landau symbol for $T \to \infty$. At nonzero displacements the cross-correlation can be written for large τ as

$$Corr(\Delta p_{t-\tau}^*, \Delta u_t) = \left(\frac{1}{2}\right)^{\tau}\left(K_2 + O\left(2^{-\tau}\right)\right)\frac{(\overline{p}-\underline{p})^2}{T} > 0, \tag{35}$$

where $K_2 = K_2(\alpha, \delta)$.

For sufficiently large T the contemporaneous cross-correlation converges to a negative constant, and all cross-correlations at nonzero displacements converge to a positive constant.

[21] A variation of this setup is the model of Diamond and Verrecchia (1987), where short selling constraints cause periods of no trading to be a noisy signal of a low latent price.

[22] This corresponds to proposition 7 in Easley and O'Hara (1992).

[23] For a derivation of these expressions see appendix D.

Keeping T fixed, the cross-correlation converges geometrically to zero at rate $1/2$ in τ.

A similar result holds for the general case, where informed traders are allowed to let a profitable trade slip away. Easley and O'Hara (1992) show that transaction prices converge to the strong form efficient price in clock time at exponential rates for large τ.[24] Denote $\beta_{\tau,\{\overline{p}\}}$ the belief at time $t+\tau$ that a high efficient price has been observed, $\beta_{\tau,\{\underline{p}\}}$ the belief that a low efficient price has been observed and $\beta_{\tau,\{\}}$ the belief that nobody has observed any signal, all conditional on $\Omega_t \cup \{q_t\}$. τ sufficiently large allows invoking a law of large numbers for the observations included in the market maker believes. The market maker sets under perfect competition

$$\begin{aligned} p_\tau^{bid} - \underline{p} &= \beta_{\tau,\{\underline{p}\}}(1-\beta_{\tau,\{\}})\underline{p} + \beta_{\tau,\{\overline{p}\}}(1-\beta_{\tau,\{\}})\overline{p} + \beta_{\tau,\{\}}\frac{\underline{p}+\overline{p}}{2} - \underline{p} \\ &= \left(\beta_{\tau,\{\overline{p}\}} + \frac{\beta_{\tau,\{\}}}{2}\right)\left(\overline{p}-\underline{p}\right). \end{aligned} \tag{36}$$

For the case that watchful traders observe a low strong form efficient price, Easley and O'Hara (1992) show that $\beta_{\tau,\{\overline{p}\}} = exp(-r_1\tau)$ and $\beta_{\tau,\{\}} = exp(-r_2\tau)$ for some $r_1, r_2 > 0$. Hence for large τ the bid price p_t^{bid} converges to $\underline{p}$ almost surely at the exponential rate $r = min(r_1, r_2)$ in clock time.

$$p_t^{bid} \stackrel{a.s.}{\rightarrow} \underline{p}. \tag{37}$$

An analogous result applies to the convergence of the ask price to $\underline{p}$.

If periods without trade are permitted, the result strictly applies only to calendar time sampling. Tick time sampling misses the no-trade periods, which reveal information to the market maker, too. During trading days in which no trader has observed the strong form efficient price there are more no-trade periods than during trading days in which some have. On such a day the convergence rate is higher, because tick time sampling drops periods without a trade, but still exponential, because information per trade shrinks at a constant proportion.

The following proposition summarizes the cross-correlations in Easley and O'Hara (1992)-type models. The calculation of cross-correlations considers only the dominant exponential learning pattern, and ignores all terms which disappear at a faster rate as τ gets large.

Proposition 2 ***(Cross-correlations in Easley-O'Hara model)***

[24]This corresponds to proposition 6 in Easley and O'Hara (1992).

The contemporaneous cross-correlation in the Easley and O'Hara (1992) model is

$$Corr\left(\Delta p_t^*, \Delta u_t\right) = -\frac{1 + e^{-r(T-1)}}{2\sqrt{K}} < 0, \tag{38}$$

and the cross-correlations at sufficiently large nonzero displacements follow

$$Corr\left(\Delta p_{t-\tau}^*, \Delta u_t\right) = \frac{e^r - 1}{2\sqrt{K}} e^{-r\tau} > 0, \ \forall \tau \in [1, T-1] \tag{39}$$

$$Corr\left(\Delta p_{t-T}^*, \Delta u_t\right) = \frac{e^{-r(T-1)}}{2\sqrt{K}} > 0, \tag{40}$$

where $K = K(r, T)$.

Proof: See appendix D.

Unsurprisingly because of the assumption of risk neutrality, the contemporaneous correlation is negative, and approaches its minimum for small r and small T. Furthermore, for $\tau \in [1, T-1]$,

$$Corr\left(\Delta p_{t-\tau}^*, \Delta u_t\right) = \left(\frac{1}{e^r}\right)^{\tau-1} Corr\left(\Delta p_{t-1}^*, \Delta u_t\right). \tag{41}$$

That is, the cross-correlation decays geometrically to zero until $\tau = T$. In the first row of Figure 1 we graph this cross-correlation function. We show the cross-correlation pattern for a convergence rate of $r = 0.5$ in the upper left panel, and for a convergence rate of $r = 2$ in the upper right panel.

[Figure 1 about here.]

3.2.3 Strategic Traders

Because the market maker cannot distinguish informed trades from uninformed ones, informed traders can act strategically. The aim of strategic behavior of informed traders is to make the signals about p_t^* conveyed by their orders as noisy as possible, while still executing the desired trades. By mimicing uninformed traders they keep the market maker unaware of new information, i.e. unaware of the change in p_t^*. Because the market maker observes order flow imbalances and uses them to detect informed trading, the informed traders stretch their orders over a long time period such that detecting any significantly abnormal trading pattern becomes difficult. The market maker will, of course, notice the imbalance in trades over time. By sequentially updating his belief about p_t^* based on the history of trades he still learns about p_t^*, but, because of the strategic behavior of traders, at the slowest possible rate.

Markets of this type have been described in Kyle (1985) and Easley and O'Hara (1987). In the following we discuss the cross-correlation function implied by the Kyle (1985) model. The strategic behavior described by Kyle (1985) requires that exactly one trader is informed, or that all informed traders build a monopoly and coordinate trading. Here, the market maker does not maximize a particular objective function, he merely ensures market efficiency, i.e. sets the market price such that it equals the expected strong form efficient price, $\tilde{\tilde{p}}_t$, given the observed aggregate trading volume from informed and uninformed traders. The only optimizer in this model is the (risk neutral) informed trader who optimally spreads his orders over the day to minimize the (unfavorable) price reaction of the market maker. Thereby he maximizes his expected total daily profit using his private information and taking the price setting rule $\Delta p_t(\Omega_t)$ of the market makers as given. Effectively, the informed trader trades most when the sensitivity of prices to trading quantity is small.

Kyle (1985) assumes a linear reaction function of the market maker, which implies $\lambda_t = \lambda$ $\forall t \in [1, T]$, and a linear reaction function for the informed trader, which implies $q_t = q$ $\forall t \in [0, T-1]$. Under these assumptions he shows that in expectation the market price approaches the latent price linearly, not exponentially as in the previous subsection. The reason for this difference is that the market maker in Easley and O'Hara (1992) updates his believes in a Bayesian manner, whereas in Kyle (1985) the market maker's actions are constrained to market clearing. The other key feature of this model is that by the end of the trading day – just before p_t^* would be revealed – the market price reflects all information.

From the continuous auction equilibrium in Kyle (1985) the price change at time t is

$$d\tilde{\tilde{p}}(t) = \frac{p^* - \tilde{\tilde{p}}(t)}{T - t} dt + \sigma dz, \;\; t \in [0, T]. \tag{42}$$

dz is white noise with $dz \sim N(0, 1)$ and reflects the price impact of uninformed traders. This stochastic differential equation has the solution

$$\tilde{\tilde{p}}(t) = \frac{t}{T} p^* + \frac{T - t}{T} \tilde{\tilde{p}}(0) + (T - t) \int_0^t \frac{\sigma}{T - s} dB_s, \tag{43}$$

where $dB_s \equiv dz$.[25] The increments of the expected price over a discrete interval of time

[25] The third term reflects uninformed trading. It has an expected value of zero, and the impact of this random component increases during the early trading day and decreases lateron – its contribution to $\tilde{\tilde{p}}(t)$ is therefore hump-shaped over time.

follow therefore

$$\Delta\tilde{\tilde{p}}_{\tau} = \frac{\Delta p_0^*}{T} + (T-\tau)\int_{\tau-1}^{\tau} \frac{\sigma}{T-s} dB_s - \int_0^{\tau-1} \frac{\sigma}{T-s} dB_s. \tag{44}$$

The following proposition presents the cross-correlations for the Kyle (1985) model.[26]

Proposition 3 ***(Cross-correlations in Kyle model)***

The contemporaneous cross-correlation in Kyle (1985) is

$$Corr\left(\Delta p_t^*, \Delta u_t\right) = -\sqrt{\frac{T}{T^2+1}}, \tag{45}$$

the cross-correlations at displacements $\tau \in [1;T]$ *are*

$$Corr\left(\Delta p_{t-\tau}^*, \Delta u_t\right) = \sqrt{\frac{1}{T(T^2+1)}}, \tag{46}$$

and all higher order cross-correlations are zero.

Proof: See appendix E.

The cross-covariance at nonzero displacements is positive because of market maker learning. It is constant because of the strategic behavior of traders, which spread new information equally over time. This maximizes the time it takes the market maker to include the entire strong form efficient price change in his quotes. The more periods, the more pronounced is the negative contemporaneous cross-correlation, and the smaller are the cross-correlations at nonzero displacements.

We plot the cross-correlation function given by Proposition 3 in the second row of Figure 1. We show the cross-correlation function under modestly frequent changes in the latent price ($T = 5$) in the left panel, and for more frequent changes ($T = 2$) in the right panel. Table 1 compares standard multiperiod market microstructure models. In contrast to markets with nonstrategic traders, which display decaying lagged cross-correlations (row 3), markets with strategic traders display constant lagged cross-correlations (row 4).

[26] These cross-correlations, and cross-correlations for a similar model in our framework, are given in appendix E.

3.3 One-period Case

In this section we consider the extreme case of markets in which p_t^* automatically becomes public information at the end of each period, i.e. $c_t = p_{t-1}^* - \tilde{p}_{t-1}$ and $T = 1$. p_{t-1}^* is thus known when the market maker decides on p_t, and $\tilde{\tilde{p}}_t = p_t^* - \sigma\varepsilon_t$. The free distribution of information removes any incentive for informed traders to behave strategically. They therefore react immediately, which implies $E(q_{t-\tau}\varepsilon_t) = 0 \ \ \forall\tau \neq 0$ and trades are serially uncorrelated, i.e. $E(q_t|q_{t-1}) = 0$. For the market maker all periods are identical, and therefore the spread and reaction parameters are both constant over time, i.e. $s_t = s$ and $\lambda_t = \lambda \ \ \forall t$.

3.3.1 General Property

Because $T = 1$ the market maker's recursive problem (17) collapses to a sequence of single period ($\delta = 0$) problems. This by itself pins down the shape of the cross-correlation function. By (32) all cross-correlations at displacements larger than one are zero. Because $E(\Delta p_{t-\tau}^* \Delta p_t^*) = 0 \ \ \forall\tau \geq 1$ we can write

$$\begin{aligned} Corr(\Delta p_{t-\tau}^*, \Delta u_t) &= -Corr(\Delta p_{t-\tau}^*, \Delta p_t^*) + Corr(\Delta p_{t-\tau}^*, \Delta p_t) \\ &= Corr(\Delta p_{t-\tau}^*, \Delta p_t), \end{aligned} \tag{47}$$

$\forall\tau \geq 1$. Because p_{t-1}^* is known at the beginning of time t, the market price in period t is p_{t-1}^* adjusted by the market maker's reaction, $R(\cdot)$, to his information $\tilde{\Omega}_t$ about p_t^*, i.e. $p_t = p_{t-1}^* + \mu_t + R(\tilde{\Omega}_t)$. Because $\tilde{\Omega}_t$ in a one-period model is unrelated to past changes in the strong form efficient price, (47) becomes for displacement $\tau = 1$

$$\begin{aligned} &Corr(\Delta p_{t-1}^*, \Delta p_t) \\ &= -Corr(\Delta p_{t-1}^*, \Delta u_{t-1}) + Corr(\Delta p_{t-1}^*, p_t - p_{t-2} - p_{t-1}^* + p_{t-2}^*) \\ &= -Corr(\Delta p_t^*, \Delta u_t) + Corr(\Delta p_{t-1}^*, p_{t-1}^* + R(\tilde{\Omega}_t) - p_{t-1}^*) \\ &= -Corr(\Delta p_t^*, \Delta u_t). \end{aligned} \tag{48}$$

From (47) and (48) we conclude that in models of one-period private information the cross-correlation at displacement one has the opposite sign and same absolute value as the contemporaneous cross-correlation. In order to pin down the contemporaneous cross-correlation, we now turn to specific models.

3.3.2 No Market Maker Information

We start with our baseline assumption that the market maker at time t has no information whatsoever about Δp_t^*. Plugging $T = 1$, $s_t = s$, and $\lambda_t = \lambda$ into the general multiperiod results derived in appendix B gives

Proposition 4 ***(Strong form cross-correlation, one period model)***

$$Corr(\Delta p_t^*, \Delta u_t) = \frac{1}{\sqrt{2}} \frac{sE\left(q_t \varepsilon_t\right) - \sigma}{\sqrt{s^2 + \sigma^2 - 2s\sigma E(q_t \varepsilon_t)}}, \tag{49}$$

Proof: We have

$$\Delta p_t^* = \sigma \varepsilon_t, \tag{50}$$

$$\Delta u_t = s(q_t - q_{t-1}) - \sigma(\varepsilon_t - \varepsilon_{t-1}) \tag{51}$$

and

$$Var(\Delta p_t^*) = \sigma^2, \tag{52}$$

$$Var(\Delta u_t) = 2s^2 + 2\sigma^2 - 4s\sigma E(q_t \varepsilon_t). \tag{53}$$

This implies for the cross-covariance

$$\begin{aligned} Cov(\Delta p_t^*, \Delta u_t) &= E\left(\sigma \varepsilon_t \left(s q_t - s q_{t-1} - \sigma \varepsilon_t + \sigma \varepsilon_{t-1}\right)\right) \\ &= s\sigma E\left(q_t \varepsilon_t\right) - \sigma^2, \end{aligned} \tag{54}$$

where we have used $E(\varepsilon_t | q_{t-1}) = 0$ and $E(\varepsilon_t | \varepsilon_{t-1}) = 0$. Using (52), (53), and (54) we immediately obtain (49). *Q.E.D.*

As the following Proposition 5 shows, the cross-correlation (49) can be bounded from above and below.

