PENROSE LIMITS, SUPERGRAVITY AND BRANE DYNAMICS

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Abstract. We investigate the Penrose limits of classical string and M-theory backgrounds. We prove that the number of (super)symmetries of a supergravity background never decreases in the limit. We classify all the possible Penrose limits of AdS $\times S$ spacetimes and of supergravity brane solutions. We also present the Penrose limits of various other solutions: intersecting branes, supersymmetric black holes and strings in diverse dimensions, and cosmological models. We explore the Penrose limit of an isometrically embedded spacetime and find a generalisation to spaces with more than one time. Finally, we show that the Penrose limit is a large tension limit for all branes including those with fields of Born–Infeld type.

Contents

1. Introduction

It has been shown recently that the maximally supersymmetric backgrounds of M-theory and IIB superstring are related by a limiting procedure, known as the *Penrose limit*. In particular, it has been found that eleven-dimensional Minkowski spacetime and the maximally supersymmetric Hpp-wave[1] of [2] (see also [1]) can arise as Penrose limits of the $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$ solutions of M-theory [3]. In addition, ten-dimensional Minkowski spacetime and the maximally supersymmetric Hpp-wave of [4] are Penrose limits of the $\mathrm{AdS}_5 \times S^5$ solution of IIB superstring theory. The other known maximally supersymmetric pp-wave solutions in four [5] and five and six dimensions [6] can also be obtained in this way.

As shown in [7], the (Green–Schwarz) IIB superstring on the maximally supersymmetric Hpp-wave solution is a free theory after gauge-fixing, and can therefore be quantised exactly. This fact together with the above relation between the maximally supersymmetric solutions of IIB supergravity

[1]The name *Hpp-wave* was coined in [1] to denote pp-waves with a homogeneous geometry, say G/H, and homogeneous (that is, G-invariant) fluxes.

were used in [8] to propose a novel derivation of the spectrum of IIB superstrings on both Minkowski and maximally supersymmetric Hpp-wave spacetimes from gauge theory using the AdS/CFT correspondence.

In [9] Penrose showed that any spacetime (i.e., any solution of the Einstein field equations) has a limiting spacetime which is a plane wave. This limit can be thought of as a "first order approximation" to the spacetime along a null geodesic. The limiting spacetime depends on the choice of null geodesic and hence a spacetime can have more than one Penrose limit. More recently, Güven [10] extended Penrose's argument to show that any solution of a supergravity theory has plane wave limits which are also solutions. This depends crucially on the local symmetries (diffeomorphisms and gauge invariance) of supergravity theories as well as the homogeneity of the supergravity action under a certain constant rescaling of the fields, features already present in the four-dimensional Einstein (or Einstein–Maxwell) action.

A novelty of the Penrose limit is that that string theory in pp-wave backgrounds simplifies dramatically due to the existence of a natural light-cone gauge, and in many cases can be quantised exactly. String theory on pp-wave NSR backgrounds has been considered before (see, for example, [11] and references therein). In fact, the various contractions of WZW models considered in [12, 13, 14, 15] are special cases of a Penrose limit.

Since the Penrose limit relates in a precise way a general supergravity background to a pp-wave, it allows us to understand string theory on many backgrounds at least at a Penrose limit. This opens the possibility to probe string theory in backgrounds that were hitherto inaccessible, for example backgrounds with RR fields, in the background of various gravitational solitons, black holes and even non-supersymmetric backgrounds. The existence of different Penrose limits is equally important, as every such limit is associated to a perturbation theory with parameter the inverse of the string tension [3] and different limits organise the perturbation theory in different ways. So it may be the case that what is a perturbative effect in one perturbation expansion, may be viewed as non-perturbative in another.

In this paper we shall investigate Penrose limits, following the foundational work of Penrose and Güven, derive some of its basic properties and explore the different Penrose limits of many supergravity backgrounds. We shall be primarily concerned with those properties of a supergravity background which are inherited by its Penrose limits. By rephrasing the Penrose limit in the more general context of "limits of spacetimes" introduced by Geroch [16], we shall show that the numbers both of symmetries and supersymmetries never decrease in a Penrose limit. We shall also show that the symmetry superalgebra of a supergravity background gets contracted in a Penrose limit. Of course, the limiting background may admit additional symmetries and supersymmetries not present in the original background and in fact it can be shown that Penrose limits of supergravity theories always preserve at least one half of the supersymmetry.

After establishing the hereditary properties of the Penrose limit, we classify the Penrose limits of AdS $\times S$ spacetimes. We will show that such spacetimes can have only two distinct Penrose limits: Minkowski spacetime or an Hpp-wave. We shall then classify the different Penrose limits of elementary brane solutions. These classifications follow by determining the orbits of the isometry group of the spacetime on the space of null geodesics and invoking the *covariance property* of the Penrose limit, which says that null geodesics which are related by an isometry yield Penrose limits which are themselves isometric.

We will then exhibit Penrose limits of many supergravity solutions, including

- M-branes, D-branes, NS-branes and their near-horizon limits;
- intersecting brane solutions, in particular intersecting M-branes;
- supersymmetric black hole solutions in four- and five-dimensions and string solutions in six dimensions as well as those of their near horizon geometries which are of AdS $\times S$ type; and
- cosmological models.

The latter case has been included to demonstrate the universality of the Penrose limit and to point out a similarity between the Penrose limit of the string solution and that of a cosmological model. We also investigate the Penrose limit of solutions that admit an isometric embedding in flat space. This naturally leads to a generalisation of the Penrose limit. Finally we shall show that the Penrose limit is a large tension limit for all brane probes in a spacetime. This includes branes with Born-Infeld type of worldvolume fields like D-branes and the M5-brane. The precise dependence of the D-brane probe actions on the string tension plays a key role.

1.1. **Contents and summary of main results.** This paper is organised as follows. In Section 2 we summarise some known facts about the Penrose limit in the context of supergravity. We define the Penrose limit, discuss the geometry of limiting spacetime and relate it to pp-waves in both Rosen and Brinkman coordinates. We discuss the physical meaning of the limit and derive the covariance property which will be crucial in later sections.

In Section 3 we introduce the concept of a *hereditary* property, modifying slightly the more general notion introduced by Geroch [16]. We discuss hereditary properties involving the curvature tensor: showing that, for example, that Einstein spaces have Ricci-flat limits, that conformal flatness is hereditary and that so is the condition of being (locally) symmetric. We show that the property of being a supergravity background is also hereditary, so that solutions yield solutions in the limit, a fact already observed by Güven. We discuss isometries from two points of view: one very explicit which shows why the symmetry algebra gets contracted in the limit, and one more abstract which shows why its dimension does not. This is achieved by paraphrasing and slightly generalising an argument originally due to Geroch. This generalisation consists in showing that the dimension of the space of parallel sections of a family of vector bundles does not decrease in the limit. This is then applied to Killing vectors and to Killing spinors to conclude, after a minor refinement, that the symmetry (super)algebra of a Penrose limit is at least as large as that of the original background.

In Section 4 we exhibit the maximally supersymmetric Hpp-waves [2, 1, 4] of eleven-dimensional and IIB supergravity as Penrose limits of the near-horizon geometries [17] of the D3 and M2/5 brane solutions. More generally we show that all AdS $\times S$ solutions have only two different Penrose limits: Minkowski spacetime and the Hpp-waves. In addition we demonstrate explicitly that the isometries are hereditary and in this way illustrate the contraction phenomenon.

Section 5 contains a classification of the possible Penrose limits of elementary brane solutions of a supergravity theory. Roughly speaking this is the "cohomogeneity one" analogue of the classification in Section 4. As a result we do not just obtain a finite number of non-isometric Penrose limits, but rather a continuous family, labelled by the angular momentum of the null geodesic along which we take the limit. We illustrate these results by explicit computations of the limits for a number of brane solutions: D-branes, fundamental strings, NS-branes and M-branes.

Section 6 is devoted to the Penrose limits of intersecting brane solutions in ten and eleven dimensions. After some general remarks on the different Penrose limits of an intersecting brane solution, we discuss explicitly the case of a triple pointlike intersection of M2-branes and the intersection of two M2 and two M5 branes, along with their near-horizon geometries.

Section 7 considers the Penrose limits of supersymmetric five- and four-dimensional black hole solutions arising in toroidal compactification of string and M-theories. We also discuss the Penrose limit of a six-dimensional string solution.

Section 8 examines the Penrose limits of cosmological models, focusing for definiteness on the four-dimensional Friedmann–Robertson–Walker spacetime. For the spatially flat cosmologies we find that the Penrose limit is a homogeneous Lorentzian spacetime.

In Section 9 we discuss the Penrose limit of a spacetime which is isometrically embedded in a flat pseudo-riemannian space. We will argue that there exists a generalisation of the notion of Penrose limit for pseudoriemmanian spaces with signature (s, t) where the limit is taken, not along null geodesics, but along totally geodesic maximally isotropic submanifolds. This is illustrated with the near-horizon geometry of the four-dimensional Reissner–Nordström black hole. A similar discussion applies *mutatis mutandis* to the near-horizon geometry of the D3 brane. We show, in passing, how to isometrically embed the Hpp-wave metrics in flat space.

Finally in section 10, we show that Penrose limits are large tension limits in the context of brane dynamics, including that of branes with fields of Born-Infeld type. To establish this for the case of D-branes, the precise dependence of D-brane actions on the string tension is used. We also find that every Penrose limit defines a perturbation theory for the brane probes, including fundamental strings. Generically two different perturbation expansions are related in a non-linear way. So a perturbative effect in one requires the summation of an infinite number of graphs (terms) in the other.

2. Penrose limit of supergravity theories

In this section we review the Penrose limit of supergravity theories along the lines of [3]. We also review the physical interpretation of the Penrose limit and we introduce a useful covariance property.

2.1. **The Penrose–Güven limit.**

In this section we will review the Penrose limit as described by Güven for backgrounds of supergravities in ten and eleven dimensions.

Let (M, g) be a D-dimensional lorentzian spacetime. According to [9, 10] in the neighbourhood of a segment of a null geodesic γ containing no conjugate points, it is possible to introduce local coordinates U, V, Y^i such that the metric takes the form

$$g = dV \left(dU + \alpha dV + \sum_i \beta_i dY^i \right) + \sum_{i,j} C_{ij} dY^i dY^j \ , \tag{2.1}$$

where α, β_i and C_{ij} are functions of all the coordinates, and where C_{ij} is a symmetric positive-definite matrix. The coordinate system breaks down as soon as $\det C = 0$, signalling the existence of a conjugate point. The coordinate U is the affine parameter along a congruence of null geodesics labelled by V and Y^i. The geodesic γ is the one for which $V = 0 = Y^i$. Notice that the metric is characterised by the conditions $g_{UV} = 1/2$ and $g_{UU} = 0 = g_{UY^i}$ (we will also frequently use coordinates in which $g_{UV} = 1$). This means that the vector field $\partial/\partial U$ is self-parallel and hence geodetic. Also its dual one-form dV is closed so that the null geodesic congruence into which γ has been embedded is twist-free.

In ten- and eleven-dimensional supergravity theories there are other fields besides the metric, such as the dilaton Φ, gauge potentials or more generally p-form potentials A_p with $(p+1)$-form field strengths. The gauge potentials are defined up to gauge transformations $A_p \mapsto A_p + d\Lambda_{p-1}$ in such a way that the field strength $F_{p+1} = dA_p$ is gauge invariant. It is possible to use this gauge freedom in order to gauge away some of the components of the p-form potentials. Indeed, one can choose a gauge in which

$$i(\partial/\partial U) A = 0 \ , \tag{2.2}$$

or in components

$$A_{Ui_1i_2\dots i_{p-1}} = A_{UVi_1i_2\dots i_{p-2}} = 0 \ . \tag{2.3}$$

The starting point of the Penrose limit is the data (M, g, Φ, A_p) defined in the neighbourhood of a conjugate-point-free segment of a null geodesic γ where g and A_p take the forms (2.1) and (2.2), respectively.

We now introduce a positive real constant $\Omega > 0$ and rescale the coordinates as follows

$$U = u \ , \qquad V = \Omega^2 v \qquad \text{and} \qquad Y^i = \Omega y^i \ . \tag{2.4}$$

Substituting these expressions in the fields of the theory we obtain an Ω-dependent family of fields $g(\Omega)$, $\Phi(\Omega)$ and $A_p(\Omega)$. Let φ_Ω denote the (local) diffeomorphism defined by (2.4) and let us define new fields

$$g_\Omega = \Omega^{-2}\varphi_\Omega^* g \qquad \Phi_\Omega = \varphi_\Omega^*\Phi \qquad A_\Omega = \Omega^{-p}\varphi_\Omega^* A \implies F_\Omega = \Omega^{-p}\varphi_\Omega^* F \ . \tag{2.5}$$

These new fields $(g_\Omega, F_\Omega, \Phi_\Omega)$ are related to the original fields (g, F, Φ) by a diffeomorphism and a rescaling, and perhaps a gauge transformation.

Explicitly, g_Ω is

$$\begin{aligned} g_\Omega =& dvdu + \sum_{i,j} C_{ij}(u, \Omega y^i, \Omega^2 v) dy^i dy^j \\ &+ \Omega \sum_i \beta_i(u, \Omega y^i, \Omega^2 v) dy^i + \Omega^2 \alpha(u, \Omega y^i, \Omega^2 v)(dv)^2 \ . \end{aligned} \tag{2.6}$$

The coordinate and gauge choices (2.1) and (2.2) ensure that the following *Penrose limit* [9] (as extended by Güven [10] to fields other than the metric) is well-defined:

$$\begin{aligned} \bar{g} &= \lim_{\Omega\to 0} g_\Omega \\ \bar{\Phi} &= \lim_{\Omega\to 0} \Phi_\Omega \\ \bar{A}_p &= \lim_{\Omega\to 0} A_\Omega \\ \bar{F}_{p+1} &= \lim_{\Omega\to 0} F_\Omega \ . \end{aligned} \tag{2.7}$$

By virtue of (2.4) the limiting fields only depend on the coordinate u, which is the affine parameter along the null geodesic. The resulting expression for the metric is

$$\bar{g} = dudv + \sum_{i,j} C_{ij}(u) dy^i dy^j \ , \tag{2.8}$$

where $C_{ij}(u) \equiv C_{ij}(u, 0, 0)$. We see that for the limit to exist it is necessary that $g_{UU} = g_{UY^i} = 0$. These conditions alone are sufficient to ensure that $X = \partial/\partial U$ is self-parallel, $\nabla_X X \propto X$, so that one may as well assume that X is geodesic, $\nabla_X X = 0$. Taken together, these conditions then lead precisely to the form of the metric given in (2.1).

An obvious property of the metric (2.8) is that it is mapped to itself under another Penrose limit along $\partial/\partial u$, and that the Penrose limit of (2.8) along $\partial/\partial v$ is isometric to the flat Minkowski metric.

The gauge potentials $\bar{A}_p$ only have components in the transverse directions y^i:

$$i(\partial/\partial u)\bar{A}_p = 0 = i(\partial/\partial v)\bar{A}_p \ , \tag{2.9}$$

and the field strengths $\bar{F}_{p+1}$ are therefore of the form

$$\bar{F}_{p+1} = du \wedge \bar{A}_p(u)' \ , \tag{2.10}$$

where $'$ denotes d/du. Notice in particular that $\bar{F}_{p+1}$ is always null.

So far we have not shown that a spacetime that emerges as the Penrose limit of a solution of a supergravity theory is a solution of the same theory. This is not apparent and it will be established later when the hereditary properties of the Penrose limit are investigated. As we shall see for the supergravity theories this follows because of the invariance of the supergravity action under diffeomorphisms and gauge transformations and the its homogeneity properties under the (overall) scaling of the fields required for the Penrose limit and a continuity argument.

The above expression for $\bar{g}$ is that of a pp-wave in Rosen coordinates. Generically it possess a $(2D-3)$-dimensional algebra of isometries, isomorphic to a Heisenberg algebra, even though for particular choices of $C_{ij}(u)$ the isometry algebra can be larger. This Lie algebra is generated by the following Killing vectors:

$$e_i = \frac{\partial}{\partial y^i} \ , \quad e_+ = \frac{\partial}{\partial v} \quad \text{and} \quad e_i^* = y^i \frac{\partial}{\partial v} - \sum_j \int C^{ij}(u) du \frac{\partial}{\partial y^j} \ , \tag{2.11}$$

where C^{ij} is the inverse of C_{ij}, which exists in the segment of the null geodesic in which our coordinates are valid. These Killing vectors satisfy a Heisenberg Lie algebra

$$[e_i, e_j] = 0 = \left[e_i^*, e_j^*\right] \qquad \left[e_i, e_j^*\right] = \delta_{ij} e_+ \tag{2.12}$$

with central element e_+.

2.2. **Penrose limits and pp-waves.** It is possible to change to Brinkman (also called harmonic) coordinates in such a way that the Penrose limit metric (2.8) metric takes the form

$$\bar{g} = 2dx^+dx^- + \left(\sum_{i,j} A_{ij}(x^-)x^i x^j\right)(dx^-)^2 + \sum_i dx^i dx^i \ . \tag{2.13}$$

When A_{ij} is constant this metric describes a lorentzian symmetric or Cahen–Wallach space [18]. Such spaces include the maximally supersymmetric

Hpp-waves of eleven-dimensional [2, 1] and IIB supergravity [4], namely

$$g_{11} = 2dx^+dx^- - \left(\sum_{i,j=1}^{3} \delta_{ij}x^i x^j + \tfrac{1}{4}\sum_{i,j=4}^{9} \delta_{ij}x^i x^j\right)(dx^-)^2 + \sum_{i=1}^{9} dx^i dx^i \tag{2.14}$$

$$g_{\text{IIB}} = 2dx^+dx^- - \left(\sum_{i,j=1}^{8} \delta_{ij}x^i x^j\right)(dx^-)^2 + \sum_{i=1}^{8} dx^i dx^i \tag{2.15}$$

(up to an overall scaling of A_{ij} by a real positive constant which can always be absorbed into a boost of (x^+, x^-)).

The explicit change of variables which takes the metric from Rosen to Brinkman form is given by

$$u = 2x^- \qquad v = x^+ - \tfrac{1}{2}\sum_{i,j} M_{ij}(x^-)x^i x^j \qquad y^i = \sum_j Q^i_j(x^-)x^j \ , \tag{2.16}$$

where Q^i_j is an invertible matrix satisfying

$$C_{ij}Q^i_k Q^j_l = \delta_{kl} \qquad \text{and} \qquad C_{ij}\left(Q'^i_j Q^j_l - Q'^i_k Q^j_l\right) = 0 \ , \tag{2.17}$$

and

$$M_{ij} = C_{kl}Q'^k_i Q^l_j \ , \tag{2.18}$$

which is symmetric by virtue of the second equation in (2.17). Here a $'$ denotes differentiation with respect to x^-. This equation guarantees that the limiting metric $\bar{g}$ has the form (2.13). The relation between C_{ij} and A_{ij} is

$$A_{ij} = -[C_{kl}Q'^l_j]'Q^k_i \ . \tag{2.19}$$

It remains to see that a Q obeying (2.17) always exists. Since C is symmetric it can be diagonalised at every point. Moreover, since C depends smoothly on the affine parameter u, we can find a matrix Q depending smoothly on u such that the first equation $QCQ^t = I$ in (2.17) is satisfied. This does not determine Q uniquely, since we can always multiply on the left by an orthogonal matrix O depending smoothly on u. Suppose that Q_0 has been chosen so that $Q_0CQ_0^t = I$. We claim it is possible to find an orthogonal matrix O such that $Q = OQ_0$ also satisfies the second equation of (2.17). This equation says that $M = Q'CQ$ is symmetric. Let us then decompose $M = S + A$ into symmetric and antisymmetric parts and similarly with $M_0 = Q_0'CQ_0 = S_0 + A_0$. The equation for O is then $A = O'O^t + OA_0O^t = 0$, which is equivalent to $O' = -A_0O$. This is a linear first-order differential equation depending smoothly on u, and hence has a unique solution for each initial value, at least for small enough u.

It is possible to rewrite the field strengths $\bar{F}_{p+1}$ given in (2.10) in terms of Brinkman coordinates, to arrive at the following expression:

$$\bar{F}_{p+1} = \sum_{i_k, j_k} \frac{d}{dx^-} \bar{A}_{i_1 i_2 \dots i_p}(2x^-) Q^{i_1}_{j_1} Q^{i_2}_{j_2} \cdots Q^{i_p}_{j_p} dx^- \wedge dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_p} \ . \tag{2.20}$$

A special case of this, frequently occurring in applications, is when C_{ij} is diagonal,

$$C_{ij}(u) = a_i^2(u)\delta_{ij} \ . \tag{2.21}$$

Then one can choose $Q^i_j = a_i(u)^{-1}\delta^i_j$ and finds

$$A_{ij}(x^-) = \frac{(a_i(x^-))''}{a_i(x^-)}\delta_{ij} \ . \tag{2.22}$$

The (p+1)-form field strength becomes

$$\begin{aligned} \bar{F}_{p+1} = \sum_{i_k} \frac{d}{dx^-} \bar{A}_{i_1 i_2 \dots i_p}(2x^-) a_{i_1}(x^-)^{-1} a_{i_2}(x^-)^{-1} \cdots a_{i_p}(x^-)^{-1} \\ dx^- \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p} \ . \end{aligned} \tag{2.23}$$

In particular we learn from this that the Penrose limit of a metric (2.1) with diagonal C_{ij} is the flat metric if and only if

$$a_i(x^-)'' = 0 \ , \tag{2.24}$$

i.e., if and only if

$$a_i(u) = b_i + c_i u \ . \tag{2.25}$$

We also learn that the Penrose limit is of Cahen–Wallach type (constant nondegenerate A_{ij}) if and only if the functions $a_i(u)$ are either trigonometric (for negative eigenvalues) or hyperbolic (for positive eigenvalues). Indeed, if

$$a_i(u) = b_i \sin \mu_i u + c_i \cos \mu_i u \tag{2.26}$$

or

$$a_i(u) = b_i \sinh \mu_i u + c_i \cosh \mu_i u \ , \tag{2.27}$$

then

$$a_i(u)'' = \mp \mu_i^2 a_i(u) \ , \tag{2.28}$$

and therefore $b_i(u)$ is constant,

$$b_i = \mp \mu_i^2 \ . \tag{2.29}$$

2.3. **The physical effect of the Penrose limit.** The physical interpretation of the Penrose limit is described by Penrose as follows [9]:

> *There is a 'physical' interpretation of the above mathematical procedure, which is the following. We envisage a succession of observers travelling in the space-time M whose world lines approach the null geodesic γ more and more closely; so we picture these observers as travelling with greater and greater speeds, approaching that of light. As their speeds increase they must correspondingly recalibrate their clocks to run faster and faster (assuming that all space-time measurements are referred to clock measurements in the standard way), so that in the limit the clocks measure the affine parameter x^0 along γ. (Without clock recalibration a degenerate space-time metric would result.) In the limit the observers measure the space-time to have the plane wave structure W_γ.*

In other words, the Penrose limit can be understood as a boost followed by a commensurate uniform rescaling of the coordinates in such a way that the affine parameter along the null geodesic remains invariant.

