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A SIGMA MODEL FIELD THEORETIC REALIZATION OF HITCHIN'S GENERALIZED COMPLEX GEOMETRY

by

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Abstract

We present a sigma model field theoretic realization of Hitchin's generalized complex geometry, which recently has been shown to be relevant in compactifications of superstring theory with fluxes. Hitchin sigma model is closely related to the well known Poisson sigma model, of which it has the same field content. The construction shows a remarkable correspondence between the (twisted) integrability conditions of generalized almost complex structures and the restrictions on target space geometry implied by the Batalin–Vilkovisky classical master equation. Further, the (twisted) classical Batalin–Vilkovisky cohomology is related non trivially to a generalized Dolbeault cohomology.

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Contents

1. Introduction

Mirror symmetry is a duality relating compactifications of type IIA and type IIB superstring theory, which result in the same four–dimensional effective theory. For Calabi–Yau compactifications, it has been known for a long time and it has played an important role in their study. Recently, more general compactifications allowing for non Ricci–flat metrics and NS and RR fluxes have become object of intense scrutiny. Therefore, it is important to investigate whether mirror symmetry generalizes to this more general class of compactifications and, if so, to analyze in depth its properties. This program was outlined originally in refs. [1,2] and was subsequently pursued with an increasing level of generality in a series of papers.

In refs. [3], it was shown that mirror symmetry can be defined on manifolds with SU(3) structure, i.e. admitting a nowhere vanishing globally defined internal spinor. In this case, the symmetry maps RR into RR fluxes, but it mixes the metric and the NS flux in a non trivial fashion.

Recently, Hitchin formulated the notion of generalized complex geometry, which, at the same time extends and unifies the customary notions of complex and symplectic geometry and incorporates a natural generalization of Calabi–Yau geometry [4]. Hitchin's ideas were further developed by Gualtieri [5]. Since, in topological string theory, mirror symmetry relates complex and symplectic manifolds [6], it is conceivable that generalized complex geometry may provide a natural framework for the study of mirror symmetry [7]. In refs. [8,9], it was shown that supersymmetric SU(3) structure manifolds are indeed generalized Calabi–Yau manifolds as defined by Hitchin. Other studies of mirror symmetry relying on generalized complex geometry can be found in refs. [10–14].

In refs. [15,16], a sigma model realization of Hitchin's generalized complex geometry closely resembling a Poisson sigma model was obtained [17,18]. In this paper, we obtain a new sigma model realization of the same geometry, whose relation to the standard Poisson sigma model is even closer and which we now briefly outline.

In ref. [19] (see also [20]), Cattaneo and Felder quantized the Poisson sigma model by using the Batalin–Vilkovisky quantization algorithm [21–23]. They showed in particular that the action of the model satisfies the Batalin–Vilkovisky classical master equation, provided the target space almost Poisson structure is actually Poisson, thus establishing a remarkable connection between Poisson geometry and quantization à la Batalin–Vilkovisky of the sigma model.

In this paper, we introduce a Hitchin sigma model, which has the same field content as the standard Poisson sigma model, but whose target space geometry is specified by a generalized almost complex structure. Proceeding in an analogous manner, we quantize the model following the Batalin–Vilkovisky quantization prescriptions. We then show that the action satisfies the Batalin–Vilkovisky classical master equation, when the generalized almost complex structure is actually a generalized complex structure. We carry out our analysis both in the twisted and in the untwisted case. Further, we find that the classical Batalin–Vilkovisky cohomology is related non trivially to a hitherto unknown generalized Dolbeault cohomology containing the deformation cohomology of generalized complex structures [5].

Up to a topological term, the Hitchin sigma model reduces to the usual Poisson sigma model, in the particular case where the generalized complex structure is actually a symplectic structure. In this way, our analysis partially generalizes and broadens the scope of Cattaneo's and Felder's.

This paper is organized as follows. In sect. 2, we review the main notions of generalized complex geometry both in the twisted and in the untwisted case. In sect. 3, we outline the de Rham superfield formalism suitable for the formulation of the Poisson sigma model and its generalizations. In sect. 4, we introduce the untwisted Hitchin sigma model and show the correspondence between the conditions on the target space geometry implied by the Batalin–Vilkovisky classical master equation and the integrability condition of the target space generalized almost complex structure. In sect. 5, we repeat the same analysis for the twisted case. Finally, in sect. 6, we analyze the Batalin–Vilkovisky cohomology and its relation to generalized complex geometry.

2. Generalized complex geometry

The notion of generalized complex structure was introduced by Hitchin in [4] and developed by Gualtieri [5] in his thesis. It encompasses the usual notions of complex and symplectic structures as special cases. It is the complex counterpart of the notion of Dirac structure, introduced by Courant and Weinstein, which unifies Poisson and symplectic geometry [24,25].

Let M be a manifold of even dimension d. Consider the vector bundle $TM \oplus T^*M$. A generic section $X + \xi \in C^\infty(TM \oplus T^*M)$ of this bundle is the direct sum of sections $X \in C^\infty(TM)$, $\xi \in C^\infty(T^*M)$ of TM, T^*M, respectively. X is a vector field, ξ is a 1–form.

$TM \oplus T^*M$ is equipped with a natural indefinite metric of signature (d, d) defined by

$$\langle X + \xi, Y + \eta \rangle = \tfrac{1}{2}(i_X \eta + i_Y \xi), \tag{2.1}$$

for $X + \xi, Y + \eta \in C^\infty(TM \oplus T^*M)$, where i_V denotes contraction with respect a vector field V. This metric has a large isometry group. This contains the full diffeomorphism group of M, acting by pull-back. It also contains the following distinguished isometries, called b transforms, defined by

$$\exp(b)(X + \xi) = X + \xi + i_X b, \tag{2.2}$$

where $b \in C^\infty(\wedge^2 T^*M)$ is a 2–form.

There is a natural bilinear pairing defined on $C^\infty(TM \oplus T^*M)$ extending the customary Lie pairing on $C^\infty(TM)$, called Courant brackets [24,25]. It is given by

$$[X + \xi, Y + \eta] = [X, Y] + l_X \eta - l_Y \xi - \tfrac{1}{2} d_M (i_X \eta - i_Y \xi). \tag{2.3}$$

with $X + \xi, Y + \eta \in C^\infty(TM \oplus T^*M)$, where l_V denotes Lie derivation with respect a vector field V and d_M is the exterior differential of M. The pairing is antisymmetric, but it fails to satisfy the Jacobi identity. However, remarkably, the Jacobi identity is satisfied when restricting the sections $X + \xi, Y + \eta \in C^\infty(L)$, where L is a subbundle of $TM \oplus T^*M$ isotropic with respect to $\langle\, ,\rangle$ and involutive (closed) under $[\, ,]$. The brackets $[\, ,]$ are covariant under the action of the diffeomorphism group. They are also covariant under b transform

$$[\exp(b)(X + \xi), \exp(b)(Y + \eta)] = \exp(b)[X + \xi, Y + \eta], \tag{2.4}$$

provided the 2–form b is closed.