Proposition 5 ***(Bounds of contemporaneous cross-correlation)***

$$-\frac{1}{\sqrt{2}} \leq Corr(\Delta p_t^*, \Delta u_t) \leq 0. \tag{55}$$

Proof: Negativity can be seen as follows. For uninformed traders, which trade randomly ($E(q_t | \varepsilon_t) = 0$), we have $sE(q_t^u \varepsilon_t) = 0$. In contrast, informed traders buy ($q_t = +1$) only when $\sigma\varepsilon_t > s$ and sell ($q_t = -1$) only when $\sigma\varepsilon_t < -s$. Thus in a market of only informed traders $\sigma q_t^i \varepsilon_t > s \geq 0\ \forall t$. Therefore we can write

$$1 = E({q_t^i}^2 \varepsilon_t^2) > E\left(\frac{s}{\sigma} q_t^i \varepsilon_t\right) > E\left(\frac{s^2}{\sigma^2}\right) > 0, \tag{56}$$

so in particular $\sigma > sE(q_t^i \varepsilon_t) > 0$. Combining informed and uninformed trades we have

$$\sigma \geq sE(q_t \varepsilon_t) > 0, \tag{57}$$

which implies that the contemporaneous cross-correlation (49) is negative.

Further, (49) is bounded from below by $-1/\sqrt{2}$, which we prove by contradiction. Suppose this was not the case, then from (49)

$$sE\left(q_t \varepsilon_t\right) - \sigma < -\sqrt{s^2 + \sigma^2 - 2s\sigma E(q_t \varepsilon_t)}. \tag{58}$$

Squaring both sides and simplifying gives the condition

$$\left[E\left(q_t \varepsilon_t\right)\right]^2 > 1, \tag{59}$$

but by Jensen's inequality

$$\left[E\left(q_t \varepsilon_t\right)\right]^2 \leq E\left(q_t^2 \varepsilon_t^2\right) = 1, \tag{60}$$

which contradicts (59). *Q.E.D.*

Note that the lower bound holds with equality for mid prices ($s = 0$).[27] The contemporaneous cross-correlation is therefore less pronounced for transaction prices than for mid prices. The contemporaneous cross-correlation for mid prices is negative, because p_t^{mid} does not react at all to the change in the strong form efficient price in the same period.[28] It differs from negative unity because market prices move in adjustment to the strong form efficient return one period earlier.

We summarize these results in the upper two rows of Table 2. Compared to the multi-period case ($T > 1$) the absolute value of the cross-correlation at lag one is large, because all information is revealed. Cross-correlations at any displacement beyond one, in contrast, are all zero.

[27] $s = 0$ must also hold by Assumption 3, if the market consisted of uninformed traders only.

[28] This is an instance of the price stickiness that Bandi and Russell (2006b) show to generate "mechanically" a negative contemporaneous cross-correlation.

[Table 2 about here.]

3.3.3 Incomplete Market Maker Information

In the previous subsection the market maker set prices without any information about the strong form efficient return in period t. Now suppose that the market maker observes a signal about the sign of Δp_t^*, namely $\{\text{sgn}(\varepsilon_t)\} \in \Omega_t$, before setting his price p_t^{mid}. This enables him to change p_t^{mid} before any informed trader reacts to the strong form efficient price change. With the signal $\{\text{sgn}(\varepsilon_t)\}$ the market maker updates his prior belief $p_t^* \sim (p_{t-1}^* + \mu_t, \sigma^2)$ summarized by the distribution $\breve{f}(\frac{\Delta p_t^*}{\sigma})$. The updated distribution $f(\cdot)$ differs from $\breve{f}(p_t^*)$ in that it is truncated from below or above at $p_t^* = p_{t-1}^* + \mu$ when $\text{sgn}(\varepsilon_t) > 0$ or $\text{sgn}(\varepsilon_t) < 0$, respectively.[29] Figure 2 illustrates what the posterior distribution looks like after observing the signal $\{\text{sgn}(\varepsilon_t) = +1\}$: If the prior is a normal distribution, the posterior is given by the half normal in the upper left panel. If the prior is a tent distribution, the posterior is given by the triangular distribution in the lower left panel.

After observing this signal and the outcomes of period $t-1$, in particular p_{t-1}^*, the market maker quotes a bid and an ask price for the following period, taking the spread s as given:

$$p_t = p_{t-1}^* + \mu_t + sq_t + R(\{\text{sgn}(\varepsilon_t)\}). \tag{61}$$

Because the market maker can adjust the mid price in response to the extra information $\{\text{sgn}(\varepsilon_t)\}$, (61) augments (5) by the market maker response function $R(\cdot)$. $R(\cdot)$ depends in particular on the market maker's risk aversion, n.[30]

An approximation[31] to the problem of choosing $p_t^{mid} \equiv p(n)$ based on loss function (15)

[29] We assume that the market makers' beliefs make proper use of the available information, in particular that $f(\cdot)$ is consistent with Assumption 1.

[30] The extra information of the market maker disconnects the direction of trade from the direction of the change in the strong form efficient price. If the informed trade, then it must be that $p_t^* > p_t^{ask}$ or that $p_t^* < p_t^{bid}$. When $R(\cdot) = 0$, as in the previous sections, the sign of the innovation, ε_t, pins down the trading direction. For example, $\varepsilon_t > 0$ implies $p_t^* > p_t^{ask}$ in periods of informed trading. In this subsection, once the market maker observes $\{\text{sgn}(\varepsilon_t)\}$, his mid price quote, p_t^{mid}, takes the expected change in p_t^* into account. Because his expectation of p_t^* could both be too high or too low, the sign of ε_t does not pin down the trading direction in periods of informed trading. (As before, uninformed trades occur no matter what p_t^*, p_t^{ask} and p_t^{bid} are.)

[31] This approximation is exact for $s = 0$ or, more generally, for

$$\int_{p(n)-s}^{p(n)} (p(n) - p^*)^n f(p^*)dp^* + \int_{p(n)}^{p(n)+s} (p^* - p(n))^n f(p^*)dp^* = 0.$$

is

$$p(n) = \underset{x\in[\underline{p},\overline{p}]}{\text{argmax}} - \int_{\underline{p}}^{x} (x-p^*)^n f(p^*)dp^* - \int_{x}^{\overline{p}} (p^*-x)^n f(p^*)dp^*. \tag{62}$$

For any density $f(\cdot)$ which has all moments we can apply Leibnitz's rule and obtain the first order condition

$$\int_{\underline{p}}^{p(n)} (p(n)-p^*)^{n-1} f(p^*)dp^* - \int_{p(n)}^{\overline{p}} (p^*-p(n))^{n-1} f(p^*)dp^* = 0. \tag{63}$$

For some values of n, explicit solutions to (63) are available, which we list in Proposition 6.[32]

Proposition 6 ***(Optimal Mid Price)***

$$p(1) = \text{Median}(p_t^*) \tag{64}$$

$$p(2) = \text{E}(p_t^*) \tag{65}$$

$$p(\infty) = \text{Midsupport}(p_t^*). \tag{66}$$

Proof: The first two equations are the well-known result that the median is the best predictor under linear (absolute) loss, whereas the mean is the best predictor under squared loss. The third equation follows from rewriting (63) as a metric

$$\begin{aligned} &\lim_{n\to\infty} \left(\int_{\underline{p}}^{p(n)} (p(n)-p^*)^{n-1} f(p^*)dp^*\right)^{1/(n-1)} \\ &= \lim_{n\to\infty} \left(\int_{p(n)}^{\overline{p}} (p^*-p(n))^{n-1} f(p^*)dp^*\right)^{1/(n-1)}, \end{aligned} \tag{67}$$

which after taking the limit degenerates to the sup norm

$$\underset{p^*\in[\underline{p},p(\infty)]}{sup} (p(\infty)-p^*) = \underset{p^*\in[p(\infty),\overline{p}]}{sup} (p^*-p(\infty)). \tag{68}$$

Hence

$$p(\infty) = \frac{\underline{p}+\overline{p}}{2}. \tag{69}$$

[32]We assume $n \geq 1$ throughout, because this implies realistic market maker preferences. However, (62) can be solved for any $n \geq 0$. In particular, $p(0)$ is the mode of $f(\cdot)$ when $s = 0$, or the highest density (connected) region when $s > 0$. For $n \notin \{1, 2, \infty\}$ no explicit solution exists, and for $n > 25$ even obtaining numerical solutions creates difficulty for non-trivial distribution functions $f(\cdot)$.

Thus, by monotonicity (69) solves (63) for $n \to \infty$. *Q.E.D.*

For distributions with finite support $p(\infty)$ is a finite number. Otherwise, it is positive or negative infinity for one-sided distributions or does not exist for distributions with positive density over the entire real line. For the halfnormal distribution shown in Figure 2, for example, we get $p(\infty) = +\infty$.

[Figure 2 about here.]

As risk aversion, n, grows, $p(n)$ moves monotonically from the median of $f(\cdot)$ to the midpoint of the support of $f(\cdot)$.[33] The upper right panel of Figure 2 illustrates this for right-skewed distributions $f(\cdot)$ with infinite support such as the halfnormal distribution. $p(n)$ increases in n, starting from the median for $n = 1$, monotonically without bound. If $f(\cdot)$ has finite support, $p(n)$ increases from the median monotonically toward an asymptote $p(\infty)$.

Analogously, for left-skewed distributions with infinite support, $p(n)$ decreases in n from the median monotonically without bound, and with finite support toward an asymptote $p(\infty)$. The asymptote is clearly visible in the lower right panel of Figure 2, in which we plot $p(n)$ for the triangular distribution defined on $[0,1]$ shown in the lower left panel of the same figure. We use these observations in the proof of the following proposition:

Proposition 7 ***(Cross-correlation under market maker information)***
If $\Omega_t = \{\text{sgn}(\varepsilon_t), p^*_{t-1}\}$ and Assumption 1 holds, then the optimal $E\left(|R(\{\text{sgn}(\varepsilon_t)\})|\right)$ strictly increases in risk aversion, $n \geq 1$, without bound. If, further, the distribution of innovations $\breve{f}$ induces beliefs with support $[\underline{p}, \overline{p}]$ satisfying condition (72), then $\exists n_0 > 1$ such that $\forall n > n_0$ it holds that $Corr(\Delta p^*_t, \Delta u_t) > 0$.

Proof: Define $R \equiv R(\{\text{sgn}(\varepsilon_t)\}) \equiv \left|p_t - (p^*_{t-1} + \mu_t + sq_t)\right|$. After $\Delta p^*_t = \sigma\varepsilon_t > 0$ we have $R > 0$ and so $\Delta p_t = -p_{t-1} + p^*_{t-1} + sq_t + R$, and after $\Delta p^*_t < 0$ we have $\Delta p_t = -p_{t-1} + p^*_{t-1} + sq_t - R$. Therefore

$$\begin{aligned} E\left(\Delta p_t \varepsilon_t\right) &= \frac{1}{2} E\left(\left(-p_{t-1} + p^*_{t-1} + sq_t + R\right)\varepsilon_t \,|\varepsilon_t > 0\right) \\ &+ \frac{1}{2} E\left(\left(-p_{t-1} + p^*_{t-1} + sq_t - R\right)\varepsilon_t \,|\varepsilon_t < 0\right) \\ &= \frac{1}{2} E\left((sq_t + R)\varepsilon_t \,|\varepsilon_t > 0\right) + \frac{1}{2} E\left((sq_t - R)\varepsilon_t \,|\varepsilon_t < 0\right) \\ &= RE\left(|\varepsilon_t|\right) + sE\left(q_t \varepsilon_t\right). \end{aligned} \tag{70}$$

[33]See appendix F for a proof.

Plugging (70) with $E(\Delta p_t \Delta p_t^*) = \sigma E(\Delta p_t \varepsilon_t)$ into (19) implies that the contemporaneous cross-covariance is positive if and only if

$$R > \frac{\sigma - sE(q_t \varepsilon_t)}{E(|\varepsilon_t|)}. \tag{71}$$

From Proposition 6 for any $p(n) \in \left[\text{Median}(p); \frac{\underline{p}+\overline{p}}{2}\right]$ there is a risk aversion level n such that market makers will – after observing the signal $\{\text{sgn}(\varepsilon_t)\}$ – quote this price as p_t^{mid}. Therefore, for all distributions $f(\cdot)$ which satisfy

$$\frac{\underline{p}+\overline{p}}{2} > \frac{Var_{\breve{f}}(\Delta p_t^*)}{E_{\breve{f}}(|\Delta p_t^*|)} = \frac{\sigma}{E(|\varepsilon_t|)} \tag{72}$$

a sufficiently large n leads to a market maker response which by (13) satisfies (71). *Q.E.D.*

Condition (72) holds, for example, for $\breve{F}$ being the normal distribution, but not for a tent distribution, which corresponds to the post-signal triangular distribution discussed earlier.

Comparing these results in the third row of Table 2 with the other model setups, it appears that even though the contemporaneous cross-correlation can be positive for high risk aversion levels, the usual case is that it is negative. For the halfnormal distribution in the upper left panel of Figure 2, for example, we need a rather high risk aversion of $n \geq 8$. Clearly though, changes in risk aversion of the market maker have a fundamental impact on the cross-correlation. Hansen and Lunde (2006) note as their "Fact IV" that "the properties of the noise have changed over time." Since they base this observation on a comparison of year 2000 with year 2004 it is possible that the underlying cause is a change in risk aversion.

The link between properties of noise and risk aversion offers itself as a way to estimate the time path of risk aversion from the market price cross-correlation patterns. In stable periods with low risk aversion the contemporaneous cross-correlation is negative, but as uncertainty shoots up, contemporaneous cross-correlation shoots up with it. In periods of crisis this can lead to the extreme case of an inverted cross-correlation pattern that we have described in this section. The negative contemporaneous cross-correlation in Hansen and Lunde (2006) indicates that during their sample period the risk aversion of market makers was rather low.