A related physical effect of the Penrose limit is to blow up the neighbourhood of a null geodesic. Indeed, up to the coordinate transformation (2.4), the metric g_Ω differs from the original spacetime metric g only by the conformal rescaling $g \to \Omega^{-2} g$. Thus up to a coordinate transformation (which becomes singular in the limit $\Omega \to 0$), the Penrose limit is a large volume limit.

In particular, consider two points P, Q is a neighbourhood of a null geodesic and the (geodesic) distance

$$d(P, Q; g) = \int_Q^P ds \tag{2.30}$$

which is defined as the length of the shortest geodesic joining P and Q, as measured with respect to the metric g. Suppose now that the geodesic joining P and Q is either timelike or spacelike, so that $d(P, Q; g) \neq 0$. The distance of the points P, Q can be written as

$$d(P, Q; g) = \Omega d(P, Q; \Omega^{-2} g(\Omega)) \ . \tag{2.31}$$

Thus since $\Omega \ll 1$, we have that

$$d(P, Q; g) = \Omega d(P, Q; \Omega^{-2} g(\Omega)) \ll d(P, Q; \Omega^{-2} g(\Omega)) \tag{2.32}$$

and therefore $d(P, Q; g) \ll d(P, Q; \bar{g})$, where $d(P, Q; \bar{g})$ is the distance of the two points as measured by the metric at the Penrose limit. Therefore, the distance of the two points P, Q as measured by the Penrose limit metric is thus much larger than that measured with respect to the original spacetime metric g. It is worth mentioning though that if the points P, Q are joined by a null geodesic, they will remain separated by a null geodesic in the Penrose limit.

2.4. **Covariance of the Penrose limit.** Let (M, g) be a lorentzian manifold. A null geodesic γ is characterised (at least for small values of the affine parameter) by specifying the initial position $\gamma(0) \in M$ and the initial velocity $\dot{\gamma}(0) \in T_{\gamma(0)}M$. In fact, the Penrose limit is only susceptible to the initial *direction* of the geodesic. Indeed, if γ_1 and γ_2 are two null geodesics starting at the same point but with collinear velocities; that is, $\dot{\gamma}_1(0) = \lambda\dot{\gamma}_2(0)$ for some nonzero constant λ, then the geodesics are related by a rescaling of the affine parameter. The resulting Penrose limits (see, e.g., (2.8)) are related by a rescaling of u, which can be reabsorbed in a reciprocal rescaling of the conjugate coordinate v. In other words, the Penrose limit depends on the actual curved traced by the geodesic and not on how it is parametrised. We conclude that the Penrose limit depends only on the data $(\gamma(0), [\dot{\gamma}(0)])$, where $\gamma(0)$ is a point in M and $[\dot{\gamma}(0)]$ is a point on the (future-pointing, say) celestial sphere of $T_{\gamma(0)}M$, which is the projectivisation of the nonzero future-pointing null vectors in $T_{\gamma(0)}M$.

A fundamental property of the Penrose limit is that if two null geodesics are related by an isometry, their Penrose limits are themselves isometric. We shall refer to this as the *covariance property* of the Penrose limit. This property is very useful in determining the possible Penrose limits in spacetimes with a large isometry group and will be used repeatedly in the bulk of the paper in classifying (up to isometry) the possible Penrose limits of supersymmetric backgrounds and their near-horizon geometries.

The covariance property holds because the isometry in question is by assumption Ω-independent and will therefore continue to exist when $\Omega = 0$. By contrast, if two metrics g_Ω and h_Ω are related by an Ω-dependent isometry, then their Penrose limits need not be isometric because the isometry between them could become singular in the limit. For example, the metrics g_Ω and $\Omega^{-2}g$ are isometric for all finite values of Ω, being related by the Ω-dependent coordinate transformation (2.4). In the limit $\Omega \to 0$, however, the limit of g_Ω is the Penrose limit, which is generically not isometric to the naive large volume limit, i.e., the limit as $\Omega \to 0$ of $\Omega^{-2}g$ (which is well defined if one combines it with the coordinate transformation $(U, V, Y^i) = \Omega(u, v, y^i)$).

We see that the notion of a Penrose limit or, more generally, that of a limit of a family of spacetimes in the sense discussed in [16] (see also Section 3.1), is not invariant under coordinate transformations depending on the parameter labelling the family of spacetimes. This accounts for the fact that the definition of the Penrose limit given in Section 2.1 looks rather non-covariant. Indeed it is, and it necessarily has to be. The situation would be different if we were interested in all possible limits of a family of spacetimes [16], but we are not.

3. Hereditary properties of Penrose limits

In this section we discuss those properties of a supergravity background that are inherited by all its Penrose limits.

3.1. **The set-up.** As mentioned in the introduction, Penrose limits of solutions of supergravity theories are also solutions by virtue of the homogeneity of the action under scaling. This observation is due to Penrose [9] originally and to Güven [10] in the supergravity context. We will re-establish this below. It is then of interest to investigate whether the Penrose limit of a solution inherits some other properties (e.g., isometries, supersymmetry) of the original solution.

The appropriate framework for addressing these questions has been introduced by Geroch in 1969 [16]. In this somewhat more general context one considers a (one-parameter) family of spacetimes (M_Ω, g_Ω) for $\Omega > 0$ and tries to make sense and study the properties of the limit spacetime as $\Omega \to 0$.

The reason for considering such a one-parameter family instead of a one-parameter family of metrics g_Ω on a fixed spacetime M is that, as we saw in Section 2.4, the limit is not invariant under Ω-dependent coordinate transformations. Hence, if one is interested in studying *all* limits of a given one-parameter family of space-times, one should not presuppose an a priori identification of points in the different M_Ω. This evidently leads to some technical complications in defining *the* limit of a family of spacetimes which are analysed and resolved in [16].

In our case, however, we can sidestep these technical problems and work with a fixed spacetime M because we are not interested in studying all possible limits but we are singling out a particular limit, namely the Penrose limit. Evidently, the coordinate transformation (2.4) provides us precisely with an identification of points between $M_{\Omega=1} = M$ and M_Ω, and then the family of spacetimes defined by (2.6) refers to a fixed spacetime M equipped with a one-parameter family of metrics g_Ω.

In either case it is convenient to consider the family (M_Ω, g_Ω) of D-dimensional spacetimes with $\Omega > 0$ as a $(D+1)$-dimensional manifold $\mathcal{M}$ equipped with a degenerate metric g_Ω and a scalar field, namely Ω, whose level surfaces are the M_Ω. Questions about the limit of (M_Ω, g_Ω) are then questions about the boundary $\partial\mathcal{M}$ of $\mathcal{M}$.

Geroch calls a property of spacetimes *hereditary* if, whenever a family of spacetimes have that property, all the limits of this family also have this property. For present purposes we find it convenient to slightly modify this definition. We will call a property of a supergravity configuration *hereditary* if, whenever a supergravity configuration has this property, all the Penrose limits of this configuration also have this property.

This definition differs in three respects from Geroch's definition:

1. First of all, instead of referring just to a property of the spacetime (M, g), the topological, causal and metric properties of the spacetime manifold, we consider all the supergravity fields and talk about the properties of a supergravity configuration (M, g, Ψ) where Ψ collectively refers to the supergravity matter fields.
2. Moreover, instead of talking about all limits, we only consider Penrose limits.
3. Finally, instead of talking about a property of a family of spacetimes, we only refer to a property of the initial supergravity configuration.

The reason for the first two modifications is obvious. The reason for the third modification is that we are usually not given a family of spacetimes but one particular supergravity configuration, and we would then like to study the properties of this configuration in all possible Penrose limits, i.e., for all null geodesics.

This means that to check if a certain property of a supergravity configuration (M, g, Ψ) is hereditary, we first have to check if it holds for $(M_\Omega = M, g_\Omega, \Psi_\Omega)$ for $\Omega > 0$. Now (g_Ω, Ψ_Ω) differ from (g, Ψ) by a scaling and a diffeomorphism (and possibly a gauge transformation), and we are only interested in generally covariant and gauge invariant properties of (M, g, Ψ). Thus in practice this amounts to checking if the property of interest is invariant under a finite scaling of the fields (g, Ψ) before investigating what happens as $\Omega \to 0$.

It is clear that any property which is hereditary in the sense of Geroch (once that definition is extended in a straightforward way to include matter fields) is also hereditary in our sense. However, as Penrose limits are a rather special class of limits of spacetimes, it is conceivable that there are hereditary properties of Penrose limits which are not hereditary in general. Even though these are likely to exist, all the hereditary properties we will discuss in this section are also hereditary in the sense of Geroch.

Certain spacetime (or supergravity) properties are obviously hereditary, for example those that can be expressed in terms of tensorial equations for the Riemann tensor—see Section 3.2 below. Other hereditary properties are less obvious. For example, when it comes to isometries, one could imagine that in the (Penrose) limit of family of spacetimes, all possessing a certain number n of Killing vectors, one finds less linearly independent Killing vectors simply because some Killing vectors which happen to be linearly independent for all $\Omega > 0$ cease to be linearly independent at $\Omega = 0$. This is at least what a direct approach to the problem would suggest as being possible.

However, a very elegant and powerful argument due to Geroch [16], which we will recall and generalise slightly below, establishes that the number of linearly independent Killing vectors can never decrease in the limit. This argument has the additional virtue of being readily applicable to Killing spinors and supersymmetries. As a consequence we will also establish that the number of supersymmetries preserved by a supergravity configuration can never decrease in the Penrose limit. It is of course possible, and it is often the case, that the Penrose limit actually has more (super-)symmetries than the original spacetime. Indeed, as we will see later, the Penrose limit preserves at least one half of the supersymmetry.

3.2. **Hereditary properties involving the curvature tensor.** One of the most elementary and basic hereditary properties of any family (M_Ω, g_Ω) of space-times is the following [16]: If there is some tensor field constructed from the Riemann tensor and its derivatives which vanishes for all $\Omega > 0$, then it also vanishes for $\Omega = 0$.

In particular, the Penrose limit[2] of a Ricci-flat space-time is Ricci-flat, and the Penrose limit of a conformally flat space-time (vanishing Weyl tensor, $\mathrm{Weyl}(g_\Omega) = 0$) is conformally flat.

However, the Penrose limit of an Einstein manifold with fixed non-zero cosmological constant or scalar curvature is not of the same type (as the Ricci scalar, unlike the Ricci tensor, is not scale-invariant). Rather, if g is Einstein, so that $\mathrm{Ric}(g) = \Lambda g$, then $\mathrm{Ric}(g_\Omega) = \Lambda\Omega^2 g_\Omega$, and in the limit $\mathrm{Ric}(\bar{g}) = 0$. In other words, the Penrose limit of an Einstein space is always Ricci-flat. Conversely, if every Penrose limit of a space-time is Ricci-flat, then the original space-time is Einstein [9].

Similarly the Penrose limit of a locally symmetric space is locally symmetric. Indeed, the vanishing of the covariant derivative of the Riemann curvature tensor is a hereditary property. The only metrics of type (2.13) which are locally symmetric are the Cahen–Wallach metrics where A_{ij} is constant. It follows that the Penrose limit of an $\mathrm{AdS} \times S$ space, which is a symmetric space, is a Cahen–Wallach spacetime but with a possibly degenerate A_{ij}; that is, it is locally isometric to a product of an indecomposable Cahen–Wallach spacetime with a flat space. Penrose limits of $\mathrm{AdS} \times S$ are discussed in Section 4, which contains a more precise statement.

3.3. **Supergravity equations of motion.** We saw in Section 2 that two of the ingredients of the Penrose limit are a change of coordinates (2.4) and a rescaling (2.7) of the fields. The change of variables is a diffeomorphism for arbitrary nonzero Ω. The limit consists in taking the limit $\Omega \to 0$, where the change of coordinates becomes singular, but having rescaled the fields in such a way that the limit exists.

What makes this limit interesting is the fact that the supergravity actions considered here are homogeneous under the rescaling. Indeed, let the metric rescale as

$$g \rightsquigarrow \Omega^{-2} g \tag{3.1}$$

[2]Of course there is no such thing as *the* Penrose limit, as it in general depends on the choice of null geodesic γ. When talking about *the Penrose limit* without specifying a particular γ, we mean *any Penrose limit*.

and the p-form gauge potentials as

$$A_p \rightsquigarrow \Omega^{-p} A_p \ . \tag{3.2}$$

This implies that the field strengths rescale as

$$F_{p+1} \rightsquigarrow \Omega^{-p} F_{p+1} \tag{3.3}$$

and also that the Hodge star $\star$ acting on p-forms in D dimensions scales as

$$\star_p \rightsquigarrow \Omega^{2p-D} \star_p \ . \tag{3.4}$$

In particular, the Hodge star operator in an even-dimensional space is conformally invariant when acting on middle-dimensional forms. This means that self-duality conditions which do not follow from an action principle, like that of the 5-form field strength in IIB supergravity, for instance, are invariant under rescaling.

Under the rescaling of the metric the D-dimensional Einstein–Hilbert action is homogeneous with degree $2-D$:

$$\int d^D x \sqrt{|\det g|} R \rightsquigarrow \Omega^{2-D} \int d^D x \sqrt{|\det g|} R \ . \tag{3.5}$$

Similarly the Maxwell action is homogeneous of the same degree:

$$\int F_{p+1} \wedge \star F_{p+1} \rightsquigarrow \Omega^{2-D} \int F_{p+1} \wedge \star F_{p+1} \ , \tag{3.6}$$

and the same is true of the dilaton action, which can be thought of as the $p=0$ case of the above action:

$$\int d\Phi \wedge \star d\Phi \rightsquigarrow \Omega^{2-D} \int d\Phi \wedge \star d\Phi \ . \tag{3.7}$$

As the dilaton scales trivially, the above results remain true when there are non-minimal couplings of the dilaton to the IIB metric (in the string frame) and the RR-fields.

Likewise, on dimensional grounds, the cubic Chern-Simons terms of supergravity in D dimensions, like those of eleven-dimensional supergravity (for $p=3$) and of five-dimensional N=2 supergravity (for $p=1$),

$$\int F_{p+1} \wedge F_{p+1} \wedge A_p \ , \tag{3.8}$$

and generically of the form

$$\int dA_p^{(1)} \wedge dA_q^{(2)} \wedge A_{D-2-p-q}^{(3)} \ , \tag{3.9}$$

are also homogeneous of the same degree,

$$\int dA_p^{(1)} \wedge dA_q^{(2)} \wedge A_{D-2-p-q}^{(3)} \rightsquigarrow \Omega^{2-D} \int dA_p^{(1)} \wedge dA_q^{(2)} \wedge A_{D-2-p-q}^{(3)} \ . \tag{3.10}$$

The fermionic terms in the supergravity action also scale homogeneously with weight $2-D$ provided that the fermionic fields scale appropriately. Consider the kinetic term of a gravitino:

$$\int d^D x \sqrt{|\det g|}\bar{\psi}_M \Gamma^{MNP} \nabla_N \psi_P \ . \tag{3.11}$$

This term scales with weight $2-D$ provided that the gravitino scales with $-1/2$. Similarly, the kinetic term of other fermions

$$\int d^D x \sqrt{|\det g|}\bar{\lambda} \Gamma^M \nabla_M \lambda \tag{3.12}$$

scales with weight $2-D$ if and only if the fermions scale with weight $1/2$. With these conventions, other fermionic terms in which fermions couple non-minimally to the field strengths F_{p+1} again scale homogeneously with weight $2-D$.

These properties imply that the field equations are homogeneous. In particular, if a bosonic configuration (g, F, Φ) satisfies the equations of motion so will the rescaled fields.

The new fields $(g_\Omega, F_\Omega, \Phi_\Omega)$ defined in (2.5) are related to the original fields (g, F, Φ) not just by a rescaling but also by a diffeomorphism and perhaps a gauge transformation to bring the fields into a form in which the limit $\Omega \to 0$ exists. But since the equations of motion are covariant under diffeomorphisms and gauge transformations as well as under the rescaling, the new fields $(g_\Omega, F_\Omega, \Phi_\Omega)$ will satisfy the equations of motion for any $\Omega > 0$ if the original fields (g, F, Φ) do. A general continuity argument now guarantees that the limiting fields

$$\bar{g} = \lim_{\Omega\to 0} g_\Omega \qquad \bar{\Phi} = \lim_{\Omega\to 0} \Phi_\Omega \qquad \bar{F} = \lim_{\Omega\to 0} F_\Omega \tag{3.13}$$

also satisfy the equations of motion[3]. Thus the Penrose limit of a supergravity solution is a (possibly new) solution of the supergravity equations of motion.

3.4. **Isometries I: preliminary considerations.** We now come to the more subtle issue of isometries (and supersymmetries). To set the stage, let ξ be a Killing vector of the metric g. Then in the rescaled coordinates (2.4), ξ acquires a dependence on Ω, $\xi \to \xi(\Omega)$ and $\xi(\Omega)$ is a Killing vector for the transformed metric $g(\Omega)$ as well as for the transformed and rescaled metric $\Omega^{-2} g(\Omega)$. Hence in the limit

$$\bar{\xi} = \lim_{\Omega\to 0} \Omega^{\Delta_\xi} \xi(\Omega) \tag{3.14}$$

[3]To make this and the various other continuity arguments below rigorous, one has to put an appropriate topological structure on the space of the various objects, like the space of solutions, that are involved in arguments and show that the limiting processes described are continuous. However we shall not attempt this here.

is a non-trivial Killing vector of the limiting metric $\bar{g}$ provided that $\Delta_\xi \in \mathbb{R}$ can be chosen so that the above limit exists, i.e., such that the limit is both non-singular and non-zero.

We will now show that such a Δ_ξ always exists. If the Killing vector ξ in the local coordinates adapted to a null geodesic is

$$\xi = \alpha(U,V,Y^i)\partial_U + \beta(U,V,Y^i)\partial_V + \gamma^i(U,V,Y^i)\partial_{Y^i} \ , \tag{3.15}$$

then $\xi(\Omega)$ is

$$\xi(\Omega) = \alpha(u,\Omega^2 v,\Omega y^i)\partial_u + \Omega^{-2}\beta(u,\Omega^2 v,\Omega y^i)\partial_v + \Omega^{-1}\gamma^i(u,\Omega^2 v,\Omega y^i)\partial_{y^i} \ . \tag{3.16}$$

For sufficiently small Ω we can expand the Killing vector about $\Omega = 0$, i.e., about the geodesic $(u, v = 0, y^i = 0)$, and find

$$\Omega^2\xi(\Omega) = \bar{\beta}(u)\partial_v + \Omega(\bar{\gamma}^i(u)\partial_{y^i} + y^i\partial_{y^i}\bar{\beta}(u)\partial_v) + \dots \ , \tag{3.17}$$

where $\bar{\beta}(u) = \beta(u,0,0)$ etc. Now let Ω^{k_ξ} be the coefficient of the first non-zero term on the right hand side of this Taylor expansion. Thus $k_\xi \geq 0$ and

$$\lim_{\Omega\to 0} \Omega^{2-k_\xi}\xi(\Omega) \tag{3.18}$$

is finite and non-zero. This shows that $\bar{\xi}$ is finite and non-zero with the choice $\Delta_\xi = 2 - k_\xi \leq 2$.

The infinitesimal symmetries of a supergravity background (M, g, Ψ) are given by Killing vectors which in addition leave invariant the other fields in the background, i.e. we require

$$L_\xi \Psi = 0 \ . \tag{3.19}$$

Without loss of generality, we can assume that each Ψ is in the Penrose-Güven gauge. Then Ψ has a well defined Penrose-Güven limit

$$\bar{\Psi} = \lim_{\Omega\to 0} \Psi_\Omega = \lim_{\Omega\to 0} \Omega^{\Delta_\Psi}\Psi \tag{3.20}$$

(which may or may not be non-zero). To show that this symmetry of the background is preserved in the Penrose limit, we observe that

$$L_{\bar{\xi}}\bar{\Psi} = \lim_{\Omega\to 0} \Omega^{\Delta_\xi + \Delta_\Psi} L_\xi \Psi = 0 \ . \tag{3.21}$$

Now consider two linearly independent Killing vectors ξ_1 and ξ_2 of (M, g). It is of course perfectly possible that $\xi_1(\Omega)$ and $\xi_2(\Omega)$ are linearly independent for $\Omega > 0$ but that their leading order terms in a small-Ω expansion are linearly dependent (see e.g., the example in Section 4.6). Let us assume that this happens and, without loss of generality, that the leading order terms are in fact equal. Then $\bar{\xi}_1 = \bar{\xi}_2$ and we appear to have lost a Killing vector upon taking the limit. However, let us now consider the difference

$$\xi_-(\Omega) = \xi_1(\Omega) - \xi_2(\Omega) \ . \tag{3.22}$$

In this difference the leading order term proportional to $\bar{\xi}_1 - \bar{\xi}_2 = 0$ drops out and the first non-vanishing term, with

$$\Delta_{\xi_-} < \Delta_{\xi_1} = \Delta_{\xi_2} \ , \tag{3.23}$$

defines a Killing vector $\bar{\xi}_-$. If this Killing vector is linearly independent of $\bar{\xi}_1$, then the procedure stops here. If not, one needs to iterate this procedure. It is possible to show that in this way one eventually ends up with two linearly independent Killing vectors of the Penrose limit spacetime $(M, \bar{g})$.

However, this procedure is not very enlightening, and the result is actually a special case of a general result by Geroch which states that the number of linearly independent Killing vectors can never decrease in the limit of a family of spacetimes possessing a fixed number of linearly independent Killing vectors. Geroch's elegant argument, which we will recall below, also generalises in a straightforward way to Killing spinors and supersymmetries.

Anticipating this result, let us make two more remarks. The first is that because different Killing vectors may have to be rescaled with different values of Δ, the isometry algebra may get contracted in the limit. We will see an example of this phenomenon in Section 4.6 when discussing the isometry algebra of Penrose limits of spacetimes of the form AdS $\times S$.

Let us also note that, even if the original metric has no isometries, the Penrose limit always does. This is because, as we saw in Section 2, D-dimensional metrics of the form (2.8) always possess a $(2D-3)$-dimensional algebra of isometries, isomorphic to a Heisenberg algebra. Since generically these isometries have no counterpart in the original space-time and only arise at $\Omega = 0$, one should then not think of these isometries as hereditary but rather as a *post mortem* effect.

Thus, starting with a spacetime (M, g) with n linearly independent Killing vectors, the number of linearly independent Killing vectors of the Penrose limit space-time is always at least as large as $\max(n, 2D-3)$.