A generalized complex structure $\mathcal{J}$ is a section of $C^\infty(\mathrm{End}(TM \oplus T^*M))$, which is an isometry of the metric $\langle\, ,\rangle$ and satisfies

$$\mathcal{J}^2 = -1. \tag{2.5}$$

The group of isometries of $\langle\, ,\rangle$ acts on $\mathcal{J}$ by conjugation. In particular, the b transform of $\mathcal{J}$ is defined by

$$\hat{\mathcal{J}} = \exp(-b)\mathcal{J}\exp(b). \tag{2.6}$$

The $\pm\sqrt{-1}$ eigenbundles of $\mathcal{J}$ are complex and, thus, their analysis requires complexifying $TM \oplus T^*M$ leading to $(TM \oplus T^*M) \otimes \mathbb{C}$. The projectors on the eigenbundles are given by

$$\Pi_\pm = \tfrac{1}{2}(1 \mp \sqrt{-1}\mathcal{J}). \tag{2.7}$$

The generalized almost complex structure $\mathcal{J}$ is integrable if its eigenbundles are involutive, i. e. if

$$\Pi_\mp[\Pi_\pm(X+\xi), \Pi_\pm(Y+\eta)] = 0, \tag{2.8}$$

for any $(X+\xi), (Y+\eta) \in C^\infty(TM \oplus T^*M)$. In that case, $\mathcal{J}$ is called a generalized complex structure. Integrability is equivalent to the single statement

$$N(X+\xi, Y+\eta) = 0, \tag{2.9}$$

for all $X+\xi, Y+\eta \in C^\infty(TM \oplus T^*M)$, where N is the generalized Nijenhuis tensor, defined by

$$\begin{aligned} N(X+\xi, Y+\eta) = {} & [X+\xi, Y+\eta] + \mathcal{J}[\mathcal{J}(X+\xi), Y+\eta] + \mathcal{J}[X+\xi, \mathcal{J}(Y+\eta)] \\ & - [\mathcal{J}(X+\xi), \mathcal{J}(Y+\eta)]. \end{aligned} \tag{2.10}$$

The b transform $\hat{\mathcal{J}}$ of a generalized complex structure $\mathcal{J}$ is again a generalized complex structure, provided the 2–form b is closed.

In practice, it is convenient to decompose a generalized almost complex structure $\mathcal{J}$ in block form as follows

$$\mathcal{J} = \begin{pmatrix} J & P \\ Q & -J^* \end{pmatrix}, \tag{2.11}$$

where $J \in C^\infty(TM \otimes T^*M)$, $P \in C^\infty(\wedge^2 TM)$, $Q \in C^\infty(\wedge^2 T^*M)$.

For later use, we write in explicit tensor notation the conditions obeyed by J, P, Q:

$$P^{ab} + P^{ba} = 0, \tag{2.12a}$$

$$Q_{ab} + Q_{ba} = 0, \tag{2.12b}$$

$$J^a{}_c J^c{}_b + P^{ac} Q_{cb} + \delta^a{}_b = 0, \tag{2.13a}$$

$$J^a{}_c P^{cb} + J^b{}_c P^{ca} = 0, \tag{2.13b}$$

$$Q_{ac} J^c{}_b + Q_{bc} J^c{}_a = 0. \tag{2.13c}$$

Under b transform, we have

$$\hat{J}^a{}_b = J^a{}_b - P^{ac} b_{cb}, \tag{2.14a}$$

$$\hat{P}^{ab} = P^{ab}, \tag{2.14b}$$

$$\hat{Q}_{ab} = Q_{ab} + b_{ac} J^c{}_b - b_{bc} J^c{}_a + P^{cd} b_{ca} b_{db}. \tag{2.14c}$$

where $b_{ab} + b_{ba} = 0$.

The integrability condition of a generalized almost complex structure $\mathcal{J}$ can be cast in the form of a set of four tensorial equations

$$A^{abc} = 0, \tag{2.15a}$$

$$B_a{}^{bc} = 0, \tag{2.15b}$$

$$C_{ab}{}^c = 0, \tag{2.15c}$$

$$D_{abc} = 0, \tag{2.15d}$$

where A, B, C, D are tensors defined by

$$A^{abc} = P^{ad} \partial_d P^{bc} + P^{bd} \partial_d P^{ca} + P^{cd} \partial_d P^{ab}, \tag{2.16a}$$

$$B_a{}^{bc} = J^d{}_a \partial_d P^{bc} + P^{bd}(\partial_a J^c{}_d - \partial_d J^c{}_a) - P^{cd}(\partial_a J^b{}_d - \partial_d J^b{}_a) - \partial_a (J^b{}_d P^{dc}), \tag{2.16b}$$

$$\begin{aligned} C_{ab}{}^c = {} & J^d{}_a \partial_d J^c{}_b - J^d{}_b \partial_d J^c{}_a - J^c{}_d \partial_a J^d{}_b + J^c{}_d \partial_b J^d{}_a \\ & + P^{cd}(\partial_d Q_{ab} + \partial_a Q_{bd} + \partial_b Q_{da}), \end{aligned} \tag{2.16c}$$

$$\begin{aligned} D_{abc} = {} & J^d{}_a(\partial_d Q_{bc} + \partial_b Q_{cd} + \partial_c Q_{db}) + J^d{}_b(\partial_d Q_{ca} + \partial_c Q_{ad} + \partial_a Q_{dc}) \\ & + J^d{}_c(\partial_d Q_{ab} + \partial_a Q_{bd} + \partial_b Q_{da}) - \partial_a(Q_{bd} J^d{}_c) - \partial_b(Q_{cd} J^d{}_a) - \partial_c(Q_{ad} J^d{}_b). \end{aligned} \tag{2.16d}$$

The above expressions in a different but equivalent form were derived in [16].

The usual complex structures J can be viewed as generalized complex structures of the special form

$$\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}. \tag{2.17}$$

Indeed, one can check an object of this form satisfies conditions (2.12a,b), (2.13a–c), (2.15a–d) precisely when J is a complex structure, i. e. its Nijenhuis tensor vanishes. Similarly, the usual symplectic structures Q can be viewed as generalized complex structures of the special form

$$\mathcal{J} = \begin{pmatrix} 0 & -Q^{-1} \\ Q & 0 \end{pmatrix}. \tag{2.18}$$

as this object satisfies (2.12a,b), (2.13a–c), (2.15a–d) precisely when Q is a symplectic structure, i. e. it is closed. As noticed by Hitchin, other exotic examples exist. In fact, there are manifolds which cannot support any complex or symplectic structure, but do admit generalized complex structures [4]. These facts explain the reason why Hitchin's construction is interesting and worthwhile pursuing.

Let $H \in C^\infty(\wedge^3 T^*M)$ be a closed 3–form. Define the H twisted Courant brackets by

$$[X+\xi, Y+\eta]_H = [X+\xi, Y+\eta] + i_X i_Y H \tag{2.19}$$

with $X+\xi, Y+\eta \in C^\infty(TM \oplus T^*M)$. [1] Under b transform with b a closed 2–form, (2.4) holds with the brackets $[\,,]$ replaced by $[\,,]_H$. More generally, for a non closed b, one has

$$[\exp(b)(X+\xi), \exp(b)(Y+\eta)]_{H+d_M b} = \exp(b)[X+\xi, Y+\eta]_H. \tag{2.20}$$

So, b transform shifts H by the exact 3–form $d_M b$:

$$\hat{H} = H + d_M b. \tag{2.21}$$

The case where $[H/2\pi]$ belongs to the image of $H^3(M,\mathbb{Z})$ in $H^3(M,\mathbb{R})$ is particular important for its relation to gerbes. In this case, b transform with $[b/2\pi]$ contained in image of $H^2(M,\mathbb{Z})$ in $H^2(M,\mathbb{R})$ represents the gerbe generalization of gauge transformation.

One can define an H twisted generalized Nijenhuis N_H tensor as in (2.10) by using the brackets $[\,,]_H$ instead of $[\,,]$. A generalized almost complex structure $\mathcal{J}$ is H integrable if

$$N_H(X+\xi, Y+\eta) = 0, \tag{2.22}$$

for all $X+\xi, Y+\eta \in C^\infty(TM \oplus T^*M)$. In such a case, we call $\mathcal{J}$ an H twisted generalized complex structure.