Note that in this section from the point of view of the market maker all periods are ex ante identical. Every period the market maker gets the same type of new information (p_{t-1}^*, and either $\text{sgn}(\varepsilon_t) = 1$, or $\text{sgn}(\varepsilon_t) = -1$), thus s_t is the same in every period. Only a small change in the model allows for time variation in spreads (Demsetz 1968). Based on our assumptions the information event $\{\text{sgn}(\varepsilon_t) = 0\}$ occurs with zero probability. If – contrary to the maintained assumptions – we assign nonzero probability mass to this event and keep $Var(\varepsilon_t) = 1$ by moving probability mass to the tails of the distribution, then observing this signal $\{\text{sgn}(\varepsilon_t) = 0\}$ ensures the market maker of no informed trading in this period. Therefore, the competitive spread in this period is zero. A subsequent $\text{sgn}(\varepsilon_t) = \pm 1$ then not only triggers a shift in p_t^{mid}, but also an increase in spread. Because of the higher probability mass on a large strong form efficient return, a smaller risk aversion than before suffices to generate a positive contemporaneous cross-correlation.

3.4 Frequent Price Changes

So far in this section we discussed models, where the old strong form efficient price becomes public information at the beginning of the trading day before any new shift in the strong form efficient price. In general, however, the efficient price may change again before the old efficient price becomes fully publicly known. In this case the old p_{t-1}^* still contains information about the new p_t^*. As p_{t-1}^* is not precisely known itself, the entire history of prices contains information about p_t^*.

Suppose that at any point in time the T most recent changes in the strong form efficient price are private information. The noise and its variance are then larger than before at any point in time. The signal $\{\text{sgn}(p_t - p_t^*)\}$ is now different from the signal $\{\text{sgn}(\varepsilon_t)\}$. Under the former signal and with $Corr(p_t^*, \Delta p_{t-\tau}^*) > 0$, for $\tau > 0$ the information set Ω_t contains information about $p_{t-\tau}^*$ not contained in Ω_{t-1}. By (21) the signs of the cross-correlations at nonzero displacements remain unchanged even if p_t^* changes frequently. But the more often p_t^* changes during $[t, t-\tau]$, the closer to zero is the cross-correlation $Corr(p_t^*, \Delta p_{t-\tau}^*)$, the less informative is the signal in t about $\Delta p_{t-\tau}^*$, and thus the closer to zero is the cross-correlation between strong form efficient returns and noise. For both signals the contemporaneous cross-correlation is dampened toward zero, because the signal $\{\text{sgn}(p_t - p_t^*)\}$ mixes up information on Δp_t^* with information on $\Delta p_{t-\tau}^*$, and the signal $\{\text{sgn}(\varepsilon_t)\}$ is related only to a small component of Δu_t. Overall, slowly decaying private information keeps the cross-correlation sign pattern unchanged, but dampens the absolute values toward zero.

In summary we have shown that many market properties leave their mark on the cross-correlation pattern: The displacement beyond which correlation is zero gives an indication of the frequency of information events. The larger the correlation is in absolute value terms the fewer unformed trades occur in the market. If contemporaneous strong form cross-correlation is high and positive, then market makers are very risk averse and have access to extra information. If the cross-correlations at nonzero displacements decay quickly, then market makers learn fast. If they do not decay at all, then informed traders act strategically.

4 Semi-strong form Correlation

Now we base the cross-correlation calculation on the semi-strong form efficient price, $\tilde{p}_t$. Equivalently we could interpret this setup as an endogenous latent price process, determined by an exogenous trading process q_t, with $q_t \in \{-1, +1\}$, because the strong-form efficient price remains unobserved and enters the model only via the informed trades. This setup is closely related to the generalized Roll (1984) bid/ask model in Hasbrouck (2007).

4.1 The General Multi-period Case

In the period of a change in the strong form efficient price, in which also when the previous strong form efficient price becomes public information, the semi-strong form efficient return is[34]

$$\begin{aligned}\Delta\tilde{p}_0 &= \lambda_0 q_0 + c_0 \\ &= \lambda_0 q_0 + \sigma\varepsilon_{-T} - \sum_{t=1}^{T} \lambda_{-t} q_{-t},\end{aligned} \tag{73}$$

where the first term reflects the market maker's guess about the new strong form efficient return based on a trade, the second term internalizes the new information about the previous return, and as a countermove the sum undoes the now obsolete guesses about the previous return. In all other periods the semi-strong form efficient price changes by

$$\Delta\tilde{p}_t = \lambda_t q_t. \tag{74}$$

From (10) we get for $\forall t$

$$\Delta u_t = -\lambda_t q_t + \lambda_{t-1} q_{t-1} + s_t q_t - s_{t-1} q_{t-1}, \tag{75}$$

where the first two terms reflect information-revealing trades, and the second two terms reflect the bid/ask bounce.

Using Assumption 2 this immediately leads to an expression for the contemporaneous

[34]We use again the shorthand notation $p_0 \equiv p_{\kappa T}$ $\forall\kappa \in Z$, and likewise $p_{-x} \equiv p_{\kappa T - x}$ $\forall\kappa, x \in Z$.

covariance

$$
\begin{aligned}
Cov(\Delta\tilde{p}_t, \Delta u_t) &= \frac{1}{T}\left\{-\lambda_0^2 + s_0\lambda_0 + \sigma\lambda_{-1}E(q_{-1}\varepsilon_{-T}) - \sigma s_{-1}E(q_{-1}\varepsilon_{-T})\right. \\
&- \sum_{i=-1}^{-T}\left(\lambda_{-1} - s_{-1}\right)\lambda_i Cov(q_i q_{-1}) \\
&+ \left.\sum_{i=1}^{T-1}\left(-\lambda_i^2 + \lambda_i\lambda_{i-1}E(q_i q_{i-1}) + s_i\lambda_i - s_{i-1}\lambda_i E(q_i q_{i-1})\right)\right\}, \qquad (76)
\end{aligned}
$$

for covariance at higher displacements $\tau \in [1, T-1]$

$$
\begin{aligned}
Cov(\Delta\tilde{p}_{t-\tau}, \Delta u_t) &= \frac{1}{T}\left\{-\lambda_0\lambda_\tau E(q_0 q_\tau) + \lambda_0\lambda_{\tau-1}E(q_0 q_{\tau-1}) + \lambda_0 s_\tau E(q_0 q_\tau)\right. \\
- &\ \lambda_0 s_{\tau-1}E(q_0 q_{\tau-1}) + \lambda_{T-\tau}\left(\lambda_{T-1} - s_{T-1}\right)E(q_{T-\tau}q_{T-1}) \\
+ &\left.\sum_{i=\tau+1}^{T-1}\left[\lambda_{i-\tau}(-\lambda_i + s_i)E(q_{i-\tau}q_i) + \lambda_{i-\tau}(\lambda_{i-1} - s_{i-1})E(q_{i-\tau}q_{i-1})\right]\right\}, \qquad (77)
\end{aligned}
$$

for covariance at displacement T

$$
Cov(\Delta\tilde{p}_{t-T}, \Delta u_t) = \frac{1}{T}\lambda_0\left(\lambda_{T-1} - s_{T-1}\right)E(q_0 q_{T-1}), \qquad (78)
$$

and for all higher order displacements $\tau > T$

$$
Cov(\Delta\tilde{p}_{t-\tau}, \Delta u_t) = 0. \qquad (79)
$$

Under semi-strong market efficiency ($s_t = \lambda_t \quad \forall t$) the cross-correlation function is zero for all displacements. The special cases we discuss in the following subsection therefore all assume lack of even this weak form of market efficiency.

4.2 Special Multi-period Cases

The cross-correlations for semi-strong form efficient prices stem from a gap between the spread, s_t, and the adverse selection parameter, λ_t. Such a gap can result from processing costs ($s_t > \lambda_t$), from legal restrictions ($s_t < \lambda_t$), or merely from suboptimal behavior of the market maker. Noisy signals or strategic behavior do not affect the semi-strong cross-correlations – all what matters is that the market maker's knowledge passes into market prices one-to-one.

In Easley and O'Hara (1992), for example, prices are semi-strong form efficient by definition, and therefore the semi-strong form cross-correlation function is zero always.

The Kyle (1985) model assumptions $\lambda_t = \lambda$ and $s_t = s\ \forall t$ give with (77)

$$Cov(\Delta\tilde{p}_{t-\tau}, \Delta u_t) = \frac{\lambda(\lambda - s)}{T}\left\{E(q_{T-\tau}q_{T-1}) + \sum_{i=\tau}^{T-1}\left[E(q_{i-\tau}q_{i-1}) - E(q_{i-\tau}q_i)\right]\right\}. \tag{80}$$

If $\lambda = 0$, then this cross-correlation is flat at zero. Likewise, if $q_t = q$, it is flat at $\frac{\lambda(\lambda-s)}{T}$. More generally, because $E(q_i q_j) > E(q_{i-\tau}q_j) > 0\ \forall i \leq j,\ \forall \tau > 0$, the cross-correlation decreases in τ.

4.3 One-period Case

The simpler case of markets in which all information is revealed after one period, i.e.

$$\Delta\tilde{p}_t = \lambda(q_t - q_{t-1}) + \sigma\varepsilon_{t-1} \tag{81}$$

offers itself again for illustration of these cross-correlation effects. In the one period case the semi-strong form efficient prices follow a martingale, but unlike their strong form counterpart the semi-strong form efficient returns do not follow a martingale difference sequence.[35] We will see in the following proposition that in contrast to the strong form correlations, the absolute value of semi-strong form cross-correlation at displacement zero and one usually differs.

Proposition 8 ***(Semi-strong form cross-correlation, one period model)***

The contemporaneous cross-correlation is

$$Corr(\Delta\tilde{p}_t, \Delta u_t) = \frac{2\lambda - \sigma E(q_t\varepsilon_t)}{\sqrt{\sigma^2 - 2\sigma\lambda E(q_t\varepsilon_t) + 2\lambda^2}}\frac{\operatorname{sgn}(s-\lambda)}{\sqrt{2}}. \tag{82}$$

The cross-correlation at displacement one equals

$$Corr(\Delta\tilde{p}_{t-1}, \Delta u_t) = \frac{-\lambda}{\sqrt{\sigma^2 - 2\sigma\lambda E(q_t\varepsilon_t) + 2\lambda^2}}\frac{\operatorname{sgn}(s-\lambda)}{\sqrt{2}}. \tag{83}$$

All cross-correlations at higher displacements are zero.

[35] In multiperiod models strong form efficient prices follow a martingale, but semi-strong form efficient prices do not.

Proof: The expressions for the cross-correlations follow directly from their multiperiod counterparts. See (76), (78), and appendix G.

Bounds on the contemporaneous cross-correlation can be obtained by assuming a specific market marker loss function and then solving for the market maker's optimal λ. Suppose the market maker has a quadratic loss function, then

$$\lambda^{opt} = \underset{\lambda}{\operatorname{argmin}}\, E\left[(\tilde{p}_t - p_t^*)^2\right], \tag{84}$$

which becomes

$$\lambda^{opt} = \underset{\lambda}{\operatorname{argmin}}\, \lambda^2 - 2\sigma\lambda E\left(q_t\varepsilon_t\right), \tag{85}$$

and therefore $\lambda^{opt} = \sigma E\left(q_t\varepsilon_t\right) > 0$. At λ^{opt} we have

$$Corr(\Delta\tilde{p}_t, \Delta u_t) = E(q_t\varepsilon_t)\frac{\operatorname{sgn}(s - \lambda^{opt})}{\sqrt{2}}, \tag{86}$$

$$Corr(\Delta\tilde{p}_{t-1}, \Delta u_t) = -E(q_t\varepsilon_t)\frac{\operatorname{sgn}(s - \lambda^{opt})}{\sqrt{2}}, \tag{87}$$

and by (13)

$$|Corr(\Delta\tilde{p}_t, \Delta u_t)| = |Corr(\Delta\tilde{p}_{t-1}, \Delta u_t)| \leq \frac{1}{\sqrt{2}}. \tag{88}$$

Both the bounds and the equality of absolute contemporaneous and lagged cross-correlations do not hold in general, but only for a quadratic market maker loss function.

Proposition 8 shows that the size of the spread matters only relative to the adverse selection parameter. The cross-correlation at displacement one, for example, is negative if and only if the spread exceeds the adverse selection cost. $s > \lambda$ is reasonable, because the spread must cover the order processing cost. It also entails, however, that the average trader in expectation incurs a loss with every transaction. Hasbrouck (2007) justifies this with the liquidity needs of traders. The sign of contemporaneous cross-correlation is ambiguous in general. As in Diebold (2006), for s sufficiently large (and $\lambda > \frac{\sigma}{2}E(q_t\varepsilon_t)$) the model predicts a cross-correlation pattern that is exactly the opposite of the empirical pattern in Hansen and Lunde (2006). We illustrate this in the last row of Figure 1, which on the left shows the cross-correlation function for a small spread ($0 \leq s < \lambda$), and on the right for a sufficiently wide spread ($s > \lambda > 0$). If sufficiently many lags are included, the Hansen and Lunde estimator is unbiased for the strong form efficient price defined as in (1) and (2), but by construction not for its semi-strong form counterpart.

Under high risk aversion the spread can become very large without violating the market maker's zero-profit condition. By the same reasoning as in section 3.3.3, there exists a minimal risk aversion level n_0 such that all $n > n_0$ generate a spread $s > \lambda$. Thus when $\lambda > \frac{\sigma}{2}E(q_t\varepsilon_t)$ there exists n_0 such that all $n > n_0$ generate a positive contemporaneous cross-correlation and a negative cross-correlation at displacement one. Note that unlike in section 3.3.3 positive contemporaneous cross-correlation obtains even though the market maker does not observe a signal. We summarize the results in the lower four rows of Table 2.

In summary, positive contemporaneous cross-correlations occur for (1) strong form efficient prices under sufficiently high risk aversion if a signal is observed, and (2) semi-strong form efficient prices for large spreads. Various market arrangements and sampling speeds can dampen the contemporaneous cross-correlation to zero, but the negative sign maintains except in the two aforementioned cases. Bandi and Russell (2006b) and Diebold (2006) rightly wonder whether a negative cross-correlation is inevitable. In contrast to Hansen and Lunde (2006), Bandi and Russell (2008) find no "obvious evidence of a significant, negative correlation." These seemingly contradictory results might stem from the inability of purely statistical estimators to clearly distinguish strong form from semi-strong form efficient prices. Without controlling for market features, which the realized volatility literature so far largely ignores, the estimate may pick up any of the two prices. As we have seen, a positive cross-correlation is of course possible, but a negative cross-correlation appears most realistic for strong form efficient prices.

