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# Exact solution of the XXZ Gaudin model with generic open boundaries

**Wen-Li Yang,** [a,b] **Yao-Zhong Zhang** [b] **and Mark D. Gould**[b]

[a] Institute of Modern Physics, Northwest University, Xian 710069, P.R. China

[b] Department of Mathematics, University of Queensland, Brisbane, QLD 4072, Australia

**Abstract**

The XXZ Gaudin model with *generic* integerable boundaries specified by generic *non-diagonal* K-matrices is studied. The commuting families of Gaudin operators are diagonalized by the algebraic Bethe ansatz method. The eigenvalues and the corresponding Bethe ansatz equations are obtained.

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# 1 Introduction

Gaudin type models [1] have played an essential role in the study of quantum systems appeared in many physical fields such as the BCS theory [2] of small metallic grains [3, 4, 5], nuclear physics theory [6, 7] and QCD theory [8, 9]. They also provide a testing ground for ideas such as the functional Bethe ansatz and general procedure of separation of variables [10, 11, 12] and the construction of integral representations of the solutions to the Knizhnik-Zamolodchikov (KZ) equation [13, 14, 15, 16, 17].

The original Gaudin's magnet Hamiltonians (or Gaudin operators) can be rewritten in terms of the quasi-classical expansion of the transfer matrices (*row-to-row transfer matrices*) of inhomogeneous spin chains with periodic boundary condition [18]. Since the elegant work of Sklyanin [19], the powerful Quantum Inverse Scattering Method (QISM) has been applied to various integrable models with non-trivial boundary conditions, which are specified by K-matrices satisfying the reflection equation and its dual [20]. The quasi-classical expansion of the corresponding transfer matrices (*double-row transfer matrices*) produces *generalized* Gaudin Hamiltonians with boundaries [16, 21]. In particular, twisted boundary conditions and open boundary conditions associated with *diagonal* K-matrices give rise to Gaudin magnets in non-uniform local magnetic fields [16] and interacting electron pairs with certain non-uniform long-range coupling strengths [4, 22, 23, 24, 21].

In this paper, we study the XXZ type Gaudin magnets with most generic boundary conditions specified by the generic *non-diagonal* K-matrices given in [25, 26]. In section 3, we construct the generalized Gaudin operators associated with the generic boundary K-matrices. The commutativity of these operators follows from the standard procedure [18, 16, 21] specializing to the inhomogeneous spin-$\frac{1}{2}$ XXZ open chain, thus ensuring the integrability of the Gaudin magnets. In section 4, we diagonalize the Gaudin operators simultaneously by means of the algebraic Bethe ansatz method. This constitutes the main new result in this paper. The diagonalization is achieved by means of the technique of the "vertex-face" transformation. In section 5, we conclude this paper by offering some discussions.

## 2 Preliminaries: the inhomogeneous spin-$\frac{1}{2}$ XXZ open chain

Throughout, $V$ denotes a two-dimensional linear space and $\sigma^{\pm}$, $\sigma^{z}$ are the usual Pauli matrices which realize the spin-$\frac{1}{2}$ representation of the Lie algebra $sl(2)$ on $V$. The spin-$\frac{1}{2}$ XXZ chain can be constructed from the well-known six-vertex model R-matrix $\overline{R}(u) \in \mathrm{End}(V \otimes V)$ [27] given by

$$\overline{R}(u) = \begin{pmatrix} a(u) & & & \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ & & & a(u) \end{pmatrix}. \tag{2.1}$$

The coefficient functions read

$$\begin{aligned} \overline{R}^{11}_{11}(u) &= \overline{R}^{22}_{22}(u) = a(u) = 1, \\ \overline{R}^{12}_{12}(u) &= \overline{R}^{21}_{21}(u) = b(u) = \frac{\sin u}{\sin(u+\eta)}, \\ \overline{R}^{12}_{21}(u) &= \overline{R}^{21}_{12}(u) = c(u) = \frac{\sin \eta}{\sin(u+\eta)}. \end{aligned} \tag{2.2}$$

Here $u$ is the spectrum parameter and $\eta$ is the so-called crossing parameter. The R-matrix satisfies the quantum Yang-Baxter equation (QYBE),

$$R_{12}(u_1-u_2)R_{13}(u_1-u_3)R_{23}(u_2-u_3) = R_{23}(u_2-u_3)R_{13}(u_1-u_3)R_{12}(u_1-u_2), \tag{2.3}$$

and the properties,

$$\text{Unitarity}: \; \overline{R}_{12}(u)\overline{R}_{21}(-u) = \mathrm{id}, \tag{2.4}$$

$$\text{Crossing-unitarity}: \; (\overline{R})^{t_2}_{21}(-u-2\eta)(\overline{R})^{t_2}_{12}(u) = \frac{\sin u \sin(u+2\eta)}{\sin(u+\eta)\sigma(u+\eta)} \,\mathrm{id}, \tag{2.5}$$

$$\text{Quasi-classical property}: \; \overline{R}_{12}(u)|_{\eta\to 0} = \mathrm{id}. \tag{2.6}$$

Here $\bar{R}_{21}(u) = P_{12}\bar{R}_{12}(u)P_{12}$ with $P_{12}$ being the usual permutation operator and $t_i$ denotes the transposition in the $i$-th space. Here and below we adopt the standard notations: for any matrix $A \in \mathrm{End}(V)$ , $A_j$ is an embedding operator in the tensor space $V \otimes V \otimes \cdots$, which acts as $A$ on the $j$-th space and as identity on the other factor spaces; $R_{ij}(u)$ is an embedding operator of R-matrix in the tensor space, which acts as identity on the factor spaces except for the $i$-th and $j$-th ones.