[1] The sign convention of the H field used in this paper is opposite to that of ref. [5].

In tensor notation, the H integrability conditions can be cast as

$$A_H{}^{abc} = 0, \tag{2.23a}$$

$$B_{H\,a}{}^{bc} = 0, \tag{2.23b}$$

$$C_{H\,ab}{}^{c} = 0, \tag{2.23c}$$

$$D_{H\,abc} = 0, \tag{2.23d}$$

where A_H, B_H C_H, D_H are tensors defined by

$$A_H{}^{abc} = A^{abc}, \tag{2.24a}$$

$$B_{H\,a}{}^{bc} = B_a{}^{bc} + P^{bd}P^{ce}H_{ade}, \tag{2.24b}$$

$$C_{H\,ab}{}^{c} = C_{ab}{}^{c} - J^d{}_a P^{ce}H_{bde} + J^d{}_b P^{ce}H_{ade}, \tag{2.24c}$$

$$D_{H\,abc} = D_{abc} - H_{abc} + J^d{}_a J^e{}_b H_{cde} + J^d{}_b J^e{}_c H_{ade} + J^d{}_c J^e{}_a H_{bde}. \tag{2.24d}$$

These expressions also where obtained in [16].

3. 2–dimensional de Rham superfields

In general, the fields of a 2–dimensional field theory are differential forms on a oriented closed 2–dimensional manifold Σ. They can be viewed as elements of the space $\mathrm{Fun}\,(\Pi T\Sigma)$ of functions on the parity reversed tangent bundle $\Pi T\Sigma$ of Σ, which we shall call de Rham superfields [19]. More explicitly, we associate with the coordinates z^α of Σ Grassmann odd partners ζ^α with

$$\deg z^\alpha = 0, \qquad \deg \zeta^\alpha = 1. \tag{3.1}$$

$\Pi T\Sigma$ is endowed with a natural differential d defined by

$$dz^\alpha = \zeta^\alpha, \qquad d\zeta^\alpha = 0. \tag{3.2}$$

A generic de Rham superfield $\psi(z,\zeta)$ is a triplet formed by a 0–, 1–, 2–form field $\psi^{(0)}(z)$, $\psi^{(1)}{}_\alpha(z)$, $\psi^{(2)}{}_{\alpha\beta}(z)$ organized as

$$\psi(z,\zeta) = \psi^{(0)}(z) + \zeta^\alpha \psi^{(1)}{}_\alpha(z) + \tfrac{1}{2}\zeta^\alpha\zeta^\beta\psi^{(2)}{}_{\alpha\beta}(z). \tag{3.3}$$

The forms $\psi^{(0)}$, $\psi^{(1)}$, $\psi^{(2)}$ are called the components of ψ. Note that, in this formalism, the exterior differential of Σ can be identified with the operator

$$d = \zeta^{\alpha}\partial/\partial z^{\alpha}. \tag{3.4}$$

The coordinate invariant integration measure of $\Pi T\Sigma$ is

$$\mu = \mathrm{d}z^1\mathrm{d}z^2\mathrm{d}\zeta^1\mathrm{d}\zeta^2. \tag{3.5}$$

Any de Rham superfield ψ can be integrated on $\Pi T\Sigma$ according to the prescription

$$\int_{\Pi T\Sigma}\mu\,\psi = \int_{\Sigma}\tfrac{1}{2}dz^{\alpha}dz^{\beta}\psi^{(2)}{}_{\alpha\beta}(z). \tag{3.6}$$

By Stokes' theorem,

$$\int_{\Pi T\Sigma}\mu\,d\psi = 0. \tag{3.7}$$

It is possible to define functional derivatives of functionals of de Rham superfields. Let ψ be a de Rham superfield and let $F(\psi)$ be a functional of ψ. We define the left/right functional derivative superfields $\delta_{l,r}F(\psi)/\delta\psi$ as follows. Let σ be a superfield of the same properties as ψ. Then,

$$\frac{d}{dt}F(\psi+t\sigma)\Big|_{t=0} = \int_{\Pi T\Sigma}\mu\,\sigma\frac{\delta_l F(\psi)}{\delta\psi} = \int_{\Pi T\Sigma}\mu\,\frac{\delta_r F(\psi)}{\delta\psi}\sigma. \tag{3.8}$$

In the applications below, the components of the relevant de Rham superfields carry, besides the form degree, also a ghost degree. We shall limit ourselves to homogeneous superfields. A de Rham superfield ψ is said homogeneous if the sum of the form and ghost degree is the same for all its components $\psi^{(0)}$, $\psi^{(1)}$, $\psi^{(2)}$ of ψ. The common value of that sum is called the (total) degree $\deg\psi$ of ψ. It is easy to see that the differential operator d and the integration operator $\int_{\Pi T\Sigma}\mu$ carry degree 1 and -2, respectively. Also, if $F(\psi)$ is a functional of a superfield ψ, then $\deg\delta_{l,r}F(\psi)/\delta\psi = \deg F - \deg\psi + 2$.

The singular chain complex of Σ can be given a parallel treatment. A singular superchain C is a triplet formed by a 0–, 1– and 2–dimensional singular chain $C_{(0)}$, $C_{(1)}$, $C_{(2)}$ organized as a formal chain sum

$$C = C_{(0)} + C_{(1)} + C_{(2)}. \tag{3.9}$$

The singular boundary operator ∂ extends to superchains in obvious fashion by setting

$$(\partial C)_{(0)} = \partial C_{(1)}, \quad (\partial C)_{(1)} = \partial C_{(2)}, \quad (\partial C)_{(2)} = 0. \tag{3.10}$$

A singular supercycle Z is a superchain such that

$$\partial Z = 0. \tag{3.11}$$

A de Rham superfield ψ can be integrated on a superchain C:

$$\int_C \psi = \int_{C_{(0)}} \psi^{(0)} + \int_{C_{(1)}} dz^\alpha \psi^{(1)}{}_\alpha(z) + \int_{C_{(2)}} \tfrac{1}{2} dz^\alpha dz^\beta \psi^{(2)}{}_{\alpha\beta}(z). \tag{3.12}$$

Stokes' theorem states that

$$\int_C d\psi = \int_{\partial C} \psi. \tag{3.13}$$

In particular,

$$\int_Z d\psi = 0, \tag{3.14}$$

if Z is a supercycle.

4. The Hitchin sigma model

In this section, we shall first briefly review the formulation of the standard Poisson sigma model based on the Batalin–Vilkovisky quantization scheme [21,22] worked out by Cattaneo and Felder in [19] (see also [20,26]). To make the treatment as simple and transparent as possible, we shall use the convenient de Rham superfield formalism outlined above. Expressions in terms of components are straightforward to obtain, though they are rather lengthy and unwieldy. Subsequently, we introduce the Hitchin sigma model as a closely related partial generalization of the former. We shall limit ourselves to the lowest order in perturbation theory, since this constraints on target space geometry following from the Batalin–Vilkovisky classical master equation lead directly to Hitchin's generalized complex geometry. Quantum corrections will presumably yield a deformation of the latter, whose study is beyond the scope of this paper. We will not attempt the gauge fixing of the field theory, which, at any rate, is expected to be essentially identical to that of the ordinary Poisson sigma model as described in [19,20]. For clarity, we shall treat first the untwisted case.

The basic fields of the standard Poisson sigma model are a degree 0 superembedding $x \in \Gamma(\Pi T\Sigma, M)$ and a degree 1 supersection $y \in \Gamma(\Pi T\Sigma, x^*\Pi T^*M)$. With respect to each local coordinate t^a of M, x, y are given as de Rham superfields x^a, y_a. Under a change of coordinates, these transform as

$$x'^a = t'^a \circ t^{-1}(x), \tag{4.1}$$

$$y'_a = \frac{\partial t^b}{\partial t'^a} \circ t^{-1}(x) y_b. \tag{4.2}$$

The resulting transformation rules of the de Rham components of $x^a(z,\zeta)$, $y_a(z,\zeta)$ are obtainable by expanding these relations in powers of ζ^α.