5 Additional Discussion of Econometric Issues

We have already drawn econometric implications insofar as we have shown that market microstructure models predict rich cross-correlation patterns between latent prices and microstructure noise, which have yet to be investigated empirically. Here we go farther, sketching some specific aspects of such empirics, including the relationship between theory-based and data-based (sample) cross-correlation functions, as well as strategies for using microstructural information to obtain improved volatility estimators.

5.1 Effects of Sampling Frequency

We have thus far focused on sampling at the rate corresponding exactly to the market maker's reaction time. Sampling at faster or slower rates will affect the shape of cross-correlation functions. This has immediate implications for the shape of empirically estimated (sample) cross-correlation functions, because the reaction speed of the market maker is generally unknown, so that econometric sampling may proceed at faster or slower rates.

Consider first the effects of sampling "too quickly." If, for example, we sample m times during an interval of no changes in both latent and market prices, then we record each latent return / noise pair m times. The cross-correlation function is then a step function, but it still has the overall shape obtained at slower sampling speeds. For example, the cross-correlations up to displacement $m-1$ all share the sign of the contemporaneous cross-correlation. A cross-correlation at displacement one with the same sign as the contemporaneous cross-correlation implies overly fast sampling (because all models predict a sign change in cross-correlations between displacement zero and displacement one).

[Table 3 about here.]

Alternatively, consider a market maker who updates p_t infrequently, for example changing p_t only every second period. After a latent price change at $t = 0$, he updates his quotes for the first time at $t = 2$, and then, observing the trades in between, again at $t = 4$, $t = 6$, and so forth. The noise pattern is therefore $-\Delta p_t^*, 0, \Delta p_2, 0, \Delta p_4, \ldots$ Trading activity during the two interim periods provides more information than during only one period, but because the quote in the interim period is fixed, the two interim periods provide less additional information than if the price were updated in every period. Whereas the variance of Δp_t^* is unchanged, the (unconditional) variance of noise shrinks to somewhat more than half the variance that obtains when the market maker updates p_t every period. The cross-correlation function therefore oscillates.

Now consider the effects of sampling too slowly. Suppose, for example, that in the one-period model of section 4.3 we sample only every n-th tick. Let $p_{t,i}$, $i = 1, \ldots, n-1$ be the unsampled market prices of the omitted ticks, and $\tilde{p}_{t,i}$ the corresponding semi-strong form efficient prices. Then (81) is replaced by

$$\Delta\hat{\tilde{p}}_t = \sum_{i=1}^{n} \Delta\tilde{p}_{t,i} = \lambda(q_t - q_{t-1}) + \sigma \sum_{i=1}^{n} \varepsilon_{t-1,i}, \tag{89}$$

whereas the noise term (75) remains unchanged. Hence, $Cov(\Delta\hat{\tilde{p}}_t, \Delta\hat{u}_t)$ and $Var(\Delta\hat{u}_t)$ also remain unchanged. The variance of semi-strong form efficient returns, however, increases to $Var(\Delta\hat{\tilde{p}}_t) = 2\lambda^2 + n\sigma^2 - 2\lambda\sigma E(q_t\varepsilon_t)$. Thus

$$\left|Corr(\Delta\hat{\tilde{p}}_t, \Delta\hat{u}_t)\right| = \left|\frac{2\lambda - \sigma E(q_t\varepsilon_t)}{\sqrt{2}\sqrt{n\sigma^2 - 2\lambda\sigma E(q_t\varepsilon_t) + 2\lambda^2}}\right| < \left|Corr(\Delta\tilde{p}_t, \Delta u_t)\right|, \tag{90}$$

so that increasing the sampling interval averages the initial market price underreaction with later price readjustments, thereby dampening the entire cross-correlation pattern toward zero. The most informative cross-correlations are therefore obtained by sampling every tick. Hansen and Lunde (2006) find a negative contemporaneous cross-correlation between returns and noise, which diminishes as more ticks are combined into one market price sample. This can stem either from the averaging effect just described, or from cross-correlations at nonzero displacements working in the opposite direction. This ambiguity could be sorted out by evaluating the entire cross-correlation function, which shows the importance of not limiting noise analysis to the contemporaneous cross-correlation.

In summary, the sampling frequency does not change the sign pattern of cross-correlations, but can severely impact its absolute values, as summarized in Table 3. At low sampling rates the cross-correlations become empirically indistinguishable from zero, which can be useful when analysis is being done on the assumption of independent noise. At higher sampling frequencies the cross-correlation structure of the noise needs to be addressed, and in the next section we suggest how to do so in parsimonious fashion by exploiting market microstructure theory.

5.2 Implications for Volatility Estimation I: Imposing Restrictions from Microstructure Theory

In the introduction we highlighted the key issue of estimation of integrated volatility using high-frequency data, the potential problems of the first-generation estimator (simple realized volatility) in the presence of MSN, and subsequent attempts to "correct" for MSN.

In an important development, Hansen and Lunde (2006) suggest making realized volatility robust to serial correlation via HAC estimation methods, which are asymptotically justified under very general conditions. That asymptotic generality is, however, not necessarily helpful in finite samples. Indeed the frequently unsatisfactory finite-sample performance of nonparametric HAC estimators leads Bandi and Russell (2006a) to suggest sophisticated alternative statistical approaches.

Here we suggest a different approach that *specializes* the estimator in accordance with the implications of market microstructure theory. As we have seen, dynamic market microstructure models imply that noise decays geometrically over time after displacement one, with two polar cases of immediate decay (as in section 3.3) and no decay (as in section 3.2.3). That knowledge could be used to construct improved volatility estimators that impose the restrictions implied by market microstructure theory.

Suppose Δp_t follows an MA(∞) process in the innovations for the latent price,

$$\Delta p_t = \alpha_0 \sigma \varepsilon_t + \alpha_1 \sigma \sum_{\tau=1}^{\infty} \alpha_2^{\tau} \varepsilon_{t-\tau}. \tag{91}$$

This form of Δp_t accommodates very persistent cross-correlations, similar to the idea behind the sequence of examples in Oomen (2006). We can decompose latent returns into market returns and noise

$$\Delta p_t^* = \Delta p_t + (1 - \alpha_0) \sigma \varepsilon_t - \alpha_1 \sigma \sum_{\tau=1}^{\infty} \alpha_2^{\tau} \varepsilon_{t-\tau} = \sigma \varepsilon_t. \tag{92}$$

Using this, we have a simple formula for the integrated volatility (IV) of latent returns,

$$IV = E\left[(\Delta p_t^*)^2\right] = E\left[(\Delta p_t - \Delta u_t)^2\right] = E(\Delta p_t^2) + (1+\alpha_0)(1-\alpha_0) IV - \alpha_1^2 \sum_{\tau=1}^{\infty} \alpha_2^{2\tau} IV. \tag{93}$$

Solving for IV, using $\sum_{\tau=1}^{\infty} \alpha_2^{2\tau} = \frac{\alpha_2^2}{1-\alpha_2^2}$, and simplifying yields

$$IV = \frac{1 - \alpha_2^2}{\alpha_0^2 (1 - \alpha_2^2) + \alpha_1^2 \alpha_2^2} \cdot E(\Delta p_t^2). \tag{94}$$

Standard RV is consistent for $E(\Delta p_t^2)$; hence a consistent estimator for IV is

$$\hat{IV} = \frac{1 - \hat{\alpha}_2^2}{\hat{\alpha}_0^2 (1 - \hat{\alpha}_2^2) + \hat{\alpha}_1^2 \hat{\alpha}_2^2} \cdot RV, \tag{95}$$

where $\hat{\alpha}_0$, $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are consistent estimators. Such estimators are easily obtained, for example, in a GMM framework using three moments.

The result is even simpler in a learning model with $T = \infty$ and frequent latent price changes, in which case we have

$$\Delta p_t \approx 0 \cdot \sigma \varepsilon_t + \sum_{\tau=1}^{\infty} \left(-e^{-r\tau} + e^{-r(\tau-1)}\right) \sigma \varepsilon_{t-\tau} = \sigma \left(e^r - 1\right) \sum_{\tau=1}^{\infty} e^{-r\tau} \varepsilon_{t-\tau}. \tag{96}$$

The integrated variance can then be consistently estimated by

$$\hat{IV} = \frac{1 - e^{-2\hat{r}}}{1 - 2e^{-\hat{r}} + e^{-2\hat{r}}} \cdot RV, \tag{97}$$

which requires a consistent estimator of only one parameter, the rate of learning r.

The expression for the integrated variance (97) offers a structural interpretation to estimates of noise and integrated variance, such as the results reported in Table 3 of Hansen and Lunde (2006). The Easley and O'Hara (1992) model predicts that the noise decreases as the learning rate of the market maker increases. Slow learning implies a very persistent cross-correlation between noise and latent returns, and hence persistent autocorrelation of noise, so that fluctuations in noise tend to dominate the integrated variance.

Figure 3 provides some perspective. It is based on the noise-to-integrated-variance ratios reported by Hansen and Lunde (2006), which are (unfortunately) derived under the assumption of independent noise. The ratio of noise to integrated variance shrinks with the number of price-changing quotes per day. If the number of times that the market maker changes his price quote during a trading day is indicative of his speed of learning, then MSN indeed decreases as the learning rate of the market maker increases. Thus, even though these ratios may not be directly applicable, they seem to support the multiperiod learning model.

Furthermore, the recent decline in noise-induced bias of realized volatility (Hansen and Lunde's fact III) suggests that the learning rate r has increased. Meddahi's (2002) finding that the standard deviation of the bias is large relative to the integrated variance suggests that the learning rate itself may have fluctuated considerably around its increasing trend.

[Figure 3 about here.]

5.3 Implications for Volatility Estimation II: Structural vs. Nonstructural Volatility Estimators

In this section we emphasize that the more the econometrician knows about the price process of relevance, the more the noise correction can be tailored to it by exploiting microstructure theory. This is important, because the price process of interest may differ across users of volatility estimates (e.g., many users are likely to be interested in price processes different from (1) and (2)), which has implications for appropriate volatility estimation. For example, the volatility of strong form efficient returns is

$$E(\Delta p_t^{*2}) = \frac{\sigma^2}{T}, \tag{98}$$

which differs both conceptually and numerically from the volatility of semi-strong form efficient returns

$$E(\Delta\tilde{p}_t^2) = \frac{1}{T}\left\{\sigma^2 + \sum_{i=0}^{T-1}\lambda_i^2 + E\left[\left(\sum_{i=-1}^{-T}\lambda_i q_i\right)^2\right] - 2\sigma\sum_{i=-1}^{-T}\lambda_i E(q_i\varepsilon_{-T})\right\}. \tag{99}$$

Consider, for example, the case of $T = 1$. Immediately, the strong form volatility (98) is σ^2 and the semi-strong volatility (99) simplifies to

$$E(\Delta\tilde{p}_t^2) = \sigma^2 + 2\lambda - 2\sigma E(q_t\varepsilon_t) \neq \sigma^2. \tag{100}$$

Now, the RV estimator promoted by Hansen and Lunde (2006) is

$$RV_{AC_1}^{1tick} = \Delta p_t^2 + \Delta p_{t-1}\Delta p_t + \Delta p_t\Delta p_{t+1}, \tag{101}$$

which for $T = 1$ is

$$\begin{aligned} E\left(RV_{AC_1}^{1tick}\right) &= E\left((sq_t - sq_{t-1} + \sigma\varepsilon_{t-1}) \times (\sigma(\varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2}) + sq_{t+1} - sq_{t-2})\right) \\ &= \sigma^2. \end{aligned} \tag{102}$$

Hence $RV_{AC_1}^{1tick}$ is unbiased for σ^2, and in general biased with ambiguous direction relative to $Var(\Delta\tilde{p}_t^2)$, because by construction a noise robust estimator wit h lag window T correctly removes *any* microstructure and other correlation effects. For this estimator to work, the latent price process of interest must follow a martingale difference sequence (MDS). Even though semi-strong form prices with $T = 1$ form a martingale, their returns are not an MDS. They are serially correlated and inevitably $RV_{AC_1}^{1tick}$ is biased relative to $Var(\Delta\tilde{p}_t^2)$. We have modeled the strong form efficient price in this paper as an MDS, and indeed this latent price series is of interest on its own. We doubt, however, that this is the unique latent price of interest in volatility estimation. Efficient prices from an informed trader's perspective could themselves be seen as the result of a learning process about the state of the economy,[36] which implies that the p_t^* of interest is often not an MDS, but instead has the properties that we have derived in this paper for the semi-strong form efficient price $\tilde{p}_t$.

Suppose, for example, that the strong form efficient prices are themselves the result of

[36] Also, they might be the result of learning about information of other market participants, as in Foster and Viswanathan (1996).

learning of informed traders about fundamentals, η_t, which follow a random walk. Then

$$\Delta p_t^* = \sigma \sum_{\tau=1}^{T} \left(-e^{-r_1\tau} + e^{-r_1(\tau-1)}\right) \eta_{t-\tau}. \tag{103}$$

Let market prices follow the usual process of market maker learning, for example

$$\Delta p_t = \sum_{\tau=1}^{T} \left(-e^{-r_2\tau} + e^{-r_2(\tau-1)}\right) \Delta p_{t-\tau}^*. \tag{104}$$

Then $RV_{AC_T}^{1tick}$ is the variance of the fundamental, not the variance of the strong form efficient price. Obviously, a purely statistical noise correction cannot distinguish between cross-correlation caused by fundamentals and cross-correlation caused by MSN. This is where market microstructure theory can contribute new insights to realized volatility estimation. By providing distinctive but flexible relationships between noise and latent returns, we can decompose the agnostic statistical noise estimate into its various components – in the previous example into MSN and fundamental correlation in the strong form efficient price. Our example uses a MA($2T$) process with only two free coefficients, but the large sample sizes typical with high frequency data can accommodate much richer specifications. Empirical work in market microstructure tends to favor extreme parametrizations, ranging from the very parsimonious as in Glosten and Harris (1988)-type regressions, to the profligate as in Hasbrouck (1996)-type vector autoregressions. For the purpose of RV noise correction the most useful parametrizations may be intermediate – imposing a general correlation pattern but avoiding highly situation-specific assumptions.

6 Concluding Remarks

The recent realized volatility literature provides statistical insights into microstructure noise (MSN) and its effects. In this paper we have provided complementary *economic* insights, treating MSN not simply as a nuisance, but rather as the result of financial economic decisions, which we seek to understand.[37] In that regard, we derived the predictions of economic theory regarding correlation between two types of latent price and MSN; we characterized and contrasted the entire cross-correlation functions corresponding to a variety of market

[37]For an interesting related perspective, see Engle and Sun (2007). Their approach and environment (conditional duration modeling), however, are very different from ours.

environments, with a variety of results.