One introduces the "row-to-row" monodromy matrix $T(u)$, which is an $2\times 2$ matrix with elements being operators acting on $V^{\otimes N}$, where $N=2M$ ($M$ being a positive integer),

$$T_0(u)=\overline{R}_{01}(u+z_1)\overline{R}_{02}(u+z_2)\cdots\overline{R}_{0N}(u+z_N). \tag{2.7}$$

Here $\{z_j|j=1,\cdots,N\}$ are arbitrary free complex parameters which are usually called inhomogeneous parameters. With the help of the QYBE (2.3), one can show that $T(u)$ satisfies the so-called "RLL" relation

$$\overline{R}_{12}(u-v)T_1(u)T_2(v)=T_2(v)T_1(u)\overline{R}_{12}(u-v). \tag{2.8}$$

Integrable open chain can be constructed as follows [19]. Let us introduce a pair of K-matrices $K^-(u)$ and $K^+(u)$. The former satisfies the reflection equation (RE) [20]

$$\begin{aligned}&\overline{R}_{12}(u_1-u_2)K_1^-(u_1)\overline{R}_{21}(u_1+u_2)K_2^-(u_2)\\ &\quad=K_2^-(u_2)\overline{R}_{12}(u_1+u_2)K_1^-(u_1)\overline{R}_{21}(u_1-u_2),\end{aligned} \tag{2.9}$$

and the latter satisfies the dual RE

$$\begin{aligned}&\overline{R}_{12}(u_2-u_1)K_1^+(u_1)\overline{R}_{21}(-u_1-u_2-2\eta)K_2^+(u_2)\\ &\quad=K_2^+(u_2)\overline{R}_{12}(-u_1-u_2-2\eta)K_1^+(u_1)\overline{R}_{21}(u_2-u_1).\end{aligned} \tag{2.10}$$

For open spin-chains, instead of the standard "row-to-row" monodromy matrix $T(u)$ (2.7), one needs to introduce the "double-row" monodromy matrix $\mathbb{T}(u)$

$$\mathbb{T}(u)=T(u)K^-(u)T^{-1}(-u). \tag{2.11}$$

Using (2.8) and (2.9), one can prove that $\mathbb{T}(u)$ satisfies

$$\overline{R}_{12}(u_1-u_2)\mathbb{T}_1(u_1)\overline{R}_{21}(u_1+u_2)\mathbb{T}_2(u_2)=\mathbb{T}_2(u_2)\overline{R}_{12}(u_1+u_2)\mathbb{T}_1(u_1)\overline{R}_{21}(u_1-u_2). \tag{2.12}$$

Then the *double-row transfer matrix* of the inhomogeneous spin-$\frac{1}{2}$ XXZ chain with open boundary is given by

$$\tau(u)=tr(K^+(u)\mathbb{T}(u)). \tag{2.13}$$

With the help of (2.3)-(2.5) and (2.9)-(2.10), one can prove that the transfer matrices with different spectral parameters commute with each other [19]:

$$[\tau(u),\tau(v)]=0. \tag{2.14}$$

This ensures the integrability of the inhomogeneous spin-$\frac{1}{2}$ XXZ chain with open boundary.

# 3 XXZ Gaudin models with generic boundaries

In this paper, we will consider a *generic* K-matrix $K^-(u)$ which is a generic solution to the RE (2.9) associated the six-vertex model R-matrix [25, 26]

$$K^-(u) = \begin{pmatrix} k_1^1(u) & k_2^1(u) \\ k_1^2(u) & k_2^2(u) \end{pmatrix} \equiv K(u). \tag{3.1}$$

The coefficient functions are

$$\begin{aligned}
k_1^1(u) &= \frac{2\cos(\lambda_1-\lambda_2) - \cos(\lambda_1+\lambda_2+2\xi)e^{-2iu}}{4\sin(\lambda_1+\xi+u)\sin(\lambda_2+\xi+u)},\\
k_2^1(u) &= \frac{-i\sin(2u)e^{-i(\lambda_1+\lambda_2)}e^{-iu}}{2\sin(\lambda_1+\xi+u)\sin(\lambda_2+\xi+u)},\\
k_1^2(u) &= \frac{i\sin(2u)e^{i(\lambda_1+\lambda_2)}e^{-iu}}{2\sin(\lambda_1+\xi+u)\sin(\lambda_2+\xi+u)},\\
k_2^2(u) &= \frac{2\cos(\lambda_1-\lambda_2)e^{-2iu} - \cos(\lambda_1+\lambda_2+2\xi)}{4\sin(\lambda_1+\xi+u)\sin(\lambda_2+\xi+u)}.
\end{aligned} \tag{3.2}$$

At the same time, we introduce the corresponding *dual* K-matrix $K^+(u)$ which is a generic solution to the dual reflection equation (2.10) with a particular choice of the free boundary parameters with respect to $K^-(u)$:

$$K^+(u) = \begin{pmatrix} k^{+1}_{\ 1}(u) & k^{+1}_{\ 2}(u) \\ k^{+2}_{\ 1}(u) & k^{+2}_{\ 2}(u) \end{pmatrix}. \tag{3.3}$$

The matrix elements are

$$\begin{aligned}
k^{+1}_{\ 1}(u) &= \frac{2\cos(\lambda_1-\lambda_2)e^{-i\eta} - \cos(\lambda_1+\lambda_2+2\xi)e^{2iu+i\eta}}{4\sin(\lambda_1+\xi-u-\eta)\sin(\lambda_2+\xi-u-\eta)},\\
k^{+1}_{\ 2}(u) &= \frac{i\sin(2u+2\eta)e^{-i(\lambda_1+\lambda_2)}e^{iu-i\eta}}{2\sin(\lambda_1+\xi-u-\eta)\sin(\lambda_2+\xi-u-\eta)},\\
k^{+2}_{\ 1}(u) &= \frac{-i\sin(2u+2\eta)e^{i(\lambda_1+\lambda_2)}e^{iu+i\eta}}{2\sin(\lambda_1+\xi-u-\eta)\sin(\lambda_2+\xi-u-\eta)},\\
k^{+2}_{\ 2}(u) &= \frac{2\cos(\lambda_1-\lambda_2)e^{2iu+i\eta} - \cos(\lambda_1+\lambda_2+2\xi)e^{-i\eta}}{4\sin(\lambda_1+\xi-u-\eta)\sin(\lambda_2+\xi-u-\eta)}.
\end{aligned} \tag{3.4}$$

The K-matrices depend on three free boundary parameters $\{\lambda_1,\ \lambda_2,\ \xi,\}$ which specify integrable boundary conditions [26]. It is very convenient to introduce a vector $\lambda = \sum_{k=1}^{2}\lambda_k\epsilon_k$ associated with the boundary parameters $\{\lambda_i\}$, where $\{\epsilon_i,\ i=1,2\}$ form the orthonormal basis of $V$ such that $\langle\epsilon_i,\epsilon_j\rangle = \delta_{ij}$. We remark that $K^-(u)$ does not depend on the crossing parameter $\eta$ but $K^+(u)$ does. They satisfy the following relation:

$$\lim_{\eta\to 0}\{K^+(u)K^-(u)\} = \lim_{\eta\to 0}\{K^+(u)\}K(u) = \text{id}. \tag{3.5}$$