3.5. **Isometries II: Geroch's argument.** We will now study the fate of isometries in the more general context of hereditary properties of limits of spacetimes. We begin with a generalisation of an argument due to Geroch for the hereditary property of isometries. This will allow us to easily extend this argument from isometries to supersymmetries. The crucial observation is that (super)symmetries of a supergravity background are in one-to-one correspondence with Killing vectors and Killing spinors subject, perhaps, to algebraic conditions; and that the condition of a vector or a spinor being Killing can be rephrased in terms of a section of a certain vector bundle being parallel: a subbundle of the tensor bundle in the case of Killing vectors and the spinor bundle in the case of Killing spinors. We therefore start by discussing what happens to parallel sections of vector bundles under limits such as the one discussed in Section 3.1.

3.5.1. *A slight generalisation of Geroch's argument.* We will be concerned here with vector bundles with connection defined on $\mathcal{M}$ (see Section 3.1) which moreover extend to the boundary M_0. More precisely, let $E_\Omega \to M_\Omega$, $\Omega > 0$ be a smooth family of rank k vector bundles with connection D^Ω. We will assume that the limit $\Omega \to 0$ exists, so that E_0 is a smooth rank-k vector bundle on M_0 with connection D^0. This hypothesis will be justified for each of the cases to which we will apply the results of this section. Indeed, the bundles (and their connections) under consideration will only depend on the metric and the other bosonic fields of the supergravity theory, which have well-defined Penrose limits.

The connection D^Ω defines a notion of parallel transport along curves $c_\Omega : I \to M_\Omega$, and by restricting to closed curves, a notion of holonomy. Parallel sections of E_Ω, if they exist, define a rank-k' subbundle $E'_\Omega \subset E_\Omega$: the fibre $E'_\Omega(p_\Omega)$ at $p_\Omega \in M_\Omega$ is the subspace of $E_\Omega(p_\Omega)$ spanned by the values at p_Ω of the parallel sections, equivalently of those sections which are invariant under the holonomy group of D^Ω at p_Ω. We would like to investigate the limit as $\Omega \to 0$ of the family (E_Ω, D^Ω) and in particular to say something about the rank of $E'_0 \subset E_0$. We will see, in fact, that the rank of E'_0 is not smaller than the rank of E'_Ω.

To see this, let us choose a point $p_0 \in M_0$ and a path $t \mapsto p_t$ in $\mathcal{M}$, such that for each $\Omega > 0$, $p_\Omega \in M_\Omega$ and such that the limit $\lim_{\Omega\to 0} p_\Omega = p_0$, as the notation suggests. We can always trivialise the bundle along the path, and in this way identify the fibres $E_\Omega(p_\Omega)$ with a fixed k-dimensional vector space E. The fibres $E'_\Omega(p_\Omega)$ define a family of k'-dimensional subspaces of E, and hence a path in the Grassmannian $\mathrm{Gr}(k', E)$ of k'-dimensional planes in E. Because the Grassmannian is compact, this path has a limit point in the Grassmannian as $\Omega \to 0$, and thus we obtain a k'-dimensional subspace $E'_0(p_0)$ of $E_0(p_0)$. Doing this for all p_0 in the boundary M_0 we obtain a rank-k' subbundle $E'_0 \subset E_0$.

We will now show that E'_0 is left invariant by the holonomy group of the connection D^0. To this end let c_0 be a closed curve through p_0 in M_0 and let c_Ω be a family of closed curves through p_Ω in M_Ω in such a way that as $\Omega \to 0$, $p_\Omega \to p_0$ and $c_\Omega \to c_0$. Now consider a basis for E'_Ω near p_Ω made out of parallel sections of E_Ω and yielding a basis for E'_0 in the limit $\Omega \to 0$. (The existence of such a basis is guaranteed by the argument in the previous paragraph.) Their parallel transport around c_Ω is trivial for all $\Omega > 0$, hence by continuity the parallel transport along c_0 of a basis for E'_0 is again trivial. This shows that all elements of $E'(p_0)$ can be integrated to parallel sections of $E_0 \to M_0$.

In practice we will be interested in parallel sections satisfying additional linear equations. For example, the infinitesimal symmetries of a supergravity background are Killing vectors which, in addition to the metric, also leave invariant the other fields in the background. These conditions single out a linear subspace of the parallel sections and we will be interested in the fate of this subspace in the limit. This requires a slight refinement of the above argument, which we now detail.

Let C_Ω be a a family of linear conditions on sections of $E_\Omega \to M_\Omega$, depending smoothly on Ω and having a well-defined limit as $\Omega \to 0$. In practice, C_Ω will depend on the supergravity fields and hence the limit $\Omega \to 0$ is well-defined by virtue of these fields having a well-defined Penrose–Güven limit. For every point p_Ω let $E''_\Omega(p_\Omega) \subset E_\Omega(p_\Omega)$ denote the linear subspace spanned by the values $\psi_\Omega(p_\Omega)$ at p_Ω of parallel sections ψ_Ω of E_Ω which in addition satisfy the condition

$$C_\Omega \psi_\Omega = 0 \ . \tag{3.24}$$

These subspaces define a rank-k'' sub-bundle $E''_\Omega \to M_\Omega$. Repeating the argument above for this subbundle we find that for every point $p_0 \in M_0$ in the boundary of $\mathcal{M}$, we obtain a k''-dimensional subspace $E''(p_0)$ of $E(p_0)$ spanned by the values at p_0 of parallel sections of $E_0 \to M_0$. Now suppose that $\{\psi_\Omega^{(1)}, \dots, \psi_\Omega^{(k'')}\}$ is a frame for $E''_\Omega \to M_\Omega$ yielding in the limit $\Omega \to 0$ a frame for E''_0. Since each $\psi_\Omega^{(i)}$ obeys equation (3.24), continuity implies that in the limit $C_0 \psi_0^{(i)} = 0$. In summary, the dimension of the space of D^Ω-parallel sections ψ_Ω of $E_\Omega \to M_\Omega$ obeying the conditions (3.24) cannot decrease in the limit $\Omega \to 0$.

We will apply this argument both to isometries and supersymmetries, by realising them as parallel sections of appropriate vector bundles with connection, perhaps subject to additional linear conditions.

3.5.2. *Killing transport.* We now describe a useful local characterisation of Killing vectors as parallel sections of a vector bundle with connection (see, for example, [19, 16]).

Let X be any vector field on a connected n-dimensional spacetime (M, g) and let A_X denote the map taking a vector field Y to $\nabla_Y X$. This map is tensorial (that is, $C^\infty(M)$-linear) and hence defines a section of $\mathrm{End}(TM) \cong T^*M \otimes TM$. A vector field ξ is a Killing vector if and only if A_ξ is skew-symmetric relative to g; indeed, a vector ξ is Killing if and only if

$$g(\nabla_X \xi, Y) = -g(X, \nabla_Y \xi) \tag{3.25}$$

for all vector fields X, Y; but this can be rewritten

$$g(A_\xi X, Y) = -g(X, A_\xi Y) \ , \tag{3.26}$$

which shows that A_ξ is skew-symmetric relative to g.

In local coordinates, the Killing vector condition is

$$\nabla_M \xi_N + \nabla_N \xi_M = 0 \ , \tag{3.27}$$

and the components of A_ξ, when thought of as a two-form, are

$$A_{MN} = \nabla_M \xi_N = -\nabla_N \xi_M \ . \tag{3.28}$$

The skew-symmetric endomorphisms define a sub-bundle $\mathfrak{so}(TM) \subset \mathrm{End}(TM)$. Therefore a Killing vector ξ gives rise to a section (ξ, A_ξ) of the bundle

$$\mathcal{E} = TM \oplus \mathfrak{so}(TM) \ . \tag{3.29}$$

This can be understood as the local decomposition of a Killing vector into a "translation" and a "rotation". Of course, given the Killing vector field ξ, A_ξ is redundant as it can be constructed from ξ. The importance of A_ξ arises from the fact that the Killing vector ξ is completely determined by specifying $(\xi(p), A_\xi(p))$ at a single point $p \in M$.

This is a consequence of the Killing identity which says that, for a Killing vector ξ and for any vector field X,

$$\nabla_X A_\xi = R(X,\xi) \ , \tag{3.30}$$

where $R(X,Y)$ is the curvature operator defined by

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \ . \tag{3.31}$$

This Killing identity[4], which in local coordinates reads

$$\nabla_L \nabla_M \xi_N = R^P_{LMN} \xi_P \ , \tag{3.33}$$

(R^P_{LMN} are the components of the Riemann curvature tensor) shows that indeed second and higher derivatives of ξ at some point $p \in M$ can be expressed recursively in terms of $\xi_M(p)$ and $(\nabla_M \xi_N)(p)$.

In other words, the Killing identity together with the definition of A_ξ defines a differential system

$$\nabla_X \xi - A_\xi X = 0 \qquad \text{and} \qquad \nabla_X A_\xi - R(X,\xi) = 0 \ , \tag{3.34}$$

whose solutions can be interpreted as parallel sections of a suitable connection on $\mathcal{E}$. Indeed, if (ξ, A) is *any* section of $\mathcal{E}$ we define its covariant derivative by

$$D_X \xi = \nabla_X \xi - A(X) \qquad \text{and} \qquad D_X A = \nabla_X A - R(X,\xi) \ . \tag{3.35}$$

It follows that the parallel sections are precisely the sections (ξ, A) where ξ is a Killing vector and $A = A_\xi$.

This means that a Killing vector ξ is uniquely specified by the value of $(\xi(p), A_\xi(p))$ at a point p, say, with the value at any other point q being determined by parallel transport along any curve joining p and q. (Recall that M is assumed to be connected.) The value thus obtained is independent on the curve because for a Killing vector, (ξ, A_ξ) is invariant under parallel transport around closed loops. Incidentally, this explains why the dimension

[4]This identity can be proven as follows. Take the covariant derivative of A_ξ and use the algebraic Bianchi identity to conclude that

$$B(X,Y,Z) := g((\nabla_X A_\xi)Y, Z) - g(R(X,\xi)Y, Z) \tag{3.32}$$

is symmetric in the first two entries. It is also skew-symmetric in the last two entries because A_ξ (and hence $\nabla_X A_\xi$) is skew-symmetric. These two properties now imply that B vanishes, resulting in the Killing identity.

of the isometry algebra of a D-dimensional spacetime is at most $D(D+1)/2$, which is the rank of the bundle $\mathcal{E}$.

3.5.3. *Isometries are hereditary.* Let (M, g) be a spacetime with n linearly independent Killing vectors. We had already seen that for $\Omega > 0$, (M_Ω, g_Ω) also has n Killing vectors and the question is what happens as $\Omega \to 0$. We will use the local characterisation of Killing vectors as parallel sections of the bundle $\mathcal{E}$ in (3.29) in order to apply the results of Section 3.5.1.

To this end, let $\mathcal{E}_\Omega = TM_\Omega \oplus \mathfrak{so}(TM_\Omega)$ and let D^Ω be the connection defined by equation (3.35). Notice that the bundle and the connection depend only on the metric. Since the limit $\Omega \to 0$ of the metric exists, so do the limits $\mathcal{E}_0$ and D^0 of the bundle and the connection. Moreover, D^0 is the connection defined by (3.35) relative to the Penrose limit metric. This is the hypothesis which allows us to apply the argument in Section 3.5.1 to immediately conclude that in the Penrose limit the number of linearly independent Killing vectors cannot decrease and that these n Killing vectors of $(M, \bar{g} = g_0)$ arise as limits of Killing vectors of (M, g).

As mentioned above, the infinitesimal symmetries of a supergravity background (M, g, Ψ) are given by Killing vectors which in addition leave invariant the other fields in the background. These conditions translate into linear equations $L_\xi \Psi = 0$ on the Killing vector ξ. Since the fields Ψ have well-defined Penrose–Güven limits, these equations are also well-defined in the limit, and using either the argument of Section 3.4 or the (refined) argument in Section 3.5.1, we conclude that the dimension of the symmetry algebra of a supergravity background does not decrease in the Penrose limit.

3.6. **Supersymmetries.** We have shown that the Penrose limit of a solution of the supergravity field equations is again a solution and moreover that it admits at least as many symmetries as the original solution. It is natural to ask whether the limiting solution preserves at least as many supersymmetries as the original solution. The equations of a bosonic supergravity background being supersymmetric translates into equations on spinors coming from the supersymmetry variations of the fermions in the theory. The variation of the gravitini yields a Killing spinor equation, whereas the variation of the other fermions in the theory yield additional algebraic conditions. As a result the above question is essentially equivalent to asking whether the property of a spinor being Killing is hereditary.

To answer this question, we can appeal to the generalisation of Geroch's argument presented in Section 3.5.1. The key point is that the condition of being Killing can be interpreted as the condition of being parallel with respect to a connection on the spinor bundle $\$$. This again implies that any Killing spinor is uniquely specified by its value at one point p in spacetime. Parallel transport will then define the Killing spinor everywhere. There may be less supersymmetries than parallel spinors because of the presence of the additional algebraic conditions, but the important point is that the supersymmetries preserved by a bosonic background are uniquely specified by their values at any one point in spacetime.

3.6.1. *Supersymmetries are hereditary.* We will first prove that the dimension of the space of Killing spinors does not decrease in the Penrose limit. This will follow once again from the argument in Section 3.5.1. Then we will show that this conclusion does not change when we incorporate the algebraic equations.

Let (M, g) be a spacetime with n linearly independent Killing spinors. It is clear that $(M_\Omega, g_\Omega, \Psi_\Omega)$ also admits n Killing spinors for all $\Omega > 0$. The question is whether this persists in the limit $\Omega \to 0$. Let ε_Ω be a Killing spinor of $(M_\Omega, g_\Omega, \Psi_\Omega)$. Then by linearity of the Killing spinor equation also $\Omega^\Delta \varepsilon_\Omega$ is a Killing spinor for any Δ. As for Killing vectors, we can always find a Δ such that the limit $\lim_{\Omega\to 0} \Omega^\Delta \varepsilon_\Omega$ exists and is non-zero (by choosing Δ to pick out the first non-zero coefficient in a Taylor expansion around $\Omega = 0$). Different Killing spinors may require different Δ's, and this can lead to a contraction of the superalgebra. To show linear independence of the Killing spinors in the limit, we adapt Geroch's argument to the case at hand.

The spinor bundle $\$$ depends on the metric,[5] and hence defines a family $\$_\Omega \to M_\Omega$ of spinor bundles. The supercovariant derivative depends on the metric and the other bosonic fields in the background, and through this dependence it defines a connection D^Ω on $\$_\Omega$. This family of bundles with connection extends to the boundary M_0, since the metric and the other bosonic fields in the supergravity theory have well-defined limits as $\Omega \to 0$. We can therefore appeal to the argument in Section 3.5.1 to conclude that there are at least as many parallel sections of $\$_0$ as there are of $\$_\Omega$ for $\Omega > 0$.

Supersymmetries of a supergravity background are in one-to-one correspondence with Killing spinors satisfying additional algebraic linear equations, coming from the supersymmetry variations of the other fermionic fields in the supergravity theory besides the gravitini. These algebraic equations depend on the bosonic fields in the theory and since they have a well-defined Penrose–Güven limit, so do the equations. This allows us to apply the refinement of Geroch's argument in Section 3.5.1 to conclude that the Penrose limit of a supergravity background (M, g, Ψ) preserves at least as many supersymmetries as (M, g, Ψ).

It should be remarked that it is crucial in this assertion that the limit D^0 of the supercovariant derivative is the supercovariant derivative evaluated at the Penrose limit. In other words, if we write the dependence on the

[5] It also depends on a choice of spin structure. However the limiting spacetime M_0, being homeomorphic to a neighbourhood of a segment of a null geodesic, is contractible and hence has a unique spin structure. More generally, the existence of a spin structure is hereditary and spin structures form a discrete set, so they cannot change continuously as we vary the parameter Ω.

bosonic fields explicitly as $D(g, \Psi)$, then we have that in the limit

$$D^0 = D(\bar{g}, \bar{\Psi}) \ . \tag{3.36}$$

Finally let us mention that even if the original background is not supersymmetric, its Penrose limits always preserve at least one-half of the supersymmetry. Indeed, the algebraic equations are satisfied on the sub-bundle $\$_-$ of spinors annihilated by Γ_+ because of the form (2.20) of the field-strengths in the Penrose limit. Furthermore, it is easy to show that on the sub-bundle $\$_-$ the connection D^0 defined above is flat.

3.6.2. *Eleven-dimensional supergravity.* Let us illustrate the above results with the example of eleven-dimensional supergravity. In some conventions, the supercovariant derivative of eleven-dimensional supergravity theory is given by

$$D_M = \nabla_M - \Omega_M \ , \tag{3.37}$$

where

$$\Omega_M = \tfrac{1}{288} F_{PQRS} \left(\Gamma^{PQRS}{}_M + 8\Gamma^{PQR}\delta^S_M \right) \ , \tag{3.38}$$

and the spin connection ∇ is

$$\nabla_M = \partial_M + \tfrac{1}{4}\omega_M{}^{ab}\Gamma_{ab} \ . \tag{3.39}$$

Suppose now that we have an eleven-dimensional background which preserves some supersymmetry, i.e., there are non-vanishing spinors ε such that

$$D_M \varepsilon = 0 \ . \tag{3.40}$$

Clearly, this is a parallel transport equation of the spinor bundle $\$$ of eleven-dimensional spacetime (M, g, F) of rank 32.

Next observe that the above Killing spinor equation is well-defined at a Penrose limit. For this, let us adopt coordinates in the neighbourhood of a null geodesic, scale the coordinates as described in section 2.1 and perform the overall scaling $g \rightsquigarrow \Omega^{-2} g$ and $A_3 \rightsquigarrow \Omega^{-3} A_3$ where A_3 is the three-form gauge potential appropriately gauge-fixed. Note that the four-form field strength $F_4 = dA_3$ is scaled as $F_4 \rightsquigarrow \Omega^{-3} F_4$. Next we notice that the Γ matrices in the frame indices do not scale, whereas the eleven-dimensional frame scales as $e^a_M \rightsquigarrow \Omega^{-1} e^a_M$. Under these scalings, the supercovariant derivative does not rescale: $D_M \rightsquigarrow D_M$. Therefore, the Penrose limit of the Killing spinor equation (3.40) is the standard eleven-dimensional supergravity Killing spinor equation evaluated at the Penrose limit of the associated spacetime. Having established this, we can use the argument of the previous section to show that any Penrose limit of a supersymmetric solution of eleven-dimensional supergravity is supersymmetric and it admits at least as many Killing spinors as the original solution. In fact, it is easy to see that the Penrose limit of any solution, even if the solution does not preserve any supersymmetry, preserves at least sixteen supersymmetries. This is because all the pp-wave type solutions in eleven dimensions preserve at least half the supersymmetry. The supersymmetry projection is $\Gamma_+\varepsilon = 0$.

4. Penrose limits of AdS $\times S$

In this section we classify all Penrose limits of space-time geometries of the form AdS $\times S$. As shown in Section 3.2, any Penrose limit of such a geometry is locally isometric to a product of an indecomposable Cahen–Wallach space with flat space. More precisely, we will now prove that the Penrose limit along any null geodesic in $\mathrm{AdS}_{p+2} \times S^n$ is either flat or a Cahen–Wallach metric with two negative eigenvalues in equal ratio to the radii of curvatures of the two factor spaces, depending on whether or not the tangent component to the sphere of geodetic vector vanishes.

In particular, we exhibit the maximally supersymmetric Hpp-wave solutions to eleven-dimensional and IIB supergravity as Penrose limits of the near horizon geometries of the M2/5 and D3 branes respectively. We then discuss some generalisations of this construction and make some comments about the fate of isometries under the Penrose limit.

4.1. **The AdS $\times S$ metrics.** We identify anti-de Sitter space AdS_{p+2} with radius of curvature R_{AdS} with the following quadric in the pseudo-euclidean space $\mathbb{E}^{2,p+1}$

$$(X^0)^2 + (X^{p+2})^2 - (X^1)^2 - \cdots - (X^{p+1})^2 = R_{\mathrm{AdS}}^2 \tag{4.1}$$

with the induced metric. Introduce the following parametrisation

$$\begin{aligned} X^0 &= R_{\mathrm{AdS}} \cos\tau \\ X^{p+2} &= R_{\mathrm{AdS}} \sin\tau \sqrt{1+r^2} \\ X^i &= R_{\mathrm{AdS}} r \sin\tau \theta^i \quad \text{for } i = 1, \dots, p+1, \end{aligned} \tag{4.2}$$

where $\sum_i (\theta^i)^2 = 1$ parametrise a p-dimensional sphere. In these coordinates, the anti-de Sitter metric becomes

$$g_{\mathrm{AdS}} = R_{\mathrm{AdS}}^2 \left[-d\tau^2 + (\sin\tau)^2 \left(\frac{dr^2}{1+r^2} + r^2 d\Omega_p^2 \right) \right] , \tag{4.3}$$

where $d\Omega_p^2$ is the p-sphere metric.

Similarly we identify the round n-sphere S^n with radius of curvature R_S with the quadratic in $(n+1)$-dimensional euclidean space $\mathbb{E}^{n+1}$

$$(X^1)^2 + (X^2)^2 + \cdots + (X^{n+1})^2 = R_S^2 \tag{4.4}$$

with the induced metric. Let ψ be the colatitude and write

$$\begin{aligned} X^{n+1} &= R_S \cos\psi \\ X^i &= R_S \sin\psi \omega^i \quad \text{for } i = 1, \dots, n, \end{aligned} \tag{4.5}$$

where $\sum_i (\omega^i)^2 = 1$ parametrise the equatorial $(n-1)$-sphere. In these coordinates, the round metric on the n-sphere becomes

$$g_S = R_S^2 \left[d\psi^2 + (\sin\psi)^2 d\Omega_{n-1}^2 \right] , \tag{4.6}$$

where $d\Omega_{n-1}^2$ is the metric on the equatorial $(n-1)$-sphere.

The metric on $\mathrm{AdS}_{p+2} \times S^{D-p-2}$ is then $g = g_{\mathrm{AdS}} + g_S$, which is given by

$$R^{-2} g = \rho^2 \left[-d\tau^2 + (\sin\tau)^2 \left(\frac{dr^2}{1+r^2} + r^2 d\Omega_p^2 \right) \right] + d\psi^2 + (\sin\psi)^2 d\Omega_{D-p-3}^2 , \tag{4.7}$$

where we have introduced the ratio $\rho := R_{\mathrm{AdS}_{p+2}} / R_{S^{D-p-2}}$ of the radii of curvature of the two factors and where R is the radius of curvature of the sphere.

In particular, the near horizon geometries of the M2-, M5- and D3-brane solutions are of the form $\mathrm{AdS}_p \times S^{D-p-2}$ where the values of p and D corresponding to each of the above branes are listed in Table 1 along with the ratio ρ.