Some results are generic. In particular, cross-correlations between strong form efficient price and MSN at displacements greater than zero have sign opposite to that of the contemporaneous correlation.

Some results are not generic but nevertheless quite robust to model choice. In particular, all models predict negative contemporaneous correlation between latent price and MSN, so long as the risk aversion of market makers is not too high.

Finally, some results are highly model-specific. In particular, the cross-correlation patterns and absolute magnitudes depend critically on the frequency of latent price changes, the presence of bid/ask bounce, the timing of information and actions, and the market maker's degree of risk aversion.

We hope that the results of this paper will help us to use data to discipline theory, and theory do discipline data. In particular, we have argued that data-based cross-correlation patterns between latent price and MSN help determine (if not definitively resolve) the comparative merits of various economic microstructure models, and conversely, that our theoretical cross-correlation results may lead to improved volatility estimators.

Looking to the future, we envision some novel uses and extensions of our results. For example, the rate of decay of cross-correlations might be used to assess the extent to which strategic traders are active in the market, and the sign and size of the contemporaneous correlation might be used to assess the degree of market maker risk aversion. Indeed market maker risk aversion might be time-varying, with associated time-varying cross-correlation structure between latent price and MSN. During crises, for example, market makers may be more risk averse, as borrowing and hedging possibilities are reduced. If so, the "normal pattern" of negative contemporaneous cross-correlation and positive higher-order cross-correlations might switch to a "crisis pattern" of positive contemporaneous cross-correlation and negative higher-order cross-correlations. Such possibilities await future empirical exploration.

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Table 1: Strong form Efficient Cross-correlations in Multi-period Models

p_t^* martingale	signal	traders strategic	ρ_0	ρ_τ $\tau \in [1, T-1]$	ρ_T	ρ_τ $\tau > T$
yes	none	n.a.	$\rho_0 < 0$	0	$-\rho_0$	0
yes	certain/ noisy	no	$\rho_0 < 0$	$\lvert\rho_{\tau-1}\rvert > \rho_\tau > 0$	$\rho_T > 0$	0
no	noisy	no	$-\frac{1+e^{-r(T-1)}}{2\sqrt{K(r,T)}}$	$\frac{-e^{-r\tau}+e^{-r(\tau-1)}}{2\sqrt{K(r,T)}}$	$\frac{e^{-r(T-1)}}{2\sqrt{K(r,T)}}$	0
yes	noisy	yes	$-\sqrt{\frac{T}{T^2+1}}$	$\sqrt{\frac{1}{T(T^2+1)}}$	$\sqrt{\frac{1}{T(T^2+1)}}$	0

The table reports $\rho_\tau = Corr(\Delta p^*_{t-\tau}, \Delta u_t)$ in multiperiod models ($T > 1$) under risk neutrality ($n = 1$).

Table 2: Cross-correlations in One-period Models

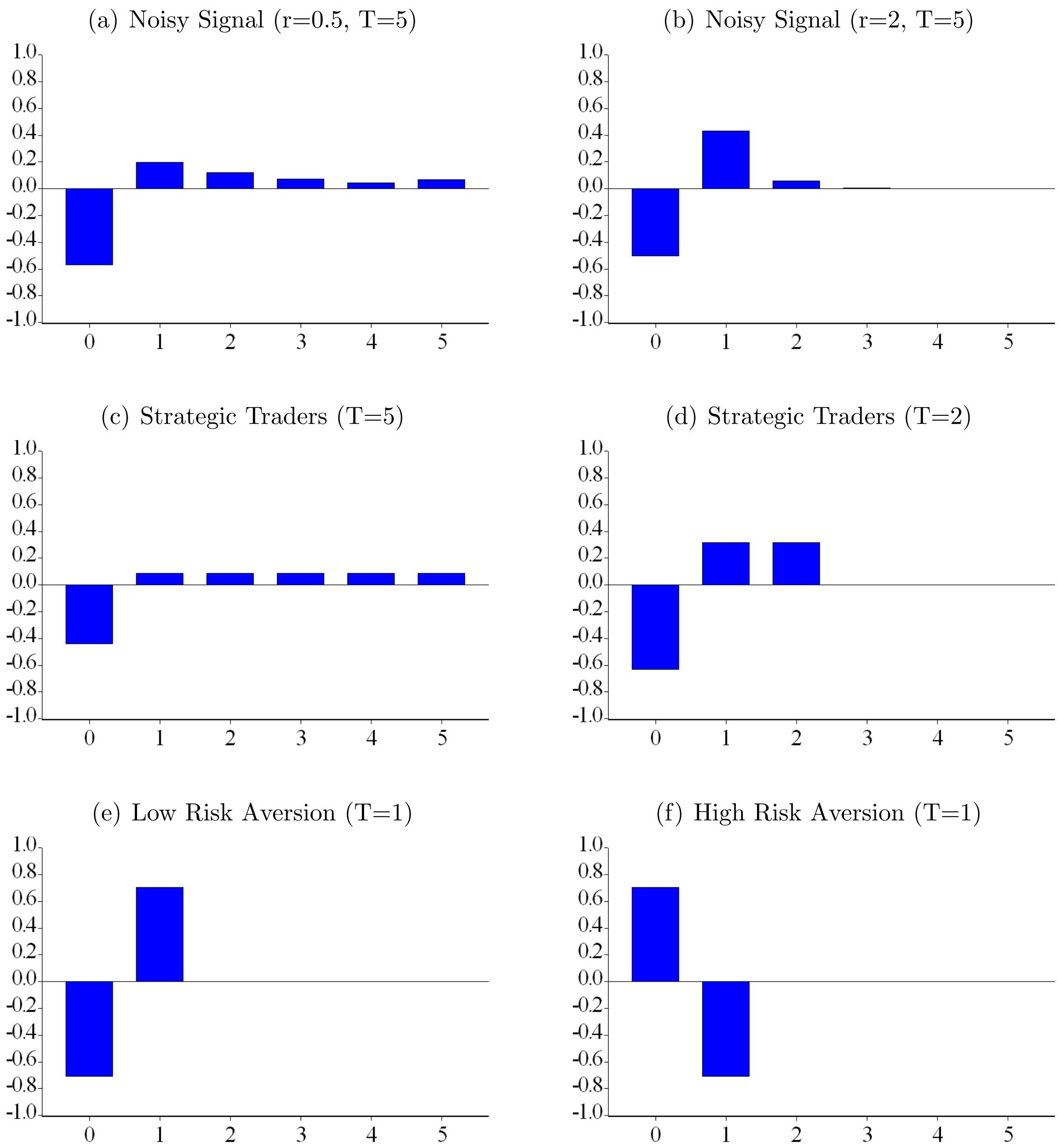
efficient price	spread	loss function	ρ_0	ρ_1	ρ_τ $\tau > 1$
strong	0	any	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
strong	≥ 0	any	$-\frac{1}{\sqrt{2}} \leq \rho_0 < 0$	$-\rho_0$	0
strong	≥ 0	high n + extra info	$\rho_0 > 0$	$-\rho_0$	0
semi-strong	≥ 0	quadratic	$-\frac{1}{\sqrt{2}} \leq \rho_0 \leq \frac{1}{\sqrt{2}}$	$-\rho_0$	0
semi-strong	$\in [0, \lambda[$	any	ambiguous	$\rho_1 > 0$	0
semi-strong	λ	any	0	0	0
semi-strong	$\geq \lambda$	any	ambiguous	$\rho_1 < 0$	0

The upper half of this table reports $\rho_\tau = Corr(\Delta p^*_{t-\tau}, \Delta u_t)$ under no extra market maker information $\tilde{\Omega}_t = \{\}$, and in row 3 under extra market maker information $\tilde{\Omega}_t = \{\text{sgn}(\varepsilon_t)\}$. The lower half of this table reports $\rho_\tau = Corr(\Delta \tilde{p}_{t-\tau}, \Delta u_t)$.

Table 3: Cross-correlation Patterns at Various Sampling Frequencies

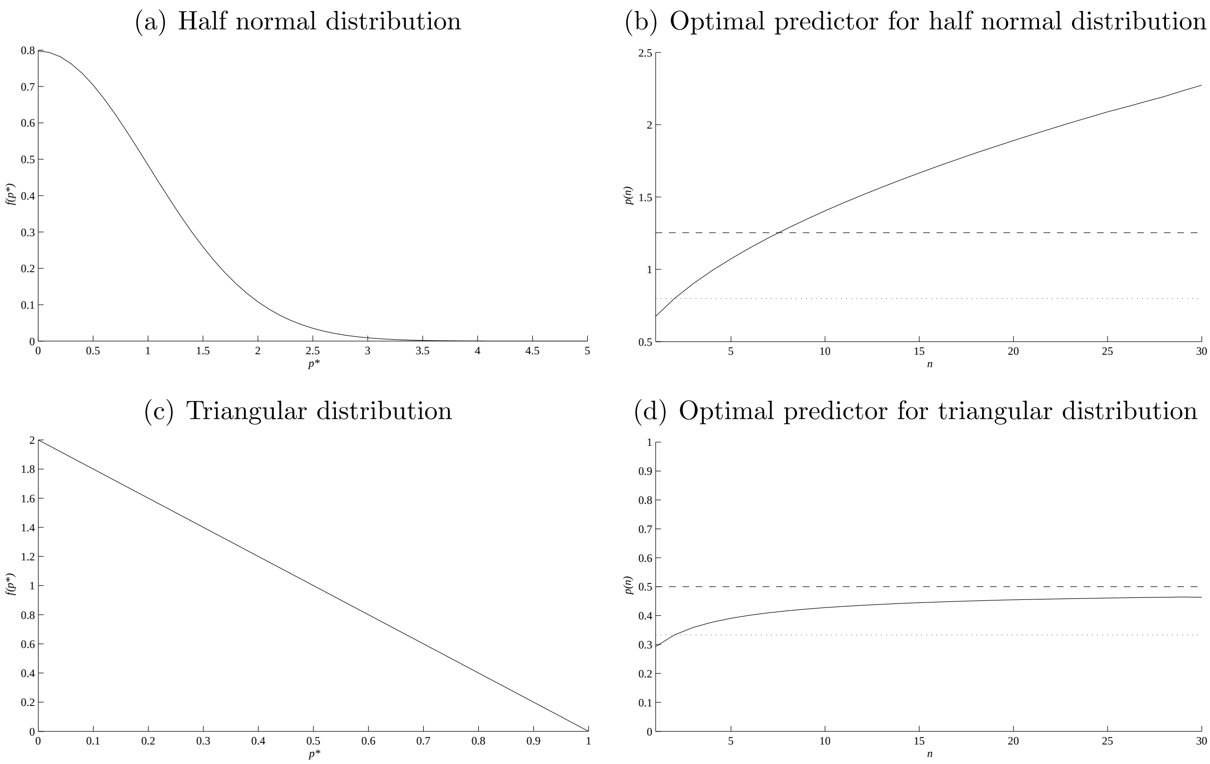
sampling rate	cross-correlation function						note
	$\tau = 0$	1	2	3	4	5	
optimal	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	
$>$ latent price frequency	ρ_0	ρ_0	ρ_1	ρ_1	ρ_2	ρ_2	
$>$ market maker update freq.	ρ_0^{MM}	0	ρ_1^{MM}	0	ρ_2^{MM}	0	$\rho_i^{MM} > \rho_i \ \forall i$
$<$ latent price frequency	ρ_0^{SL}	ρ_1^{SL}	ρ_2^{SL}	ρ_3^{SL}	ρ_4^{SL}	ρ_5^{SL}	$\rho_i^{SL} < \rho_i \ \forall i$

Figure 1: Cross-correlation Functions of Strong form Efficient Price



The graphs show the cross-correlation functions $\rho(\tau)$ of the strong form efficient price. The top row shows the typical cross-correlation pattern for an Easley-O'Hara (1992)-type model ($K = 1$, $T = 5$) under learning rate $r = 0.5$ in the left panel, and under faster learning ($r = 2$) in the right panel. The second row shows the cross-correlation pattern in a Kyle (1985)-type setup, under frequent changes in the strong form efficient price ($T = 5$) in the left panel, and under more frequent changes ($T = 2$) in the right panel. The left panel in the last row shows the typical cross-correlation pattern of strong form efficient prices in a one period model with modest risk aversion, and the right panel with higher risk aversion. The graphs in the last row apply as well to semi-strong form efficient prices. In this case, the left panel shows the cross-correlation under a relatively small spread ($0 \leq s < \lambda$), and the right panel for a typical spread ($s > \lambda > 0$).