We identify the fields and antifields with x^a and y_b, respectively. The Batalin–Vilkovisky odd symplectic form is

$$\Omega_{BV} = \int_{\Pi T\Sigma} \mu\, \delta x^a \delta y_a. \tag{4.3}$$

Therefore, the Batalin–Vilkovisky antibrackets are given by

$$(F, G) = \int_{\Pi T\Sigma} \mu \left[\frac{\delta_r F}{\delta x^a} \frac{\delta_l G}{\delta y_a} - \frac{\delta_r F}{\delta y_a} \frac{\delta_l G}{\delta x^a} \right], \tag{4.4}$$

for any two functionals F, G of x^a, y_a.

The target space geometry of the standard Poisson sigma model is specified by an almost Poisson structure, that is a 2–vector P. The action of the model is

$$S = \int_{\Pi T\Sigma} \mu \left[y_a dx^a + \tfrac{1}{2} P^{ab}(x) y_a y_b \right]. \tag{4.5}$$

The consistent quantization of the model requires tha S satisfies the classical Batalin–Vilkovisky master equation

$$(S, S) = 0. \tag{4.6}$$

By a straightforward computation one finds

$$(S, S) = 2 \int_{\Pi T\Sigma} \mu \left[-\tfrac{1}{6} A^{abc}(x) y_a y_b y_c \right], \tag{4.7}$$

where A is given by (2.16a). Hence, S satisfies (4.6), if (2.15aa) holds. As is well–known, condition (2.15a) ensures the almost Poisson structure P is actually Poisson, so that M is a Poisson manifold [27].

The Batalin–Vilkovisky variations are

$$\delta_{BV} x^a = (S, x^a), \tag{4.8a}$$

$$\delta_{BV} y_a = (S, y_a) \tag{4.8b}$$

[21,22]. From (4.4), (4.5), one finds easily that

$$\delta_{BV} x^a = dx^a + P^{ab}(x) y_b, \tag{4.9a}$$

$$\delta_{BV} y_a = dy_a + \tfrac{1}{2}\partial_a P^{bc}(x) y_b y_c. \tag{4.9b}$$

δ_{BV} is nilpotent:

$${\delta_{BV}}^2 = 0 \tag{4.10}$$

if (2.15a) holds, as follows from the Batalin–Vilkovisky theory or by direct verification. By construction,

$$\delta_{BV} S = 0. \tag{4.11}$$

We construct the Hitchin sigma model as follows. The fields of the model, the Batalin–Vilkovisky odd symplectic form and the associated antibrackets are the same as those of the Poisson sigma model given by eqs. (4.3), (4.4). The target space geometry is specified by a generalized almost complex structure $\mathcal{J}$ (cf. sect. 2). In the representation (2.11), the action of the model reads

$$S = \int_{\Pi T\Sigma} \mu \Big[y_a dx^a + \tfrac{1}{2} P^{ab}(x) y_a y_b + \tfrac{1}{2} Q_{ab}(x) dx^a dx^b + J^a{}_b(x) y_a dx^b \Big]. \tag{4.12}$$

We now verify under which conditions S satisfies the classical Batalin–Vilkovisky master equation (4.6). By a straightforward computation, one finds

$$\begin{aligned}(S,S) = 2\int_{\Pi T\Sigma} \mu \Big[&- \tfrac{1}{6} A^{abc}(x) y_a y_b y_c + \tfrac{1}{2} B_a{}^{bc}(x) dx^a y_b y_c \\ &- \tfrac{1}{2} C_{ab}{}^c(x) dx^a dx^b y_c + \tfrac{1}{6} D_{abc}(x) dx^a dx^b dx^c \Big],\end{aligned} \tag{4.13}$$

where the tensors A, B, C, D are given by (2.16a–d). Hence, S satisfies the classical Batalin–Vilkovisky master equation (4.6), if conditions (2.15a–d) hold. (2.15a–d) ensure that $\mathcal{J}$ is a generalized complex structure so that M is a generalized complex manifold.

This shows that *there is a non trivial connection between generalized complex geometry and quantization à la Batalin–Vilkovisky of the sigma model.*

(2.15a–d) are sufficient but not necessary conditions for the fulfillment of the master equation (4.6). In fact, as $dx^a dx^b dx^c = 0$ identically on a 2–dimensional manifold Σ, the last term in (4.13) vanishes identically so that condition (2.15d) could be dropped. Remarkably, however, a formal implementation of the antibracket algebra yields precisely this term. At any rate, this opens the possibility that the Hitchin sigma model might consistently be defined for a class of target space geometries wider than that of generalized complex structures. This would somewhat parallel what happens for the Poisson sigma model, which makes sense for the class of Poisson target space geometries, which strictly contains that of symplectic geometries. We will not elaborate further on this point.

The Batalin–Vilkovisky variations, defined by (4.8a,b), can be easily obtained using (4.4), (4.12). They are given by

$$\delta_{BV} x^a = dx^a + P^{ab}(x) y_b + J^a{}_b(x) dx^b, \tag{4.14a}$$

$$\begin{aligned}\delta_{BV} y_a = dy_a &+ \tfrac{1}{2}\partial_a P^{bc}(x) y_b y_c + \tfrac{1}{2}(\partial_a Q_{bc} + \partial_b Q_{ca} + \partial_c Q_{ab})(x) dx^b dx^c \\ &+ (\partial_a J^b{}_c - \partial_c J^b{}_a)(x) y_b dx^c + J^b{}_a(x) dy_b.\end{aligned} \tag{4.14b}$$

One can check that (4.10), (4.11) hold, if eqs. (2.15a–d) are fulfilled.

When the generalized complex structure $\mathcal{J}$ is a symplectic structure Q, one has $dQ = 0$, $J = 0$, $P = -Q^{-1}$ (cf. eq. (2.18)). In this case, as is readily verified, the Hitchin sigma model action equals the Poisson sigma model action up to a topological term, while the Batalin–Vilkovisky variations of the two models are the same. For this reason, Hitchin sigma model is only a partial generalization of the Poisson sigma model, since it can reproduce the latter only in the particular case where the target manifold Poisson structure is symplectic.

It is interesting to see how the action S behaves under a b transform of the underlying generalized almost complex structure $\mathcal{J}$ of the form (2.14a–c). It turns out that a meaningful comparison of the resulting action $\hat{S}$ and the original action S requires that the superfields x^a, y_a also must undergo a b transform of the form

$$\hat{x}^a = x^a, \tag{4.15a}$$

$$\hat{y}_a = y_a + b_{ab}(x) dx^b. \tag{4.15b}$$

It is simple to verify that

$$\hat{\Omega}_{BV} = \Omega_{BV} + \int_{\Pi T\Sigma} \mu \, \tfrac{1}{2}\big(\partial_a b_{bc} + \partial_b b_{ca} + \partial_c b_{ab}\big)(x)\delta x^a dx^b \delta x^c. \tag{4.16}$$

Hence, when the 2–form b is closed, the b transform is canonical, i. e. it leaves the Batalin–Vilkovisky odd symplectic form (4.3) invariant,

$$\hat{\Omega}_{BV} = \Omega_{BV}. \tag{4.17}$$

Under the b transform, one has

$$\hat{S} = S - \int_{\Pi T\Sigma} \mu \, b_{ab}(x) dx^a dx^b. \tag{4.18}$$

Remarkably, S, $\hat{S}$ differ by a topological term. If $b/2\pi$ has integer periods and, so, describes a gerbe gauge transformation, one has $\exp(\sqrt{-1}\hat{S}) = \exp(\sqrt{-1}S)$ in the quantum path integral. So, *gerbe gauge transformation is a duality symmetry of the quantum Hitchin sigma model.* It must be remarked that the result holds also when $b/2\pi$ has half integer periods. This is due to a factor 1/2 mismatch of the normalization of the b field in Hitchin's formulation of generalized complex geometry and the standard conventional normalization of the Poisson sigma model action. We would suggest to change the normalization of the action by a factor 1/2, but, for the time being, we stick to the normalization conventions mostly used in the physical literature.