Let us introduce the generalized XXZ Gaudin operators [1] $\{H_j|j = 1, 2, \cdots, N\}$ associated with the spin-$\frac{1}{2}$ XXZ model with generic boundaries specified by the boundary K-matrices in (3.1) and (3.3):

$$H_j = \Gamma_j(z_j) + \sum_{k\neq j}^{2M} \frac{1}{\sin(z_j - z_k)} \left\{\sigma_k^+\sigma_j^- + \sigma_k^-\sigma_j^+ + \cos(z_j - z_k)\frac{\sigma_k^z\sigma_j^z - 1}{2}\right\}$$
$$+ \sum_{k\neq j}^{2M} \frac{K_j^{-1}(z_j)}{\sin(z_j + z_k)} \left\{\sigma_j^+\sigma_k^- + \sigma_j^-\sigma_k^+ + \cos(z_j + z_k)\frac{\sigma_j^z\sigma_k^z - 1}{2}\right\} K_j(z_j), \quad (3.6)$$

where $\Gamma_j(u) = \frac{\partial}{\partial\eta}\{\bar{K}_j(u)\}|_{\eta=0}K_j(u)$, $j = 1, \cdots, N$, with $\bar{K}_j(u) = tr_0\left\{K_0^+(u)\overline{R}_{0j}(2u)P_{0j}\right\}$, and $\{z_j\}$ correspond to the inhomogeneous parameters of the inhomogeneous spin-$\frac{1}{2}$ XXZ chain with generic open boundary. For a generic choice of the boundary parameters $\{\lambda_1,\ \lambda_2,\ \xi\}$, $\Gamma_j(u)$ is an non-diagonal matrix, in contrast to that of [21].

The XXZ Gaudin operators (3.6) are obtained by expanding the double-row transfer matrix $\tau(u)$ (2.13) at the point $u = z_j$ around $\eta = 0$:

$$\tau(z_j) = \tau(z_j)|_{\eta=0} + \eta H_j + O(\eta^2), \quad j = 1, \cdots, N, \quad (3.7)$$
$$H_j = \frac{\partial}{\partial\eta}\tau(z_j)|_{\eta=0}. \quad (3.8)$$

The relations (2.6) and (3.5) imply that the first term $\tau(z_j)|_{\eta=0}$ in the expansion (3.7) is equal to an identity, namely,

$$\tau(z_j)|_{\eta=0} = \text{id}. \quad (3.9)$$

Then the commutativity of the transfer matrices $\{\tau(z_j)\}$ (2.14) for a generic $\eta$ implies

$$[H_j, H_k] = 0, \quad i, j = 1, \cdots, N. \quad (3.10)$$

Thus the Gaudin system defined by (3.6) is integrable. Moreover, the fact that the Gaudin operators $\{H_j\}$ (3.6) can be expressed in terms of the transfer matrix of the inhomogeneous spin-$\frac{1}{2}$ XXZ open chain enables us to exactly diagonalize the operators by the algebraic Bethe ansatz method with the help of the "vertex-face" correspondence technique, as can be seen in the next section. The aim of this paper is to find the common eigenvectors and eigenvalues of the operators (3.6).

# 4 Eigenvalues and Bethe ansatz equations

## 4.1 Six-vertex SOS R-matrix and face-vertex correspondence

The simple root $\alpha$ and fundamental weight $\Lambda_1$ of $sl(2)$ are given in terms of the orthonormal basis $\{\epsilon_i\}$ as: $\alpha = \epsilon_1 - \epsilon_2$, $\Lambda_1 = \frac{\alpha}{2}$. Set

$$\hat{\imath} = \epsilon_i - \overline{\epsilon}, \quad \overline{\epsilon} = \frac{1}{2}\sum_{k=1}^{2}\epsilon_k, \quad i = 1, 2, \quad \text{then} \quad \sum_{i=1}^{2}\hat{\imath} = 0. \tag{4.1}$$

For each dominant weight $\Lambda = a\Lambda_1$ , $a \in \mathbb{Z}^+$(the set of non-negative integer), there exists an irreducible highest weight finite-dimensional representation $V_\Lambda$ of $A_1$ with the highest vector $|\Lambda\rangle$. For example the fundamental vector representation is $V_{\Lambda_1}$.

Let $\mathfrak{h}$ be the Cartan subalgebra of $A_1$ and $\mathfrak{h}^*$ be its dual. A finite dimensional diagonalisable $\mathfrak{h}$-module is a complex finite dimensional vector space $W$ with a weight decomposition $W = \oplus_{\mu\in\mathfrak{h}^*}W[\mu]$, so that $\mathfrak{h}$ acts on $W[\mu]$ by $x\,v = \mu(x)\,v$, $(x \in \mathfrak{h}, \quad v \in \ W[\mu])$. For example, the fundamental vector representation $V_{\Lambda_1} = V$, the non-zero weight spaces $W[\hat{\imath}] = \mathbb{C}\epsilon_i$, $i = 1, 2$.

For a generic $m \in V$, define

$$m_i = \langle m, \epsilon_i\rangle, \quad m_{ij} = m_i - m_j = \langle m, \epsilon_i - \epsilon_j\rangle, \quad i, j = 1, 2. \tag{4.2}$$