Figure 2: Optimal Predictor $p(n)$ Under a Half-normal and a Triangular Distribution



(a) Half normal distribution

(b) Optimal predictor for half normal distribution

(c) Triangular distribution

(d) Optimal predictor for triangular distribution

The left panels show two possible expectations of the market maker about the strong form efficient price after observing $\text{sgn}(\varepsilon_t) = +1$. The upper left panel shows the normal distribution case. It shows the density after observing the signal, which is the upper halfnormal distribution $f(p_t^*) = 2\varphi\left(\frac{p_t^* - p_{t-1}^* - \mu}{\sigma}\right)$, plotted with $p_{t-1}^* + \mu = 0$ and $\sigma^2 = 1$. The lower left panel shows the tent distribution case. It shows the density after observing the signal, which is the density of the right-skewed triangular distribution $f(p_t^*) = \frac{2(\overline{p} - p_t^*)}{(\overline{p} - p_{t-1}^* - \mu)^2}$ with support $[p_{t-1}^* + \mu, \overline{p}]$ plotted with $p_{t-1}^* + \mu = 0$ and $\overline{p} = p_{t-1}^* + \mu + \sigma/\sqrt{3} = 1$.
The right panels show the corresponding optimal predictors, $p(n)$, as a function of risk aversion n. The dotted line marks $E(\Delta p_t^*)$, the dashed line marks $\frac{Var(\Delta p_t^*)}{E(|\Delta p_t^*|)}$.
In particular, in the lower right panel, the solid line is the solution to

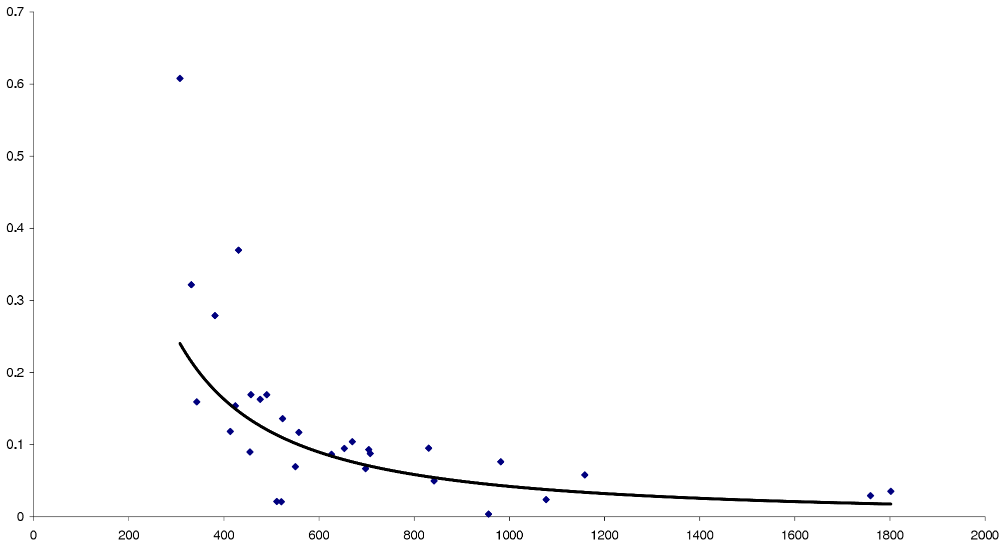
$$p(n) = \underset{x \in [\underline{p}, \overline{p}]}{\operatorname{argmax}} - \int_{\underline{p}}^{x} (x - p^*)^n f(p^*) dp^* - \int_{x}^{\overline{p}} (p^* - x)^n f(p^*) dp^*.$$

Evaluated at $\underline{p} = 0$ and $\overline{p} = 1$ for the triangular distribution on $[0, 1]$, this reduces to

$$p(n) = \underset{x \in [0,1]}{\operatorname{argmax}} - \int_{0}^{x} 2(1 - p^*)(x - p^*)^n dp^* - \int_{x}^{1} 2(1 - p^*)(p^* - x)^n dp^*,$$

which has the solutions $p(1) = \text{Median}(p_t^*) = 0.29$, $p(2) = \text{E}(p_t^*) = 0.33$, $\lim_{n \to \infty} p(n) = 0.5$.

Figure 3: Relationship between Noise-to-Integrated-Variance Ratio and Quotes per Day



The vertical axis measures the noise to signal ratio as 100 times noise divided by integrated variance under the assumption of independent noise. The horizontal axis is the number of quotes per day with a price change. Data is for the year 2000 for 30 stocks listed on NYSE and NASDAQ. Data are from Hansen and Lunde (2006), Tables 1 and 3. The solid line is a fitted power trend line.

A Model-free Cross-correlations

The unconditional expectations of noise and latent price changes are zero ($E(\Delta u_t) = 0$, $E(\Delta p_t^*) = \sigma E(\varepsilon_t) = 0$), and therefore the *contemporaneous cross-covariance* between strong form efficient returns and noise is

$$\begin{aligned} Cov(\Delta p_t^*, \Delta u_t) &= E(\Delta p_t^*(\Delta p_t(\Omega_t, \Omega_{t-1}) - \Delta p_t^*)) \\ &= E(\Delta p_t(\Omega_t, \Omega_{t-1})\Delta p_t^*) - E(\Delta p_t^{*2}) \\ &= E(\Delta p_t(\Omega_t, \Omega_{t-1})\Delta p_t^*) - Var(\Delta p_t^*). \end{aligned} \tag{105}$$

By the definition of the correlation, this immediately implies the first part of Proposition 1. The result for semi-strong form efficient prices is analogous, because $E(\tilde{p}_t) = 0$.

Similarly, the cross-covariance at nonzero displacements between latent returns $\tau \geq 1$ periods ago and noise is

$$\begin{aligned} Cov(\Delta p_{t-\tau}^*, \Delta u_t) &= E(\Delta p_{t-\tau}^*(\Delta p_t - \Delta p_t^*)) \\ &= E(\Delta p_{t-\tau}^*\Delta p_t) - E(\Delta p_{t-\tau}^*\Delta p_t^*). \end{aligned} \tag{106}$$

The result for semi-strong form efficient prices is analogous, and both together imply the second part of Proposition 1. For strong form efficient prices we can simplify (106) further to

$$\begin{aligned} Cov(\Delta p_{t-\tau}^*, \Delta u_t) &= E(\Delta p_{t-\tau}^*\Delta p_t) \\ &= \sigma E(\varepsilon_{t-\tau}\Delta p_t). \end{aligned} \tag{107}$$

B Strong form Cross-correlations

In the multi-period setup of section 3.1 the strong form efficient price has the unconditional variance

$$Var(\Delta p_t^*) = \frac{1}{T} Var(\sigma\varepsilon_0) = \frac{\sigma^2}{T} \tag{108}$$

and the corresponding noise has an unconditional variance of

$$Var(\Delta u_t) = \frac{1}{T}\sum_{i=0}^{T-1} Var(\Delta u_i)$$

$$
\begin{aligned}
&= \frac{1}{T}\Bigg\{2\sigma^2 + \sum_{i=0}^{T-1}\left(s_i^2 + s_{i-1}^2\right) - 2\sigma s_{T-1}E(q_{T-1}\varepsilon_0) - 2\sigma\sum_{i=0}^{T-2}\lambda_i E(q_i\varepsilon_0) \\
&- 2\sigma s_0 E(q_0\varepsilon_0) + 2s_{T-1}\sum_{i=0}^{T-2}\lambda_i E(q_i q_{T-1}) + E\left[\left(\sum_{i=0}^{T-2}\lambda_i q_i\right)^2\right] \\
&+ \sum_{i=1}^{T-1}\left(\lambda_{i-1}^2 + 2s_i(\lambda_{i-1} - s_{i-1})E(q_{i-1}q_i) - 2s_{i-1}\lambda_{i-1}\right)\Bigg\}. \qquad (109)
\end{aligned}
$$

Using (29) the contemporaneous cross-correlation is

$$
Corr(\Delta p_t^*, \Delta u_t) = \frac{s_0(E(q_0\varepsilon_0) - \sigma)}{\sqrt{T\,Var(\Delta u_t)}}. \qquad (110)
$$

All other cross-correlations can be obtained analogously using (30) to (32).

For $T = 1$, spread and adverse selection parameter are constants, i.e. $s_t = s$ and $\lambda_t = \lambda$ $\forall t$, and the variance term radically simplifies.

$$
Var(\Delta u_t) = 2(\sigma^2 + s^2) - 4s\sigma E(q_t\varepsilon_t). \qquad (111)
$$

Thus

$$
Corr(\Delta p_t^*, \Delta u_t) = \frac{sE(q_t\varepsilon_t) - \sigma}{\sqrt{2}\sqrt{\sigma^2 + s^2 - 2s\sigma E(q_t\varepsilon_t)}}. \qquad (112)
$$

C Example of Optimal Learning

Assuming no discounting ($\delta = 1$), risk neutrality ($n = 1$), zero spread ($s_t = 0$) the general recursive problem (18) simplifies to

$$
\begin{aligned}
V(\underline{p}, \overline{p}) &= \max_p \quad \Bigg[-\int_{\underline{p}}^{p}(p - p^*)f(p^*)dp^* \\
&+ V(\underline{p}, p)F(p) + V(p, \overline{p})\,(1 - F(p)) - \int_{p}^{\overline{p}}(p^* - p)f(p^*)dp^*\Bigg]. \qquad (113)
\end{aligned}
$$

To simplify the problem further, we assume as in the example in Aghion et al. (1991) that $f(\cdot)$ is uniform. Then, the location of the interval $[\underline{p}, \overline{p}]$ does not matter, but only the length of it, $m = \overline{p} - \underline{p}$, is relevant. (113) becomes

$$
\begin{aligned}
V(m) &= \max_{\alpha} \quad -\frac{1}{m}\frac{1}{2}(\alpha m)^2 + V(m)\alpha \\
&- \frac{1}{m}\frac{1}{2}\left((1-\alpha)m\right)^2 + V(m)(1-\alpha) \\
&= \max_{\alpha} \quad V(m) - \frac{m}{2}\left(\alpha^2 + 1 - 2\alpha + \alpha^2\right) \\
&= \max_{\alpha} \quad V(m) - m\left(\alpha^2 - \alpha + \frac{1}{2}\right). \qquad (114)
\end{aligned}
$$

From the first order condition we find the maximum

$$\alpha = 1/2, \qquad (115)$$

thus optimal learning is achieved by repeated bisections.

This result is driven by uniformity, which ensures that Assumption 1 holds in every period, in particular that $f(\cdot)$ in (113) is always symmetric. Thereby, the term in brackets in (113) is symmetric around the symmetry point of $f(\cdot)$ as well, and the optimal p^{mid} equals the median, and the midpoint of the support of $f(\cdot)$. For non-uniform $f(\cdot)$ the solution path over time is specific to the shape of $f(\cdot)$ and has to be determined numerically.

D Cross-correlations under a Noisy Signal

In this appendix we derive the cross-correlation properties of the Easley and O'Hara (1992) model. The notation is as in Easley and O'Hara (1992): α is the probability of an information event, δ is the probability of a low signal, and μ denotes the probability of an informed trade if a profit opportunity appears. We start with solving for the cross-correlations in a market in which the informed traders always trade if a profit opportunity appears. Easley and O'Hara (1992) discuss this case ($\mu = 1$) in their proposition 7. By our assumption that uninformed traders trade always in periods of no informed trading we have $\varepsilon^B = \varepsilon^S = 1$, and because uninformed traders buy and sell with equal probability $\gamma = 1/2$. Strong form efficient returns and noise in the case $p_t^* = \overline{p}$ are

$$
\Delta p_0^* = \begin{cases} 0 & \text{if } p_{t-1}^* = \overline{p} \\ \overline{p} - \underline{p} & \text{if } p_{t-1}^* = \underline{p} \\ \delta(\overline{p} - \underline{p}) & \text{if } p_{t-1}^* = \delta\underline{p} + (1-\delta)\overline{p} \end{cases} \qquad (116)
$$

and for $t \neq \kappa T$

$$\Delta p_t^* = 0. \tag{117}$$

Note that in contrast to all other models we discuss, the strong form efficient price process in Easley and O'Hara (1992) is not a martingale. Its variance is

$$Var(\Delta p_t^*) = \frac{(\overline{p} - \underline{p})^2}{T} K_3 \tag{118}$$

where $K_3 = K_3(\alpha, \delta)$.

The noise in the period of a change in the strong form efficient price is

$$\begin{aligned} \Delta u_0 &= \Delta p_0 - \Delta p_0^* \\ &= \frac{-(1-\alpha)\delta(\overline{p} - \underline{p})}{2\alpha(1-\delta) + 1 - \alpha} + \left(\overline{p} - \underline{p}\right) \times \begin{cases} \frac{(1-\alpha)\delta}{\alpha(1-\delta)2^T + 1 - \alpha} & \text{w. prob. } \alpha(1-\delta) \\ \frac{(1-\alpha)(1-\delta)}{\alpha\delta 2^T + 1 - \alpha} & \text{w. prob. } \alpha\delta \\ \frac{1}{2}\left[\frac{\alpha(1-\delta)\delta}{\alpha(1-\delta)2^T + 1 - \alpha} + \frac{\alpha\delta(1-\delta)}{\alpha\delta 2^T + 1 - \alpha}\right] & \text{w. prob. } 1 - \alpha \end{cases} \end{aligned}$$

and otherwise for $t \neq \kappa T$

$$\Delta u_t = \frac{\alpha(1-\alpha)\delta(1-\delta)(\overline{p} - \underline{p})}{\left[\alpha(1-\delta)2^{t+1} + 1 - \alpha\right]\left[\alpha(1-\delta) + (1-\alpha)\left(\frac{1}{2}\right)^t\right]} > 0. \tag{119}$$

The noise variance is

$$Var(\Delta u_t) = \frac{(\overline{p} - \underline{p})^2}{T} O(1). \tag{120}$$

Aggregating over $p_t^* = \overline{p}$ and $p_t^* = \underline{p}$, the general contemporaneous cross-covariance becomes

$$\begin{aligned} Cov(\Delta p_t^*, \Delta u_t) &= \frac{1}{T} E\left(\Delta p_0^* \Delta u_0\right) \\ &= \frac{\overline{p} - \underline{p}}{T} \Big\{ \alpha(1-\delta)\left[\alpha\delta E\left(\Delta u_0^{\underline{p}\overline{p}}\right) + (1-\alpha) E\left(\delta \Delta u_0^{\underline{p}\overline{p}}\right)\right] \\ &+ \alpha\delta\left[-\alpha(1-\delta) E\left(\Delta u_0^{\overline{p}\underline{p}}\right) - (1-\alpha) E\left((1-\delta)\Delta u_0^{\underline{p}\underline{p}}\right)\right] \\ &+ (1-\alpha)\left[-\alpha(1-\delta) E\left(\delta \Delta u_0^{\overline{p}\underline{p}}\right) + \alpha\delta E\left((1-\delta)\Delta u_0^{\underline{p}\underline{p}}\right)\right]\Big\} \\ &= -\alpha(1-\alpha)\delta(1-\delta)\frac{(\overline{p} - \underline{p})^2}{T}\left[\frac{\delta}{2\alpha(1-\delta) + 1 - \alpha} + \frac{1-\delta}{2\alpha\delta + 1 - \alpha}\right. \\ &+ \left.\frac{1-\delta}{2^T\alpha\delta + 1 - \alpha} + \frac{\delta}{2^T\alpha(1-\delta) + 1 - \alpha}\right] < 0, \end{aligned} \tag{121}$$

or in condensed form

$$Cov(\Delta p_t^*, \Delta u_t) = -\left[K_4 + O\left(2^{-T}\right)\right] \frac{(\overline{p} - \underline{p})^2}{T} < 0, \tag{122}$$

where $K_4 = K_4(\alpha, \delta)$.