5. The twisted Hitchin sigma model

The natural question arises whether it is possible to construct a Hitchin sigma model for twisted generalized complex structures. We are going to study this issue next.

Let H be a closed 3–form. Consider a H twisted generalized complex structure $\mathcal{J}$. Then, the tensors A_H, B_H, C_H, D_H all vanish (cf. eqs. (2.23*a*–*d*)).

Since H is closed, it can be trivialized locally. So, there are locally defined 2–forms B such that

$$H = d_M B. \tag{5.1}$$

We can use the local 2–forms B to carry out local B transforms of $\mathcal{J}$. In this way, for each local 2–form B, we have a local generalized almost complex structure $\tilde{\mathcal{J}}$ given by

$$\tilde{J}^a{}_b = J^a{}_b - P^{ac} B_{cb}, \tag{5.2a}$$

$$\tilde{P}^{ab} = P^{ab}, \tag{5.2b}$$

$$\tilde{Q}_{ab} = Q_{ab} + B_{ac}J^c{}_b - B_{bc}J^c{}_a + P^{cd}B_{ca}B_{db} \tag{5.2c}$$

in the representation (2.11) (cf. eqs. (2.14*a*–*c*)). It is straightforward to verify that the H integrability of $\mathcal{J}$ implies that these local $\tilde{\mathcal{J}}$ are integrable: the corresponding local tensors $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ all vanish (cf. eqs. (2.15*a*–*d*)).

Let us assume that the appropriate fields/antifields for the twisted version of Hitchin sigma model are the same as those of the untwisted model, viz the degree 0 superembedding $x \in \Gamma(\Pi T\Sigma, M)$ and the degree 1 supersection $y \in \Gamma(\Pi T\Sigma, x^*\Pi T^*M)$. This is reasonable, since in the limit case $H = 0$, one should recover the untwisted model. In the previous section, we learned that these fields behave non trivially under b transform (cf. eqs. (4.15*a*,*b*)). Thus, it seems appropriate to define B transformed fields $\tilde{x}^a$, $\tilde{y}_a$ by

$$\tilde{x}^a = x^a, \tag{5.3a}$$

$$\tilde{y}_a = y_a + B_{ab}(x)dx^b. \tag{5.3b}$$

Since the 2–form B is only locally defined, the superfields $\tilde{x}^a$, $\tilde{y}_a$ do not have the global meaning that the original superfields x^a, y_a do.

From the above discussion, it would seem that, at a heuristic level, one may define the twisted Hitchin sigma model with target space geometry specified by the data H, $\mathcal{J}$ and basic superfields x, y as the untwisted Hitchin sigma model with target space geometry specified by the data $\tilde{\mathcal{J}}$ and basic superfields $\tilde{x}$, $\tilde{y}$. However, it is clear that this way of proceeding cannot work in general, because $\tilde{\mathcal{J}}$, $\tilde{x}$, $\tilde{y}$ have only a local nature.

There is however a particular case where this can be done, namely when the closed 3–form H is exact. In that case, there is a globally defined 2–form B such that (5.1) holds. Then, $\tilde{\mathcal{J}}$ is a globally defined generalized complex structure and $\tilde{x}$, $\tilde{y}$ have the same global meaning as the original fields x, y. In this way, we can construct an untwisted Hitchin sigma model using $\tilde{\mathcal{J}}$, $\tilde{x}$, $\tilde{y}$, which now we describe. As we shall see, from this analysis, we can learn much on the H twisted Hitchin sigma model for non exact H.

The odd symplectic form of the H twisted model is defined by the relation

$$\Omega_{BVH} = \tilde{\Omega}_{BV}, \tag{5.4}$$

where $\tilde{\Omega}_{BV}$ is given by (4.3) with x^a, y_a replaced by $\tilde{x}^a$, $\tilde{y}_a$. A straightforward calculation shows that

$$\Omega_{BVH} = \int_{\Pi T\Sigma} \mu \Big[\delta x^a \delta y_a + \tfrac{1}{2} H_{abc}(x) \delta x^a dx^b \delta x^c\Big]. \tag{5.5}$$

Ω_{BVH} is not of the canonical form (4.3). Hence, x^a, y_a are not canonical fields/antifields. However, Ω_{BVH} is a degree 1 closed functional form, since it is related by a field redefinition to $\tilde{\Omega}_{BV}$, which is. In this way, one can define H twisted antibrackets $(,)_H$ in standard fashion. The resulting expression is

$$(F,G)_H = \int_{\Pi T\Sigma} \mu \left[\frac{\delta_r F}{\delta x^a} \frac{\delta_l G}{\delta y_a} - \frac{\delta_r F}{\delta y_a} \frac{\delta_l G}{\delta x^a} - H_{abc}(x) \frac{\delta_r F}{\delta y_a} dx^b \frac{\delta_l G}{\delta y_c} \right], \tag{5.6}$$

for any two functionals F, G of x^a, y_a.

Similarly, the action of the H twisted model is defined by the relation

$$S_H = \tilde{S}, \tag{5.7}$$

where $\tilde{S}$ is given by (4.12) with J, P, Q and x^a, y_a replaced by $\tilde{J}$, $\tilde{P}$, $\tilde{Q}$ and $\tilde{x}^a$, $\tilde{y}_a$, respectively. A straightforward calculation shows that

$$S_H = \int_{\Pi T\Sigma} \mu \left[y_a dx^a + \tfrac{1}{2} P^{ab}(x) y_a y_b + \tfrac{1}{2} Q_{ab}(x) dx^a dx^b + J^a{}_b(x) y_a dx^b \right] - 2 \int_\Gamma x^{(0)*} H. \tag{5.8}$$

Here, Γ is a 3–fold such that $\partial\Gamma = \Sigma$ and $x^{(0)} : \Gamma \to M$ is an embedding such that $x^{(0)}|_\Sigma$ equals the lowest degree 0 component of the superembedding x, whose choice is immaterial.

It is easy to see that the H twisted action S_H obeys the H twisted Batalin–Vilkovisky classical master equation

$$(S_H, S_H)_H = 0. \tag{5.9}$$

This can be verified directly, but it is obvious by itself, since Ω_{BVH}, S_H are related to $\tilde{\Omega}_{BV}$, $\tilde{S}$ by the same field redefinition, via (5.4), (5.7), respectively, and the action $\tilde{S}$ obeys the master equation (4.6).