Let $R(u, m) \in \mathrm{End}(V \otimes V)$ be the R-matrix of the six-vertex SOS model, which is trigonometric limit of the eight-vertex SOS model [28] given by

$$R(u,m)=\sum_{i=1}^{2}R^{ii}_{ii}(u,m)E_{ii}\otimes E_{ii}+\sum_{i\neq j}\left\{R^{ij}_{ij}(u,m)E_{ii}\otimes E_{jj}+R^{ji}_{ij}(u,m)E_{ji}\otimes E_{ij}\right\}, \tag{4.3}$$

where $E_{ij}$ is the matrix with elements $(E_{ij})^l_k = \delta_{jk}\delta_{il}$. The coefficient functions are

$$R^{ii}_{ii}(u,\lambda) = 1, \quad R^{ij}_{ij}(u,\lambda) = \frac{\sin u \sin(m_{ij} - \eta)}{\sin(u + \eta)\sin(m_{ij})}, \quad i \neq j, \tag{4.4}$$

$$R^{ji}_{ij}(u,m) = \frac{\sin\eta \sin(u + m_{ij})}{\sin(u + \eta)\sin(m_{ij})}, \quad i \neq j, \tag{4.5}$$

and $m_{ij}$ is defined in (4.2). The R-matrix satisfies the dynamical (modified) quantum Yang-Baxter equation

$$\begin{aligned} &R_{12}(u_1 - u_2, m - \eta h^{(3)})R_{13}(u_1 - u_3, m)R_{23}(u_2 - u_3, m - \eta h^{(1)}) \\ &\quad = R_{23}(u_2 - u_3, m)R_{13}(u_1 - u_3, m - \eta h^{(2)})R_{12}(u_1 - u_2, m). \end{aligned} \tag{4.6}$$

We adopt the notation: $R_{12}(u, m-\eta h^{(3)})$ acts on a tensor $v_1 \otimes v_2 \otimes v_3$ as $R(u, m-\eta\mu) \otimes id$ if $v_3 \in W[\mu]$.

Define the following functions

$$\theta^{(1)}(u) = e^{-iu}, \quad \theta^{(2)}(u) = 1. \tag{4.7}$$

Let us introduce an intertwiner, i.e. an 2-component column vector $\phi_{m,m-\eta\hat{j}}(u)$ whose $k$-th element is

$$\phi^{(k)}_{m,m-\eta\hat{j}}(u) = \theta^{(k)}(u + 2m_j). \tag{4.8}$$

Using the intertwiner, one can derive the following face-vertex correspondence relation [29]

$$\begin{aligned} &\overline{R}_{12}(u_1-u_2)\phi_{m,m-\eta\hat{\imath}}(u_1) \otimes \phi_{m-\eta\hat{\imath},m-\eta(\hat{\imath}+\hat{\jmath})}(u_2) \\ &\quad = \sum_{k,l} R(u_1-u_2, m)^{kl}_{ij} \phi_{m-\eta\hat{l},\lambda-\eta(\hat{l}+\hat{k})}(u_1) \otimes \phi_{m,m-\eta\hat{l}}(u_2). \end{aligned} \tag{4.9}$$

Then the QYBE (2.3) of for the vertex-type R-matrix $\overline{R}(u)$ is equivalent to the dynamical Yang-Baxter equation (4.6) of the SOS R-matrix $R(u, m)$. For a generic $m$, we can introduce other types of intertwiners $\bar{\phi}$, $\tilde{\phi}$ satisfying the conditions,

$$\sum_{k=1}^{2} \bar{\phi}^{(k)}_{m,m-\eta\hat{\mu}}(u)\ \phi^{(k)}_{m,m-\eta\hat{\nu}}(u) = \delta_{\mu\nu}, \tag{4.10}$$

$$\sum_{k=1}^{2} \tilde{\phi}^{(k)}_{m+\eta\hat{\mu},m}(u)\ \phi^{(k)}_{m+\eta\hat{\nu},\lambda}(u) = \delta_{\mu\nu}, \tag{4.11}$$

from which one can derive the following relations:

$$\sum_{\mu=1}^{2} \bar{\phi}^{(i)}_{m,m-\eta\hat{\mu}}(u)\ \phi^{(j)}_{m,m-\eta\hat{\mu}}(u) = \delta_{ij}, \tag{4.12}$$

$$\sum_{\mu=1}^{2} \tilde{\phi}^{(i)}_{m+\eta\hat{\mu},m}(u)\ \phi^{(j)}_{m+\eta\hat{\mu},m}(u) = \delta_{ij}. \tag{4.13}$$

Through straightforward calculations, we find the K-matrices $K^{\pm}(u)$ given by (3.1) and (3.3) can be expressed in terms of the intertwiners and *diagonal* matrices $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ as follows

$$K^{-}(u)^s_t = \sum_{i,j} \phi^{(s)}_{\lambda-\eta(\hat{\imath}-\hat{\jmath}),\ \lambda-\eta\hat{\imath}}(u)\mathcal{K}(\lambda|u)^j_i \bar{\phi}^{(t)}_{\lambda,\ \lambda-\eta\hat{\imath}}(-u), \tag{4.14}$$

$$K^{+}(u)^s_t = \sum_{i,j} \phi^{(s)}_{\lambda,\ \lambda-\eta\hat{\jmath}}(-u)\tilde{\mathcal{K}}(\lambda|u)^j_i \tilde{\phi}^{(t)}_{\lambda-\eta(\hat{\jmath}-\hat{\imath}),\ \lambda-\eta\hat{\jmath}}(u). \tag{4.15}$$

Here the two *diagonal* matrices $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ are given by

$$\mathcal{K}(\lambda|u) \equiv \mathrm{Diag}(k(\lambda|u)_1,\, k(\lambda|u)_2) = \mathrm{Diag}(\frac{\sin(\lambda_1+\xi-u)}{\sin(\lambda_1+\xi+u)},\, \frac{\sin(\lambda_2+\xi-u)}{\sin(\lambda_2+\xi+u)}), \tag{4.16}$$

$$\begin{aligned}\tilde{\mathcal{K}}(\lambda|u) &\equiv \mathrm{Diag}(\tilde{k}(\lambda|u)_1,\, \tilde{k}(\lambda|u)_2)\\ &= \mathrm{Diag}(\frac{\sin(\lambda_{12}-\eta)\sin(\lambda_1+\xi+u+\eta)}{\sin\lambda_{12}\sin(\lambda_1+\xi-u-\eta)},\, \frac{\sin(\lambda_{12}+\eta)\sin(\lambda_2+\xi+u+\eta)}{\sin\lambda_{12}\sin(\lambda_2+\xi-u-\eta)}).\end{aligned} \tag{4.17}$$

Moreover, one can check that the matrices $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ satisfy the SOS type reflection equation and its dual, respectively [30]. Although the K-matrices $K^{\pm}(u)$ given by (3.1) and (3.3) are generally non-diagonal (in the vertex form), after the face-vertex transformations (4.14) and (4.15), the face type counterparts $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ *simultaneously* become diagonal. This fact enables us to apply the generalized algebraic Bethe ansatz method developed in [31] for SOS type integrable models to diagonalize the transfer matrices $\tau(u)$ (2.13).