At nonzero displacements $\tau \in [1; T-1]$ the cross-covariance is

$$\begin{aligned}
Cov(\Delta p_{t-\tau}^*, \Delta u_t) &= \frac{1}{T} E\left(\Delta p_0^* \Delta u_\tau\right) \\
&= \frac{\overline{p} - \underline{p}}{T} \left\{\alpha(1-\delta)\left[\alpha\delta E\left(\Delta u_\tau^{\overline{p}}\right) + (1-\alpha) E\left(\delta \Delta u_\tau^{\overline{p}}\right)\right]\right. \\
&+ \alpha\delta\left[-\alpha(1-\delta) E\left(\Delta u_\tau^{\underline{p}}\right) - (1-\alpha) E\left((1-\delta)\Delta u_\tau^{\underline{p}}\right)\right] \\
&+ \left.(1-\alpha)\left[-\alpha(1-\delta) E\left(\delta \Delta u_\tau^{\underline{p}}\right) + \alpha\delta E\left((1-\delta)\Delta u_\tau^{\underline{p}}\right)\right]\right\} \\
&= \alpha^2(1-\alpha)\delta^2(1-\delta)^2 \frac{(\overline{p} - \underline{p})^2}{T} \left(\frac{1}{2}\right)^\tau \\
&\times \left\{\frac{1}{\left[2\alpha(1-\delta) + (1-\alpha)\left(\frac{1}{2}\right)^\tau\right]\left[\alpha(1-\delta) + (1-\alpha)\left(\frac{1}{2}\right)^\tau\right]}\right. \\
&+ \left.\frac{1}{\left[2\alpha\delta + (1-\alpha)\left(\frac{1}{2}\right)^\tau\right]\left[\alpha\delta + (1-\alpha)\left(\frac{1}{2}\right)^\tau\right]}\right\} > 0,
\end{aligned} \tag{123}$$

or in condensed form

$$Cov(\Delta p_{t-\tau}^*, \Delta u_t) = \left(\frac{1}{2}\right)^\tau \left[K_5 + O\left(2^{-\tau}\right)\right] \frac{(\overline{p} - \underline{p})^2}{T} > 0, \tag{124}$$

where $K_5 = K_5(\alpha, \delta)$. Comparing this with the variance terms (118) and (120) we see that although the cross-covariances (121) and (123) approach zero as T becomes large, the contemporaneous cross-correlation converges for fixed τ to a negative constant, and all cross-correlations at nonzero displacements converge to a positive constant. Keeping T fixed, the cross-correlation converges geometrically to zero at rate $1/2$ in τ.

We now turn to the general case, in which the informed traders trade only with a probability $\mu > 0$ if a profit opportunity appears. Easley and O'Hara (1992) discuss this in their proposition 6. The expressions for the cross-correlation in the general case are quite complex. Because our focus is on the correlation pattern, we discuss here a very stylized version of this general case, which allows us to derive again an explicit expression for the cross-correlations.

Suppose the strong form efficient price process switches between two states of equal probability

$$p_t^* = \begin{cases} \overline{p} & \text{with} \quad \text{probability} \quad 1/2 \\ \underline{p} & \text{with} \quad \text{probability} \quad 1/2 \end{cases} . \tag{125}$$

Therefore,

$$\Delta p_0^* = \begin{cases} \overline{p} - \underline{p} & \text{with} \quad \text{probability} \quad 1/4 \\ 0 & \text{with} \quad \text{probability} \quad 1/2 \\ \underline{p} - \overline{p} & \text{with} \quad \text{probability} \quad 1/4 \end{cases} \tag{126}$$

with the properties

$$E((\Delta p_t^*)^2) = \frac{1}{T}\left[\frac{(\overline{p}-\underline{p})^2}{4} + \frac{(\underline{p}-\overline{p})^2}{4}\right] = \frac{(\overline{p}-\underline{p})^2}{2T} \equiv \frac{\sigma^2}{T}, \tag{127}$$

$$E(\Delta p_t^* \Delta p_t) = 0, \tag{128}$$

$$E(\Delta p_{t-\tau}^* \Delta p_t^*) = 0. \tag{129}$$

Using the result from Easley and O'Hara (1992) that transaction prices converge to the strong form efficient price at an exponential rate we get

$$\Delta p_0 = \frac{\overline{p}-\underline{p}}{2}\left(e^{-r(T-1)} - 1\right) \operatorname{sgn}\left(p_{-T}^* - \frac{\overline{p}+\underline{p}}{2}\right) \tag{130}$$

$$\Delta p_\tau = \frac{\overline{p}-\underline{p}}{2}\left(e^{-r(\tau-1)} - e^{-r\tau}\right) \operatorname{sgn}\left(p_0 - \frac{\overline{p}+\underline{p}}{2}\right) \tag{131}$$

$$\Delta u_0 = \frac{\overline{p}-\underline{p}}{2}\left(e^{-r(T-1)} - 1\right) \operatorname{sgn}\left(p_{-T}^* - \frac{\overline{p}+\underline{p}}{2}\right) - \Delta p_0^* \tag{132}$$

$$\Delta u_\tau = \frac{\overline{p}-\underline{p}}{2}\left(e^{-r(\tau-1)} - e^{-r\tau}\right) \operatorname{sgn}\left(p_0 - \frac{\overline{p}+\underline{p}}{2}\right) \tag{133}$$

The contemporaneous cross-covariance ($\tau = 0$) is

$$\begin{aligned} Cov\left(\Delta p_t^*, \Delta u_t\right) &= \frac{1}{T} E\left(\Delta p_0^* \Delta u_0\right) \\ &= -\frac{\sigma^2}{2T}\left[1 + e^{-r(T-1)}\right]. \end{aligned} \tag{134}$$

The second term inside the brackets is an artifact of p_t^* not following a martingale. In the period of the efficient price change it is optimal for the market maker to set p_t to the unconditional mean of p_t^*, thereby offsetting the effect of all previous learning, which the efficient price change rendered obsolete.

The cross-covariance for $\tau \in [1; T-1]$ is

$$\begin{aligned} Cov\left(\Delta p^*_{t-\tau}, \Delta u_t\right) &= \frac{1}{T} E\left(\Delta p^*_0 \Delta u_\tau\right) \\ &= \frac{\sigma^2}{2T}\left(-e^{-r\tau} + e^{-r(\tau-1)}\right), \end{aligned} \tag{135}$$

and for $\tau = T$ we have

$$\begin{aligned} Cov\left(\Delta p^*_{t-T}, \Delta u_t\right) &= \frac{1}{T} E\left(\Delta p^*_{-T} \Delta u_0\right) \\ &= \frac{\sigma^2}{2T} e^{-r(T-1)}. \end{aligned} \tag{136}$$

The variance of the noise is

$$\begin{aligned} Var(\Delta u_t) &= \frac{1}{T}\left[\frac{(\overline{p}-\underline{p})^2}{T}\left(e^{-r(T-1)}-1\right)^2 + \sigma^2 + 2\frac{(\overline{p}-\underline{p})^2}{4}\left(e^{-r(T-1)}-1\right)\right. \\ &+ \left.\sum_{\tau=1}^{T-1}\frac{(\overline{p}-\underline{p})^2}{4}\left(-e^{-r\tau}+e^{-r(\tau-1)}\right)^2\right] \\ &= \frac{\sigma^2}{T}\left[\frac{1}{2}e^{-2r(T-1)} + \frac{1}{2} + \frac{1}{2}(-e^r+1)^2\frac{(e^{-2r})^{T-1}-1}{e^{-2r}-1}\right]. \end{aligned} \tag{137}$$

Denoting the term in brackets by $K = K(r,T)$ we get for the contemporaneous cross-correlation

$$Corr\left(\Delta p^*_t, \Delta u_t\right) = -\frac{1+e^{-r(T-1)}}{2\sqrt{K}}, \tag{138}$$

for the cross-correlation at displacements $\tau \in [1; T-1]$

$$Corr\left(\Delta p^*_{t-\tau}, \Delta u_t\right) = \frac{-e^{-r\tau}+e^{-r(\tau-1)}}{2\sqrt{K}}, \tag{139}$$

and for the cross-correlation at displacement T

$$Corr\left(\Delta p^*_{t-T}, \Delta u_t\right) = \frac{e^{-r(T-1)}}{2\sqrt{K}}. \tag{140}$$

E Cross-correlations with Strategic Traders

In this appendix we derive cross-correlations for the model of Kyle (1985). In order to present a closed form solution we use continuous time, $t \in [0, T]$, but note that Kyle (1985) discussed the discrete time case as well. The discussion is based on the assumption of Kyle (1985) that the reaction functions for quantity demanded and prices are linear, i.e. that $\lambda_t = \lambda$, and $s_t = s$. Nonlinear solutions might nevertheless exist as well.

We assume semi-strong market efficiency, and so $s = \lambda$. We get from (29)

$$Cov\left(\Delta p_t^*, \Delta u_t\right) = -\frac{\sigma}{T}\left(\lambda E(q\varepsilon_0) - \sigma\right) < 0. \tag{141}$$

From (30) the cross-covariance function at nonzero displacements

$$Cov\left(\Delta p_{t-\tau}^*, \Delta u_t\right) = \frac{\sigma}{T}\lambda E(q\varepsilon_0) > 0 \tag{142}$$

is constant $\forall t \in [1, T-1]$, and zero $\forall t \geq T$.

More specifically, we derive based on (44) for the noise (assuming zero spread)

$$\Delta u_0 = \frac{\Delta p_{-T}^*}{T} - \int_0^{T-1} \frac{\sigma}{T-s} dB_s - \Delta p_0^* \tag{143}$$

and for $\tau \in [1, T-1]$

$$\Delta u_\tau = \frac{\Delta p_0^*}{T} + (T-\tau)\int_{\tau-1}^{\tau} \frac{\sigma}{T-s} dB_s - \int_0^{\tau-1} \frac{\sigma}{T-s} dB_s. \tag{144}$$

The variance of the noise is therefore

$$\begin{aligned} Var(\Delta u_t) &= \frac{1}{T}\left[E(\Delta u_0^2) + \sum_{t=1}^{T-1} E(\Delta u_t^2)\right] \\ &= \frac{\sigma^2}{T}\left[\frac{T+1}{T} + \frac{T-1}{T} + \frac{(T-1)^2}{T}\right] \\ &= \frac{\sigma^2}{T^2}\left(T^2+1\right). \end{aligned} \tag{145}$$

The covariances are simply, at displacement zero

$$Cov(\Delta p_t^*, \Delta u_t) = \frac{1}{T} Cov(\Delta p_0^*, -\Delta p_0^*) = \frac{-\sigma^2}{T}, \tag{146}$$

and at higher order displacements

$$Cov(\Delta p_{t-\tau}^*, \Delta u_t) = \frac{1}{T} Cov(\Delta p_0^*, \frac{\Delta p_0^*}{T}) = \frac{\sigma^2}{T^2}, \tag{147}$$

which leads directly to the cross-correlations given by Proposition 3.

In the remainder of this section we describe for comparison a Kyle-type setup within our discrete time framework. Linear information revelation in our framework implies $\lambda_t = \frac{\sigma}{T}\frac{\varepsilon_0}{q_t} = \frac{\sigma}{T}|\varepsilon_0|$. Discretizing (44), i.e. integrating over a time interval of unit length, and dropping the zero-mean diffusion term we get

$$\Delta\tilde{\tilde{p}}_\tau = \frac{1}{T}(p^* - \tilde{\tilde{p}}(0)) = \frac{1}{T}\Delta p_0^*. \tag{148}$$

Therefore for $\tau \in [1; T-1]$ we have $\Delta p_t = \frac{\Delta p_0^*}{T}$, and for $\tau = T$ we get $\Delta p_T = \frac{\Delta p_0^*}{T} + \lambda_T q_T - \lambda_0 q_0$. From (27)

$$\begin{aligned} \Delta u_0 &= \sigma(\varepsilon_{-T} - \varepsilon_0) + s_0 q_0 - s_{-1} q_{-1} - s_{-1} q_{-1}(T-1) \\ &= \sigma\varepsilon_0 \frac{1-T}{T}, \end{aligned} \tag{149}$$

and from (28)

$$\Delta u_t = s q_t = \frac{\sigma\varepsilon_0}{T}. \tag{150}$$

Further,

$$E((\Delta p_t^*)^2) = \frac{\sigma^2}{T} \tag{151}$$

$$E(\Delta p_t^* \Delta p_t) = \frac{\sigma^2}{T^2} \tag{152}$$

$$E(\Delta p_0^* \Delta p_T) = 0. \tag{153}$$

The contemporaneous cross-covariance ($\tau = 0$) is therefore

$$Cov\left(\Delta p_t^*, \Delta u_t\right) = \frac{1}{T} Cov\left(\sigma\varepsilon_0, \sigma\varepsilon_0 \frac{1-T}{T}\right) = \frac{\sigma^2}{T}\frac{1-T}{T}, \tag{154}$$

and the cross-covariance at nonzero displacements $\tau \in [1; T-1]$ is

$$Cov\left(\Delta p_{t-\tau}^*, \Delta u_t\right) = \frac{1}{T} Cov\left(\sigma\varepsilon_0, \sigma\varepsilon_0 \frac{1}{T}\right) = \frac{\sigma^2}{T^2}. \tag{155}$$

The variance of the noise is

$$Var(\Delta u_t) = \frac{1}{T} E(\Delta u_0^2) + \frac{T-1}{T} E(\Delta u_t^2)$$

$$
\begin{aligned}
&= \sigma^2\frac{(T-1)^2}{T^3}+\sigma^2(T-1)\frac{1}{T^3}\\
&= \frac{T-1}{T^2}\sigma^2. \qquad (156)
\end{aligned}
$$

Therefore the contemporaneous cross-correlation is

$$
Corr\left(\Delta p_t^*, \Delta u_t\right) = \frac{-\frac{\sigma^2}{T}\left(1-\frac{1}{T}\right)}{\sqrt{\frac{\sigma^2}{T}}\sqrt{\frac{\sigma^2}{T}\left(1-\frac{1}{T}\right)}} = -\sqrt{1-\frac{1}{T}}. \qquad (157)
$$

The cross-correlation at nonzero displacements is for $\tau \in [1; T-1]$

$$
Corr\left(\Delta p_{t-\tau}^*, \Delta u_t\right) = \frac{\frac{\sigma^2}{T^2}}{\sqrt{\frac{\sigma^2}{T}}\sqrt{\frac{\sigma^2}{T}\left(1-\frac{1}{T}\right)}} = \frac{1}{\sqrt{T(T-1)}} < \frac{1}{\sqrt{2}}, \qquad (158)
$$

and zero for $\tau \geq T$.