Brane	p	D	ρ
M2	2	11	$\frac{1}{2}$
D3	3	10	1
M5	5	11	2

TABLE 1. Dimensions and radii of curvature

4.2. **Penrose limits of** AdS**.** As a first step towards studying the Penrose limits of the AdS $\times S$ geometries in general, we determine the Penrose limits of AdS space-times. In fact, we will see that any Penrose limit of AdS is flat Minkowski space. This could be deduced as a by-product of our more general calculations below, establishing this result for particular null geodesics, and then extended to all null geodesics using the maximal symmetry of AdS and the covariance property of Penrose limits.

However, there is also a simpler and more general argument requiring no calculation and using only the hereditary properties of Penrose limits. Indeed, we had seen in Section 3.2 that the Penrose limit of any Einstein manifold is Ricci-flat. We had also seen that the Penrose limit of a conformally flat space-time (vanishing Weyl tensor) is necessarily conformally flat.

In particular, therefore, the Penrose limit of a maximally symmetric (conformally flat, Einstein) manifold, either de Sitter or anti-de Sitter space-time, has vanishing Ricci and Weyl tensors. This implies that the Riemann curvature tensor is zero and hence that the Penrose limit is isometric to Minkowski space-time.

4.3. **Classification of Penrose limits of** AdS $\times S$**.** Because of the covariance property of Penrose limits, in order to classify the possible Penrose limits of these space-times we must investigate the orbits of the isometry group G of $M = \mathrm{AdS}_{p+2} \times S^n$ acting on the space of pairs $(\gamma(0), [\dot{\gamma}(0)])$ consisting of a point in M and a future-pointing null direction at that point. Because M is homogeneous,

$$M = \frac{G}{H} = \frac{\mathrm{SO}(2,p+1) \times \mathrm{SO}(n+1)}{\mathrm{SO}(1,p+1) \times \mathrm{SO}(n)} \ , \tag{4.8}$$

the isometry group G acts transitively on points. Once we have fixed a point $\gamma(0)$, the subgroup fixing that point is isomorphic to the stabiliser H and it remains to investigate the action of H on the celestial sphere of null directions at a point.

It is easy to see that there are two orbits. The "big" orbit is $B^{p+1} \times S^{n-1}$, where the ball B^{p+1} is the space of future-pointing time-like directions at a point in AdS_{p+2} and the sphere S^{n-1} is the space of directions at a point in S^n. There is also a smaller orbit diffeomorphic to S^p, the celestial sphere at a point in AdS_{p+2}, corresponding to null geodesics γ for which the component of $\dot{\gamma}$ tangent to S^n vanishes.

By the covariance property there are then at most two non-isometric Penrose limits corresponding to the two orbits. We will show that the small orbit gives rise to a flat Penrose limit - this is almost obvious from the fact we established above that the Penrose limit of pure AdS is flat. We will then show that the Penrose limit corresponding to the large orbit (which is hence generic) gives rises to a Cahen–Wallach spacetime where the matrix A_{ij} has two negative eigenvalues with multiplicities $p+1$ and $n-1$, respectively, commensurate with the radii of curvature of AdS_{p+2} and S^n. In particular, this will establish that the Penrose limits of the near horizon geometries of the D3-, M2-, and M5-branes are the maximally supersymmetric Hpp-waves of 10- and 11-dimensional supergravity.

4.4. **The non-generic orbit: null geodesics in** AdS**.** We consider null geodesics of AdS $\times S$ tangent to AdS. Because of this and because the metric is the product metric, we can investigate the fate of the two factors separately. We had already seen above that the Penrose limit of the AdS factor is flat. It thus remains to consider what happens to the metric on the sphere S in the Penrose limit. We know from Section 2.3 that one consequence of the Penrose limit is to blow up the geodesic distance between any two points on S. As in the present case this blowing up is uniform on all of S, clearly this implies that in the Penrose limit the metric on S is the infinite radius flat metric.

One can also see this directly in local coordinates. Let the line-element on the sphere S be

$$g_S = g_{ab}(Y^c)dY^a dY^b \ . \tag{4.9}$$

As the Y^a are among the transverse coordinates Y^i, in the Penrose limit they scale as (see (2.4)) $Y^a = \Omega y^a$ so that

$$g_S(\Omega) = \Omega^2 g_{ab}(\Omega y^c) dy^a dy^b \ . \tag{4.10}$$

It thus follows from (2.7) that the metric in the Penrose limit is the constant (hence flat) metric

$$\bar{g}_S = \lim_{\Omega\to 0} \Omega^{-2} g_S(\Omega) = g_{ab}(0) dy^a dy^b \ . \tag{4.11}$$

Putting this together we deduce that the Penrose limit of AdS $\times S$ for any null geodesic in the non-generic orbit, i.e., tangent to AdS, is Minkowski space-time.

4.5. **Generic null geodesics.** We now consider null geodesics in AdS $\times S$ with a non-zero component along S. There are two ways to approach the calculation of the Penrose limit. One is to solve the geodesic equation for a suitable initial condition and then to find a coordinate transformation which puts the metric into the form (2.1) adapted to the null geodesic congruence. The other is to forego the determination of a null geodesic and directly find a coordinate transformation which puts the metric into the canonical form (2.1) which then exhibits $\partial/\partial U$ as the null geodesic. While ultimately both methods are equivalent, the former is occasionally more transparent while the latter may be quicker. We will illustrate both methods in the following, starting with the second method.

We follow closely the discussion in [3] where the emphasis was on the near horizon geometries $\mathrm{AdS}_5 \times S^5$ and $\mathrm{AdS}_{4|7} \times S^{7|4}$. Recall the AdS $\times S$ metric (4.7),

$$R^{-2} g = \rho^2 \left[-d\tau^2 + (\sin\tau)^2 \left(\frac{dr^2}{1+r^2} + r^2 d\Omega_p^2 \right) \right] + d\psi^2 + (\sin\psi)^2 d\Omega_{D-p-3}^2 \ . \tag{4.12}$$

Let us now change coordinates in the (ψ, τ) plane to

$$u = \psi + \rho\tau \qquad v = \psi - \rho\tau \ , \tag{4.13}$$

in terms of which, the metric g becomes

$$R^{-2} g = du dv + \rho^2 \sin((u-v)/2\rho)^2 \left(\frac{dr^2}{1+r^2} + r^2 d\Omega_p^2 \right) + \sin((u+v)/2)^2 d\Omega_{D-p-3}^2 \ . \tag{4.14}$$

We now take the Penrose limit along the null geodesic parametrised by u. In practice this consists in dropping the dependence on other coordinates but u. Doing so we find

$$R^{-2}\bar{g} = dudv + \rho^2 \sin(u/2\rho)^2 ds^2(\mathbb{E}^{p+1}) + (\sin(u/2))^2 ds^2(\mathbb{E}^{D-p-3}) \ , \quad (4.15)$$

which is, as we have seen in Section 2.2, the metric of a Cahen–Wallach symmetric space in Rosen coordinates (compare with [20] for the $d = 11$ solution).

To determine the resulting Cahen–Wallach space more explicitly, let us introduce coordinates y^a for $a = 1, \dots, D-2$ in such a way that the metric (4.15) becomes

$$R^{-2}\bar{g} = dudv + \sum_{a=1}^{D-2} \frac{(\sin \lambda_a u)^2}{(2\lambda_a)^2} dy^a dy^a \ , \quad (4.16)$$

where

$$\lambda_a = \begin{cases} 1/2\rho & a = 1, \dots, p+1 \\ 1/2 & a = p+2, \dots, D-2 \ . \end{cases} \quad (4.17)$$

We change coordinates to (x^+, x^-, x^a) where

$$x^- = u/2 \ , \quad x^+ = v - \tfrac{1}{4} \sum_a y^a y^a \frac{\sin(2\lambda_a u)}{2\lambda_a} \ , \quad x^a = y^a \frac{\sin(\lambda_a u)}{2\lambda_a} \ , \quad (4.18)$$

so that the metric now becomes

$$R^{-2}\bar{g} = 2dx^+dx^- - 4 \left(\sum_a \lambda_a^2 x^a x^a \right) (dx^-)^2 + \sum_a dx^a dx^a \ , \quad (4.19)$$

which we recognise as a Cahen–Wallach metric (2.13) whose matrix A_{ij} is constant and diagonal with negative eigenvalues $\{-\lambda_a^2\}$. For λ_a given as in (4.17) we obtain [3], if $\rho = \frac{1}{2}$ or $\rho = 2$, precisely the metrics of the maximally supersymmetric Hpp-waves of eleven-dimensional supergravity (2.14) discovered in [2] (see also [1]), and if $\rho = 1$ the maximally supersymmetric Hpp-wave of IIB supergravity (2.15) discovered in [4]. In [3] it was also shown that the limits of the corresponding $(D-p-2)$-form field strengths agree with those of the Hpp-waves [3].

We conclude that the maximally supersymmetric Hpp-waves of [2, 1] and [4] appear as Penrose limits along generic null geodesics of the near horizon geometries of the M2/5 and D3 branes, respectively [3]. Likewise, the maximally supersymmetric Hpp-waves in four [5] and five and six dimensions [6] appear as Penrose limits of the $AdS_2 \times S^2$, $\mathrm{AdS}_2 \times S^3$ or $\mathrm{AdS}_3 \times S^2$, and $\mathrm{AdS}_3 \times S^3$ solutions of the corresponding supergravity theories. The latter is the near-horizon limit of the six-dimensional self-dual string [21] whose Penrose limits we will discuss in Section 7.3.

In the light of the results of section 3.6, this derivation also provides an alternative proof that these solutions are indeed maximally supersymmetric.

Moreover we now understand the (originally somewhat puzzling) fact that the IIB Hpp-wave is characterised by a matrix A_{ij} with a single eigenvalue with multiplicity eight, whereas the Hpp-wave of eleven-dimensional supergravity has two distinct eigenvalues with multiplicities 3 and 6 respectively. It is is related to the fact that the two curvature radii of the IIB $\mathrm{AdS}\times S$ solution are equal whereas those of the solutions of eleven-dimensional supergravity are not.

We also see that we can obtain any indecomposable Cahen–Wallach metric with A_{ij} having at most two (negative) eigenvalues as the Penrose limit of a product $\mathrm{AdS}_m \times S^n$ by appropriate choices of m, n and the ratio of radii of curvature of the two factors. More generally, other Cahen–Wallach metrics with multiple eigenvalues of any sign can be obtained as the Penrose limit of products involving one (anti) de Sitter space and multiple spheres and hyperbolic spaces of appropriate dimensions and radii of curvature.

4.6. **Isometries revisited.** In this section we illustrate the discussion in Sections 3.4 and 3.5 about the fate of isometries under the Penrose limit with some examples of the form $\mathrm{AdS}\times S$.

We start with a "toy model" corresponding to the near horizon geometry of the Reissner–Nordström black hole in four-dimensional $N{=}2$ supergravity, namely $\mathrm{AdS}_2 \times S^2$ with equal radii of curvature. The $\mathrm{SO}(2,1)\times \mathrm{SO}(3)$ isometry group of this space is most easily exhibited by embedding $\mathrm{AdS}_2 \times S^2$ as the intersection of two quadrics in $\mathbb{E}^{2,4}$. This is the case $p=0$, $D=4$ (and hence $n=2$) in the notation of Section 4.1. An explicit parametrisation is given by

$$\begin{aligned} X^0 &= R\cos\tau\ , \quad X^1 = R\sinh\beta\sin\tau \quad \text{and} \quad X^2 = R\cosh\beta\sin\tau \\ X^3 &= R\cos\psi\ , \quad X^4 = R\sin\psi\cos\theta \quad \text{and} \quad X^5 = R\sin\psi\sin\theta\ , \end{aligned} \tag{4.20}$$

where R is the common radius of curvature of the two spaces. In terms of the ambient coordinates, a basis for Killing vectors of $\mathrm{AdS}_2 \times S^2$ is given by the six vector fields

$$\begin{aligned} X^1\partial_i + X^i\partial_1 &\qquad \text{for } i=0,2, \text{ and} \\ X^i\partial_j - X^j\partial_i &\qquad \text{for } i,j=3,4,5 \text{ and } i,j=0,2, \end{aligned} \tag{4.21}$$

where $\partial_i = \partial/\partial X^i$. In terms of the embedding coordinates, these Killing vectors are explicitly given by

$$\begin{aligned} E_1 &= \frac{\partial}{\partial\theta} & E_2 &= \frac{\partial}{\partial\beta} \\ E_1^* &= \sin\theta\frac{\partial}{\partial\psi} + \cos\theta\cot\psi\frac{\partial}{\partial\theta} & E_2^* &= -\sinh\beta\frac{\partial}{\partial\tau} + \cosh\beta\cot\tau\frac{\partial}{\partial\beta} \\ \xi_1 &= \cos\theta\frac{\partial}{\partial\psi} - \sin\theta\cot\psi\frac{\partial}{\partial\theta} & \xi_2 &= -\cosh\beta\frac{\partial}{\partial\tau} + \sinh\beta\cot\tau\frac{\partial}{\partial\beta} \end{aligned} \tag{4.22}$$

The Penrose limit starts by rescaling the coordinates as follows

$$\psi = \tfrac{1}{2}(u + \Omega^2 v) \ , \quad \tau = \tfrac{1}{2}(u - \Omega^2 v) \ , \quad \theta = \Omega y^1 \quad \text{and} \quad \beta = \Omega y^2 \ . \tag{4.23}$$

In terms of the new variables, the Killing vectors acquire Ω dependence. To leading order in Ω one finds, respectively,

$$\begin{aligned} E_1(\Omega) &= \Omega^{-1}\frac{\partial}{\partial y^1} & E_2(\Omega) &= \Omega^{-1}\frac{\partial}{\partial y^2} \\ E_1^*(\Omega) &= \Omega^{-1}\left(y^1\frac{\partial}{\partial v} + \cot\tfrac{1}{2}u\frac{\partial}{\partial y^1}\right) & E_2^*(\Omega) &= \Omega^{-1}\left(y^2\frac{\partial}{\partial v} + \cot\tfrac{1}{2}u\frac{\partial}{\partial y^2}\right) \\ \xi_1(\Omega) &= \Omega^{-2}\frac{\partial}{\partial v} & \xi_2(\Omega) &= \Omega^{-2}\frac{\partial}{\partial v} \end{aligned} \tag{4.24}$$

which are to be compared with the expression (2.11) for the generic isometries of a Penrose limit.

We observe the phenomena described in Section 3.4. First of all, we see that the rescaled Killing vectors

$$\Omega E_i(\Omega) \qquad \Omega E_i^*(\Omega) \qquad \Omega^2\xi_i(\Omega) \tag{4.25}$$

have well-defined limits as $\Omega \to 0$. Secondly, we notice that two linearly independent Killing vectors, namely $\xi_1(\Omega)$ and $\xi_2(\Omega)$ are equal up to subleading terms in Ω. Since we know that we cannot lose any Killing vectors in the limit, we are thus led to consider the linear combinations

$$\xi_\pm(\Omega) = \xi_1(\Omega) \pm \xi_2(\Omega) \ . \tag{4.26}$$

To leading order in Ω one has

$$\xi_+(\Omega) = 2\Omega^{-2}\frac{\partial}{\partial v} \tag{4.27}$$

$$\xi_-(\Omega) = 2\frac{\partial}{\partial u} - \tfrac{1}{2}|y|^2\frac{\partial}{\partial v} - \sum_i y^i \cot\tfrac{1}{2}u\frac{\partial}{\partial y^i} \ , \tag{4.28}$$

so that $\Omega^2\xi_+(\Omega)$ and $\xi_-(\Omega)$ are well defined (and linearly independent) in the limit. Comparing with (2.11), we see that

$$e_i := \lim_{\Omega\to 0}\Omega E_i(\Omega) \qquad e_i^* := \lim_{\Omega\to 0}\Omega E_i^*(\Omega) \qquad e_+ := \tfrac{1}{2}\lim_{\Omega\to 0}\Omega^2\xi_+(\Omega) \tag{4.29}$$

are precisely the Killing vectors of a generic Penrose limit pp-wave spacetime satisfying the Heisenberg algebra (2.12), only that in this case they arise from isometries already present in the original spacetime.

Moreover, in this case, since the Penrose limit is actually an Hpp-wave, there is another Killing vector, namely $\partial/\partial x^-$ in Brinkman coordinates (as A_{ij} is constant, independent of x^-) or

$$\frac{\partial}{\partial x^-} = 2\frac{\partial}{\partial u} - \tfrac{1}{2}|y|^2\frac{\partial}{\partial v} - \sum_i y^i \cot\tfrac{1}{2}u\frac{\partial}{\partial y^i} \tag{4.30}$$

in Rosen coordinates. We see that this agrees precisely with

$$e_- := \lim_{\Omega\to 0} \xi_-(\Omega) \ , \tag{4.31}$$

so that this Killing vector is again inherited from an isometry of the original $\mathrm{AdS}\times S$ spacetime.

We also see that the Killing vectors have to be rescaled by different powers of Ω to have a well-defined limit, exactly as in the rescaling (2.4) of the coordinates, by Ω for the transverse directions and by Ω^0 and Ω^2 for the directions corresponding to $\partial/\partial u$ and $\partial/\partial v$. Therefore the isometry algebra will get contracted in the limit. Here $\mathfrak{so}(1,2)$ and $\mathfrak{so}(3)$ both get contracted to Heisenberg algebras and furthermore their central elements are identified in the limit to become the common central element of the combined Heisenberg algebras. $\xi_-(\Omega)$, on the other hand, becomes an outer automorphism of the Heisenberg algebra.

Moreover in this case we have an additional isometry because of the fact that the two spaces AdS_2 and S^2 have equal radii of curvature. This accidental isometry manifests itself as rotations in the y^1, y^2 plane: $y^1\partial/\partial y^2 - y^2\partial/\partial y^1$. These extra isometries commute with each other and act on the generic isometries as outer automorphisms of the Heisenberg algebra.

More generally, we can consider the Penrose limit of $\mathrm{AdS}_{p+2}\times S^{D-p-2}$ with radii of curvature $R_{\mathrm{AdS}} = \rho R$ and $R_S = R$, respectively. The Penrose limit along the null geodesic considered above is a Cahen–Wallach metric where the (constant) matrix A_{ij} has two eigenvalues with ratio ρ and multiplicities $p+1$ and $D-p-3$. Under the Penrose limit, the isometry algebra $\mathfrak{so}(2,p+1)\oplus\mathfrak{so}(D-p-1)$ of $\mathrm{AdS}_{p+2}\times S^{D-p-2}$ undergoes the following contraction. The $\mathfrak{so}(2,p+1)$ factor contracts to $\mathfrak{h}(p+1)\rtimes\mathfrak{so}(p+1)$, where $\mathfrak{h}(p+1)$ is a Heisenberg algebra with $2p+3$ generators, whose $p+1$ creation and $p+1$ annihilation operators transform as vectors under $\mathfrak{so}(p+1)$. Similarly the $\mathfrak{so}(D-p-1)$ factor contracts to $\mathfrak{h}(D-p-3)\rtimes\mathfrak{so}(D-p-3)$. The central element in both Heisenberg algebras coincide. This means that there are two Killing vectors $\xi_1(\Omega)$ and $\xi_2(\Omega)$ agreeing to leading order in Ω and hence agreeing in the limit. This prompts us to consider the linear combinations $\xi_\pm(\Omega) = \xi_1(\Omega)\pm\xi_2(\Omega)$. These vector fields must be rescaled differently for their limits to exist: $\xi_+(\Omega)$ becomes in the limit the common central element of the combined Heisenberg algebra $\mathfrak{h}(D-2)$, whereas $\xi_-(\Omega)$ becomes an outer automorphism commuting with $\mathfrak{so}(p+1)\oplus\mathfrak{so}(D-p-3)$. In Brinkman coordinates, $\bar{\xi}_\pm$ are realised as $\partial/\partial x^\pm$. We see, therefore, that the isometry algebra $\mathfrak{so}(2,p+1)\oplus\mathfrak{so}(D-p-1)$ of $\mathrm{AdS}_{p+2}\times S^{D-p-2}$ contracts to a semidirect product

$$\mathfrak{h}(D-2)\rtimes(\mathfrak{so}(p+1)\oplus\mathfrak{so}(D-p-3)\oplus\mathbb{R}) \ . \tag{4.32}$$

When the radii of curvature are equal there is an additional symmetry enhancement, and the subalgebra $\mathfrak{so}(p+1)\oplus\mathfrak{so}(D-p-3)$ is enlarged to the full $\mathfrak{so}(D-2)$. This however has no counterpart in the original metric.

4.7. **Generic null geodesics: another example.** Our general arguments on covariance of the Penrose limit, combined with the above explicit calculations, now tell us that we know the Penrose limit for any null geodesic in AdS $\times S$. Nevertheless, for illustrative purposes we will now look at another example which allows us to see more explicitly what happens to the Penrose limit of a generic geodesic as the angular momentum of the geodesic along the sphere vanishes (and hence the geodesic approaches a non-generic geodesic).

To be specific, we consider $\mathrm{AdS}_5 \times S^5$ with AdS in Poincaré coordinates times the sphere in standard spherical coordinates,

$$ds^2 = r^{-2}dr^2 + r^2(-dt^2 + ds^2(\mathbb{E}^3)) + d\psi^2 + (\sin\psi)^2 ds^2(S^4) \ , \tag{4.33}$$

and look at null geodesics in the (r, t, ψ)-direction. Thus the metric we will actually be working with is

$$ds^2_{(3)} = r^{-2}dr^2 - r^2dt^2 + d\psi^2 \ . \tag{4.34}$$

The null condition gives

$$r^{-2}\dot{r}^2 + \dot{\psi}^2 = r^2\dot{t}^2 \ . \tag{4.35}$$

Energy and angular momentum conservation (t- and ψ-independence of the metric) lead to

$$r^2\dot{t} = E \ , \qquad \dot{\psi} = \ell \ . \tag{4.36}$$

Hence one obtains

$$\dot{r}^2 + \ell^2 r^2 = E^2 \ , \tag{4.37}$$

which is solved by

$$r(\tau) = \ell^{-1}E\sin\ell\tau \ , \tag{4.38}$$

and therefore

$$t(\tau) = -E^{-1}\ell\cot\ell\tau \qquad \text{and} \qquad \psi(\tau) = \ell\tau \ . \tag{4.39}$$

Here without loss of generality we have set all integration constants to zero (they will, in any case, reappear below as the transverse coordinates parametrising the congruence of null geodesics). Without loss of generality we can also choose $E = 1$ by a rescaling of τ.