## 4.2 Algebraic Bethe ansatz

Using the relations (4.12) and (4.13), the decomposition of $K^+(u)$ (4.15) and the diagonal property (4.17), the transfer matrix $\tau(u)$ (2.13) can be recasted into the following face type form

$$\begin{aligned}\tau(u) &= tr(K^+(u)\mathbb{T}(u))\\ &= \sum_{\mu,\nu} tr\left(K^+(u)\phi_{\lambda-\eta(\hat{\mu}-\hat{\nu}),\lambda-\eta\hat{\mu}}(u)\tilde{\phi}_{\lambda-\eta(\hat{\mu}-\hat{\nu}),\lambda-\eta\hat{\mu}}(u)\ \mathbb{T}(u)\phi_{\lambda,\lambda-\eta\hat{\mu}}(-u)\bar{\phi}_{\lambda,\lambda-\eta\hat{\mu}}(-u)\right)\\ &= \sum_{\mu,\nu} \bar{\phi}_{\lambda,\lambda-\eta\hat{\mu}}(-u)K^+(u)\phi_{\lambda-\eta(\hat{\mu}-\hat{\nu}),\lambda-\eta\hat{\mu}}(u)\ \tilde{\phi}_{\lambda-\eta(\hat{\mu}-\hat{\nu}),\lambda-\eta\hat{\mu}}(u)\ \mathbb{T}(u)\phi_{\lambda,\lambda-\eta\hat{\mu}}(-u)\\ &= \sum_{\mu,\nu} \tilde{\mathcal{K}}(\lambda|u)^{\mu}_{\nu}\mathcal{T}(\lambda|u)^{\nu}_{\mu} = \sum_{\mu} \tilde{k}(\lambda|u)_{\mu}\mathcal{T}(\lambda|u)^{\mu}_{\mu}.\end{aligned} \tag{4.18}$$

Here we have introduced the face-type double-row monodromy matrix $\mathcal{T}(\lambda|u)$

$$\begin{aligned}\mathcal{T}(\lambda|u)^{\nu}_{\mu} &= \tilde{\phi}_{\lambda-\eta(\hat{\mu}-\hat{\nu}),\lambda-\eta\hat{\mu}}(u)\ \mathbb{T}(u)\phi_{\lambda,\lambda-\eta\hat{\mu}}(-u)\\ &\equiv \sum_{i,j} \tilde{\phi}^{(j)}_{\lambda-\eta(\hat{\mu}-\hat{\nu}),\lambda-\eta\hat{\mu}}(u)\ \mathbb{T}(u)^j_i\phi^{(i)}_{\lambda,\lambda-\eta\hat{\mu}}(-u).\end{aligned} \tag{4.19}$$

This face-type double-row monodromy matrix can be expressed in terms of the face type R-matrix $R(\lambda|u)$ (4.3) and K-matrix $\mathcal{K}(\lambda|u)$ (4.16) (for the details, see equation (4.19) of

[31]). Moreover from (2.12), (4.9) and (4.13) one can derive the following exchange relations among $\mathcal{T}(\lambda|u)_\mu^\nu$:

$$\begin{aligned}
&\sum_{i_1,i_2}\sum_{j_1,j_2} R(u_1-u_2,\lambda)_{i_1,j_1}^{i_0,j_0}\mathcal{T}(\lambda+\eta(\hat{j}_1+\hat{i}_2)|u_1)_{i_2}^{i_1} \\
&\qquad\times R(u_1+u_2,\lambda)_{j_2,i_3}^{j_1,i_2}\mathcal{T}(\lambda+\eta(\hat{j}_3+\hat{i}_3)|u_2)_{j_3}^{j_2} \\
&=\sum_{i_1,i_2}\sum_{j_1,j_2} \mathcal{T}(\lambda+\eta(\hat{j}_1+\hat{i}_0)|u_2)_{j_1}^{j_0}R(u_1+u_2,\lambda)_{i_1,j_2}^{i_0,j_1} \\
&\qquad\times \mathcal{T}(\lambda+\eta(\hat{j}_2+\hat{i}_2)|u_1)_{i_2}^{i_1}R(u_1-u_2,\lambda)_{j_3,i_3}^{j_2,i_2}. \qquad (4.20)
\end{aligned}$$

As in [31], let us introduce a set of standard notions for convenience: [1]

$$\mathcal{A}(\lambda|u)=\mathcal{T}(\lambda|u)_1^1,\quad \mathcal{B}(\lambda|u)=\frac{\mathcal{T}(\lambda|u)_2^1}{\sigma(\lambda_{12})}, \qquad (4.21)$$

$$\mathcal{D}(\lambda|u)=\frac{\sigma(\lambda_{12}+\eta)}{\sigma(\lambda_{12})}\{\mathcal{T}(\lambda|u)_2^2-R(2u,\lambda+\eta\hat{1})_{1\,2}^{2\,1}\mathcal{A}(\lambda|u)\}. \qquad (4.22)$$

Hereafter, we adopt the convention: $\lambda_{ij}=\lambda_i-\lambda_j$, introduced in (4.2). After tedious calculations, we find the commutation relations among $\mathcal{A}(\lambda|u)$, $\mathcal{D}(\lambda|u)$ and $\mathcal{B}(\lambda|u)$. Here we give the relevant ones for our purpose,

