

On Reductions of Noncommutative Anti-Self-Dual Yang-Mills Equations

Masashi Hamanaka¹

*Graduate School of Mathematics, Nagoya University,
Chikusa-ku, Nagoya, 464-8602, JAPAN*

Abstract

In this paper, we show that various noncommutative integrable equations can be derived from noncommutative anti-self-dual Yang-Mills equations in the split signature, which include noncommutative versions of Korteweg-de Vries, Non-Linear Schrödinger, N -wave, Davey-Stewartson and Kadomtsev-Petviashvili equations. $U(1)$ part of gauge groups for the original Yang-Mills equations play crucial roles in noncommutative extension of Mason-Sparling's celebrated discussion. The present results would be strong evidences for noncommutative Ward's conjecture and imply that these noncommutative integrable equations could have the corresponding physical pictures such as reduced D-brane configurations of D0-D4 brane systems in open $N=2$ string theories. The BPS conditions of the reduced D-brane configurations would lead to integrabilities of the corresponding noncommutative equations. Possible applications to the D-brane dynamics are also discussed.

¹E-mail: hamanaka@math.nagoya-u.ac.jp

1 Introduction

Non-Commutative (NC) extension of field theories has been studied intensively for the last several years. NC gauge theories are equivalent to ordinary gauge theories in the presence of background magnetic fields and succeeded in revealing various aspects of them [1]. NC solitons especially play important roles in the study of D-brane dynamics, such as the confirmation of Sen's conjecture on tachyon condensation [2]. One of the distinguished features of NC theories is resolution of singularities. This gives rise to various new physical objects such as $U(1)$ instantons and makes it possible to analyze singular configurations as usual [3].

NC extension of integrable equations is also one of the hot topics and has been studied from various viewpoints by many authors [5]-[47]. These equations imply no gauge field and NC extension of them perhaps might have no physical picture or no good property on integrabilities. To make matters worse, NC extension of $(1+1)$ -dimensional equations introduces infinite number of time derivatives, which makes it hard to discuss or define the integrability. However, some of them actually possess integrable properties, such as the existence of infinite number of conserved quantities and the linearizability which are widely accepted as definition of complete integrability of equations. Furthermore, a few of them can be derived from NC (anti-)self-dual Yang-Mills (ASDYM) equations by suitable reductions. This fact may give some physical meanings and good properties to the lower-dimensional NC field equations and makes us expect that Ward's conjecture [48] still holds on NC spaces.

Confirmation of NC Ward conjecture is very important to give physical pictures into lower-dimensional integrable equations and to apply analysis of NC solitons to that of the corresponding D-branes. Origin of the integrable-like properties would be also explained from the viewpoints of NC twistor theory and the preserved supersymmetry in the D-brane systems.

In this paper, we show that various NC integrable equations can be derived from NC ASDYM equations on $(2+2)$ -dimensional flat spaces. The signature of metric is $(+, +, -, -)$ and called the *split signature*. In section 3, we extend Mason-Sparling's celebrated discussion [49] to NC spaces and derive NC Korteweg-de Vries (KdV) and Non-Linear Schrödinger (NLS) equations in NC $(1+1)$ -dimensional space-time. We can prove that the gauge groups in original gauge theories must be $G = GL(2, \mathbf{R})$ not $G = SL(2, \mathbf{R})$ and $G = U(2)$ not $G = SU(2)$, respectively. Hence we conclude that $U(1)$ part of the gauge groups play crucial roles as usual in NC gauge theories. In section 4, we derive NC N -wave, Davey-Stewartson (DS) and Kadomtsev-Petviashvili (KP) equations on NC $(2+1)$ -dimensional space-time where the Lie algebras are operator algebras. The present results imply that these NC integrable equations could have actually the corresponding

physical pictures such as reduced D-brane configurations of D0-D4 brane systems in open N=2 string theories in background of B -fields [50]. The BPS conditions of the reduced D-brane configurations would lead to integrabilities of the corresponding NC equations. Possible applications to the D-brane dynamics are also discussed in section 5.

2 NC ASDYM Equations

NC spaces are defined by the noncommutativity of the coordinates:

$$[x^i, x^j] = i\theta^{ij}, \quad (2.1)$$

where θ^{ij} are real constants and called the *NC parameters*.

NC field theories are given by the exchange of ordinary products in the commutative field theories for the star-products and realized as deformed theories from the commutative ones.