Having obtained this congruence of null geodesics, the next step is to change coordinates to an adapted coordinate system

$$(r, t, \psi) \to (u, v, \phi)$$

where u is the parameter τ along the null geodesics, i.e., ∂_u is the null geodesic vector field with $g_{uu} \equiv g(\partial_u, \partial_u) = 0$, and otherwise characterised by $g_{uv} = 1$ and $g_{u\phi} = 0$. A possible choice is

$$\begin{aligned}\partial_u &= \dot{r}\partial_r + \dot{t}\partial_t + \dot{\psi}\partial_\psi \\ &= (1-\ell^2 r^2)^{1/2}\partial_r + r^{-2}\partial_t + \ell\partial_\psi \\ \partial_v &= -\partial_t \\ \partial_\phi &= \partial_\psi + \ell\partial_t \ .\end{aligned} \tag{4.40}$$

This integrates to

$$\begin{aligned} r(u,v,\phi) =& \ell^{-1}\sin\ell u \\ t(u,v,\phi) =& -\ell\cot\ell u - v + \ell\phi \\ \psi(u,v,\phi) =& \phi + \ell u \ , \end{aligned} \tag{4.41}$$

so that (v, ϕ) have the interpretation of constants of integration in the geodesic equation parametrising the congruence of null geodesics.

The next step is to express the metric in the new variables. By construction, $g_{uu} = g_{u\phi} = 0$, and one finds

$$\begin{aligned} ds^2 = 2dudv &+ 2\ell^{-1}(\sin\ell u)^2 dv\,d\phi - \ell^{-2}(\sin\ell u)^2(dv)^2 + (\cos\ell u)^2\ (d\phi)^2 \\ &+ \ell^{-2}(\sin\ell u)^2 ds^2(\mathbb{E}^3) + (\sin(\ell u + \phi))^2 ds^2(S^4)\ . \end{aligned} \tag{4.42}$$

We will now show that the Penrose limit of the above metric is flat Minkowski space if and only if $\ell = 0$ and the maximally supersymmetric Hpp-wave of IIB supergravity otherwise.

In order to establish this, it is useful to recall the result (2.22) on the form of $A_{ij}(x^-)$ for metrics with diagonal $C_{ij}(u) = a_i(u)^2\delta_{ij}$ in the Penrose limit in Rosen coordinates. We had seen that $A_{ij} = 0$ (i.e., the Penrose limit is flat) if and only if $a_i(u) = b_i + c_i u$.

Moreover, since for constant A_{ij} we can always absorb a positive constant μ^2 into A_{ij} by a scaling of (x^+, x^-), we learn that a metric in Rosen coordinates with diagonal C_{ij} is equivalent to the maximally supersymmetric IIB Hpp-wave (2.15) if and only if

$$a_i(u)'' = -\mu^2 a_i(u) \tag{4.43}$$

for an i-independent non-zero constant μ, i.e., if and only if

$$a_i(u) = b_i \sin\mu u + c_i \cos\mu u\ . \tag{4.44}$$

Having established this, it is now easy to determine what is the Penrose limit of the metric (4.42). Clearly for $\ell \neq 0$ the metric is of the required form (with $\mu = \ell$) to give the Hpp wave solution: the second and third term disappear and the remaining terms are of the required trigonometric form.

In the limit $\ell \to 0$, on the other hand, even before taking the Penrose limit, the second (rotation) term goes to zero and the fourth and sixth term combine to the line element on $\mathbb{E}^5$. Thus the metric becomes

$$ds^2_{\ell=0} = 2du\,dv + u^2(-(dv)^2 + ds^2(\mathbb{E}^3)) + ds^2(\mathbb{E}^5)\ , \tag{4.45}$$

which is the ten-dimensional Minkowski spacetime in the Penrose limit.

5. Penrose limits of branes

In this and the next sections we will investigate the Penrose limits of supergravity brane solutions. We will limit our discussion in this section to elementary p-brane solutions for which the isometry group of the D-dimensional spacetime metric is $\mathrm{ISO}(1,p)\times\mathrm{SO}(D{-}p{-}1)$. Intersecting brane configurations will be discussed in the next section.

5.1. Classification of null geodesics.

As discussed above for the near horizon geometries, one way to classify the different null geodesics, and hence the different Penrose limits, is to use the covariance property, by which two null geodesics which are related by an isometry induce isometric Penrose limits. We therefore need to study the orbits of the isometry group on the space of null geodesics, which is the space of pairs consisting of a point in the space time (the "initial" point of the geodesic) and a null direction at that point.

The typical metric for a supergravity brane solution in D dimensions is

$$\begin{aligned} ds^2 &= A^2(r) ds^2(\mathbb{E}^{(1,p)}) + B^2(r) ds^2(\mathbb{E}^{D-p-1}) \\ F_{p+2} &= \mathrm{dvol}(\mathbb{E}^{1,p}) \wedge dC(r) \\ \phi &= \phi(r) \end{aligned} \tag{5.1}$$

where r is the radial transverse coordinate, $\mathbb{E}^{(1,p)}$ is the worldvolume of the brane and $\mathbb{E}^{D-p-1}$ is the transverse space. The components A, B and C as well as the scalar ϕ for a single brane or for many branes located at the same point depend only on the radial coordinate r.

Unlike the near horizon geometries, which are homogeneous, the isometry group $G = \mathrm{ISO}(1,p) \times \mathrm{SO}(D{-}p{-}1)$ acts on the brane solutions with cohomogeneity one. The orbits are labelled by the radial distance r transverse to the brane: for $r > 0$ they are diffeomorphic to $\mathbb{R}^{p+1} \times S^{D-p-2}$ and have codimension one. (As $r \to 0$ one recovers the near horizon geometry which, at least for the M2, D3 and M5 branes, was already discussed above.)

The isotropy subgroup of a point P a distance $r > 0$ away from the brane is isomorphic to $H = \mathrm{SO}(1,p) \times \mathrm{SO}(D{-}p{-}2)$. To study how this group acts on the null directions, we break the tangent space T_PM at P to the spacetime manifold (M,g) describing the brane into three orthogonal subspaces

$$T_PM = T_PB \oplus T_PR \oplus T_PS\ , \tag{5.2}$$

where T_PB are those vectors tangent to the brane, T_PR is the radial component and T_PS are those vectors tangent to the transverse sphere. The metrics on each of the factors depend only on the radial distance r. A null geodesic in M is specified by the initial point P and by a null direction at P; that is, (the projectivisation of) a null vector in $T_P M$. Let V be one such vector and let $V = V_B + V_R + V_S$ be its decomposition relative to the above splitting. Notice that V_R is determined up to a sign by V_B and V_S, since V is null. We must therefore distinguish two cases:

1. *V_B is null; hence $V_R = V_S = 0$.* Since $\mathrm{SO}(1,p)$ acts transitively on the celestial sphere of $T_P B$, all geodesics starting at P in the direction of V are equivalent.
2. *V_B is timelike.* We must distinguish between two subcases:
   (a) *$V_S = 0$.* $\mathrm{SO}(1,p)$ acts transitively on timelike directions in $T_P B$. We simply normalise V_B appropriately and V_R is determined up to a sign; but this sign can be changed by changing the sign of the affine parameter along the geodesic and using a Lorentz transformation in $\mathrm{SO}(1,p)$ to reverse the sign of V_B.
   (b) *$V_S \neq 0$.* H acts transitively on timelike directions in $T_P B$ and on directions in $T_P S$; but now the relative scale matters. Therefore there is a free parameter in this case: the ratio of the norms of V_B and V_S. Any two choices with the same ratio are equivalent under H. Again, the radial component V_R is determined up to a sign by the condition that V is null. As above, this sign is immaterial.

We will call geodesics in 1 *longitudinal* and those in 2(a) *radial*, whereas those in 2(b) are *generic* and we will now in turn discuss the associated Penrose limits.

5.2. **Longitudinal null geodesics.** Without loss of generality (covariance of the Penrose limit) we can choose the longitudinal null geodesic to lie in the (t,x)-plane where x is any of the longitudinal worldvolume coordinates. A coordinate system adapted to such a null geodesic sitting at a fixed regular (non-zero) value r_0 of the transverse radial coordinate r is

$$\begin{aligned} v &= x - t \\ u &= \tfrac{1}{2}(x+t)A(r)^2 \\ \rho &= r - r_0 \end{aligned} \tag{5.3}$$

with all the other coordinates unchanged. In terms of these coordinates, the metric in (5.1) reads

$$\begin{aligned} ds^2 = 2dudv &- 4A(\rho+r_0)'A(\rho+r_0)^{-3}udvd\rho \\ &+ A^2(\rho+r_0)ds^2(\mathbb{E}^{p-1}) + B^2(\rho+r_0)ds^2(\mathbb{E}^{D-p-1}) \end{aligned} \tag{5.4}$$

which is indeed of the required form (2.1). In the Penrose limit one finds

$$ds^2 = 2dudv + A^2(r_0)ds^2(\mathbb{E}^{p-1}) + B^2(r_0)ds^2(\mathbb{E}^{D-p-1}) \ . \tag{5.5}$$

This is isometric to the flat Minkowski metric on $\mathbb{E}^{(1,D-1)}$ for any $r_0 > 0$. In addition the other fields associated with brane solutions either vanish or become constant in the limit. In particular the various form field strengths vanish and the scalars become constant.

5.3. **Radial null geodesics.** We will now provide a fairly complete analysis of the Penrose limit of brane solutions along radial null geodesics, dealing with D-branes, M-branes and NS-branes. In this case explicit formulae can be given for the various brane solutions in the limit.

5.3.1. *Some general remarks on radial null geodesics.* We can choose the timelike component V_B to lie in the time-direction. Thus we shall investigate the Penrose limit involving the worldvolume time coordinate t and the radial transverse coordinate r. We shall focus mainly on the limit involving the metric. The Penrose limit for the rest of the fields will be described at the end. To achieve this, we first write the metric in (5.1) as

$$ds^2 = A^2(r) ds^2(\mathbb{E}^{(1,p)}) + B^2(r) \left(dr^2 + r^2 ds^2(S^{D-p-2}) \right) . \tag{5.6}$$

To adapt coordinates appropriate for taking the Penrose limit we have to find coordinates (u, v) to rewrite the two-dimensional metric

$$ds^2_{(2)} = -A^2(r) dt^2 + B^2(r) dr^2 \tag{5.7}$$

as

$$ds^2 = 2du dv + D^2(u, v) dv^2 \tag{5.8}$$

This implies that the vector $\partial_u \equiv \partial/\partial u$ is null and geodetic. For this we consider the coordinate transformations

$$\begin{aligned} v &= t + a(r) \\ u &= -t + b(r) \end{aligned} \tag{5.9}$$

Using these coordinate transformations and comparing the two expressions for the metric in the two coordinate systems, we find

$$\begin{aligned} D^2 &= 2 - A^2 \\ b' &= (A^2 - 1) a' \\ (a')^2 &= B^2/A^2 \end{aligned} \tag{5.10}$$

where $'$ denotes differentiation with respect to the coordinate r. This can be rewritten as

$$\begin{aligned} D^2 &= 2 - A^2 \\ b' &= \pm (A^2 - 1) B/A \\ a' &= \pm B/A \end{aligned} \tag{5.11}$$

In particular observe that $u + v = a + b$ and $(a + b)' = \pm AB$. This is a key equation because it gives the transformation between the $u + v$ coordinate and r and so $r = r(u + v)$.

Next rewriting the brane metric in the above coordinate system, we get

$$ds^2 = 2dudv + D^2(u,v)dv^2 + A^2\left(r(u+v)\right)ds^2(\mathbb{E}^p) \\ + B^2\left(r(u+v)\right)r^2(u+v)ds^2(S^{D-p-2}) \quad (5.12)$$

Taking the Penrose limit, we find

$$ds^2 = 2dudv + A^2\left(r(u)\right)ds^2(\mathbb{E}^p) + B^2\left(r(u)\right)r^2(u)ds^2(\mathbb{E}^{D-p-2})\ . \quad (5.13)$$

It remains now to put this metric in Hpp-wave form. For this we write $ds^2(\mathbb{E}^p) = \sum_a d\tilde{x}^a d\tilde{x}^a$ and $ds^2(\mathbb{E}^{D-p-2}) = \sum_i d\tilde{y}^i d\tilde{y}^i$. Then we perform the following coordinate transformations

$$\begin{aligned} u &= x^- \\ v &= x^+ + \tfrac{1}{2}\frac{\partial_- A(x^-)}{A(x^-)}x^2 + \tfrac{1}{2}\frac{\partial_-\big(r(x^-)B(x^-)\big)}{r(x^-)B(x^-)}y^2 \\ \tilde{x}^a &= \frac{1}{A(x^-)}x^a \\ \tilde{y}^i &= \frac{1}{r(x^-)B(x^-)}y^i\ , \end{aligned} \quad (5.14)$$

where $\partial_- = \frac{d}{dx^-}$, $x^2 = \delta_{ab}x^a x^b$ and $y^2 = \delta_{ij}y^i y^j$. The metric in the new coordinate system is

$$ds^2 = 2dx^+dx^- + \left[\frac{\partial_-^2 A(x^-)}{A(x^-)}x^2 + \frac{\partial_-^2\left(r(x^-)B(x^-)\right)}{r(x^-)B(x^-)}y^2\right](dx^-)^2 \\ + ds^2(\mathbb{E}^p) + ds^2(\mathbb{E}^{D-p-2})\ . \quad (5.15)$$

It is sometimes complicated to express explicitly the non-trivial component of the metric in the Penrose limit in terms of the x^- coordinate. This is because it is difficult to find the explicit expression for the transformation $r = r(x^-)$. However, it is straightforward to express it in terms of the original r coordinate as follows. Define

$$\mathcal{A}(x^-,x,y) = \frac{\partial_-^2 A(x^-)}{A(x^-)}x^2 + \frac{\partial_-^2\left(r(x^-)B(x^-)\right)}{r(x^-)B(x^-)}y^2 \quad (5.16)$$

Then we can use the chain rule and write

$$\partial_- f(x^-) = \partial_- r\partial_r f(r) = \pm\frac{1}{AB}\partial_r f(r) \quad (5.17)$$

where we have use that in the Penrose limit $\partial_r x^- = \pm AB$ and $f(x^-) = f(r(x^-))$. In particular we find that

$$\mathcal{A}(r,x,y) = \left[\frac{\partial_r^2 A}{A^3B^2} - \frac{(\partial_r A)^2}{A^4B^2} - \frac{\partial_r A\partial_r B}{A^3B^3}\right]x^2 \\ + \left[\frac{\partial_r^2 B}{A^2B^3} + \frac{\partial_r B}{rA^2B^3} - \frac{\partial_r A}{rA^3B^2} - \frac{(\partial_r B)^2}{A^2B^4} - \frac{\partial_r A\partial_r B}{A^3B^3}\right]y^2\ . \quad (5.18)$$

Turning to investigate the Penrose limit of the form-field strengths, we remark that they vanish at the limit. To see this observe that in the $(u, v, \tilde{x}^a, \tilde{y}^i)$ coordinate system

$$F_{p+2} = 2(-1)^p du \wedge dv \wedge \mathrm{dvol}(\mathbb{E}^p)(\partial_u C)(r(u+v)) \tag{5.19}$$

An appropriate choice for a gauge potential is

$$C_{p+1} = 2(-1)^p dv \wedge \mathrm{dvol}(\mathbb{E}^p) C(r(u+v)) \ . \tag{5.20}$$

The Penrose limit of the above C_{p+1} is zero since, as we had seen in (2.9), $i(\partial/\partial v)\bar{C}_{p+1} = 0$. Thus $F_{p+2} = dC_{p+1} = 0$ in the Penrose limit. For the scalar ϕ in the Penrose limit we find $\phi = \phi(r(x^-))$. We remark that if a solution has non-vanishing scalars, then it is possible to choose different frames to describe the metric (5.6). However the formulae that we have presented above for the description of the Penrose limit do not depend on the choice of frame. So they can easily be adapted to any choice of frame.

Next we shall investigate the D-branes, NS-branes and M-branes separately.

5.3.2. *D-branes.* The spacetime metric of a Dp-brane in the string frame [22, 23, 24, 25] is

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}} ds^2(\mathbb{E}^{1,p}) + H^{\frac{1}{2}} ds^2(\mathbb{E}^{9-p}) \\ F_{p+2} &= \mathrm{dvol}(\mathbb{E}^{1,p}) \wedge dH^{-1} \\ e^{2\phi} &= H^{\frac{3-p}{2}} \end{aligned} \tag{5.21}$$

where

$$H = 1 + \frac{Q_p}{r^{7-p}} \tag{5.22}$$

is a harmonic function on the transverse space $\mathbb{E}^{9-p}$, r is the radial coordinate in $\mathbb{E}^{9-p}$ the and Q_p is the charge of Dp-brane in some units. Note that the form field strength associated of the D3-brane is self-dual and so one has to project onto the self-dual component of the field strength presented above.

We shall focus on the Penrose limit of the Dp-brane metric. Then at the end we shall give all non-vanishing fields of the solution. Since $A = H^{-\frac{1}{4}}$ and $B = H^{\frac{1}{4}}$, then $u + v = \pm r$; the integration constant has been absorbed in the definition of the (u, v) coordinates. Writing the Dp-brane metric in these coordinates, we have

$$\begin{aligned} ds^2 = 2dudv + \big(2 - H^{-\frac{1}{2}}(u+v)\big)du^2 + H^{-\frac{1}{2}}(u+v) ds^2(\mathbb{E}^p) \\ + (u+v)^2 H^{\frac{1}{2}}(u+v) ds^2(S^{8-p}) \end{aligned} \tag{5.23}$$

In the Penrose limit we get

$$ds^2 = 2dudv + H(u)^{-\frac{1}{2}} ds^2(\mathbb{E}^p) + u^2 H(u)^{\frac{1}{2}} ds^2(\mathbb{E}^{8-p}) \ . \tag{5.24}$$

It remains to put this metric in Brinkman form. For this we change coordinates again as

$$\begin{aligned} u &= x^- \\ v &= x^+ - \tfrac{1}{8}H^{-1}(x^-)H'(x^-)x^2 + \left(\tfrac{1}{8}H^{-1}(x^-)H'(x^-) + \tfrac{1}{2}(x^-)^{-1}\right)y^2 \\ \tilde{x}^a &= H^{\frac{1}{4}}(x^-)x^a \\ \tilde{y}^i &= (x^-)^{-1}H^{-\frac{1}{4}}(x^-)y^i \end{aligned} \tag{5.25}$$

where $ds^2(\mathbb{E}^p) = \sum_a d\tilde{x}^a d\tilde{x}^a$ and $ds^2(\mathbb{E}^{8-p}) = \sum_i d\tilde{y}^i d\tilde{y}^i$ as in the general case. Performing this coordinate transformation, we get

$$\begin{aligned} ds^2 &= 2dx^-dx^+ + \mathcal{A}(x^-, x, y)(dx^-)^2 + ds^2(\mathbb{E}^8) \\ e^{2\phi} &= H^{\frac{3-p}{2}}(x^-) \end{aligned} \tag{5.26}$$

where

$$\begin{aligned} \mathcal{A}(x^-, x, y) = &\left(-\tfrac{1}{4}H^{-1}H'' + \tfrac{5}{16}H^{-2}(H')^2\right)x^2 \\ &+ \left(\tfrac{1}{4}H^{-1}H'' - \tfrac{3}{16}H^{-2}(H')^2 + \tfrac{1}{2}(x^-)^{-1}H^{-1}H'\right)y^2 \end{aligned} \tag{5.27}$$

and we have added the expression for the dilaton at the limit for completeness.

For example let us consider the D3-brane separately. For a D3-brane, $H = 1 + \frac{Q_3}{r^4}$. Then A can be easily computed and yields

$$\mathcal{A}(x^-, x, y) = -5\frac{Q_3(x^-)^2}{((x^-)^4 + Q_3)^2}x^2 + 3\frac{Q_3(x^-)^2}{((x^-)^4 + Q_3)^2}y^2 \,. \tag{5.28}$$

Observe that the D3-brane metric is Ricci flat in the Penrose limit and so solves the Einstein equations without active form-field strengths or scalars. In the near horizon limit, $H = Q_3/r^4$, $A = 0$ and the Penrose limit is the ten-dimensional Minkowski spacetime.

The Penrose limit for Dp-branes was investigated in the string frame. The above analysis can easily been done in other frames like for example the Einstein frame. In fact in such a case the result cannot be presented in a closed form for $p \neq 3$ because it is not straightforward to give the coordinate transformation $u = u(r)$ in a closed form. However the general formulae given in the previous section, in particular (5.15) with (5.18), can be used to find the Penrose limit metric.

5.3.3. *Fundamental strings and NS5-branes.* The NS5-brane solution [26] is

$$\begin{aligned} ds^2 &= ds^2(\mathbb{E}^{1,5}) + H(r)ds^2(\mathbb{E}^4) \\ F_7 &= \mathrm{dvol}(\mathbb{E}^{1,5}) \wedge dH^{-1} \\ e^{2\phi} &= H \end{aligned} \tag{5.29}$$

where $H = 1 + \frac{Q_5}{r^2}$ is a harmonic function in $\mathbb{E}^4$ and Q_5 is the charge of NS5-brane. So in this case $(a+b)' = \pm H^{\frac{1}{2}}$. Unfortunately, it is not possible to find the transformation $r = r(u+v)$ explicitly. However we can still express the non-trivial component of the metric in the Penrose limit in terms of the r coordinate as it has been explained. Indeed we find that the Penrose limit of the NS5-brane is

$$\begin{aligned} ds^2 &= 2dx^+dx^- + \mathcal{A}(r,x,y)(dx^-)^2 + ds^2(\mathbb{E}^5) + ds^2(\mathbb{E}^3) \\ e^{2\phi} &= H \end{aligned} \tag{5.30}$$

where

$$\mathcal{A}(r,x,y) = \left[\tfrac{1}{2}H^{-2}\partial_r^2 H + \frac{1}{2r}H^{-2}\partial_r H - \tfrac{1}{2}H^{-3}(\partial_r H)^2\right] y^2 \ . \tag{5.31}$$

In the near horizon case, we have

$$(a+b)' = \pm|Q_5|^{\frac{1}{2}} r^{-1} \tag{5.32}$$

and so $r = \exp\left(\pm(u+v)/|Q_5|^{\frac{1}{2}}\right)$. Choosing the plus sign and substituting this into the metric and dilaton and taking the Penrose limit, we find that

$$\begin{aligned} ds^2 &= 2dx^+dx^- + ds^2(\mathbb{E}^8) \\ e^{2\phi} &= Q_5 e^{-2x^-/|Q_5|^{\frac{1}{2}}} \ . \end{aligned} \tag{5.33}$$

This is flat space with linear dilaton solution of type II (or heterotic) supergravity and preserves sixteen (or eight) supersymmetries.