$$\begin{aligned}
\mathcal{A}(\lambda|u)\mathcal{B}(\lambda-2\eta\hat{1}|v)&=\frac{\sin(u+v)\sin(u-v-\eta)}{\sin(u+v+\eta)\sin(u-v)}\mathcal{B}(\lambda-2\eta\hat{1}|v)\mathcal{A}(\lambda-2\eta\hat{1}|u) \\
&-\frac{\sin\eta\sin 2v}{\sin(u-v)\sin(2v+\eta)}\frac{\sin(u-v-\lambda_{12}+\eta)}{\sin(\lambda_{12}-\eta)}\mathcal{B}(\lambda-2\eta\hat{1}|u)\mathcal{A}(\lambda-2\eta\hat{1}|v) \\
&-\frac{\sin\eta}{\sin(u+v+\eta)}\frac{\sin(u+v+\lambda_{21}+2\eta)}{\sin(\lambda_{21}+\eta)}\mathcal{B}(\lambda-2\eta\hat{1}|u)\mathcal{D}(\lambda-2\eta\hat{1}|v), \qquad (4.23)
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}(\lambda|u)\mathcal{B}(\lambda-2\eta\hat{1}|v)&=\frac{\sin(u-v+\eta)\sin(u+v+2\eta)}{\sin(u-v)\sin(u+v+\eta)}\mathcal{B}(\lambda-2\eta\hat{1}|v)\mathcal{D}(\lambda-2\eta\hat{1}|u) \\
&-\frac{\sin\eta\sin(2u+2\eta)\sin(u-v+\lambda_{12}-\eta)}{\sin(u-v)\sin(2u+\eta)\sin(\lambda_{12}-\eta)}\mathcal{B}(\lambda-2\eta\hat{1}|u)\mathcal{D}(\lambda-2\eta\hat{1}|v) \\
&+\frac{\sin\eta\sin 2v\sin(2u+2\eta)\sin(u-v+\lambda_{12})}{\sin(u+v+\eta)\sin(2v+\eta)\sin(2u+\eta)\sin(\lambda_{12}-\eta)} \\
&\quad\times\mathcal{B}(\lambda-2\eta\hat{1}|u)\mathcal{A}(\lambda-2\eta\hat{1}|v), \qquad (4.24)
\end{aligned}$$

$$\mathcal{B}(\lambda-2\eta\hat{1}|u)\mathcal{B}(\lambda-4\eta\hat{1}|v)=\mathcal{B}(\lambda-2\eta\hat{1}|v)\mathcal{B}(\lambda-4\eta\hat{1}|u). \qquad (4.25)$$

Here we have used the identity $\hat{2}=-\hat{1}$ which can be derived from (4.1).

[1] The scalar factors in the definitions of the operators $\mathcal{B}(\lambda|u)$ and $\mathcal{D}(\lambda|u)$ are to make the relevant commutation relations as concise as (4.23)-(4.25).

In order to apply the algebraic Bethe ansatz method, in addition to the relevant commutation relations (4.23)-(4.25), one needs to construct a pseudo-vacuum state (also called reference state) which is the common eigenstate of the operators $\mathcal{A}$, $\mathcal{D}$ and is annihilated by the operator $\mathcal{C}$. In contrast to the case of the spin-$\frac{1}{2}$ XXZ open chain with *diagonal* $K^{\pm}(u)$ [19], for the open chain with generic *non-diagonal* K-matrices (3.1) and (3.3), the usually highest-weight state

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

is no longer the pseudo-vacuum state. However, after the face-vertex transformations (4.14) and (4.15), the face type counterparts K-matrices $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ *simultaneously* become diagonal. This suggests that one can transfer the spin-$\frac{1}{2}$ XXZ open chain with generic non-diagonal K-matrices into the corresponding SOS model with *diagonal* K-matrices $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ given by (4.16)-(4.17) (possibly after some local gauge transformations [29]). Then it is easy to construct the pseudo-vacuum in the "face language" and use the generalized algebraic Bethe ansatz method [31] to diagonalize the transfer matrix. We shall develop this game in the following.

Let us introduce the corresponding pseudo-vacuum state $|\Omega\rangle$

$$|\Omega\rangle = \phi_{\lambda-(N-1)\eta\hat{1},\lambda-N\eta\hat{1}}(-z_1) \otimes \phi_{\lambda-(N-2)\eta\hat{1},\lambda-(N-1)\eta\hat{1}}(-z_2) \cdots \otimes \phi_{\lambda,\lambda-\eta\hat{1}}(-z_N). \tag{4.26}$$

The state does only depend on the boundary parameters $\{\lambda_1, \lambda_2\}$ and the inhomogeneous parameters $\{z_j\}$. Using the technique developed in [31], after tedious calculations, we find that the pseudo-vacuum state given by (4.26) satisfies the following equations, as required,

$$\mathcal{A}(\lambda - N\eta\hat{1}|u)|\Omega\rangle = \frac{\sin(\lambda_1 + \xi - u)}{\sin(\lambda_1 + \xi + u)}|\Omega\rangle, \tag{4.27}$$

$$\mathcal{D}(\lambda - N\eta\hat{1}|u)|\Omega\rangle = \frac{\sin 2u \sin(\lambda_1 + \xi + u + \eta)\sin(\lambda_2 + \xi - u - \eta)}{\sin(2u + \eta)\sin(\lambda_1 + \xi + u)\sin(\lambda_2 + \xi + u)} \times \left\{ \prod_{k=1}^{N} \frac{\sin(u + z_k)\sin(u - z_k)}{\sin(u + z_k + \eta)\sin(u - z_k + \eta)} \right\} |\Omega\rangle, \tag{4.28}$$

$$\mathcal{C}(\lambda - N\eta\hat{1}|u)|\Omega\rangle = 0, \tag{4.29}$$

$$\mathcal{B}(\lambda - N\eta\hat{1}|u)|\Omega\rangle \neq 0. \tag{4.30}$$

Then the so-called Bethe states can be constructed by applying the creation operator $\mathcal{B}$ on the pseudo-vacuum state

$$|v_1, \cdots, v_M\rangle = \mathcal{B}(\lambda - 2\eta\hat{1}|v_1)\mathcal{B}(\lambda - 4\eta\hat{1}|v_2) \cdots \mathcal{B}(\lambda - 2M\eta\hat{1}|v_M)|\Omega\rangle. \tag{4.31}$$

From (4.18), (4.21) and (4.22) we can rewrite the transfer matrices $\tau(u)$ (2.13) in terms of the operators $\mathcal{A}$ and $\mathcal{D}$

$$\tau(u) = \frac{\sin(\lambda_2+\xi-u)\sin(\lambda_1+\xi+u)\sin(2u+2\eta)}{\sin(\lambda_2+\xi-u-\eta)\sin(\lambda_1+\xi-u-\eta)\sin(2u+\eta)}\mathcal{A}(\lambda|u) + \frac{\sin(\lambda_2+\xi+u+\eta)}{\sin(\lambda_2+\xi-u-\eta)}\mathcal{D}(\lambda|u). \tag{4.32}$$