The star-product is defined for ordinary fields on commutative spaces. For Euclidean spaces, it is explicitly given by

$$\begin{aligned} f \star g(x) &:= \exp\left(\frac{i}{2}\theta^{ij}\partial_i^{(x')}\partial_j^{(x'')}\right) f(x')g(x'')\Big|_{x'=x''=x} \\ &= f(x)g(x) + \frac{i}{2}\theta^{ij}\partial_i f(x)\partial_j g(x) + O(\theta^2), \end{aligned} \quad (2.2)$$

where $\partial_i^{(x')} := \partial/\partial x'^i$ and so on. This explicit representation is known as the *Moyal product* [51]. The star-product has associativity: $f \star (g \star h) = (f \star g) \star h$, and returns back to the ordinary product in the commutative limit: $\theta^{ij} \rightarrow 0$. The modification of the product makes the ordinary spatial coordinate “noncommutative,” that is, $[x^i, x^j]_\star := x^i \star x^j - x^j \star x^i = i\theta^{ij}$.

We note that the fields themselves take c-number values as usual and the differentiation and the integration for them are well-defined as usual. A nontrivial point is that NC field equations contain infinite number of derivatives in general. Hence the integrability of the equations are not so trivial as commutative cases. Nevertheless, NC ASDYM equations are integrable in some sense. (See, e.g. [19, 11, 12].)

Let us consider Yang-Mills theories on $(2+2)$ -dimensional NC spaces. Here, we follow the convention in [52]. NC ASDYM equation is derived from compatibility condition of the following linear system:

$$(D_w - \zeta D_{\bar{z}}) \star \Psi = 0, \quad (D_z - \zeta D_{\bar{w}}) \star \Psi = 0, \quad (2.3)$$

where $(z, \tilde{z}, w, \tilde{w})$ and $D_z, D_w, D_{\tilde{z}}, D_{\tilde{w}}$ denote the coordinates of the original $(2 + 2)$ -dimensional space and covariant derivatives in the Yang-Mills theory, respectively. Explicit representations are as follows:

$$\begin{aligned} \partial_w A_z - \partial_z A_w + [A_w, A_z]_\star &= 0, & \partial_{\tilde{w}} A_{\tilde{z}} - \partial_{\tilde{z}} A_{\tilde{w}} + [A_{\tilde{w}}, A_{\tilde{z}}]_\star &= 0, \\ \partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z + \partial_{\tilde{w}} A_w - \partial_w A_{\tilde{w}} + [A_z, A_{\tilde{z}}]_\star - [A_w, A_{\tilde{w}}]_\star &= 0. \end{aligned} \quad (2.4)$$

where $A_z, A_w, A_{\tilde{z}}, A_{\tilde{w}}$ denote the gauge fields, respectively. We note that the commutator part should remain even though the gauge group is $U(1)$ because the elements of the gauge group could be operators and the gauge group could be considered to be non-abelian: $U(\infty)$. This $U(1)$ part actually plays important roles as NC gauge theories and give rise to new physical objects [1, 3].

Taking various reduction conditions of NC ASDYM equations for, such as, choice of gauge groups, translation invariance, gauge fixings, and further constraints on gauge fields, we can get NC integrable equations in $(2 + 1)$ and $(1 + 1)$ -dimensional spaces.

3 NC KdV and NLS Equations from ASDYM

In this section, we discuss reductions of ASDYM equation to $(1 + 1)$ -dimensional space-time, including NC NLS eq., NC KdV equation For commutative discussions, see e.g. [52, 53, 54, 55]. Relation to hyperCR Einstein-Weyl structure has been also discussed by Dunajski and Sparling [56]. In this section, we follow the convention in [52].

Let us consider the following NC ASDYM equation with which is dimensionally reduced to 2-dimensional space time with the coordinate $(t, x) = (z, w + \tilde{w})$:

$$Q' = 0, \quad \dot{Q} + A'_w + [A_z, Q]_\star = 0, \quad A'_z - \dot{A}_w + [A_w, A_z]_\star = 0. \quad (3.1)$$

where Q, A_w and A_z are 2×2 matrices and denote the original gauge fields.

- NC KdV equation

Now let us take further reduction on the gauge fields in the ASDYM equation (3.1) as follows:

$$\begin{aligned} Q &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_w = \begin{pmatrix} q & -1 \\ q' + q \star q & -q \end{pmatrix}, \\ A_z &= \begin{pmatrix} \frac{1}{2}q'' + aq \star q' + (1-a)q' \star q & -q' \\ \phi & -\frac{1}{2}q'' - bq' \star q - (1-b)q \star q' \end{pmatrix}, \end{aligned} \quad (3.2)$$

where a and b are constants. This conditions reduces to those in Mason-Sparling's paper [49] in the commutative limit.