The Batalin–Vilkovisky variations $\tilde{\delta}_{BV}\tilde{x}^a$, $\tilde{\delta}_{BV}\tilde{y}_a$ are given by (4.14*a*,*b*) with J, P, Q and x^a, y_a replaced by $\tilde{J}$, $\tilde{P}$, $\tilde{Q}$ and $\tilde{x}^a$, $\tilde{y}_a$, respectively. Using (5.3*a*,*b*), we can then derive the expressions of the H twisted Batalin–Vilkovisky variations $\delta_{BVH}x^a$, $\delta_{BVH}y_a$. The result is

$$\delta_{BVH} x^a = dx^a + P^{ab}(x) y_b + J^a{}_b(x) dx^b, \tag{5.10a}$$

$$\delta_{BVH} y_a = dy_a + \tfrac{1}{2}\partial_a P^{bc}(x) y_b y_c + \tfrac{1}{2}(\partial_a Q_{bc} + \partial_b Q_{ca} + \partial_c Q_{ab})(x) dx^b dx^c \tag{5.10b}$$

$$+ (\partial_a J^b{}_c - \partial_c J^b{}_a)(x) y_b dx^c + J^b{}_a(x) dy_b$$

$$+\tfrac{1}{2}(H_{abd}J^d{}_c - H_{acd}J^d{}_b)(x)dx^b dx^c + H_{adc}P^{db}(x)y_b dx^c.$$

It is straightforward to see that

$$\delta_{BVH}x^a = (S_H, x^a)_H, \tag{5.11a}$$

$$\delta_{BVH}y_a = (S_H, y_a)_H \tag{5.11b}$$

and that

$$\delta_{BVH}{}^2 = 0. \tag{5.12}$$

Again, this can be verified directly, but it is obvious by itself, for reasons explained below eq. (5.9). By a similar reasoning, one has

$$\delta_{BVH}S_H = 0. \tag{5.13}$$

In the above analysis, we assumed that the closed 3–form H was exact, in order to have a globally defined 2–form B. However, the expressions obtained depend on B through H, which is globally defined anyway. This provides crucial clues about how to proceed for a non exact H.

We can use (5.5) as a definition of the H twisted odd symplectic form Ω_{BVH} in the general case. Ω_{BVH} so defined has degree 1 and is closed, as required. So, H twisted antibrackets $(,)_H$ can be introduced. They are given again by eq. (5.6).

Similarly, we can use (5.8) as a definition of the H twisted action S_H in the general case. At this stage, $\mathcal{J}$ can be assumed to be a generalized almost complex structure. The last H dependent term is a Wess–Zumino like term. A similar term was added to the action of the standard Poisson sigma model in ref. [28]. Its value depends on the embedding $x^{(0)} : \Gamma \to M$. In the quantum theory, in order to have a well defined weight $\exp(\sqrt{-1}S_H)$ in the path integral, it necessary to require that H has integer periods.

A computation analogous to the one leading to (4.13) furnishes

$$\begin{aligned}(S_H, S_H)_H = 2\int_{\Pi T\Sigma}\mu\Big[&-\tfrac{1}{6}A_H{}^{abc}(x)y_a y_b y_c + \tfrac{1}{2}B_{H\,a}{}^{bc}(x)dx^a y_b y_c \\ &-\tfrac{1}{2}C_{H\,ab}{}^c(x)dx^a dx^b y_c + \tfrac{1}{6}D_{H\,abc}(x)dx^a dx^b dx^c \\ &-\tfrac{1}{3}H_{abc}(x)dx^a dx^b dx^c\Big],\end{aligned} \tag{5.14}$$

where the tensors A_H, B_H, C_H, D_H are given by (2.24a–d). Hence, S_H satisfies the H twisted classical Batalin–Vilkovisky master equation (5.9), if (2.23a–d) hold, i.e. when $\mathcal{J}$ is an H twisted generalized complex structure. (Recall that $dx^a dx^b dx^c = 0$ on a 2–dimensional manifold Σ.) This shows that *the non trivial connection between generalized complex geometry and quantization à la Batalin–Vilkovisky of the sigma model continues to hold also in the twisted case.* As for the untwisted Hitchin model, (2.24a–d) are sufficient but not necessary conditions for the fulfillment of the master equation (5.9).

The twisted Batalin–Vilkovisky variations $\delta_{BVH}x^a$, $\delta_{BVH}y_a$ are defined by (5.11a,b). By explicit computation, one can verify that they are still given by (5.10a,b). This is obviously so in view of the above reasoning, since these are local expressions anyway. Similarly, (5.12), (5.13) continue to hold.

Under a b transform (2.14a–c) of the underlying generalized almost complex structure and (4.15a,b) of the superfields x^a, y_a, with b a closed 2–form, the H twisted odd symplectic form Ω_{BVH} and action S_H behave as their untwisted counterparts, that is (4.17), (4.18) hold with Ω_{BV}, $\hat{\Omega}_{BV}$ S, $\hat{S}$ replaced by S_H, $\hat{S}_H$, Ω_{BVH}, $\hat{\Omega}_{BVH}$.

6. Batalin–Vilkovisky cohomology and generalized complex geometry

In this final section we shall analyze the classical Batalin–Vilkovisky cohomology and its relation to Hitchin's (twisted) generalized complex geometry. Though we do not have a full computation of the cohomology, we have found a interesting subset of it related in a non trivial fashion to the underlying generalized complex structure.

We consider first the untwisted case for simplicity. We call a de Rham superfield X local, if it is a local functional of the basic superfields x^a, y_a. Let X be some local superfield. Suppose there is another local superfield Y such that

$$\delta_{BV}X = dY. \tag{6.1}$$

Thus, X defines a mod d Batalin–Vilkovisky cohomology class. Then, if Z is a singular supercycle (cf. sect. 3), one has

$$\delta_{BV}\int_Z X = \int_Z dY = 0, \tag{6.2}$$

by (3.14). It follows that

$$\langle Z, X\rangle = \int_Z X \tag{6.3}$$

defines a Batalin–Vilkovisky cohomology class. A standard analysis shows that this class depends only on the mod d Batalin–Vilkovisky cohomology class of X. So, one may obtain Batalin–Vilkovisky cohomology classes by constructing local superfields X satisfying (6.1).

We recall that

$$d\delta_{BV} + \delta_{BV} d = 0. \tag{6.4}$$

Define

$$\partial = \tfrac{1}{2}\big[d - \sqrt{-1}(\delta_{BV} - d)\big] \tag{6.5}$$

and its complex conjugate $\overline{\partial}$. From (4.10), (6.4), it is immediate to check that

$$\partial^2 = 0, \tag{6.6a}$$

$$\overline{\partial}^2 = 0, \tag{6.6b}$$

$$\partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{6.6c}$$

From (6.5), one has further

$$d = \partial + \overline{\partial}, \tag{6.7a}$$

$$\delta_{BV} = \partial + \overline{\partial} + \sqrt{-1}(\partial - \overline{\partial}). \tag{6.7b}$$

Consider the operator $\overline{\partial}$. It acts on the space of local superfields, it carries degree 1, by (6.5), and it squares to 0, by (6.6b). Therefore, one can define a $\overline{\partial}$ local superfield cohomology in obvious fashion.

Let X be a local superfield such that

$$\overline{\partial} X = 0. \tag{6.8}$$

X defines a $\overline{\partial}$ local superfield cohomology class. By (6.7a, b), X satisfies (6.1) with $Y = (1 + \sqrt{-1})X$. So, as shown above, for any supercycle Z, $\langle Z, X\rangle$ defines a Batalin–Vilkovisky cohomology class. If $X = \overline{\partial} U$ for some local superfield U, so that the corresponding $\overline{\partial}$ cohomology class is trivial, then $\langle Z, X\rangle = \frac{1}{2}\sqrt{-1}\delta_{BV}\langle Z, U\rangle$, by (6.5), (3.14), and, so, the corresponding Batalin–Vilkovisky is trivial as well. Therefore, for any singular supercycle Z, there is a well-defined homomorphism from the $\overline{\partial}$ superfield cohomology into the Batalin–Vilkovisky cohomology. This homomorphism depnds only on singular homology class of Z.

The above constructions may appear somewhat arbitrary. Their meaningfulness will become clear upon computing the operator $\overline{\partial}$ and analyzing the space of solutions of eq. (6.8).