F Effect of Risk Aversion on Optimal Price

In this appendix we show that high risk aversion pushes the optimal price toward the midpoint of the support. In other words, if $f(\cdot)$ is without loss of generality right-skewed, then $p(n)$ is increasing in n, $\forall n \geq 1$. First, note that $p(n)$, $p(n) \in [\underline{p}, \overline{p}]$, is continuous. If $\underline{p}$ or $\overline{p}$ are infinite, we replace these bounds with a function of n, thereby making the domain of p compact. As $f(\cdot)$ and all components of the integral are continuous functions, the theorem of the maximum gives continuity of $p(n)$.

Next, to evaluate how the optimal price $p(n)$ responds to changes in risk aversion n, take the total differential of (63) and rearrange to obtain

$$
\begin{aligned}
\frac{dp(n)}{dn} = \frac{1}{n-1} \times &\left\{ -\int_{\underline{p}}^{p(n)} \left(p(n)-p^*\right)^{n-1} \ln\left(p(n)-p^*\right) f(p^*)dp^* \right.\\
&+ \left. \int_{p(n)}^{\overline{p}} \left(p^*-p(n)\right)^{n-1} \ln\left(p^*-p(n)\right) f(p^*)dp^* \right\} \Big/ \\
&\left\{ \int_{\underline{p}}^{p(n)} \left(p(n)-p^*\right)^{n-2} f(p^*)dp^* + \int_{p(n)}^{\overline{p}} \left(p^*-p(n)\right)^{n-2} f(p^*)dp^* \right\}. \qquad (159)
\end{aligned}
$$

In the following argument we use that $f(\cdot)$ is monotone and assume without loss of generality that $f(\cdot)$ is monotonically decreasing. This means $f(\cdot)$ is right-skewed on $[\underline{p}, \overline{p}]$, which occurs if the market maker has some information that the strong form efficient price has increased. Under this assumption (159) is positive. To see this, note first that both terms in the denominator are positive. To economize notation we replace $p \equiv p(n)$, $d \equiv p(n) - \underline{p}$ and $x \equiv p^*$. The numerator can be broken up into three parts:

$$\begin{aligned}
& - \int_{\underline{p}}^{p} (p-x)^{n-1} \ln(p-x) f(x)dx + \int_{p}^{\overline{p}} (x-p)^{n-1} \ln(x-p) f(x)dx \\
= & - \int_{p-d}^{p-1} (p-x)^{n-1} \ln(p-x) f(x)dx + \int_{p+1}^{p+d} (x-p)^{n-1} \ln(x-p) f(x)dx \\
& - \int_{p-1}^{p} (p-x)^{n-1} \ln(p-x) f(x)dx + \int_{p}^{p+1} (x-p)^{n-1} \ln(x-p) f(x)dx \\
& + \int_{p+d}^{\overline{p}} (x-p)^{n-1} \ln(x-p) f(x)dx. \qquad (160)
\end{aligned}$$

The first term, which exists only for $d > 1$, gives

$$\begin{aligned}
& - \int_{p-d}^{p-1} (p-x)^{n-1} \ln(p-x) f(x)dx + \int_{p+1}^{p+d} (x-p)^{n-1} \ln(x-p) f(x)dx \\
= & - \int_{p+1}^{p+d} (x-p)^{n-1} \ln(x-p) f(2p-x)dx \\
& + \int_{p+1}^{p+d} (x-p)^{n-1} \ln(x-p) f(x)dx \\
= & \int_{p+1}^{p+d} (x-p)^{n-1} \ln(x-p) \left[-f(2p-x) + f(x)\right] dx \\
\geq & \int_{p+1}^{p+d} (x-p)^{n-1} \ln(d) \left[-f(2p-x) + f(x)\right] dx
\end{aligned}$$

$$= -\int_{p-d}^{p-1} (p-x)^{n-1} \ln(d)\, f(x)dx + \int_{p+1}^{p+d} (x-p)^{n-1} \ln(d)\, f(x)dx. \qquad (161)$$

The second term is for $d \geq 1$

$$\begin{aligned}
& - \int_{p-1}^{p} (p-x)^{n-1} \ln(p-x)\, f(x)dx + \int_{p}^{p+1} (x-p)^{n-1} \ln(x-p)\, f(x)dx \\
= & -\int_{p}^{p+1} (x-p)^{n-1} \ln(x-p)\, f(2p-x)dx \\
& + \int_{p}^{p+1} (x-p)^{n-1} \ln(x-p)\, f(x)dx \\
= & \int_{p}^{p+1} (x-p)^{n-1} \ln(x-p) \left[f(x) - f(2p-x)\right] dx \geq 0. \qquad (162)
\end{aligned}$$

For $d < 1$ the last inequality of the calculations for the second term be replaced by

$$\begin{aligned}
& \int_{p}^{p+1} (x-p)^{n-1} \ln(x-p) \left[f(x) - f(2p-x)\right] dx \\
\geq & \int_{p}^{p+d} (x-p)^{n-1} \left[f(x) - f(2p-x)\right] dx \ln(d) \geq 0. \qquad (163)
\end{aligned}$$

And for the last term we can write

$$- \int_{p+d}^{\overline{p}} (x-p)^{n-1} \ln(x-p)\, f(x)dx > - \int_{p+d}^{\overline{p}} (x-p)^{n-1} \ln(d)\, f(x)dx. \qquad (164)$$

Using (161), (162) and (164), (160) becomes

$$(160) \quad > \left[- \int_{p-d}^{p-1} (p-x)^{n-1} f(x)dx + \int_{p+1}^{\underline{p}} (x-p)^{n-1} f(x)dx \right] \ln(d)$$

$$
\begin{aligned}
> & \quad \left[-\int_{p-d}^{p-1} (p-x)^{n-1} f(x)dx - \int_{p-1}^{p} (p-x)^{n-1} f(x)dx \right. \\
+ & \quad \left. \int_{p}^{p+1} (x-p)^{n-1} f(x)dx + \int_{p+1}^{\overline{p}} (x-p)^{n-1} f(x)dx \right] \ln(d) \\
= & \quad \left[-\int_{p-d}^{p} (p-x)^{n-1} f(x)dx + \int_{p}^{\overline{p}} (x-p)^{n-1} f(x)dx \right] \ln(d) \\
= & \quad 0, \qquad (165)
\end{aligned}
$$

where the inequality follows from the monotonicity of $f(\cdot)$, and the last equality follows from the first order condition (63).

Likewise, for $d < 1$ we have

$$
\begin{aligned}
(160) \; > & \quad \left[-\int_{p-d}^{p} (p-x)^{n-1} f(x)dx + \int_{p}^{\underline{p}} (x-p)^{n-1} f(x)dx \right] \ln(d) \\
= & \quad 0. \qquad (166)
\end{aligned}
$$

Therefore the numerator is positive and

$$
\frac{dp(n)}{dn} > 0 \qquad (167)
$$

for right-skewed distributions. Combining this with the fact that $p(1) = \text{Median}(p^*)$ and $p(\infty) = \text{Midsupport}(p^*)$ we conclude that $p(n)$ monotonically increases from the median to the midpoint of the support of the efficient price distribution $f(\cdot)$, if $f(\cdot)$ is right-skewed. Analogously, for left-skewed $f(\cdot)$, $p(n)$ monotonically *de*creases from the median to the midpoint of the support.

G Semi-strong form Cross-correlations

In the multi-period setup of section 4.1 the semi-strong form efficient price has the unconditional variance

$$Var(\Delta\tilde{p}_t) = \frac{1}{T}\left\{\sigma^2 + \sum_{i=0}^{T-1}\lambda_i^2 + E\left[\left(\sum_{i=-1}^{-T}\lambda_i q_i\right)^2\right] - 2\sigma\sum_{i=-1}^{-T}\lambda_i E(q_i\varepsilon_{-T})\right\} \tag{168}$$

and the corresponding noise has an unconditional variance of

$$\begin{aligned} Var(\Delta u_t) &= \frac{1}{T}\sum_{t=0}^{T-1} Var(\Delta u_t) \\ &= \frac{1}{T}\left\{\sum_{i=0}^{T-1}\left[(\lambda_i - s_i)^2 + (\lambda_{i-1} - s_{i-1})^2\right]\right. \\ &- \left. 2\sum_{i=1}^{T-1} E(q_t q_{t-1})(\lambda_i - s_i)(\lambda_{i-1} - s_{i-1})\right\}. \end{aligned} \tag{169}$$

The contemporaneous cross-correlation is

$$Corr(\Delta\tilde{p}_t, \Delta u_t) = \frac{Cov(\Delta\tilde{p}_t, \Delta u_t)}{\sqrt{Var(\Delta\tilde{p}_t)Var(\Delta u_t)}}. \tag{170}$$

where $Cov(\Delta\tilde{p}_t, \Delta u_t)$ is given by (76). All other cross-correlation can be obtained analogously.

For $T = 1$, spread and adverse selection parameter are constants, i.e. $s_t = s$ and $\lambda_t = \lambda$ $\forall t$, and the variance terms (168) and (169) simplify radically to

$$Var(\Delta\tilde{p}_t) = \sigma^2 - 2\sigma\lambda E(q_t\varepsilon_t) + 2\lambda^2, \tag{171}$$

$$Var(\Delta u_t) = 2(s - \lambda)^2, \tag{172}$$

where we have used that q_t is serially uncorrelated. Thus for $T = 1$

$$Corr(\Delta\tilde{p}_t, \Delta u_t) = \frac{2\lambda - \sigma E(q_t\varepsilon_t)}{\sqrt{\sigma^2 - 2\sigma\lambda E(q_t\varepsilon_t) + 2\lambda^2}}\frac{\text{sgn}(s - \lambda)}{\sqrt{2}}, \tag{173}$$

and

$$Corr(\Delta\tilde{p}_{t-1}, \Delta u_t) = \frac{-\lambda}{\sqrt{\sigma^2 - 2\sigma\lambda E(q_t\varepsilon_t) + 2\lambda^2}} \frac{\text{sgn}(s-\lambda)}{\sqrt{2}}. \tag{174}$$

H Zero Profit Condition under Perfect Competition

If the market maker in a one period model can quote bid and ask prices independently, or equivalently p^{mid} and s, he can condition on q_t and therefore on $\text{sgn}(\Delta p_t^*)$. For the ask price his optimization problem is

$$p_n^{ask} = \underset{x\in[\underline{p},\overline{p}]}{\text{argmax}} - \int_x^{\overline{p}} (p^* - x)^n f(p^*) dp^*, \tag{175}$$

which is maximized at $p_n^{ask} = \infty$, because absent any competition the market maker has no incentive to do any loss-bringing trades with informed traders. But competition with other market makers drives profit down to zero. Assuming that the zero profit condition must hold in expectation for each buy and each sell independently, the spread $2s$ from (15), (16) and (17) is determined by

$$-\int_{p^{ask}}^{\overline{p}} \left(p^* - p^{ask}\right)^n f(p^*) dp^* + \pi(s) = 0. \tag{176}$$

Clearly, as in section 3.3.3 as $n \to \infty$ the ask price p^{ask} grows as well, without bound if $f(\cdot)$ has unbounded support. The expression for the bid price is analogous; the bid price falls with n.

In the following example we show that s is uniquely determined as a competitive outcome. For simplicity, we assume risk neutrality $n = 1$. Under perfect competition the zero profit condition requires the market maker's losses in trades with informed traders to exactly offset the spread earned from trades with uninformed traders. An increase in the spread benefits the market maker in two ways: it increases his spread income from uninformed traders and reduces his losses to the informed traders. This can be written as

$$-\int_{-\infty}^{p^{bid}} \left(p^* - p^{bid}\right) f(p^*) dp^* - \int_{p^{ask}}^{\infty} \left(p^{ask} - p^*\right) f(p^*) dp^*$$

$$= \left(\int_{p^{bid}}^{p^{ask}} f(p^*) dp^*\right) \frac{p^{ask} - p^{bid}}{2}. \tag{177}$$

Using the symmetry of the expected density $f(p^*)$ around p^{mid}, (177) becomes a problem of setting p^{mid} and s.

$$-\int_{-\infty}^{p^{mid}-s} \left(p^* - p^{mid} + s\right) f(p^*)dp^* = \left(\int_{p^{mid}-s}^{p^{mid}} f(p^*)dp^*\right) s. \tag{178}$$

Then, with F denoting the cumulative density function of f (whose expected value is assumed to exist)

$$-\left(E(p^*)\left|_{-\infty}^{p^{mid}-s}\right. \cdot F(p^{mid}-s) - (p^{mid}-s)F(p^{mid}-s)\right)$$
$$= \left(F(p^{mid}) - F(p^{mid}-s)\right) s, \tag{179}$$

where $E(p^*)\left|_{-\infty}^{p}\right.$ denotes the expectation of p^* over the distribution $f(p^*)$ restricted to the interval $[-\infty, p]$. p^{mid} is given by the optimal learning rule. (179) is one equation in the one unknown, s.

$$p^{mid} - E(p^*)\left|_{-\infty}^{p^{mid}-s}\right. = \frac{sF(p^{mid})}{F(p^{mid}-s)} \tag{180}$$

The left-hand side (LHS) is monotonically increasing in s from some positive number to positive infinity. The right-hand side (RHS) is monotonically increasing in s, from 0 to positive infinity. It can be shown that the RHS increases faster than the LHS and that this difference in slope does not go to zero as s becomes larger. Differentiating (178) using Leibnitz's rule, we get

$$-\left(p^{mid} - s - p^{mid} + s\right) f(p^{mid}-s)\left(\frac{-1}{2}\right) - \int_{-\infty}^{p^{mid}-s} \frac{1}{2} f(p^*)dp^*$$
$$< -f(p^{mid}-s)\left(\frac{-1}{2}\right) s + \frac{1}{2}\left(\int_{p^{mid}-s}^{p^{mid}} f(p^*)dp^*\right) \tag{181}$$

and

$$-\frac{1}{2}F(p^{mid}-s) < \frac{s}{2} f(p^{mid}-s) + \frac{1}{2}\left(F(p^{mid}) - F(p^{mid}-s)\right).$$

Therefore

$$0 < sf(p^{mid}-s) + F(p^{mid}), \tag{182}$$

which holds always by definition. This shows two things: Firstly, the RHS in (178) is increasing faster than the LHS. And secondly, the difference in slope is always at least

$F(p^{mid}) > 0$. Hence we have proven that there is a single crossing and s is determined uniquely.

If the LHS in (180) is very small because the support of the distribution became very small by learning, then s must be small as well. Hence, as market makers learn, the spread s in the market shrinks. If some market maker learned slower than his peers, he would make losses at least until the next change in p^*.