The first equation of (3.1) is trivially satisfied. The second equation of (3.1) becomes

$$\begin{pmatrix} 0 & 0 \\ (a-b)[q', q]_\star & 0 \end{pmatrix} = 0.$$

Hence $a = b$. Then, the third equation of (3.1) leads to

$$\begin{aligned} \phi &= \frac{1}{2}q''' + 2q' \star q' - \dot{q} + a \{q \star q, q'\}_\star + (1-2a)q \star q' \star q \\ &\quad + aq \star q'' + (1-a)q'' \star q + \frac{1}{2}[q, q'']_\star \end{aligned} \quad (3.3)$$

$$\begin{aligned} \phi &= \frac{1}{2}q''' + 2q' \star q' - \dot{q} + a \{q \star q, q'\}_\star + (1-2a)q \star q' \star q \\ &\quad + aq'' \star q + (1-a)q \star q'' - \frac{1}{2}[q, q'']_\star \end{aligned} \quad (3.4)$$

$$\begin{aligned} \phi' &= \dot{q}' - \dot{q} \star q - q \star \dot{q} - \{c, q\}_\star + \frac{1}{2} \{q'', q \star q + q'\}_\star + a \{q \star q \star q, q'\}_\star \\ &\quad + (1-a) \{q, q' \star q' + q \star q' \star q\}_\star + 2aq' \star q \star q', \end{aligned} \quad (3.5)$$

where $\{A, B\}_\star := A \star B + B \star A$.

If we take

$$\phi = \frac{1}{4}q''' + \frac{1}{2}q' \star q' + \frac{1}{2} \{q, q''\}_\star + a[q, q'']_\star + a \{q \star q, q'\}_\star + (1-2a)q \star q' \star q,$$

Eqs. (3.3) and (3.4) become a NC potential KdV equation:

$$\dot{q} = \frac{1}{4}q''' + \frac{3}{2}q' \star q'. \quad (3.6)$$

This is derived from the NC KdV equation

$$\dot{u} = \frac{1}{4}u''' + \frac{3}{4}(u' \star u + u \star u'). \quad (3.7)$$

by setting $q' = \frac{1}{2}u$. Then Eq. (3.5) becomes

$$\begin{aligned} &\frac{1}{4}q'''' + \left(\frac{3}{2} + a\right)q' \star q'' + \left(\frac{3}{2} - a\right)q'' \star q' - \dot{q}' + 2a(2q' \star q \star q' + [q'' \star q, q]_\star) \\ &\quad + q \star \left(-\dot{q} + \left(\frac{1}{4} + a\right)q''' + \left(\frac{3}{2} - 2a\right)q' \star q'\right) \\ &\quad + \left(-\dot{q} + \left(\frac{1}{4} - a\right)q''' + \left(\frac{3}{2} - 2a\right)q' \star q'\right) \star q = 0. \end{aligned} \quad (3.8)$$

Only when $a = 0$, this equation leads to the NC potential KdV equation as follows

$$\left(-\dot{q} + \frac{1}{4}q''' + \frac{3}{2}q' \star q'\right)' + q \star \left(-\dot{q} + \frac{1}{4}q''' + \frac{3}{2}q' \star q'\right) + \left(-\dot{q} + \frac{1}{4}q''' + \frac{3}{2}q' \star q'\right) \star q = 0.$$

In summary, only when $a = b = 0$, we can obtain the NC KdV equation (3.7) by reduction of NC ASDYM equation under the reduction conditions:

$$\begin{aligned} Q &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \Phi_w = \begin{pmatrix} q & -1 \\ q' + q \star q & -q \end{pmatrix}, \\ \Phi_z &= \begin{pmatrix} \frac{1}{2}q'' + q' \star q & -q' \\ \frac{1}{4}q''' + \frac{1}{2}q' \star q' + \frac{1}{2}(q \star q'' + q'' \star q) + q \star q' \star q & -\frac{1}{2}q'' - q \star q' \end{pmatrix}, \end{aligned} \quad (3.9)$$

We note that the gauge group is not $SL(2, \mathbf{R})$ but $GL(2, \mathbf{R})$ on NC spaces because the matrix A_z is not traceless. This result also reflects the importance of $U(1)$ part of the original gauge group.

- NC NLS equation

Now let us take another further reduction on the gauge fields in the ASDYM equation (3.1) as follows:

$$Q = \kappa \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_w = \begin{pmatrix} 0 & \psi \\ \tilde{\psi} & 0 \end{pmatrix}, \quad (3.10)$$

$$A_z = \frac{1}{2\kappa} \begin{pmatrix} a\tilde{\psi} \star \psi + (1-a)\psi \star \tilde{\psi} & \psi' \\ -\tilde{\psi}' & -b\psi \star \tilde{\psi} - (1-b)\tilde{\psi} \star \psi \end{pmatrix}. \quad (3.11)$$