Consider a local superfiled X_Ξ of the form

$$X_\Xi = \sum_{p,q\ge 0} \tfrac{1}{p!q!}\Xi^{a_1\ldots a_p}{}_{b_1\ldots b_q}(x) y_{a_1}\cdots y_{a_p} dx^{b_1}\cdots dx^{b_q}, \tag{6.9}$$

where $\Xi \in \oplus_{p,q\ge 0} C^\infty(\wedge^p TM\otimes \wedge^q T^*M\otimes \mathbb{C})$ is a formal sum of biantisymmetric complex tensor fields on M of varying bidegree (p,q). Then, from (4.14a,b), through a tedious but totally straightforward computation, one finds the following expression:

$$\begin{aligned}\overline{\partial} X_\Xi = &\sum_{p,q\ge 0} \tfrac{1}{p!q!}\overline{\partial}_M\Xi^{a_1\ldots a_p}{}_{b_1\ldots b_q}(x) y_{a_1}\cdots y_{a_p} dx^{b_1}\cdots dx^{b_q}\\ &+\sum_{p,q\ge 0} \tfrac{1}{p!q!}K_M{}^a\Xi^{a_1\ldots a_p}{}_{b_1\ldots b_q}(x) dy_a y_{a_1}\cdots y_{a_p} dx^{b_1}\cdots dx^{b_q},\end{aligned} \tag{6.10}$$

where

$$\begin{aligned}\overline{\partial}_M\Xi^{a_1\ldots a_p}{}_{b_1\ldots b_q} = \tfrac{1}{2}\Big\{&(-1)^p q\Big[\partial_{[b_1}\Xi^{a_1\ldots a_p}{}_{b_2\ldots b_q]}+\sqrt{-1}\Big(J^c{}_{[b_1}\partial_{|c|}\Xi^{a_1\ldots a_p}{}_{b_2\ldots b_q]}\\ &-p(\partial_c J^{[a_1}{}_{[b_1}-\partial_{[b_1}J^{[a_1}{}_{|c|})\Xi^{|c|a_2\ldots a_p]}{}_{b_2\ldots b_q]}\\ &-(q-1)\partial_{[b_1}J^c{}_{b_2}\Xi^{a_1\ldots a_p}{}_{|c|b_3\ldots b_q]}\Big)\Big]-p\sqrt{-1}\Big[P^{[a_1|c|}\partial_c\Xi^{a_2\ldots a_p]}{}_{b_1\ldots b_q}\\ &-\tfrac{1}{2}(p-1)\partial_c P^{[a_1a_2}\Xi^{|c|a_3\ldots a_p]}{}_{b_1\ldots b_q}+q\partial_{[b_1}P^{[a_1|c|}\Xi^{a_2\ldots a_p]}{}_{|c|b_2\ldots b_q]}\Big]\\ &+\tfrac{1}{2}q(q-1)\sqrt{-1}\Big[\partial_c Q_{[b_1b_2}+\partial_{[b_1}Q_{b_2|c|}+\partial_{[b_2}Q_{|c|b_1}\Big]\Xi^{ca_1\ldots a_p}{}_{b_3\ldots b_q]}\Big\},\end{aligned} \tag{6.11a}$$

$$K_M{}^a\Xi^{a_1\ldots a_p}{}_{b_1\ldots b_q} = \tfrac{1}{2}\Big\{\big(\delta^a{}_c+\sqrt{-1}J^a{}_c\big)\Xi^{ca_1\ldots a_p}{}_{b_1\ldots b_q}+(-1)^p\sqrt{-1}P^{ac}\Xi^{a_1\ldots a_p}{}_{cb_1\ldots b_q}\Big\}, \tag{6.11b}$$

the brackets $[\cdots]$ denoting full antisymmetrization of all enclosed indices except for those between bars $|\cdots|$.

At first glance, it would appear that eq. (6.8) is equivalent to the equations

$$\overline{\partial}_M\Xi^{a_1\ldots a_p}{}_{b_1\ldots b_q} = 0, \tag{6.12a}$$

$$K_M{}^a\Xi^{a_1\ldots a_p}{}_{b_1\ldots b_q} = 0, \tag{6.12b}$$

at least for $q \leq 2$. This is indeed the case, though there are some subtleties involved. The $K_M{}^a \Xi^{a_1 \dots a_p}{}_{b_1 \dots b_q}$ are manifestly the components of a tensor field. Eq. (6.12b) is therefore fully covariant. Conversely, as is straightforward to check, the $\overline{\partial}_M \Xi^{a_1 \dots a_p}{}_{b_1 \dots b_q}$ are not, as they do not transform covariantly under coordinate changes. [2] So, eq. (6.12a) is not covariant in itself. However, it can be checked that, when restricting on the subspace of tensors Ξ satisfying the covariant constraint (6.12b), $\overline{\partial}_M \Xi^{a_1 \dots a_p}{}_{b_1 \dots b_q}$ do transform covariantly and, thus, they are the components of a tensor field. For such Ξ, eq. (6.12a) is therefore fully covariant.

Suppose first our generalized complex structure is an ordinary complex structure J, so that $P = 0$, $Q = 0$ (cf. eq. (2.17)). By (6.11b), eq. (6.12b) implies that Ξ is a formal sum of q–forms with values in the holomorphic vector bundles $\wedge^p T^{1,0}M$ with varying (p, q). Further, by inspection of (6.11a), one recognize $\overline{\partial}_M$ as the customary nilpotent Dolbeault operator on the space of these tensor fields Ξ. Hence, in this special case, for fixed p, eqs. (6.12a,b) define the customary Dolbeault cohomology with values in $\wedge^p T^{1,0}M$, $H^{*,*}(M, \wedge^p T^{1,0}M)$. For this reason, we claim that, in the general case, eqs. (6.12a,b) define a notion of generalized Dolbeault cohomology. The claim will be substantiated next.

Denote by $\mathscr{X}^*$ the subspace of $\oplus_{p,q \geq 0} C^\infty(\wedge^p TM \otimes \wedge^q T^*M \otimes \mathbb{C})$ spanned by those Ξ satisfying the constraint (6.12b). $\mathscr{X}^*$ is a graded vector space: for $n \in \mathbb{Z}$, $\mathscr{X}^n$ is the subspace of $\mathscr{X}^*$ contained in $\oplus_{p,q,p+q=n} C^\infty(\wedge^p TM \otimes \wedge^q T^*M \otimes \mathbb{C})$.

Exploiting (2.15a–d), by a very lengthy algebraic verification, one finds that

$$K_M{}^a \overline{\partial}_M = 0 \qquad \text{on } \mathscr{X}^*, \tag{6.13a}$$

$$\overline{\partial}_M{}^2 = 0 \qquad \text{on } \mathscr{X}^*. \tag{6.13b}$$

By relation (6.13a), $\mathscr{X}^*$ is invariant under $\overline{\partial}_M$. Relation (6.13b), in turn, states that $\overline{\partial}_M$ squares to 0 on $\mathscr{X}^*$. Further, $\overline{\partial}_M$ maps $\mathscr{X}^n$ into $\mathscr{X}^{n+1}$, as is easy to check, and, so, has degree 1. Thus, the pair $(\mathscr{X}^*, \overline{\partial}_M)$ is a cochain complex, with which there is associated a cohomology $H^*(\mathscr{X}^*, \overline{\partial}_M)$. For reasons explained in the previous paragraph, we call this generalized Dolbeault cohomology of M.

$\mathscr{X}^*$ is actually a graded algebra and $\overline{\partial}_M$ is a derivation on this algebra. As a consequence, $H^*(\mathscr{X}^*, \overline{\partial}_M)$ has an obvious ring structure.