Acting the above expression of the transfer matrices on the Bethe states $|v_1,\cdots,v_M\rangle$ (4.31) and repeatedly using the relevant commutation relations (4.23)-(4.25), we obtain

$$\tau(u)|v_1,\cdots,v_M\rangle = t(u)|v_1,\cdots,v_M\rangle + \frac{\sin(2u+2\eta)\sin\eta}{\sin(\lambda_2+\xi-u-\eta)\sin(\lambda_1+\xi-u-\eta)} \times\left\{\sum_{\alpha=1}^{M} F_\alpha|v_1,\cdots,v_{\alpha-1},u,v_{\alpha+1},\cdots,v_M\rangle\right\}. \tag{4.33}$$

Here, the term associated with function $t(u)$ is the so-called *wanted* term which gives rise to the eigenvalues and the term associated with $\{F_\alpha|\alpha=1,\cdots,M\}$ is the so-called *unwanted* term. They are given, respectively, by

$$\begin{aligned} t(u) = {} & \frac{\sin(\lambda_2+\xi-u)\sin(\lambda_1+\xi-u)\sin(2u+2\eta)}{\sin(\lambda_2+\xi-u-\eta)\sin(\lambda_1+\xi-u-\eta)\sin(2u+\eta)} \\ & \times\prod_{k=1}^{M}\frac{\sin(u+v_k)\sin(u-v_k-\eta)}{\sin(u+v_k+\eta)\sin(u-v_k)} \\ & + \frac{\sin(\lambda_2+\xi+u+\eta)\sin(\lambda_1+\xi+u+\eta)\sin 2u}{\sin(\lambda_2+\xi+u)\sin(\lambda_1+\xi+u)\sin(2u+\eta)} \\ & \times\prod_{k=1}^{M}\frac{\sin(u+v_k+2\eta)\sin(u-v_k+\eta)}{\sin(u+v_k+\eta)\sin(u-v_k)} \\ & \times\prod_{k=1}^{2M}\frac{\sin(u+z_k)\sin(u-z_k)}{\sin(u+z_k+\eta)\sin(u-z_k+\eta)}, \end{aligned} \tag{4.34}$$

and

$$\begin{aligned} F_\alpha = {} & \frac{\sin 2v_\alpha\sin(\lambda_2+\xi-v_\alpha)\sin(\lambda_1+\xi-v_\alpha)}{\sin(2v_\alpha+\eta)\sin(u+v_\alpha+\eta)\sin(u-v_\alpha)} \\ & \times\left\{\prod_{k\neq\alpha}^{M}\frac{\sin(v_\alpha+v_k)\sin(v_\alpha-v_k-\eta)}{\sin(v_\alpha+v_k+\eta)\sin(v_\alpha-v_k)}\right. \\ & -\frac{\sin(\lambda_2+\xi+v_\alpha+\eta)\sin(\lambda_2+\xi-v_\alpha-\eta)\sin(\lambda_1+\xi+v_\alpha+\eta)\sin(\lambda_1+\xi-v_\alpha-\eta)}{\sin(\lambda_2+\xi+v_\alpha)\sin(\lambda_2+\xi-v_\alpha)\sin(\lambda_1+\xi+v_\alpha)\sin(\lambda_1+\xi-v_\alpha)} \\ & \times\prod_{k\neq\alpha}^{M}\frac{\sin(v_\alpha+v_k+2\eta)\sin(v_\alpha-v_k+\eta)}{\sin(v_\alpha+v_k+\eta)\sin(v_\alpha-v_k)} \end{aligned}$$

$$\times \prod_{k=1}^{2M} \frac{\sin(v_\alpha + z_k)\sin(v_\alpha - z_k)}{\sin(v_\alpha + z_k + \eta)\sin(v_\alpha - z_k + \eta)} \Bigg\}, \quad \alpha = 1, \cdots, M. \tag{4.35}$$

The relation (4.33) tells us that the Bethe states $|v_1, \cdots, v_M\rangle$ are eigenstates of the transfer matrices $\tau(u)$ if the *unwanted* terms vanish: $F_\alpha = 0$, $\alpha = 1, \cdots, M$. This leads to the Bethe ansatz equations,

$$\frac{\sin(\lambda_2 + \xi + v_\alpha)\sin(\lambda_2 + \xi - v_\alpha)\sin(\lambda_1 + \xi + v_\alpha)\sin(\lambda_1 + \xi - v_\alpha)}{\sin(\lambda_2+\xi+v_\alpha+\eta)\sin(\lambda_2+\xi-v_\alpha-\eta)\sin(\lambda_1+\xi+v_\alpha+\eta)\sin(\lambda_1+\xi-v_\alpha-\eta)}$$
$$= \prod_{k\neq\alpha}^{M} \frac{\sin(v_\alpha + v_k + 2\eta)\sin(v_\alpha - v_k + \eta)}{\sin(v_\alpha + v_k)\sin(v_\alpha - v_k - \eta)}$$
$$\times \prod_{k=1}^{2M} \frac{\sin(v_\alpha + z_k)\sin(v_\alpha - z_k)}{\sin(v_\alpha + z_k + \eta)\sin(v_\alpha - z_k + \eta)}, \quad \alpha = 1, \cdots, M. \tag{4.36}$$

We have checked that the Bethe ansatz equations indeed ensure that the eigenvalues (4.34) of transfer matrices $\tau(u)$ are entire functions. Our result recovers that of [29] for the very special case $z_k = 0$, $k = 1, \cdots, 2M$.

## 4.3 Eigenstates and the corresponding eigenvalues

The relation (3.8) between $\{H_j\}$ and $\{\tau(z_j)\}$ and the fact that the first term of (3.7) is a c-number enable us to extract the eigenstates of the Gaudin operators and the corresponding eigenvalues from the results obtained in last subsection.