These conditions automatically solve the first and the second equations in (3.1) and the third equation reduces to

$$\begin{aligned} \frac{a-1}{2\kappa} (\psi' \star \tilde{\psi} + \psi \star \tilde{\psi}' - \tilde{\psi}' \star \psi - \tilde{\psi} \star \psi') &= 0, \\ \frac{1-b}{2\kappa} (\tilde{\psi}' \star \psi + \tilde{\psi} \star \psi' - \psi' \star \tilde{\psi} - \psi \star \tilde{\psi}') &= 0, \\ -\dot{\psi} + \frac{1}{2\kappa} (\psi'' - (a+b)\psi \star \tilde{\psi} \star \psi - (1-a)\tilde{\psi} \star \psi \star \psi - (1-b)\psi \star \psi \star \tilde{\psi}) &= 0, \\ -\dot{\tilde{\psi}} + \frac{1}{2\kappa} (-\tilde{\psi}'' + (a+b)\tilde{\psi} \star \psi \star \tilde{\psi} + (1-a)\tilde{\psi} \star \tilde{\psi} \star \psi + (1-b)\psi \star \tilde{\psi} \star \tilde{\psi}) &= 0. \end{aligned}$$

We can also see that only when $a = b = 1$, these equations reduce to

$$2\kappa\dot{\psi} = \psi'' - 2\psi \star \tilde{\psi} \star \psi, \quad 2\kappa\dot{\tilde{\psi}} = -\tilde{\psi}'' + 2\tilde{\psi} \star \psi \star \tilde{\psi}. \quad (3.12)$$

By taking $\kappa = \frac{i}{2}$, $\tilde{\psi} = \bar{\psi}$, we get the NC repulsive NLS equation with $G = U(1, 1)$:

$$i\dot{\psi} = \psi'' - 2\psi \star \bar{\psi} \star \psi, \quad (3.13)$$

by taking $\kappa = \frac{i}{2}$, $\tilde{\psi} = -\bar{\psi}$, the NC attractive NLS equation [8] with $G = U(2)$:

$$i\dot{\psi} = \psi'' + 2\psi \star \bar{\psi} \star \psi. \quad (3.14)$$

In summary, only when $a = b = 1$, we can obtain the NC NLS equation by reduction of NC ASDYM equation under the reduction conditions:

$$Q = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_w = \begin{pmatrix} 0 & \psi \\ \varepsilon \bar{\psi} & 0 \end{pmatrix}, \quad A_z = i \begin{pmatrix} \bar{\psi} \star \psi & \psi' \\ -\varepsilon \bar{\psi}' & -\psi \star \bar{\psi} \end{pmatrix}, \quad (3.15)$$

where $\varepsilon = \pm 1$. The case for $\varepsilon = -1$ coincides with results by Legaré [15].

We note that the gauge group must be $U(2)$ or $U(1, 1)$ not $SU(2)$ or $SU(1, 1)$ on NC spaces because the matrix A_z is not traceless. We prove that the choice by Legaré [15] is unique for the reduction and $U(1)$ part of the gauge group is crucial in this case, also.

Here we would like to point out that *every* NC conservation laws on $(1+1)$ -dimensional space-time can be derived from NC ASDYM equations by a special reduction. Let us consider the following $G = GL(2, \mathbf{R})$ NC ASDYM equation, which is dimensionally reduced to 2-dimensional space time with the coordinate $(t, x) = (z, w)$ and a fixed gauge $A_z = A_{\bar{w}} = 0$:

$$\partial_x A_z - \partial_t A_w + [A_w, A_z]_\star = 0. \quad (3.16)$$

By taking further reduction

$$A_z = \begin{pmatrix} 0 & 0 \\ J & 0 \end{pmatrix}, \quad A_w = \begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix}, \quad (3.17)$$

we get an *arbitrary* conservation law

$$\partial_t \sigma = \partial_x J, \quad (3.18)$$

because σ and J can be taken arbitrarily. For example, $\sigma = 4u$, $J = u'' + 3u \star u$ yields the NC KdV equation.

4 NC N -wave, DS and KP Equations from ASDYM

In this section, we discuss reductions of NC ASDYM equation where the components of gauge fields are elements of an infinite dimensional Lie algebra of formal matrix differential operators in an auxiliary variable ∂_y :

$$g = \{a_0 + a_1 \partial_y + a_2 \partial_y^2\}, \quad (4.1)$$

where a_0, a_1, a_2 belong to a ring of $N \times N$ matrix functions of y . Here we follow the convention in [54].