The fundamental string solution [27] is

$$\begin{aligned} ds^2 &= H^{-1}ds^2(\mathbb{E}^{1,1}) + ds^2(\mathbb{E}^8) \\ F_3 &= \mathrm{dvol}(\mathbb{E}^{1,1}) \wedge dH^{-1} \\ e^{2\phi} &= H^{-1} \end{aligned} \tag{5.34}$$

where $(a+b)' = H^{-\frac{1}{2}}$. As in the previous case, the coordinate transformation $r = r(u+v)$ cannot be found explicitly. Nevertheless we can write the Penrose limit as

$$\begin{aligned} ds^2 &= 2dx^+dx^- + \mathcal{A}(r,x,y)(dx^-)^2 + ds^2(\mathbb{E}^5) + ds^2(\mathbb{E}^3) \\ e^{2\phi} &= H^{-1} \end{aligned} \tag{5.35}$$

where

$$\mathcal{A}(r,x,y) = \left[\tfrac{1}{2}H^{-1}(\partial_r H)^2 - \tfrac{1}{2}\partial_r^2 H\right] x^2 + \frac{1}{2r}\partial_r H y^2 \ . \tag{5.36}$$

For the near horizon case, we have $(a+b)' = \pm r^3/|Q_1|^{\frac{1}{2}}$, so that $r^2 = 2|Q_1|^{\frac{1}{4}}(u+v)^{\frac{1}{2}}$; we have chosen the plus sign. Taking the Penrose limit, the metric becomes

$$ds^2 = 2dudv + 8|Q_1|^{-\frac{1}{4}}u^{\frac{3}{2}}ds^2(\mathbb{E}) + 2|Q_1|^{\frac{1}{4}}u^{\frac{1}{2}}ds^2(\mathbb{E}^7) \tag{5.37}$$

In Brinkman coordinates, the Penrose limit of the fundamental string solution in the near horizon limit is

$$\begin{aligned} ds^2 &= 2dx^- dx^+ + \mathcal{A}(x^-, x, y)(dx^-)^2 + ds^2(\mathbb{E}) + ds^2(\mathbb{E}^7) \\ e^{2\phi} &= 8|Q_1|^{-\frac{1}{4}}(x^-)^{\frac{3}{2}} \ , \end{aligned} \tag{5.38}$$

where

$$\mathcal{A}(x^-, x, y) = -\tfrac{3}{16}(x^-)^{-2}x^2 - \tfrac{3}{16}(x^-)^{-2}y^2 \ . \tag{5.39}$$

The metric is actually Lorentzian homogeneous (cf. the discussion in Section 8). Namely, in addition to the $2D-3=17$ Heisenberg algebra Killing vectors (2.12) of a generic pp-wave spacetime, there is the $SO(8)$ rotation symmetry of the transverse coordinates, and there is the scale invariance $(x^+, x^-) \to (cx^+, c^{-1}x^-)$ corresponding to the Killing vector $x^+\partial_+ - x^-\partial_-$. Since the dilaton depends non-trivially on x^-, however, only the 45-dimensional subgroup of the isometry group generated by the Heisenberg algebra and the transverse $SO(8)$ is a symmetry of the solution. This group does not act transitively on the spacetime.

5.3.4. *M-branes.* There are two cases to consider, the M2-brane and the M5-brane. The supergravity solution for the M2-brane [28] is

$$\begin{aligned} ds^2 &= H^{-\frac{2}{3}} ds^2(\mathbb{E}^{1,2}) + H^{\frac{1}{3}} ds^2(\mathbb{E}^8) \\ F_4 &= \mathrm{dvol}(\mathbb{E}^{1,3}) \wedge dH^{-1} \end{aligned} \tag{5.40}$$

Thus we have

$$(a+b)' = H^{-\frac{1}{6}} \ . \tag{5.41}$$

This differential equation cannot be easily integrated. Nevertheless we can write the Penrose limit metric as

$$ds^2 = 2dx^+dx^- + \mathcal{A}(r, x, y)(dx^-)^2 + ds^2(\mathbb{E}^2) + ds^2(\mathbb{E}^7) \tag{5.42}$$

where

$$\begin{aligned} \mathcal{A}(r, x, y) = &\left[\tfrac{7}{18} H^{-\frac{5}{3}} (\partial_r H)^2 - \tfrac{1}{3} H^{-\frac{2}{3}} \partial_r^2 H\right] x^2 \\ &+ \left[\tfrac{1}{6} H^{-\frac{2}{3}} \partial_r^2 H + \tfrac{1}{2r} H^{-\frac{2}{3}} \partial_r H - \tfrac{1}{9} H^{-\frac{5}{3}} (\partial_r H)^2\right] y^2 \ . \end{aligned} \tag{5.43}$$

In the near horizon limit, we can easily find that

$$(u+v) = \pm\tfrac{1}{2}|Q|^{-\frac{1}{6}} r^2 \tag{5.44}$$

The near horizon M2-brane metric becomes

$$\begin{aligned} ds^2 = 2dudv &+ \left(2 - 4|Q_2|^{-\frac{1}{3}}(v+u)^2\right) du^2 \\ &+ 4|Q_2|^{-\frac{1}{3}}(v+u)^2 ds^2(\mathbb{E}^2) + |Q_2|^{\frac{1}{3}} ds^2(S^7) \ . \end{aligned} \tag{5.45}$$

The Penrose limit gives

$$ds^2 = 2dudv + 4|Q_2|^{-\frac{1}{3}} u^2 ds^2(\mathbb{E}^2) + |Q_2|^{\frac{1}{3}} ds^2(\mathbb{E}^7) \ , \tag{5.46}$$

This in fact is the eleven-dimensional Minkowski spacetime as it can be easily seen by writing the metric in Brinkman coordinates. This is in agreement with the general result obtained in Section 4.4 that the Penrose limit of any AdS $\times S$ space along a radial geodesic is flat.

Turning now to investigate the M5-brane. The supergravity solution for the M5-brane [29] is

$$\begin{aligned} ds^2 &= H^{-\frac{1}{3}} ds^2(\mathbb{E}^{1,5}) + H^{\frac{2}{3}} ds^2(\mathbb{E}^5) \\ F_7 &= \mathrm{dvol}(\mathbb{E}^{1,5}) \wedge dH^{-1} \end{aligned} \tag{5.47}$$

Thus we have that $(a+b)' = H^{\frac{1}{6}}$. This differential equation cannot be easily integrated. Nevertheless we can write the Penrose limit metric as

$$ds^2 = 2dx^+dx^- + \mathcal{A}(r,x,y)(dx^-)^2 + ds^2(\mathbb{E}^5) + ds^2(\mathbb{E}^4) \tag{5.48}$$

where

$$\begin{aligned} \mathcal{A}(r,x,y) = &\left[-\tfrac{1}{6} H^{-\frac{4}{3}} \partial_r^2 H + \tfrac{2}{9} H^{-\frac{7}{3}} (\partial_r H)^2 \right] x^2 \\ &+ \left[\tfrac{1}{3} H^{-\frac{4}{3}} \partial_r^2 H + \tfrac{1}{2r} H^{-\frac{4}{3}} \partial_r H - \tfrac{5}{18} H^{-\frac{7}{3}} (\partial_r H)^2 \right] y^2 \ . \end{aligned} \tag{5.49}$$

Again in the near horizon limit we can find

$$(u+v)^2 = 4|Q_5|^{\frac{1}{3}} r \tag{5.50}$$

Substituting this back into near horizon geometry and taking the Penrose limit we find

$$ds^2 = 2dudv + |Q_5|^{-\frac{2}{3}} u^2 ds^2(\mathbb{E}^5) + |Q_5|^{\frac{2}{3}} ds^2(\mathbb{E}^4) \tag{5.51}$$

Noting that the non-trivial coefficients of the metric are at most quadratic in u and using (2.22), or changing coordinates as for the M2-brane above, we see that this metric is in fact, as expected from our general arguments, eleven-dimensional Minkowski spacetime.

5.4. **Generic null geodesics.** We now consider the Penrose limit of brane solutions along generic null geodesics; that is, geodesics whose tangent vectors have a component tangent to the transverse sphere.

Symmetry considerations allow us to single out any direction on the transverse sphere. To this end we will write the sphere metric as

$$d\Omega^2_{D-p-2} = d\psi^2 + (\sin\psi)^2 d\Omega^2_{D-p-3} \ , \tag{5.52}$$

where ψ is a colatitude and $d\Omega^2_{D-p-3}$ is the metric on the corresponding equator. The brane metric becomes

$$ds^2 = A^2 \left(-dt^2 + ds^2(\mathbb{E}^p) \right) + B^2 dr^2 + B^2 r^2 \left(d\psi^2 + (\sin\psi)^2 d\Omega^2_{D-p-3} \right) \ . \tag{5.53}$$

We will consider null geodesics in the (t, r, ψ) space with metric

$$ds^2 = -A^2dt^2 + B^2dr^2 + B^2r^2d\psi^2 \ . \tag{5.54}$$

Let us change coordinates to (u, v, z) adapted to the null geodesic:

$$u = u(r) \qquad v = t + \ell\psi + a(r) \qquad z = \psi + b(r) \ , \tag{5.55}$$

where ℓ is a constant, and such that the metric takes the form

$$ds^2 = 2dudv + Kdv^2 + Ldvdz + Mdz^2 \ . \tag{5.56}$$

This choice of reparametrisation is consistent with a null geodesic with tangent vector

$$\frac{\partial}{\partial u} = f(r)\frac{\partial}{\partial r} + g(r)\frac{\partial}{\partial \psi} + h(r)\frac{\partial}{\partial t} \tag{5.57}$$

which has r-dependent components in all three directions. The constant parameter ℓ can be understood as the angular momentum of the massless particle whose motion is described by the null geodesic. Comparing the forms of the metric in both coordinate systems we can determine K, L and M, the functions f,g and h, the function $u(r)$ and the functions a, b in terms of the parameter ℓ and the functions A and B appearing in the metric.

After some calculation, and letting primes denote differentiation with respect to r, we find the following

$$f^2 = \frac{1}{B^2}\left(\frac{1}{A^2} - \frac{\ell^2}{B^2r^2}\right) \qquad h = -\frac{1}{A^2} \qquad g = \frac{\ell}{B^2r^2} \tag{5.58}$$

$$K = -A^2 \qquad L = 2\ell A^2 \qquad M = B^2r^2 - \ell A^2 \tag{5.59}$$

$$a' = \pm\sqrt{\frac{B^2}{A^2} - \frac{\ell^2}{r^2}} \qquad b' = \mp\frac{\ell/r^2}{\sqrt{\frac{B^2}{A^2} - \frac{\ell^2}{r^2}}} \ , \tag{5.60}$$

and

$$\frac{du}{dr} = \pm\frac{B^2}{\sqrt{\frac{B^2}{A^2} - \frac{\ell^2}{r^2}}} = \pm\frac{rAB^2}{\sqrt{r^2B^2 - \ell^2A^2}} \equiv Q \tag{5.61}$$

So $u(r)$ is defined up to an inconsequential constant of integration by the following integral

$$u(r) = \pm\int^r \frac{B^2dr}{\sqrt{\frac{B^2}{A^2} - \frac{\ell^2}{r^2}}} \ . \tag{5.62}$$

This equation can be inverted to give an implicit relation $r(u)$ which will play an important role below. All the above signs are correlated and come from choosing the sign of the square root of f^2.

In the Penrose limit we obtain the following spacetime metric in Rosen coordinates:

$$ds^2 = 2dudv + \left(B^2r^2 - \ell^2 A^2\right) dz^2 + A^2 ds^2(\mathbb{E}^p) \\ + B^2r^2(\sin b)^2 ds^2(\mathbb{E}^{D-p-3}) \ , \quad (5.63)$$

where the dependence on u is implicit through the dependence on r. In the limit $\ell \to 0$, we recover the result for the Penrose limits associated with the radial null geodesics investigated in the previous sections provided that we choose $b = \pi/2$.

The field strength for a p-brane solution is of the form

$$F_{p+2} = \mathrm{dvol}(\mathbb{E}^{1,p}) \wedge dC(r) \ , \quad (5.64)$$

for some function $C(r)$. Changing coordinates and taking the Penrose limit, we find the following field strength

$$\bar{F}_{p+2} = \pm C' \frac{\ell}{B} \sqrt{\frac{1}{A^2} - \frac{\ell^2}{B^2r^2}} du \wedge dy^1 \wedge \cdots \wedge dy^p \wedge dz \ , \quad (5.65)$$

which is nonzero provided that ℓ is different from zero.

It remains to write the metric in pp-wave form. For this we write $ds^2(\mathbb{E}^p) = \sum_a d\tilde{x}^a d\tilde{x}^a$, $ds^2(\mathbb{E}^{D-p-3}) = \sum_i d\tilde{y}^i d\tilde{y}^i$ and $\tilde{z} = z$. Then we perform the following coordinate transformations

$$\begin{aligned} u &= x^- \\ v &= x^+ + \tfrac{1}{2}\frac{\partial_- A(x^-)}{A(x^-)} x^2 + \tfrac{1}{2}\frac{\partial_-\left(r(x^-)B(x^-)\sin b\right)}{r(x^-)B(x^-)\sin b} y^2 \\ &\quad + \tfrac{1}{2}\frac{\partial_-(\sqrt{B^2r^2 - \ell^2A^2})}{\sqrt{B^2r^2-\ell^2A^2}} z^2 \\ \tilde{x}^a &= \frac{1}{A(x^-)} x^a \\ \tilde{y}^i &= \frac{1}{r(x^-)B(x^-)\sin b} y^i \\ \tilde{z} &= \frac{1}{\sqrt{B^2r^2-\ell^2A^2}} z \ , \end{aligned} \quad (5.66)$$

where $\partial_- = \frac{d}{dx^-}$. The metric in the new coordinate system is

$$ds^2 = 2dx^+dx^- + \mathcal{A}(x^-, x, y, z)(dx^-)^2 + ds^2(\mathbb{E}^p) + ds^2(\mathbb{E}^{D-p-3}) + dz^2 \ , \quad (5.67)$$

where

$$\mathcal{A}(x^-,x,y,z)=\frac{\partial_-^2 A(x^-)}{A(x^-)}x^2+\frac{\partial_-^2\,(r(x^-)B(x^-)\sin b)}{r(x^-)B(x^-)\sin b}y^2 + \frac{\partial_-^2\sqrt{B^2r^2-\ell^2A^2}}{\sqrt{B^2r^2-\ell^2A^2}}z^2\ . \quad (5.68)$$

As in (5.18) we can express the non-trivial component of the above metric as a function of the original coordinate r using the formula

$$\partial_-^2 f(x^-)=-Q^{-3}\partial_r Q\partial_r f(r)+Q^{-2}\partial_r^2 f(r)\ , \quad (5.69)$$

where Q was defined in (5.61). The resulting metric is

$$ds^2=2dx^+dx^-+\mathcal{A}(r,x,y,z)(dx^-)^2+ds^2(\mathbb{E}^{D-2}) \quad (5.70)$$

where

$$\mathcal{A}(r,x,y,z)=(B^2r^2-\ell^2A^2)^{-\frac{1}{2}}\big[-Q^{-3}\partial_r Q\partial\sqrt{B^2r^2-\ell^2A^2} + Q^{-2}\partial_r^2\sqrt{B^2r^2-\ell^2A^2}\big]z^2+A^{-1}\big[-Q^{-3}\partial_r Q\partial_r A+Q^{-2}\partial_r^2 A\big]x^2 + (Br\sin b)^{-1}\big[-Q^{-3}\partial_r Q\partial_r(Br\sin b)+Q^{-2}\partial_r^2(Br\sin b)\big]y^2 \quad (5.71)$$

Unfortunately, there does not seem to be a straightforward way to simplify this equation. So we shall not investigate each brane case separately as we have done for the case of radial null geodesics. We remark though that in the limit that $\ell\to 0$, the above calculations and formulae reduce to those we have found for the radial null geodesics. In addition in the near horizon limit for the M-branes and D3-brane, one can show after some computation that the above calculations reduce to those we have done for the Penrose limit along a generic null geodesic of the associated AdS $\times S$ spaces. The near horizon geometries of the various branes have been investigated in [17].

6. Penrose limits of intersecting branes

The methods developed to investigate the Penrose limits of branes can be easily adapted to investigate the Penrose limit of intersecting brane solutions [30, 31, 32]. Since the solutions for intersecting branes localised at the same point in the overall transverse space are of cohomogeneity one, the same symmetry considerations that have been employed for branes apply to classify all possible directions of null geodesics. There are three types depending on the direction V of the null geodesic at a point. The tangent bundle of an intersecting brane solution can split as $T_PM=T_PB\oplus T_PR\oplus T_PS$, where T_PB spans the common intersection and relative transverse directions, T_PR spans the direction along the radial coordinate and T_PS spans the directions along the overall transverse sphere. Thus we have to consider the cases of (i) longitudinal null geodesics, (ii) radial null geodesics and (iii) generic null geodesics whose definitions are parallel to those in the case of branes. The only subtlety is that the isotropy group of the point P does not act transitively on the timelike directions in $T_P B$. This complicates the classification of Penrose limits, as we have to distinguish between different classes of generic null geodesics, depending on whether the velocity vector has a component (or not) along each of relative transverse directions. We will only consider Penrose limits along those generic null geodesics whose velocity component in TB is tangent to the common intersection. The general case is straightforward.

It can be easily seen that the Penrose limit of an intersecting brane solution along longitudinal null geodesics is Minkowski spacetime; the form-field strengths vanish and the scalars are constant. So it remains to investigate Penrose limits along the radial null geodesics and the generic null geodesics. There is a large number of intersecting brane configurations. Therefore instead of doing the analysis for each case separately, we shall focus on general formulae adapted to the types of solution associated with intersecting branes. Then we give some representative examples associated with supersymmetric intersecting M-branes.

6.1. **General formulae for intersecting branes.** We begin with some general formulae for the Penrose limits of intersecting branes along radial and generic null geodesics.

6.1.1. *Radial null geodesics.* The metric of a typical intersecting brane solution is

$$ds^2 = A^2 ds^2(\mathbb{E}^{1,p}) + \sum_{i=1}^{k} A_i^2 ds_i^2(\mathbb{E}^{n_i}) + B^2 ds^2(\mathbb{E}^{D-p-n-1}) \tag{6.1}$$

where $\mathbb{E}^{1,p}$ are the coordinates along the common intersection, $\mathbb{E}^{n_i}$ are the relative transverse spaces, $\mathbb{E}^{D-p-n-1}$ is the overall transverse space and $n = \sum_i n_i$. We shall consider solutions for which all components A^2, A_i^2, B^2 depend on the radial coordinate r of the overall transverse space B^2. To continue we rewrite the metric as

$$ds^2 = A^2 ds^2(\mathbb{E}^{1,p}) + \sum_{i=1}^{k} A_i^2 ds_i^2(\mathbb{E}^{n_i}) + B^2(dr^2 + r^2 ds^2(S^{D-p-n-2})) \ . \tag{6.2}$$

The investigate the Penrose limit for radial null geodesics for intersecting branes is similar to that for the standard brane solutions investigated in the previous sections. The relevant part of the metric is again the two-dimensional metric

$$ds^2_{(2)} = -A^2 dt^2 + B^2 dr^2 \ . \tag{6.3}$$

After performing a coordinate transformation similar to that in the case of branes and taking the Penrose limit, we find that the metric becomes

$$ds^2 = 2dudv + A^2 ds^2(\mathbb{E}^p) + \sum_{i=1}^{k} A_i^2 ds_i^2(\mathbb{E}^{n_i}) + B^2 r^2 ds^2(\mathbb{E}^{D-p-n-2}) \tag{6.4}$$

where the components of the metric depend on u, and u is related to the radial coordinate r by

$$\frac{du}{dr} = A^2 B^2 \tag{6.5}$$

To put the metric in the standard pp-wave form, we write $ds^2(\mathbb{E}^p) = \sum_a (d\tilde{x}^a)^2$, $ds^2(\mathbb{E}^{n_i}) = \sum_{a_i} (d\tilde{x}_i^{a_i})^2$ and $ds^2 = \sum_p (d\tilde{y}^p)^2$ and perform the coordinate transformation

$$\begin{aligned}
u &= x^- \\
v &= x^+ + \tfrac{1}{2}\frac{\partial_- A(x^-)}{A(x^-)}x^2 + \tfrac{1}{2}\sum_i \frac{\partial_- A_i(x^-)}{A_i(x^-)}x_i^2 + \tfrac{1}{2}\frac{\partial_-\big(r(x^-)B(x^-)\big)}{r(x^-)B(x^-)} \\
\tilde{x}^a &= \frac{1}{A(x^-)}x^a \\
\tilde{x}_i^{a_i} &= \frac{1}{A_i(x^-)}x_i^{a_i} \\
\tilde{y}^p &= \frac{1}{r(x^-)B(x^-)}y^p \ ,
\end{aligned} \tag{6.6}$$

where $\partial_- = \frac{d}{dx^-}$. The metric in the new coordinate system is

$$ds^2 = 2dx^+dx^- + \mathcal{A}(x^-, x, x_i, y)(dx^-)^2 + ds^2(\mathbb{E}^{D-2}) \ , \tag{6.7}$$

where

$$\mathcal{A}(x^-, x, x_i, y) = \frac{\partial_-^2 A(x^-)}{A(x^-)}x^2 + \sum_i \frac{\partial_-^2 A_i(x^-)}{A_i(x^-)}x_i^2 + \frac{\partial_-^2\left(r(x^-)B(x^-)\right)}{r(x^-)B(x^-)}y^2 \ . \tag{6.8}$$

Again, it is sometimes complicated to express explicitly the non-trivial component of the metric in the Penrose limit in terms of the x^- coordinate. This is because it is difficult to find the explicit expression for the transformation $r = r(x^-)$. However using the chain rule, it is straightforward to express $\mathcal{A}$ in terms of the original r coordinate as follows:

$$\begin{aligned}
\mathcal{A}(r, x, x_i, y) = &\left[\frac{\partial_r^2 A}{A^3B^2} - \frac{(\partial_r A)^2}{A^4B^2} - \frac{\partial_r A \partial_r B}{A^3B^3}\right]x^2 \\
&+ \sum_i \left[\frac{\partial_r^2 A_i}{A^2B^2A_i} - \frac{\partial_r(AB)\partial_r A_i}{A^3B^3A_i}\right]x_i^2 \\
&+ \left[\frac{\partial_r^2 B}{A^2B^3} + \frac{\partial_r B}{rA^2B^3} - \frac{\partial_r A}{rA^3B^2} - \frac{(\partial_r B)^2}{A^2B^4} - \frac{\partial_r A\partial_r B}{A^3B^3}\right]y^2 \ .
\end{aligned} \tag{6.9}$$

6.1.2. *Generic null geodesics.* We now consider the Penrose limit of intersecting brane solutions along generic null geodesics; that is, geodesics whose tangent vectors have a component tangent to the overall transverse sphere. The investigation of the Penrose limit along such geodesics is similar to that explained for brane solutions. So we shall not elaborate. As mentioned above, we will only consider null geodesics whose velocity component in TB is tangent to the common intersection.