Let us introduce the state $|\Omega^{(0)}\rangle$,

$$|\Omega^{(0)}\rangle = \begin{pmatrix} e^{i(z_1-2\lambda_1)} \\ 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} e^{i(z_N-2\lambda_1)} \\ 1 \end{pmatrix} \tag{4.37}$$

This state can be obtained from the pseudo-vacuum state $|\Omega\rangle$ (4.26) by taking the limit: $|\Omega^{(0)}\rangle = \lim_{\eta\to 0}|\Omega\rangle$. Let us introduce the matrix $C(u) \in \mathrm{End}(V)$ associated the intertwiner vector $\phi$

$$C(u) = \begin{pmatrix} e^{-i(u+2\lambda_1)} & e^{-i(u+2\lambda_2)} \\ 1 & 1 \end{pmatrix}, \tag{4.38}$$

and the associated gauged Pauli operator $\sigma^-(u) \in \mathrm{End}(V)$,

$$\sigma^-(u) = C(u)\sigma^- C(u)^{-1}. \tag{4.39}$$

Then we define the states $\Psi(v_1,\cdots,v_M)$ and $\Phi_\alpha$:

$$\begin{aligned}\Psi(v_1,\cdots,v_M) = \prod_{\alpha=1}^{M}\Bigg(\sum_{k=1}^{2M}\Bigg\{&\frac{\sin(\lambda_1+\xi-v_\alpha)\sin(v_\alpha-z_k+\lambda_{12})}{\sin(\lambda_1+\xi+v_\alpha)\sin(v_\alpha-z_k)}\\ &-\frac{\sin(\lambda_2+\xi-v_\alpha)\sin(v_\alpha+z_k-\lambda_{12})}{\sin(\lambda_2+\xi+v_\alpha)\sin(v_\alpha+z_k)}\Bigg\}\sigma_k^-(-z_k)\Bigg)|\Omega^{(0)}\rangle, \quad (4.40)\end{aligned}$$

$$\begin{aligned}\Phi_\alpha = \prod_{\beta\neq\alpha}^{M}\Bigg(\sum_{k=1}^{2M}\Bigg\{&\frac{\sin(\lambda_1+\xi-v_\beta)\sin(v_\beta-z_k+\lambda_{12})}{\sin(\lambda_1+\xi+v_\beta)\sin(v_\beta-z_k)}\\ &-\frac{\sin(\lambda_2+\xi-v_\beta)\sin(v_\beta+z_k-\lambda_{12})}{\sin(\lambda_2+\xi+v_\beta)\sin(v_\beta+z_k)}\Bigg\}\sigma_k^-(-z_k)\Bigg)|\Omega^{(0)}\rangle. \quad (4.41)\end{aligned}$$

Noting the relations (3.8), (3.7) and (4.33)-(4.35), and using the same method as in [16], we derive the following relations

$$\begin{aligned}H_j\Psi(v_1,\cdots,v_M) = E_j\Psi(v_1,\cdots,v_M) + \sum_{\alpha=1}^{M}\Bigg\{&\frac{\sin 2z_j\sin 2v_\alpha}{\sin(z_j+v_\alpha)\sin(z_j-v_\alpha)}\\ &\times\frac{\sin(\lambda_2+\xi-v_\alpha)\sin(\lambda_1+\xi-v_\alpha)}{\sin(\lambda_2+\xi-z_j)\sin(\lambda_1+\xi+z_j)}f_\alpha\sigma_j^-(-z_j)\Phi_\alpha\Bigg\}. \quad (4.42)\end{aligned}$$

The functions $E_j$ and $f_\alpha$ are

$$E_j = \cot 2z_j + \sum_{j=1}^{2}\cot(\lambda_j+\xi-z_j) + \sum_{k=1}^{M}\frac{\sin 2z_j}{\sin(v_k-z_j)\sin(v_k+z_j)}, \quad (4.43)$$

$$\begin{aligned}f_\alpha = &\sum_{j=1}^{2}\frac{1}{\sin(\lambda_j+\xi-v_\alpha)\sin(\lambda_j+\xi+v_\alpha)} + \sum_{k=1}^{2M}\frac{1}{\sin(v_\alpha+z_k)\sin(v_\alpha-z_k)}\\ &-2\sum_{k\neq\alpha}^{M}\frac{1}{\sin(v_\alpha+v_k)\sin(v_\alpha-v_k)}. \quad (4.44)\end{aligned}$$

The equation (4.42) suggests that the state $\Psi(v_1,\cdots,v_M)$ is an eigenstate of the Gaudin operators $\{H_j\}$ if $\{v_j|j=1,\cdots,M\}$ is set to satisfy $f_\alpha=0$ for $\alpha=1,\cdots,M$. This leads to the corresponding Bethe ansatz equations

$$\begin{aligned}&\sum_{j=1}^{2}\frac{1}{\sin(\lambda_j+\xi-v_\alpha)\sin(\lambda_j+\xi+v_\alpha)} + \sum_{k=1}^{2M}\frac{1}{\sin(v_\alpha+z_k)\sin(v_\alpha-z_k)}\\ &\quad = 2\sum_{k\neq\alpha}^{M}\frac{1}{\sin(v_\alpha+v_k)\sin(v_\alpha-v_k)}, \quad \alpha=1,\cdots,M. \quad (4.45)\end{aligned}$$

## 5 Conclusions

We have studied the XXZ Gaudin model with generic boundaries specified by the non-diagonal K-matrices $K^{\pm}(u)$, (3.1) and (3.3). In addition to the inhomogeneous parameters $\{z_j\}$, the associated Gaudin operators $\{H_j\}$, (3.6), have three free parameters $\{\lambda_1, \lambda_2, \xi\}$, which give rise to three-parameter $(\lambda_1, \lambda_2, \xi)$ generalizations of those in [4] and two-parameter $(\lambda_1, \lambda_2)$ generalizations of those in [16, 21]. As seen from section 4, although the "vertex type" K-matrices $K^{\pm}(u)$ (3.1) and (3.3) are *non-diagonal*, the compositions, (4.14) and (4.15), lead to the *diagonal* "face-type" K-matrices, (4.16) and (4.17), after the face-vertex transformation. This enables us to successfuly construct the corresponding pseudo-vacuum state $|\Omega\rangle$ (4.26) and apply the algebraic Bethe ansatz method developed in [31] for the SOS type models to diagonalize the transfer matrix $\tau(u)$ of the inhomogeneous spin-$\frac{1}{2}$ XXZ open chain with the generic non-diagonal K-matrices. Furthermore, we have exactly diagonalized the Gaudin operators $\{H_j\}$, and derived their common eigenstates (4.40) and eigenvalues (4.43) as well as the associated Bethe ansatz equations (4.45).

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