- NC N -wave equation

Let us suppose that all gauge fields depend on $t = z$ and $x = w$ only, which is a simple dimensional reduction. The linear system (2.3) is reduced as follows :

$$\partial_x \Psi = (A_w + \zeta A_{\bar{z}}) \star \Psi, \quad \partial_t \Psi = (A_z + \zeta A_{\bar{w}}) \star \Psi, \quad (4.2)$$

where Ψ is a function of x, y, t and ζ . Now let us take further reduction condition as $A_w = U + A\partial_y$, $A_z = V + B\partial_y$, $A_{\bar{z}} = A_{\bar{w}} = 0$, where B, C are constant, commuting $N \times N$ matrices: $[A, B] = 0$. By taking $\Psi = \Phi e^{-\zeta y}$, we get a reduced linear system:

$$\partial_x \Phi = (U + A\partial_y) \star \Phi, \quad \partial_t \Phi = (V + B\partial_y) \star \Phi. \quad (4.3)$$

The compatible condition for the reduced linear system gives rise to a reduced ASDYM equation:

$$U_t - V_x + [U, V]_{\star} + A \star V_y - B \star U_y = 0, \quad [A, V]_{\star} = [B, U]_{\star}. \quad (4.4)$$

which results a non-linear equation:

$$[A, Q_t]_{\star} - [B, Q_x]_{\star} + [[A, Q]_{\star}, [B, Q]_{\star}]_{\star} + A \star [B, Q_y]_{\star} - B \star [A, Q_y]_{\star} = 0, \quad (4.5)$$

where $U = [A, Q]_{\star}$, $V = [B, Q]_{\star}$.

If we take A, B as diagonal matrices: $A_{ij} = a_i \delta_{ij}$, $B_{ij} = b_i \delta_{ij}$, we get the NC $(2+1)$ -dimensional N -wave equation which is new:

$$\frac{\partial B_{ij}}{\partial t} = \lambda_{ij} \frac{\partial B_{ij}}{\partial x} + \mu_{ij} \frac{\partial B_{ij}}{\partial y} + \sum_{k=1}^N (\lambda_{ik} - \lambda_{kj}) B_{ik} B_{kj}, \quad (4.6)$$

where $(a_i - a_j)Q_{ij} = B_{ij}$, $\lambda_{ij} = -\frac{b_i - b_j}{a_i - a_j}$, $\mu_{ij} = a_i \lambda_{ij} - b_i$, ($i \neq j$). In commutative limit, this equation reduces to the $(2+1)$ -dimensional N -wave equation. It is interesting to study whether NC N -wave equation still admits a configuration of N interacting waves as commutative case [57].

- NC Davey-Stewartson equation

From now on, we assume $N = 2$ and that all gauge fields depend on $t = z$ and $x = w + \tilde{w}$ only, which is another dimensional reduction. The linear system (2.3) is reduced as follows:

$$\partial_x \Psi = (A_w + \zeta A_{\bar{z}}) \star \Psi, \quad \partial_t \Psi = (A_z + \zeta(A_w + A_{\tilde{w}}) + \zeta^2 A_{\bar{z}}) \star \Psi, \quad (4.7)$$

where the coefficient matrices depend on x, y, t . Now let us take $\Psi = \Phi e^{-\zeta y}$ and further reduction condition: $A_w = U + A\partial_y$, $A_z = V + U\partial_y + A\partial_y^2$, $A_{\bar{z}} = A$, $A_{\bar{w}} = A\partial_y$, we get a reduced linear system:

$$\partial_x \Phi = (U + A\partial_y) \star \Phi, \quad \partial_t \Phi = (V + U\partial_y + A\partial_y^2) \star \Phi. \quad (4.8)$$

By taking further reduction condition:

$$U = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V = \frac{1}{2} \begin{pmatrix} 2R_1 & \kappa(\partial_x + \partial_y)q \\ -\kappa(\partial_x - \partial_y)r & 2R_2 \end{pmatrix}, \quad (4.9)$$

The compatibility condition in some scale transformation yields a reduced ASDYM equation which is the NC Davey-Stewartson system:

$$\begin{aligned} \dot{q} &= \frac{\kappa}{2}(\partial_x^2 + \partial_y^2)q + R_1 \star q - q \star R_2, \\ \dot{r} &= -\frac{\kappa}{2}(\partial_x^2 + \partial_y^2)r + R_2 \star r - r \star R_1, \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} (\partial_x - \partial_y)R_1 &= -\frac{\kappa}{2}(\partial_x + \partial_y)(q \star r), \\ (\partial_x + \partial_y)R_2 &= \frac{\kappa}{2}(\partial_x - \partial_y)(q \star r). \end{aligned} \quad (4.11)$$

These are also new equations. By taking a dimensional reduction $\partial_y = 0$, this discussion reduces to that of NC NLS equation.