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[2] Note, however, that the combination of the two terms in the right hand side of (6.10) is fully covariant.

It is easy to see that *eq. (6.9) defines a homomorphism of the generalized Dolbeault cohomology into the* $\overline{\partial}$ *superfield cohomology.* Recall that the latter is embedded in the classical Batalin–Vilkoviski cohomology. Therefore, *the classical Batalin–Vilkoviski cohomology is related non trivially to the generalized Dolbeault cohomology of the target manifold* M.

The above analysis generalizes verbatim to the twisted case, by replacing the Batalin–Vilkovisky operator δ_{BV} by its twisted counterpart δ_{BVH} with H a closed 3–form (cf. eqs (5.10a, b)). Only the explicit expression of the twisted generalized Dolbeault operator $\overline{\partial}_{MH}$ is different,

$$\overline{\partial}_{MH}\Xi^{a_1\ldots a_p}{}_{b_1\ldots b_q} = \overline{\partial}_{M}\Xi^{a_1\ldots a_p}{}_{b_1\ldots b_q} - \tfrac{1}{2}\sqrt{-1}q\Big\{(-1)^p pH_{cd[b_1}P^{d[a_1}\Xi^{|c|a_2\ldots a_p]}{}_{b_2\ldots b_q]} + \tfrac{1}{2}(q-1)\Big[H_{cd[b_1}J^d{}_{b_2} - H_{cd[b_2}J^d{}_{b_1}\Big]\Xi^{ca_1\ldots a_p}{}_{b_3\ldots b_q]}\Big\}, \tag{6.14}$$

where the first term in the right hand side is given by (6.11a). Using(2.23a–d), one can verify that relations (6.13a,b) still hold with $\overline{\partial}_M$ replaced by $\overline{\partial}_{MH}$. So, a twisted cochain complex $(\mathscr{X}^*, \overline{\partial}_{MH})$ and an associated twisted generalized Dolbeault cohomology $H^*(\mathscr{X}^*, \overline{\partial}_{MH})$ can be defined. Expectedly, *this is closely related to the twisted classical Batalin–Vilkovisky cohomology.*

M. Gualtieri suggested to us that the generalized Dolbeault cohomology found above could be related to the cohomology of the deformation complex of untwisted generalized complex structure [5]. We have found out that this is indeed the case, as we now show.

Let $\mathscr{D}^*$ be the subspace of $\oplus_{p,q\geq 0}C^\infty(\wedge^p TM \otimes \wedge^q T^*M \otimes \mathbb{C})$ spanned by those Ξ satisfying the constraint (6.12b) and the further constraint

$$L_{Ma}\Xi^{a_1\ldots a_p}{}_{b_1\ldots b_q} = 0, \tag{6.15}$$

where

$$L_{Ma}\Xi^{a_1\ldots a_p}{}_{b_1\ldots b_q} = \tfrac{1}{2}\Big\{(-1)^p\big(\delta^c{}_a - \sqrt{-1}J^c{}_a\big)\Xi^{a_1\ldots a_p}{}_{cb_1\ldots b_q} + \sqrt{-1}Q_{ac}\Xi^{ca_1\ldots a_p}{}_{b_1\ldots b_q}\Big\}. \tag{6.16}$$

Then, $\mathscr{D}^* \subseteq \mathscr{X}^*$. An algebraic verification similar to that yielding to (6.13a) shows that $L_{Ma}\overline{\partial}_M = 0$ on $\mathscr{D}^*$. Thus, $(\mathscr{D}^*, \overline{\partial}_M)$ is a subcomplex of $(\mathscr{X}^*, \overline{\partial}_M)$.

Next, suppose that we perform a *complex* deformation of the generalized complex structure $\mathscr{J}$ of the form

$$J'^a{}_b = J^a{}_b + \Xi^a{}_b, \qquad P'^{ab} = P^{ab} + \Xi^{ab}, \qquad Q'_{ab} = Q_{ab} + \Xi_{ab}, \tag{6.17}$$

where $\Xi^a{}_b$, Ξ^{ab}, Ξ_{ab} are the components of some element $\Xi \in \oplus_{p,q,p+q=2} C^\infty(\wedge^p TM \otimes \wedge^q T^*M \otimes \mathbb{C})$. Now, (6.17) defines a complex generalized almost complex structure $\mathcal{J}'$ to linear order in Ξ provided $\Xi \in \mathscr{D}^2$. Indeed, imposing (2.13a–c) to first order in Ξ yields the equations

$$K_M{}^a \Xi_b + L_{Mb} \Xi^a = 0, \qquad (6.18a)$$

$$K_M{}^a \Xi^b + K_M{}^b \Xi^a = 0, \qquad (6.18a)$$

$$L_{Ma} \Xi_b + L_{Mb} \Xi_a = 0, \qquad (6.18a)$$

which hold if $\Xi \in \mathscr{D}^2$. It is straightforward though lengthy to show that $\mathcal{J}'$ is integrable to linear order in Ξ if Ξ is a 2–cocycle of $(\mathscr{D}^*, \overline{\partial}_M)$. For, imposing (2.15$a$–$d$) to first order in Ξ and using (6.11a), (6.12b), (6.15) yields a set of equations for Ξ that can be cast as

$$\overline{\partial}_M \Xi^{abc} = 0, \qquad (6.19a)$$

$$\overline{\partial}_M \Xi^{ab}{}_c = 0, \qquad (6.19b)$$

$$\overline{\partial}_M \Xi^a{}_{bc} = 0, \qquad (6.19c)$$

$$\overline{\partial}_M \Xi_{abc} = 0, \qquad (6.19d)$$

so that $\overline{\partial}_M \Xi = 0$. Finally, suppose that $\Upsilon \in \mathscr{D}^1$ and that $\Xi = \overline{\partial}_M \Upsilon$, so that

$$\Xi^{ab} = \overline{\partial}_M \Upsilon^{ab}, \qquad (6.20a)$$

$$\Xi^a{}_b = \overline{\partial}_M \Upsilon^a{}_b, \qquad (6.20b)$$

$$\Xi_{ab} = \overline{\partial}_M \Upsilon_{ab}. \qquad (6.20c)$$

By explicit computation using (6.11a), (6.12b), (6.15), one finds

$$\Xi^{ab} = \tfrac{1}{2}\sqrt{-1}\, l_X P^{ab}, \qquad (6.21a)$$

$$\Xi^a{}_b = \tfrac{1}{2}\sqrt{-1}\big\{ l_X J^a{}_b - P^{ac}(\partial_c \xi_b - \partial_b \xi_c) \big\}, \qquad (6.21b)$$

$$\Xi_{ab} = \tfrac{1}{2}\sqrt{-1}\big\{ l_X Q_{ab} + J^c{}_a(\partial_c \xi_b - \partial_b \xi_c) - J^c{}_b(\partial_c \xi_a - \partial_a \xi_c) \big\}, \qquad (6.21c)$$

where $X^a = \Upsilon^a$ $\xi_a = \Upsilon_a$. Thus, Ξ is a combination of a complex infinitesimal diffeomorphism X and a complex infinitesimal b transform with $b = d_M \xi$ (cf. eqs. (2.14a–c))

and, so, represents a trivial deformation [5]. This conclusively shows that $(\mathscr{D}^*, \overline{\partial}_M)$ can be identified with the deformation complex of generalized complex structures.

The above analysis can be carried out also in the twised case without much further effort. Since the deformation theory for the twisted case was not carried out in [5], we cannot perform any comparison.

Further clarification of these matters should be left to the mathematicians. For us, it is sufficient to have found a remarkable connections between the Batalin–Vilkovisky cohomology of the Hitchin sigma model and various aspects of Hitchin's generalized complex geometry.

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