First we write the metric of the intersecting brane solution as follows:

$$\begin{aligned} ds^2 = A^2\left(-dt^2 + ds^2(\mathbb{E}^p)\right) + \sum_i A_i^2(r) ds_i^2(\mathbb{E}^{n_i}) + B^2 dr^2 \\ + B^2 r^2 \left(d\psi^2 + (\sin\psi)^2 d\Omega^2_{D-p-3}\right) \ . \end{aligned} \tag{6.10}$$

Then we consider null geodesics in the (t, r, ψ) space with induced metric

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + B^2 r^2 d\psi^2 \ . \tag{6.11}$$

and change coordinates to (u, v, z) adapted to the null geodesic as

$$u = u(r) \qquad v = t + \ell\psi + a(r) \qquad z = \psi + b(r) \ , \tag{6.12}$$

where ℓ is a constant, and such that the metric takes the form

$$ds^2 = 2dudv + K dv^2 + L dv dz + M dz^2 \ . \tag{6.13}$$

Since the relevant three dimensional metric is the same as that of Penrose limits for branes, the various formulae for the coordinate transformations are the same. The final result is

$$ds^2 = 2dx^+ dx^- + \mathcal{A}(x^-, x, y, z)(dx^-)^2 + ds^2(\mathbb{E}^p) + ds^2(\mathbb{E}^{D-p-3}) + dz^2 \ , \tag{6.14}$$

where and

$$\begin{aligned} \mathcal{A}(x^-, x, x_i, y, z) = & \frac{\partial_-^2 A(x^-)}{A(x^-)} x^2 + \sum_i \frac{\partial_-^2 A_i(x^-)}{A_i(x^-)} x_i^2 \\ & + \frac{\partial_-^2 \left(r(x^-)B(x^-)\sin b\right)}{r(x^-)B(x^-)\sin b} y^2 + \frac{\partial_-^2 \sqrt{B^2 r^2 - \ell^2 A^2}}{\sqrt{B^2 r^2 - \ell^2 A^2}} z^2 \ . \end{aligned} \tag{6.15}$$

As in (5.18) we can express the non-trivial component of the above metric as a function of the original coordinate r using the formula

$$\partial_-^2 f(x^-) = -Q^{-3} \partial_r Q \partial_r f(r) + Q^{-2} \partial_r^2 f(r) \ , \tag{6.16}$$

where Q was defined in (5.61). The resulting metric is

$$ds^2 = 2dx^+ dx^- + \mathcal{A}(r, x, y, z)(dx^-)^2 + ds^2(\mathbb{E}^{D-2}) \tag{6.17}$$

where

$$\begin{aligned}\mathcal{A}(r,x,y,z) = &(B^2r^2-\ell^2A^2)^{-\frac{1}{2}}\big[-Q^{-3}\partial_r Q\partial\sqrt{B^2r^2-\ell^2A^2}\\ &+Q^{-2}\partial_r^2\sqrt{B^2r^2-\ell^2A^2}\big]z^2\\ &+A^{-1}\big[-Q^{-3}\partial_r Q\partial_r A+Q^{-2}\partial_r^2 A\big]x^2\\ &+\sum_i A_i^{-1}\big[-Q^{-3}\partial_r Q\partial_r A_i+Q^{-2}\partial_r^2 A_i\big]x_i^2\\ &+(Br\sin b)^{-1}\big[-Q^{-3}\partial_r Q\partial_r(Br\sin b)+Q^{-2}\partial_r^2(Br\sin b)\big]y^2\end{aligned} \tag{6.18}$$

As in the case of branes, there does not seem to be a straightforward way to simplify this equation. We remark though that in the limit that $\ell \to 0$, the above calculations and formulae reduce to those we have found for the radial null geodesics. In addition for those intersecting brane configurations for which the near horizon limit is AdS $\times S\times \mathbb{E}$, the Penrose limit along a generic geodesics is $P\times \mathbb{E}$ where P is the Penrose limit of the associated AdS $\times S$ spacetime.

6.2. **Three M2-branes intersecting on a 0-brane.** The intersecting brane configuration that we shall consider is

$$\begin{array}{cccccccc} M2 & 0 & 1 & 2 & & & & \\ M2 & 0 & & & 3 & 4 & & \\ M2 & 0 & & & & & 5 & 6 \end{array} \tag{6.19}$$

Associating a harmonic function with each brane involved in the configuration, the spacetime solution is

$$\begin{aligned} ds^2 &= H_1^{\frac{1}{3}}H_2^{\frac{1}{3}}H_3^{\frac{1}{3}}\big(-H_1^{-1}H_2^{-1}H_3^{-1}dt^2+H_1^{-1}ds_1(\mathbb{E}^2)\\ &\quad +H_2^{-1}ds_2(\mathbb{E}^2)+H_3^{-1}ds_3(\mathbb{E}^2)+ds^2(\mathbb{E}^4)\big)\\ F_4 &= dt\wedge[\mathrm{dvol}_1(\mathbb{E}^2)\wedge dH_1^{-1}+\mathrm{dvol}_2(\mathbb{E}^2)\wedge dH_2^{-1}+\mathrm{dvol}_3(\mathbb{E}^2)\wedge dH_3^{-1}]\end{aligned} \tag{6.20}$$

where $H_i = 1+\frac{|Q_i|}{r^2}$ are harmonic functions on $\mathbb{E}^4$.

In this case $(a+b)' = H_1^{-\frac{1}{6}}H_2^{-\frac{1}{6}}H_3^{-\frac{1}{6}}$. The transformations $r=r(u+v)$ cannot be found explicitly but the non-trivial component of the associated pp-wave metric is given by (6.9) for

$$
\begin{aligned}
A &= H_1^{-\frac{1}{3}} H_2^{-\frac{1}{3}} H_3^{-\frac{1}{3}} \\
A_1 &= H_1^{-\frac{1}{3}} H_2^{\frac{1}{6}} H_3^{\frac{1}{6}} \\
A_2 &= H_1^{\frac{1}{6}} H_2^{-\frac{1}{3}} H_3^{\frac{1}{6}} \\
A_3 &= H_1^{\frac{1}{6}} H_2^{\frac{1}{6}} H_3^{-\frac{1}{3}} \\
B &= H_1^{\frac{1}{6}} H_2^{\frac{1}{6}} H_3^{\frac{1}{6}}
\end{aligned}
\tag{6.21}
$$

The calculation can be performed in a closed form in the special case where $H_1 = H_2 = H_3 = H$. The metric in this case is

$$
ds^2 = -H^{-2}dt^2 + ds^2(\mathbb{E}^2) + ds^2(\mathbb{E}^2) + ds^2(\mathbb{E}^2) + H ds^(\mathbb{E}^4) \tag{6.22}
$$

where $H = 1 + \frac{|Q|}{r^2}$. In angular coordinates this metric can be written as

$$
ds^2 = -H^{-2}dt^2 + ds^2(\mathbb{E}^2) + ds^2(\mathbb{E}^2) + ds^2(\mathbb{E}^2) + H(dr^2 + r^2 ds^2(S^3)) \tag{6.23}
$$

To take the Penrose limit again we have to change coordinates as in the case of branes from (t, r) to (u, v). This is easily done in this case because

$$
\frac{d}{dr}(u+v) = \pm H^{-\frac{1}{2}} \tag{6.24}
$$

and so $(u+v)^2 = r^2 + |Q|$. The metric in the Penrose limit is

$$
ds^2 = 2dudv + ds^2(\mathbb{E}^2) + ds^2(\mathbb{E}^2) + ds^2(\mathbb{E}^2) + u^2 ds^2(\mathbb{E}^3) \tag{6.25}
$$

This in fact is the Minkowski metric in eleven dimension. The Penrose limit for the near horizon geometry of (6.20) is again the Minkowski spacetime.

Now instead of taking the Penrose limit along a radial null geodesic, we take the Penrose limit of the same configuration along a generic null geodesic at the near horizon limit. The near horizon geometry of this intersecting brane solution is $\mathrm{AdS}_2 \times S^3 \times \mathbb{E}^6$. The Penrose limit of this geometry is $\mathrm{CW}_5 \times \mathbb{E}^6$, where CW_5 is the Cahen-Wallace space associated which is the Penrose limit of $\mathrm{AdS}_2 \times S^3$.

6.3. **Two M2-branes and two M5-branes intersection on a 0-brane.** The intersecting brane configuration that we shall consider is

$M2$	0	1	2					
$M2$	0			3	4			
$M5$	0	1		3		5	6	7
$M5$	0		2		4	5	6	7

(6.26)

Associating a harmonic function with each brane involved in the configuration, the spacetime solution is

$$\begin{aligned} ds^2 &= H_1^{\frac{1}{3}} H_2^{\frac{1}{3}} H_3^{\frac{2}{3}} H_4^{\frac{2}{3}} \big(- H_1^{-1} H_2^{-1} H_3^{-1} H_4^{-1} dt^2 + H_1^{-1} H_3^{-1} ds_1^2(\mathbb{E}) \\ &\quad + H_1^{-1} H_4^{-1} ds_2^2(\mathbb{E}) + H_2^{-1} H_3^{-1} ds_3(\mathbb{E}) + H_2^{-1} H_4^{-1} ds_4(\mathbb{E}) \\ &\quad + H_3^{-1} H_4^{-1} ds^2(\mathbb{E}^3) + ds^2(\mathbb{E}^3)\big) \\ F_4 &= dt \wedge [\mathrm{dvol}_1(\mathbb{E}^2) \wedge dH_1^{-1} + \mathrm{dvol}_2(\mathbb{E}^2) \wedge dH_2^{-1} + \mathrm{dvol}_3(\mathbb{E}^2) \wedge dH_3^{-1}] \end{aligned} \tag{6.27}$$

where $H_i = 1+\frac{Q_i}{r}$ are harmonic functions on $\mathbb{E}^3$ and $\star$ is the Hodge operation in $\mathbb{E}^3$.

In this case $(a+b)' = H_1^{-\frac{1}{6}} H_2^{-\frac{1}{6}} H_3^{\frac{1}{}} H_4^{\frac{1}{}}$. The transformations $r = r(u+v)$ cannot be found explicitly as in the previous intersecting branes example. However we can express the non-trivial component of the Penrose limit metric using (6.9). In particular, we take

$$\begin{aligned} A &= H_1^{-\frac{1}{3}} H_2^{-\frac{1}{3}} H_3^{-\frac{1}{6}} H_4^{-\frac{1}{6}} \\ A_1 &= H_1^{-\frac{1}{3}} H_2^{\frac{1}{6}} H_3^{-\frac{1}{6}} H_4^{\frac{1}{3}} \\ A_2 &= H_1^{-\frac{1}{3}} H_2^{\frac{1}{6}} H_3^{\frac{1}{3}} H_4^{-\frac{1}{6}} \\ A_3 &= H_1^{\frac{1}{6}} H_2^{-\frac{1}{3}} H_3^{-\frac{1}{6}} H_4^{\frac{1}{3}} \\ A_4 &= H_1^{\frac{1}{6}} H_2^{-\frac{1}{3}} H_3^{\frac{1}{3}} H_4^{-\frac{1}{6}} \\ A_5 &= H_1^{\frac{1}{6}} H_2^{\frac{1}{6}} H_3^{-\frac{1}{6}} H_4^{-\frac{1}{6}} \\ B &= H_1^{\frac{1}{6}} H_2^{\frac{1}{6}} H_3^{\frac{1}{3}} H_4^{\frac{1}{3}} \end{aligned} \tag{6.28}$$

The calculation can be performed in a closed form in the special case where $H_1 = H_2 = H_3 = H_4 = H$. (In fact it suffices to take $H_1 = H_3$ and $H_2 = H_4$.) The metric in eleven dimensions is

$$ds^2 = -H^{-2} dt^2 + ds^2(\mathbb{E}^7) + H^2 ds^2(\mathbb{E}^3) \tag{6.29}$$

where $H = 1 + \frac{|Q|}{r}$. Changing coordinates from (t, r) to (u, v), we find that $u + v = \pm r$. Taking the limit, we find the metric

$$ds^2 = 2dudv + ds^2(\mathbb{E}^7) + u^2(1 + \frac{|Q|}{u})^2 ds^2(\mathbb{E}^2) \tag{6.30}$$

Again this is the flat Minkowski eleven-dimensional metric. The Penrose limit of the associated near horizon geometry is also the Minkowski spacetime.

Now instead of taking the Penrose limit along a radial null geodesic, we take the Penrose limit of the same configuration along a generic null geodesic at the near horizon limit. The near horizon geometry of this intersecting brane solution is $\mathrm{AdS}_2 \times S^2 \times \mathbb{E}^7$. The Penrose limit of this geometry is

$\mathrm{CW}_4 \times \mathbb{E}^7$, where CW_4 is the Cahen-Wallace space associated which is the Penrose limit of $\mathrm{AdS}_2 \times S^2$.

7. Penrose limits of supersymmetric black holes and strings in lower dimensions

We shall mainly focus on the Penrose limits of supersymmetric black holes that arise in toroidal compactifications of M- and string theories. These solutions are again of cohomogeneity one. For black holes, there are two types of null geodesics to consider the following: (i) radial null geodesics and (ii) generic null geodesics. The Penrose limit and various formulae in this case are similar to those for $p = 0$ branes. In what follows we shall mainly focus on the metric of these black holes.

For string solutions that arise in toroidal compactifications of M- and string theories there are three different choices of null geodesics. These are precisely those that occur for the p-brane, $p > 0$, solutions.

7.1. **Five-dimensional black holes.**

The five-dimensional black holes that arise in toroidal compactifications of M- and string theories can be thought off as reduction to five-dimensions of intersecting brane configurations in ten and eleven dimensions [30, 33]. We shall use the form of such four- and five-dimensional black hole solutions as summarised in [34]. The relevant five-dimensional black hole metric can be written in the form

$$ds^2 = -(H_1H_2H_3)^{-\frac{2}{3}}dt^2 + (H_1H_2H_3)^{\frac{1}{3}}ds^2(\mathbb{E}^4) \ , \tag{7.1}$$

where $H_i = 1 + \frac{Q_i}{r^2}$. The harmonic functions H_i are inherited from those of the intersecting brane configuration in ten or eleven dimensions.

For the black hole metric above, the Penrose limit along a radial null geodesic cannot be given in a closed form because the coordinate transformation $u = u(r)$ is not known explicitly. However, the non-trivial component of the Penrose limit metric in the pp-wave form can expressed in terms of the radial coordinate r. The metric is

$$\begin{aligned} ds^2 = 2dx^+dx^- + \big[\tfrac{1}{6}(H_1H_2H_3)^{-\frac{2}{3}}\partial_r^2(H_1H_2H_3) \\ + \frac{1}{2r}(H_1H_2H_3)^{-\frac{2}{3}}\partial_r(H_1H_2H_3) \\ - \tfrac{1}{9}(H_1H_2H_3)^{-\frac{5}{3}}(\partial_r(H_1H_2H_3))^2\big]y^2(dx^-)^2 + ds^2(\mathbb{E}^3) \ . \end{aligned} \tag{7.2}$$

The near horizon geometry of this black hole is $\mathrm{AdS}_2 \times S^3$. Thus the Penrose limit in this case is (i) five-dimensional Minkowski spacetime if it is taken along a radial null geodesic or (ii) CW_5 if it is taken along a generic null geodesic.

7.2. **Four-dimensional black holes.** The relevant metric of four-dimensional black holes which arise in toroidal M- and string theory compactifications is

$$ds^2 = -(H_1H_2H_3H_4)^{-2}dt^2 + (H_1H_2H_3H_4)^2 ds^2(\mathbb{E}^3)\ , \tag{7.3}$$

where $H_i = 1 + \frac{Q_i}{r}$.

In this case it is easy to see that if the limit is taken along a radial null geodesic $u = \pm r$. So the Penrose limit metric can be given explicitly. In particular, we find that

$$\begin{aligned} ds^2 = 2dx^+dx^- + \big[&(H_1H_2H_3H_4)^{-1}\partial_-^2(H_1H_2H_3H_4) \\ &+ (x^-H_1H_2H_3H_4)^{-1}\partial_-(H_1H_2H_3H_4)\big]y^2(dx^-)^2 + ds^2(\mathbb{E}^2)\ . \end{aligned} \tag{7.4}$$

The near horizon geometry of this black hole is $\mathrm{AdS}_2 \times S^2$. Thus the Penrose limit in this case is (i) four-dimensional Minkowski spacetime if it is taken along a radial null geodesic or (ii) CW_4 if it is taken along a generic null geodesic.

7.3. **A string in six dimensions.** A string solution in six dimensions that arise in toroidal M- and string theory compactifications is

$$ds^2 = (H_1H_2)^{-\frac{1}{2}} ds^2(\mathbb{E}^2) + (H_1H_2)^{\frac{1}{2}} ds^2(\mathbb{E}^4)\ , \tag{7.5}$$

where $H_i = 1 + \frac{Q_i}{r^2}$. The self-dual string solution arises for $H_1 = H_2$ [21].

For Penrose limits along longitudinal null geodesics, the geometry at the limit is six-dimensional Minkowski spacetime. In the case of radial null geodesics, $u = \pm r = x^-$ and so the Penrose limit metric can be given explicitly. In particular we find

$$\begin{aligned} ds^2 = 2dx^+dx^- + \big[\big(&\frac{5}{16}(H_1H_2)^{-2}(\partial_-(H_1H_2))^2 - \frac{1}{4}(H_1H_2)^{-1}\partial_-^2(H_1H_2)\big)x^2 \\ -\frac{3}{16}(H_1H_2)^{-2}&(\partial_-(H_1H_2))^2 + \frac{1}{4}(H_1H_2)^{-1}\partial_-^2(H_1H_2)\big)y^2\big](dx^-)^2 + ds^2(\mathbb{E}^4)\ . \end{aligned} \tag{7.6}$$

The near horizon geometry of this string solution is $\mathrm{AdS}_3 \times S^3$. Thus the Penrose limit in this case is (i) six-dimensional Minkowski spacetime if it is taken along a radial null geodesic or (ii) CW_6 if it is taken along a generic null geodesic. This is the solution obtained in [6].

8. Cosmological Penrose limits

So far we have considered Penrose limits of space-times corresponding to (intersecting) branes and their near horizon limits. But of course the Penrose limit construction is more general than that. We will now consider Penrose limits of cosmological Friedmann-Robertson-Walker (FRW) space-times. For definiteness we consider four-dimensional models but qualitatively nothing changes in other dimensions.

The FRW metric is

$$-dt^2 + a(t)^2 d\tilde{s}^2 \ , \tag{8.1}$$

where $d\tilde{s}^2$ is the line element of a maximally symmetric space which can be written as

$$d\tilde{s}^2 = dr^2 + f_k(r)^2 d\Omega_2^2 \ , \tag{8.2}$$

where $f_k(r) = r, \sin r, \sinh r$ for $k = 0, \pm 1$ respectively.

A FRW cosmological model is determined by specifying k, the cosmological constant Λ, and the perfect fluid matter content, usually characterised by the equation of state $p = w\rho$ relating the energy-density ρ and the pressure-density p of the perfect fluid, with $0 \leq w \leq 1$ for not too exotic matter.

The cosmic scale factor $a(t)$ is then determined by solving the Friedmann equations. Typically these equations cannot be solved in closed form when both Λ and ρ are non-zero. We will therefore set $\Lambda = 0$ (for $\Lambda \neq 0$ but $\rho = 0$ we obtain (A)dS space-times whose Penrose limits we already know).

For $k = 0$ and any w, the solution is

$$k = 0 \Rightarrow a(t) = \beta t^\alpha \ , \tag{8.3}$$

where

$$\alpha = \tfrac{2}{3}(1 + w)^{-1} \tag{8.4}$$

and β is a constant.

Without loss of generality (covariance of the Penrose limit and isotropy of space), one can choose the null geodesics to lie in the (t, r)-plane. These null geodesics are characterised by the equations (an overdot now denotes differentiation with respect to the affine parameter τ which will become the coordinate u)

$$\dot{t}^2 = a^2 \dot{r}^2 \tag{8.5}$$

$$\dot{r} = c a^{-2} \ , \tag{8.6}$$

where c is a constant which without loss of generality can be set equal to $c = 1$ by rescaling the affine parameter. This leads to

$$\dot{t} = a^{-1} \ . \tag{8.7}$$

With $a(t)$ as above, this integrates to

$$t(\tau) = T\tau^{\frac{1}{\alpha+1}} \tag{8.8}$$

$$r(\tau) = R\tau^{\frac{1-\alpha}{1+\alpha}} \tag{8.9}$$

$$a(\tau) = A\tau^{\frac{\alpha}{\alpha+1}} \ , \tag{8.10}$$

where T, R, A are constants satisfying a quadratic relationship following from (8.5) whose precise numerical values are irrelevant for the following.

Having obtained this congruence of null geodesics we now change coordinates $(t, r) \to (u, v)$ such that

$$\begin{aligned} \partial_u &= \dot{t}\partial_t + \dot{r}\partial_r \\ &= a(t)^{-1}\partial_t + a(t)^{-2}\partial_r \ , \end{aligned} \tag{8.11}$$

and such that $g_{uv} = 1$ and $[\partial_u, \partial_v] = 0$. A possible choice is

$$\partial_v = \partial_r \ . \tag{8.12}$$

This integrates to

$$t(u,v) = \int \frac{du'}{a(u')} \tag{8.13}$$

$$r(u,v) = \int \frac{du'}{a(u')^2} + v \ , \tag{8.14}$$

the FRW metric (8.1) takes the desired form

$$ds^2 = 2dudv + a(u)^2(dv)^2 + a(u)^2 r(u,v)^2 d\Omega_2^2 \ , \tag{8.15}$$

and the Penrose limit is (in Rosen coordinates)

$$ds^2 = 2dudv + a(u)^2 r(u,0)^2 ds^2(\mathbb{E}^2) \ . \tag{8.16}$$

Performing the integral defining $r(u)$ one finds that, up to an irrelevant scaling of the $\tilde{x}$-coordinates of $\mathbb{E}^2$,

$$ds^2 = 2dudv + u^{\frac{2}{1+\alpha}} ds^2(\mathbb{E}^2) \ . \tag{8.17}$$

To pass to Brinkman coordinates, one sets

$$u = x^- \tag{8.18}$$

$$v = x^+ + \tfrac{1}{2(1+\alpha)}(x^-)^{-1}x^2 \tag{8.19}$$

$$\tilde{x}^i = (x^-)^{-\frac{1}{1+\alpha}} x^i \ , \tag{8.20}$$

and then finds

$$ds^2 = 2dx^-dx^+ + A(x^-, x)(dx^-)^2 + ds^2(\mathbb{E}^2) \ , \tag{8.21}$$

where

$$A(x^-, x) = -[\tfrac{1}{1+\alpha} - \tfrac{1}{(1+\alpha)^2}](x^-)^{-2}x^2 \ . \tag{8.22}$$

We note that $A(x^-, x)$ has the characteristic $(x^-)^{-2}$-dependence also encountered in the Penrose limit of the fundamental string (5.39). For all 'physical' values of α, $1/3 \leq \alpha \leq 2/3$, $A(x^-, x)$ is negative definite. The α-dependence (i.e., the dependence on the equation of state parameter w) of this metric cannot be scaled away by scaling (x^-, x^+) because precisely for $A(x^-, x) \sim (x^-)^{-2}$, the combination $A(x^-, x)(dx^-)^2$ is scale invariant.