- NC KP equation

NC KP equation can be derived from NC ASDYM equation in similar discussion to NC DS equation. Let us start from the reduced system (4.8), and take another further reduction on the gauge fields as follows:

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}, \quad V = \frac{1}{4} \begin{pmatrix} \alpha' + u' & -2u \\ -u'' - 2u \star u + u_y & \alpha' - u' \end{pmatrix}, \quad (4.12)$$

Then the compatibility condition gives rise to a NC reduced ASDYM equation:

$$\begin{pmatrix} -\alpha' - 3u_y & 0 \\ 4\dot{u} - u''' + 3u' \star u + 3u \star u' + \alpha_y + [u, \alpha]_{\star} & -\alpha' - 3u_y \end{pmatrix} = 0. \quad (4.13)$$

Hence we get $\alpha = -3\partial_x^{-1}u_y := -3 \int^x u_y(x')dx'$, and the NC KP equation [6, 17]:

$$\dot{u} = \frac{1}{4} \left(u''' - 3(u' \star u + u \star u') - 3\partial_x^{-1}u_{yy} - 3[u, \partial_x^{-1}u_y]_{\star} \right). \quad (4.14)$$

If we neglect y -dependence, the present discussion gives rise to another reduction to NC KdV equation from ASDYM with $G = SL(2, \mathbf{R})$, which is reduced to discussion by Bakas and Depireux [58] in commutative limit.

Reduction of NC ASDYM hierarchies to NC integrable hierarchies would be possible in similar ways. The detailed discussion would be reported somewhere.

5 Conclusion and Discussion

In the present manuscript, we proved that various NC integrable equations can be derived from NC ASDYM equations in the split signature, which include NC versions of KdV, NLS, N -wave, DS and KP equations. $U(1)$ part of gauge groups for the original Yang-Mills equations play crucial roles in NC extension of Mason-Sparling's beautiful results. Existence of these reductions guarantees the lower-dimensional integrable equations actually have the corresponding physical situations, such as, reduced D0-D4 D-brane systems. Then, Analysis of exact NC soliton solutions could be applied to that of D-brane dynamics in the special reduced situations.

In reductions of ASDYM equations, we mainly should take the split signature. NC Yang-Mills theories with the split signature can be embedded [50] in $N = 2$ string theories [59]. Simple reductions of them and relation to the string theories have been studied intensively [18, 22, 23, 29, 47]. This guarantees that NC integrable equations would have physical meanings and might lead to various successful applications to the corresponding D-brane dynamics and so on.

One of the next step is to study supersymmetric extension (e.g [15, 33]) of these reductions of ASDYM equations in order to clarify what symmetries in reductions guarantee integrabilities of lower dimensional integrable equations from the viewpoint of supersymmetries. We can judge what reductions are better or worse from the viewpoint of supersymmetry and classify list of the symmetry reductions. This might exclude the special reductions to any conservation laws as we commented in the end of section 3, from a class of symmetry reductions enough to preserve integrable properties.

The BPS equations in some D-brane configurations would just correspond to NC integrable equations and the soliton solutions correspond to lower-dimensional D-branes. In these situations, we can expect that similar applications to D-brane dynamics would be possible, such as, tachyon condensations. NC solitons are sometimes very easy to treat, such as Gopakumar-Minwalla-Strominger (GMS) solitons [60] in infinite noncommutativity limit. GMS-like solitons would exist in our situations in some limit and we could construct exact soliton solutions and analyze energy densities of them and fluctuation spectrum around them and so on. This would be a hint to reveal the corresponding D-brane configurations which might be new BPS states.

Acknowledgments

The author would like to thank H. Kanno, F. Müller-Hoissen, O. Lechtenfeld, L. Mason, S. Moriyama, A. Popov and K. Takasaki for fruitful discussions and useful comments. He is also grateful to O. Lechtenfeld, F. Müller-Hoissen and K. Lee for financial support and

hospitality during stays at the Institute for Theoretical Physics, University of Hannover, and the Max Planck Institute for dynamics and self-organization, Göttingen, and the Korea Institute for Advanced Study, on March 2005. This work was partially supported by the Daiko Foundation (#9095).

References

- [1] J. A. Harvey, “Komaba lectures on noncommutative solitons and D-branes,” hep-th/0102076; M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73** (2002) 977 [hep-th/0106048]; R. J. Szabo, Phys. Rept. **378** (2003) 207 [hep-th/0109162]; C. S. Chu, “Non-commutative geometry from strings,” hep-th/0502167 and references therein.
- [2] A. Sen, “Tachyon dynamics in open string theory,” hep-th/0410103 and references therein.
- [3] K. Furuuchi, “Topological charge of U(1) instantons on noncommutative \mathbf{R}^4 ,” hep-th/0010006; N. A. Nekrasov, “Trieste lectures on solitons in noncommutative gauge theories,” hep-th/0011095; A. Konechny and A. Schwarz, Phys. Rept. **360** (2002) 353 [hep-th/0107251]; M. Hamanaka, “Noncommutative solitons and D-branes,” Ph. D thesis, hep-th/0303256; F. A. Schaposnik, Braz. J. Phys. **34** (2004) 1349 [hep-th/0310202]; O. Lechtenfeld, Fortsch. Phys. **52** (2004) 596 [hep-th/0401158] and references therein.
- [4] M. Hamanaka, “Noncommutative solitons and integrable systems,” hep-th/0504001; L. Tamassia, “Noncommutative supersymmetric / integrable models and string theory,” Ph. D thesis, hep-th/0506064.
- [5] T. Curtright, D. Fairlie and C. K. Zachos, Phys. Lett. B **405** (1997) 37 [hep-th/9704037]; C. K. Zachos, D. Fairlie and T. Curtright, “Matrix membranes and integrability,” hep-th/9709042.
- [6] B. Kupershmidt, *KP or mKP : noncommutative mathematics of Lagrangian, Hamiltonian, and integrable systems* (AMS, 2000) [ISBN/0821814001].
- [7] A. Dimakis and F. Müller-Hoissen, Int. J. Mod. Phys. B **14** (2000) 2455 [hep-th/0006005]; Lett. Math. Phys. **54** (2000) 123 [hep-th/0007160]; J. Phys. A **34** (2001) 9163 [nlin.si/0104071].
- [8] A. Dimakis and F. Müller-Hoissen, “A noncommutative version of the nonlinear Schroedinger equation,” hep-th/0007015; Czech. J. Phys. **51** (2001) 1285.

- [9] A. Dimakis and F. Müller-Hoissen, Phys. Lett. A **278** (2000) 139 [hep-th/0007074].
- [10] A. Dimakis and F. Müller-Hoissen, J. Phys. A **34** (2001) 2571 [nlin.si/0008016].
- [11] A. Kapustin, A. Kuznetsov and D. Orlov, Commun. Math. Phys. **221** (2001) 385 [hep-th/0002193].
- [12] K. Takasaki, J. Geom. Phys. **37** (2001) 291 [hep-th/0005194].
- [13] D. J. Gross and N. A. Nekrasov, JHEP **0007** (2000) 034 [hep-th/0005204].
- [14] N. A. Nekrasov, Commun. Math. Phys. **241** (2003) 143 [hep-th/0010017].
- [15] M. Legare, “Noncommutative generalized NS and super matrix KdV systems from a noncommutative version of (anti-)self-dual Yang-Mills equations,” hep-th/0012077; J. Phys. A **35** (2002) 5489.
- [16] O. Lechtenfeld, A. D. Popov and B. Spindig, JHEP **0106** (2001) 011 [hep-th/0103196].
- [17] L. D. Paniak, “Exact noncommutative KP and KdV multi-solitons,” hep-th/0105185.
- [18] O. Lechtenfeld and A. D. Popov, JHEP **0111** (2001) 040 [hep-th/0106213]; Phys. Lett. B **523** (2001) 178 [hep-th/0108118].
- [19] K. C. Hannabuss, Lett. Math. Phys. **58** (2001) 153 [hep-th/0108228].
- [20] O. Lechtenfeld and A. D. Popov, JHEP **0203** (2002) 040 [hep-th/0109209].
- [21] M. T. Grisaru and S. Penati, Nucl. Phys. B **655** (2003) 250 [hep-th/0112246].
- [22] S. Bieling, J. Phys. A **35** (2002) 6281 [hep-th/0203269].
- [23] M. Wolf, JHEP **0206** (2002) 055 [hep-th/0204185].
- [24] K. Toda, “Extensions of soliton equations to non-commutative $(2 + 1)$ dimensions,” Proceedings of workshop on integrable theories, solitons and duality, Sao Paulo, Brazil, 1-6 July 2002 [JHEP PRHEP-unesp2002/038].
- [25] Z. Horváth, O. Lechtenfeld and M. Wolf, JHEP **0212** (2002) 060 [hep-th/0211041].
- [26] K. Furuta, T. Inami and M. Yamamoto, “Topics in nonlinear sigma models in $D = 3$,” hep-th/0211129.
- [27] M. Hamanaka and K. Toda, Phys. Lett. A **316** (2003) 77 [hep-th/0211148].