According to [35, §§10.2,21.5], a pp-wave with this profile is generically of type G_6 and V_4. This means that there is a six-dimensional group of isometries with four-dimensional orbits, i.e., with the isometry group acting transitively on space-time, in contrast to the FRW case which is of cohomogeneity one. Thus the limiting space-time is, while not Lorentzian symmetric, at least Lorentzian homogeneous.

Actually this particular metric has a seven-dimensional isometry group generated by the six Killing vectors inherited from the FRW space-time, and the additional Killing vector

$$x^+\partial_+ - x^-\partial_- \tag{8.23}$$

generating the isometry $(x^+, x^-) \to (cx^+, c^{-1}x^-)$, i.e., an infinitesimal boost. From the point of view of a general pp-wave spacetime, these seven Killing vectors arise as the $2D-3=5$ Heisenberg algebra Killing vectors (2.12) of a generic pp-wave, plus the above scale invariance, plus the 'accidental' rotation symmetry in the two transverse dimensions.

9. The Penrose limit and isometric embeddings

In this section we will make some preliminary remarks about Penrose limits and isometric embeddings. We will show, in the context of a toy model, that the Penrose limit of a spacetime is induced by a generalised Penrose limit in a space with two times. This hints at the existence of a generalised Penrose limit for pseudoriemannian spaces with arbitrary signature. In the process we show how to embed any Cahen–Wallach space with signature $(1, D-1)$ isometrically as the intersection of two quadrics in a flat space with signature $(2, D)$.

9.1. **A toy model.** We start with a "toy model" corresponding to the near horizon geometry of the Reissner–Nordström black hole in four-dimensional $N{=}2$ supergravity, namely $\mathrm{AdS}_2 \times S^2$ with equal radii of curvature. As discussed in Section 4, we can embed $\mathrm{AdS}_2 \times S^2$ as the intersection of two quadrics in $\mathbb{E}^{2,4}$. Indeed, this case corresponds to $p = 0$, $D = 4$ (and hence $n = 2$) in the notation of Section 4.1. An explicit parametrisation is given by

$$\begin{aligned} X^0 &= R\cos\tau\ , \quad X^1 = R\sinh\beta\sin\tau \quad \text{and} \quad X^2 = R\cosh\beta\sin\tau \\ X^3 &= R\cos\psi\ , \quad X^4 = R\sin\psi\cos\theta \quad \text{and} \quad X^5 = R\sin\psi\sin\theta\ , \end{aligned} \tag{9.1}$$

where R is the common radius of curvature of the two spaces. The Penrose limit starts by rescaling the coordinates as follows

$$\psi = \tfrac{1}{2}(u + \Omega^2 v)\ , \quad \tau = \tfrac{1}{2}(u - \Omega^2 v)\ , \quad \theta = \Omega y^1 \quad \text{and} \quad \beta = \Omega y^2\ , \tag{9.2}$$

in terms of which the ambient coordinates develop a dependence on the scaling parameter Ω, as follows

$$\begin{aligned} X^0(\Omega) &= R\cos\tfrac{1}{2}(u-\Omega^2 v)\\ X^1(\Omega) &= R\sinh(\Omega y^2)\sin\tfrac{1}{2}(u-\Omega^2 v)\\ X^2(\Omega) &= R\cosh(\Omega y^2)\sin\tfrac{1}{2}(u-\Omega^2 v)\\ X^3(\Omega) &= R\cos\tfrac{1}{2}(u+\Omega^2 v)\\ X^4(\Omega) &= R\cos(\Omega y^1)\sin\tfrac{1}{2}(u+\Omega^2 v)\\ X^5(\Omega) &= R\sin(\Omega y^1)\sin\tfrac{1}{2}(u+\Omega^2 v)\ . \end{aligned} \tag{9.3}$$

This suggests the following rescaling of the ambient coordinates

$$\begin{aligned} X^3+X^0 &= U^1 & X^4+X^2 &= U^2\\ X^3-X^0 &= \Omega^2 V^1 & X^4-X^2 &= \Omega^2 V^2\ ,\\ X^1 &= \Omega Y^1 & X^5 &= \Omega Y^2 \end{aligned} \tag{9.4}$$

which induces a homothety of the flat metric in $\mathbb{E}^{2,4}$. The new coordinates $\{U^\mu, V^\mu, Y^i\}$ depend on Ω. To leading order in Ω the embedding becomes

$$\begin{aligned} U^1 &= 2R\cos\tfrac{1}{2}u & U^2 &= 2R\sin\tfrac{1}{2}u\\ V^1 &= -Rv\sin\tfrac{1}{2}u & V^2 &= R\left(v\cos\tfrac{1}{2}u - \tfrac{1}{2}|y|^2\sin\tfrac{1}{2}u\right)\\ Y^1 &= Ry^1\sin\tfrac{1}{2}u & Y^2 &= Ry^2\sin\tfrac{1}{2}u\ , \end{aligned} \tag{9.5}$$

where $|y|^2 = (y^1)^2 + (y^2)^2$ and where all the omitted terms go to zero as $\Omega \to 0$. In the limit we obtain an embedded submanifold of $\mathbb{E}^{2,4}$ with ambient coordinates $\{U^\mu, V^\mu, Y^i\}$ and metric

$$G = dU^1 dV^1 + dU^2 dV^2 + (dY^1)^2 + (dY^2)^2\ , \tag{9.6}$$

corresponding to the intersection of two quadrics

$$(U^1)^2 + (U^2)^2 = 4R^2 \quad \text{and} \quad (Y^1)^2 + (Y^2)^2 + U^1V^1 + U^2V^2 = 0\ . \tag{9.7}$$

The induced metric is none other but a Cahen–Wallach Hpp-wave (in Rosen coordinates)

$$R^{-2}\bar{g} = dudv + (\sin\tfrac{1}{2}u)^2\left((dy^1)^2 + (dy^2)^2\right)\ . \tag{9.8}$$

In terms of Brinkman coordinates, the embedding takes the simpler form

$$\begin{aligned} U^1 &= 2R\cos x^- & U^2 &= 2R\sin x^-\\ V^1 &= -Rx^+\sin x^- - \tfrac{1}{2}R|x|^2\cos x^- & V^2 &= Rx^+\cos x^- - \tfrac{1}{2}R|x|^2\sin x^-\\ Y^1 &= Rx^1 & Y^2 &= Rx^2\ , \end{aligned} \tag{9.9}$$

where $|x|^2 = (x^1)^2 + (x^2)^2$ and the induced metric is

$$R^{-2}\bar{g} = 2dx^+dx^- - |x|^2(dx^-)^2 + (dx^1)^2 + (dx^2)^2\ . \tag{9.10}$$

9.2. **Isometric embeddings of Cahen–Wallach spaces.** This toy model teaches us how to embed a general D-dimensional Cahen–Wallach metric (Hpp-wave with constant A_{ij}) isometrically[6] in $\mathbb{E}^{2,D}$. Indeed, consider the metric

$$g = 2dx^+dx^- + A(x)(dx^-)^2 + \sum_i dx^i dx^i \ , \tag{9.11}$$

where $A(x) = \sum_{i,j} A_{ij} x^i x^j$, where A_{ij} is a constant symmetric matrix, not necessarily positive-definite or even non-degenerate. Let us introduce coordinates U^μ, V^μ, Y^i for $\mu = 1, 2$ and $i = 1, 2, \ldots, D-2$ for $\mathbb{E}^{2,D}$ in terms of which the flat metric is written as

$$G = \sum_{\mu=1}^{2} dU^\mu dV^\mu + \sum_{i=1}^{D-2} (dY^i)^2 \ . \tag{9.12}$$

Consider the codimension-two submanifold of $\mathbb{E}^{2,D}$ cut out by the intersection of the two quadrics

$$\sum_{\mu=1}^{2} (U^\mu)^2 = 4 \qquad \text{and} \qquad A(Y) - \sum_{\mu=1}^{2} U^\mu V^\mu = 0 \ , \tag{9.13}$$

where $A(Y) = \sum_{i,j} A_{ij} Y^i Y^j$. We can parametrise this submanifold in terms of coordinates $x^\pm, x^i$ for $i = 1, 2, \ldots, D-2$ as follows

$$\begin{gathered} U^1 = 2\cos x^- \qquad U^2 = 2\sin x^- \qquad Y^i = x^i \\ V^1 = -x^+ \sin x^- + \tfrac{1}{2} A(x) \cos x^- \qquad V^2 = x^+ \cos x^- + \tfrac{1}{2} A(x) \sin x^- \ , \end{gathered} \tag{9.14}$$

and one sees that the induced metric coincides with (9.11).

9.3. **A generalised Penrose limit.** The above toy model teaches us another thing, namely it suggests the existence of a generalisation of the Penrose limit in spaces with two times. (In fact, it should clear from our discussion below that this admits a straight-forward generalisation to spaces with more than two times.) Consider the following diagram illustrating the relationships between $\mathrm{AdS}_2 \times S^2$, its Penrose limit denoted by CW_4 and the

[6]The existence of this embedding is stated in [18], but the suggested embedding in that paper does not seem to work. The isometric embedding of pp-waves is itself not new. It has been discussed in the four-dimensional context in [36, 37] and is reviewed in [35, §32].

embedding space

$$\begin{array}{ccc} \mathbb{E}^{2,4} & & \mathbb{E}^{2,4} \\ \uparrow & & \uparrow \\ \mathrm{AdS}_2 \times S^2 & \xrightarrow{\text{Penrose limit}} & \mathrm{CW}_4 \end{array} \tag{9.15}$$

where the vertical arrows are the isometric embeddings described above. A natural question is whether there exists some limiting procedure in $\mathbb{E}^{2,4}$ which induces the Penrose limit when restricted to $\mathrm{AdS}_2 \times S^2$. In other words,

Is there an arrow $\mathbb{E}^{2,4} \xrightarrow{?} \mathbb{E}^{2,4}$ *which completes the above diagram?*

From the discussion above it seems clear that this question has a positive answer. This limiting procedure involves, not a null geodesic as in lorentzian signature, but rather a maximally isotropic totally geodesic submanifold—in this case, a maximally isotropic plane.[7]

We can choose coordinates U^μ, V^μ, Y^i for $\mu = 1, 2$ and $i = 1, 2, \ldots, D-2$ for $\mathbb{E}^{2,D}$ in a neighbourhood of a totally geodesic, maximally isotropic submanifold in such a way that the flat metric of $\mathbb{E}^{2,D}$ takes the form

$$G = \sum_\mu dV^\mu \left(dU^\mu + \sum_\nu \alpha_{\mu\nu} dV^\nu + \sum_i \beta_{\mu i} dY^i \right) + \sum_{i,j} C_{ij} dY^i dY^j \ , \tag{9.16}$$

where α, β and C can depend in principle on all the coordinates. We also demand that C be positive-definite; although this may limit the domain of validity of this coordinate system as with the original Penrose limit. The coordinates U^μ parametrise a family of maximally isotropic surfaces labelled by V^μ and Y^i. The vectors $\partial/\partial U^\mu$ are null, orthogonal and self-parallel, hence geodetic. We can now rescale the coordinates in the following way:

$$U^\mu = u^\mu \ , \quad V^\mu = \Omega^2 v^\mu \quad \text{and} \quad Y^i = \Omega y^i \ . \tag{9.17}$$

Substituting this into the metric G yields a metric which depends on Ω and such that the limiting metric

$$\overline{G} = \lim_{\Omega \to 0} \Omega^{-2} G(\Omega) \tag{9.18}$$

[7] An isotropic submanifold is one on which the restriction of the metric is identically zero, and a maximally isotropic submanifold is an isotropic submanifold of maximal dimension. In signature (p, q) a maximally isotropic submanifold has dimension $\min\{p, q\}$, whence in $(2, D)$ signature it has dimension 2 (for $D \geq 2$).. We will focus on this case, as it is the relevant case for our discussion.

is well defined. Taking the limit we find

$$\overline{G} = \sum_{\mu} du^\mu dv^\mu + \sum_{i,j} \overline{C}_{ij}(u) dy^i dy^j \ . \qquad (9.19)$$

Now suppose that M is an isometrically embedded submanifold of $\mathbb{E}^{2,D}$ with signature $(1, D-1)$. A maximally isotropic plane will cut M generically in a null curve γ, if they meet at all. One might be surprised at this fact, since a two-plane and a codimension-two submanifold will meet generically at a point, if they meet at all; but in this case, since the surface is maximally isotropic and the manifold has lorentzian signature, they must have a direction in common at any point in their intersection. If the curve γ is a null geodesic, then the generalised Penrose limit of $\mathbb{E}^{2,D}$ induces the Penrose limit of M along γ.

We saw this above for the case of $\mathrm{AdS}_2 \times S^2$ embedded in $\mathbb{E}^{2,4}$. In this case, the maximally isotropic surface was an affine plane. The same procedure works also for the Penrose limit of $\mathrm{AdS}_3 \times S^3$, which is the near horizon geometry of the self-dual string in $D = 6$ $(2,0)$ supergravity, and for $\mathrm{AdS}_5 \times S^5$, which is the near horizon geometry of the D3 brane in IIB supergravity. It is conceivable that this also works in more generality and if so, we will discuss this elsewhere.

A special case of this generalised Penrose limit has appeared in [13] in the context of exact string backgrounds given by WZW models on non-semisimple Lie groups admitting a bi-invariant metric (see, e.g., [38]). As shown in [13], the Lie algebras of some of these Lie groups can be obtained by a contraction of semisimple Lie algebras. This contraction is a special case of the generalised Penrose limit alluded to above. Indeed, let G be a compact simple Lie group and $H \subset G$ a subgroup. The Cartan–Killing form defines on G a bi-invariant metric g, which restricts nondegenerately to a bi-invariant metric $h = g\big|_H$ on H. Now consider the product group $G \times H$ with the product metric $g \oplus -h$, where we have changed the sign of the metric in the second factor. We remark that this metric is again bi-invariant. Consider now the submanifold $H \subset H \times H \subset G \times H$ given by the diagonal embedding. Since it is a Lie subgroup, it is totally geodesic (in fact, relative to any bi-invariant metric), and by virtue of our choice of metric it is maximally isotropic. If we now perform the generalised Penrose limit of $G \times H$ along H we obtain a non-semisimple Lie group with a bi-invariant metric. This is essentially the construction in [13], which, as shown in [39], is an example of a more general construction of Lie groups with bi-invariant metrics first discussed in [40].

10. Worldvolume dynamics and Penrose limits

The effect of the Penrose limit on the dynamics of brane probes with worldvolume and Wess-Zumino couplings, like the fundamental string [41] and the M2-brane [42], has been investigated in [3]. It was found that the

Penrose limit is a large tension limit for the probe. Here we shall extend this analysis for branes with fields of Born-Infeld type, such as D-branes and the M5-brane.

The Dp-brane tension in terms string units as a function of p is $T_p = \frac{1}{k_p(\alpha')^{\frac{p+1}{2}}}$ for some constant k_p which depends of the string coupling constant. The bosonic part of a typical action of a Dp-brane in a curved background [43, 44, 45, 46] is

$$I_p[g,B,C] = T_p \left(\int d^{p+1}\sigma e^{-\Phi} \sqrt{g+\mathcal{F}} + \int e^{\mathcal{F}} \wedge C \right) , \tag{10.1}$$

with $\mathcal{F} = \alpha' F + B$ and $C = \sum_k C_k$, where F is the Born–Infeld two-form field strength, B is the NSNS two-form gauge potential and C_k are the RR k-form gauge potentials. Next we set $\alpha' \rightsquigarrow \Omega^2 \alpha'$. Assuming that the Born–Infeld field F scales as $F \rightsquigarrow F$, the Dp-brane action can be written as

$$\begin{aligned} I_p[\Omega^{-2}g, \Omega^{-2}B, \Omega^{-k}C] = \frac{1}{k_p(\alpha')^{\frac{p+1}{2}}} \Bigg(\int d^{p+1}\sigma e^{-\Phi} \sqrt{\Omega^{-2}(g+B)+\alpha' F} \\ + \int \Bigg[\sum_k e^{\Omega^{-2}B+\alpha' F} \wedge \Omega^{-k} C_k \Bigg]_{p+1} \Bigg) . \end{aligned} \tag{10.2}$$

Adapting coordinates for the Penrose limit and taking $\Omega \ll 1$, the Dp-brane action can be expanded as

$$I_p[\Omega^{-2}g, \Omega^{-2}B, \Omega^{-k}C] = I_p[\bar{g}, \bar{B}, \bar{C}] + O(\Omega) , \tag{10.3}$$

where $\bar{g}, \bar{B}, \bar{C}$ are the fields at the Penrose limit. Therefore we conclude that Dp-branes at the large tension limit propagate in the Penrose limit of the associated spacetime. Since there are different Penrose limits depending on the choice of the null geodesic, there are different ways of taking the large tension limit of a Dp-brane.

Let us elaborate on this. We consider two null geodesics γ^k, $k = 1, 2$ with distinct Penrose limits. From the starting metric g we pass to the two distinct adapted metrics g^k (and likewise for the fields B and C) and from there to the two one-parameter families of metrics g^k_Ω. For all $\Omega > 0$, g^1_Ω and g^2_Ω are isometric, being related by a coordinate transformation which will in general depend in a complicated way on Ω. Nevertheless, because the brane probe action is generally covariant (and gauge invariant), we have

$$I_p[g^1_\Omega, B^1_\Omega, C^1_\Omega] = I_p[g^2_\Omega, B^2_\Omega, C^2_\Omega] \tag{10.4}$$

for all $\Omega > 0$. However, this coordinate transformation becomes singular as $\Omega \to 0$ and the Ω- (or large tension) expansion is around two distinct backgrounds. Moreover, even though, as we have just seen, the two actions are non-perturbatively (i.e., for finite Ω) equivalent, the higher order terms in Ω will not agree order by order in the perturbative expansion because the two actions are related by an Ω-dependent coordinate transformation. Hence what is a perturbative effect in one perturbation expansion, may be non-perturbative in another.

Combining the above result with that of [3], we conclude that a string theory in a supergravity background at large tension as taken by a Penrose limit is equivalent to string theory at a Penrose limit of that background. Since it is simpler to investigate string theory in pp-wave type of spacetimes, it is expected that string theory can be solved exactly at some Penrose limits. For the maximally supersymmetric case of [3], this has already been done [7] and some of the consequences have subsequently been explored in [8].

Next we turn to investigate the Penrose limit of a M5-brane probe in an eleven-dimensional spacetime [47, 48, 49, 50, 51]. To describe the field equations of the M5-brane in a general supergravity background as given in [49], we begin with some definitions. We denote with $\gamma_{\mu\nu} = e^a_\mu e^b_\nu n_{ab}$ the induced metric on the worldvolume. Next $\mathcal{E}^{\underline{a}}_\mu = \partial_\mu X^M E^{\underline{a}}_M$, where X is the embedding of map of the M5-brane into spacetime and $E^{\underline{a}}_M$ is the spacetime frame. Let $\mathcal{H}$ be the self-dual three form and $\mathcal{C} = \kappa_2 F_3 + C_3$, where $d\mathcal{C} = -\frac{1}{4}F_4$, i.e., F_3 is the worldvolume field on the M5-brane, κ_2 is the inverse of the M2-brane tension and C_3 is the eleven-dimensional supergravity three-form gauge potential. We also have

$$\begin{aligned} \mathcal{C}_{abc} &= m_a{}^d m_b{}^e \mathcal{H}_{cde} \\ m_a{}^b &= \delta_a{}^b - 2\mathcal{H}_{acd}\mathcal{H}^{bcd} \end{aligned} \tag{10.5}$$

and

$$G^{\mu\nu} = (m^2)^{ab} e^\mu_a e^\nu_b \ . \tag{10.6}$$

The field equations of the M5-brane are

$$\begin{aligned} &G^{\mu\nu}\nabla_\mu \mathcal{C}_{\nu\rho\sigma} = 0 \\ &G^{\mu\nu}\nabla_\mu \mathcal{E}^{\underline{a}} = (1 - \frac{2}{3}\mathrm{trk}^2)\epsilon^{\nu_1\ldots\nu_6} \\ &\qquad (\tfrac{1}{720}{}^*F^{\underline{a}}{}_{\nu_1\ldots\nu_6} + \frac{2}{3}F^{\underline{a}}{}_{\nu_1\ldots\nu_3}\mathcal{C}_{\nu_4\nu_5\nu_6})(\delta^{\underline{a}}_{\underline{c}} - \mathcal{E}^\mu_{\underline{c}}\mathcal{E}^{\underline{a}}_\mu) \end{aligned} \tag{10.7}$$

where ∇ is the covariant derivative with respect to the induced metric and $k_a{}^b = \mathcal{H}_{acd}\mathcal{H}^{bcd}$.

To take the Penrose limit, we adopt the appropriate light cone coordinates and scale the spacetime fields as usual $g \rightsquigarrow \Omega^{-2}g$ and $C_3 \rightsquigarrow \Omega^{-3}C_3$. Consequently, $E^{\underline{a}} \rightsquigarrow \Omega^{-1}E^{\underline{a}}$, the induced metric scales as $\gamma \rightsquigarrow \Omega^{-2}\gamma$ and $e^a \rightsquigarrow \Omega^{-1}e^a$. It is then easy to see that the metric G scales as $G \rightsquigarrow \Omega^{-2}G$. Provided that $\kappa_2 \rightsquigarrow \Omega^3\kappa_2$ and $F_3 \rightsquigarrow F_3$, the field equations for the embedding scalars scales with weight 1 while the field equation for the two-forms gauge potential scales with weight -1. Therefore a solution of the M5-brane field equations remains a solution in the Penrose limit.

Combining the above result with that of [3], we have found that the Penrose limit is a large membrane tension limit in M-theory. Placing the M-branes in an eleven-dimensional background and taking the large M2-brane tension limit and so that of the M5-brane[8], the dynamics of the M-branes is that of M-branes in a Penrose limit of the original background. Again there are different ways of taking the large M2-brane tension limit depending on the choice of null geodesic associated with the Penrose limit.

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