- [28] I. Cabrera-Carnero and M. Moriconi, Nucl. Phys. B **673** (2003) 437 [hep-th/0211193]; hep-th/0303168.
- [29] M. Ihl and S. Uhlmann, Int. J. Mod. Phys. B **18** (2003) 4889 [hep-th/0211263].
- [30] M. Hamanaka and K. Toda, J. Phys. A **36** (2003) 11981 [hep-th/0301213].
- [31] L. Martina and O. K. Pashaev, “Burgers’ equation in non-commutative space-time,” hep-th/0302055.
- [32] O. Lechtenfeld and A. D. Popov, JHEP **0401** (2004) 069 [hep-th/0306263].
- [33] H. Nishino and S. Rajpoot, Phys. Lett. B **572** (2003) 91 [hep-th/0306290].
- [34] N. Wang and M. Wadati, J. Phys. Soc. Jap. **72** (2003) 1366; J. Phys. Soc. Jap. **72** (2003) 1881; J. Phys. Soc. Jap. **72** (2003) 3055. J. Phys. Soc. Jap. **73** (2004) 1689.
- [35] A. Zuevsky, J. Phys. A **37** (2004) 537.
- [36] M. Hamanaka and K. Toda, Proc. Inst. Math. NAS Ukraine (2004) 404 [hep-th/0309265].
- [37] M. T. Grisaru, L. Mazzanti, S. Penati and L. Tamassia, JHEP **0404** (2004) 057 [hep-th/0310214].
- [38] M. Hamanaka, J. Math. Phys. **46** (2005) 052701 [hep-th/0311206].
- [39] A. Dimakis and F. Müller-Hoissen, J. Phys. A **37** (2004) 4069 [hep-th/0401142]; J. Phys. A **37** (2004) 10899 [hep-th/0406112]; “Extension of Moyal-deformed hierarchies of soliton equations,” nlin.si/0408023.
- [40] K. Shigechi, M. Wadati and N. Wang, Nucl. Phys. B **706** (2005) 518 [hep-th/0404249].
- [41] K. M. Lee, JHEP **0408** (2004) 054 [hep-th/0405244].
- [42] O. Lechtenfeld, L. Mazzanti, S. Penati, A. D. Popov and L. Tamassia, hep-th/0406065.
- [43] M. Sakakibara, J. Phys. A **37** (2004) L599 [nlin.si/0408002]
- [44] A. Dimakis and F. Müller-Hoissen, J. Phys. A **38** (2005) 5453 [nlin.si/0501003]; “Algebraic identities associated with KP and AKNS hierarchies,” math-ph/0507024.
- [45] H. Blas, H. L. Carrion and M. Rojas, JHEP **0503** (2005) 037 [hep-th/0502051].

- [46] I. Cabrera-Carnero, “Abelian Toda field theories on the noncommutative plane,” hep-th/0503147.
- [47] C. S. Chu and O. Lechtenfeld, “Time-space noncommutative abelian solitons,” hep-th/0507062.
- [48] R. S. Ward, Phil. Trans. Roy. Soc. Lond. A **315** (1985) 451; “Multidimensional integrable systems,” Lect. Notes Phys. **280** (Springer, 1986) 106 “Integrable systems in twistor theory,” in *Twistors in Mathematics and Physics* (Cambridge UP, 1990) 246 [ISBN/0-521-39783-9].
- [49] L. J. Mason and G. A. J. Sparling, Phys. Lett. A **137** (1989) 29.
- [50] O. Lechtenfeld, A. D. Popov and B. Spindig, Phys. Lett. B **507** (2001) 317 [hep-th/0012200].
- [51] J. E. Moyal, Proc. Cambridge Phil. Soc. **45** (1949) 99; H. J. Groenewold, Physica **12** (1946) 405.
- [52] L. J. Mason and N. M. Woodhouse, *Integrability, Self-Duality, and Twistor Theory* (Oxford UP, 1996) [ISBN/0-19-853498-1].
- [53] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge UP, 1991) Chapter 6.5 [ISBN/0-521-38730-2].
- [54] M. J. Ablowitz, S. Chakravarty and L. A. Takhtajan, Commun. Math. Phys. **158** (1993) 289;
- [55] T. A. Ivanova and A. D. Popov, Theor. Math. Phys. **102** (1995) 280 [Teor. Mat. Fiz. **102** (1995) 384]; S. Chakravarty, S. L. Kent and E. T. Newman, J. Math. Phys. **36** (1995) 763; M. J. Ablowitz, S. Chakravarty and R. G. Halburd, J. Math. Phys. **44** (2003) 3147.
- [56] M. Dunajski and G. Sparling, to appear in Phys. Lett. A [nlin.si/0503030].
- [57] M. Ablowitz and R. Haberman, J. Math. Phys. **16** (1975) 2301.
- [58] I. Bakas and D. A. Depireux, Mod. Phys. Lett. A **6** (1991) 399.
- [59] H. Ooguri and C. Vafa, Mod. Phys. Lett. A **5** (1990) 1389; Nucl. Phys. B **361** (1991) 469; Nucl. Phys. B **367** (1991) 83.
- [60] R. Gopakumar, S. Minwalla and A. Strominger, JHEP **0005** (2000) 020 [hep-th/0003160].