ITP 92-59
December 10, 1992

E_7 as $D = 10$ space-time symmetry — Origin of the twistor transform

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ABSTRACT

Massless particle dynamics in $D = 10$ Minkowski space is given an E_7-covariant formulation, including both space-time and twistor variables. E_7 contains the conformal algebra as a subalgebra. Analogous constructions apply to $D = 3, 4$ and 6.

Submitted to *Physics Letters B*

It is well known that massless particle actions are conformal invariant (with the possible exception of $D=10$ superparticles, in which case the matter is not clearly understood). However, there is a still larger symmetry present, as will be demonstrated in this paper. Only bosonic particles will be treated. Explicit calculations apply to $D=10$ – completely analogous constructions are valid in $D=3,4$ and 6. Some technical details and conventions are found in the appendix.

One has traditionally two options for manifestly conformal formulations of particle dynamics, the space-time picture and, in $D=3,4,6$ or 10, the twistor picture. The space-time formulation, on one hand, with $P_mP^m\approx 0$ as only constraint, is easily made conformally covariant by enlarging the vectors X^m, P^m of $SO(1,9)$ to become vectors X^μ, P^μ of $SO(2,10)$. The constraints are taken to be $X_\mu X^\mu\approx 0$, $X_\mu P^\mu\approx 0$ and $P_\mu P^\mu\approx 0$. With gauge choices $X^\oplus=c$ (=constant) and $P^\oplus=0$, the ordinary space-time picture is recovered (this of course applies to any dimensionality).

The twistor picture[1-4], on the other hand, is reached via the twistor transform

$$\begin{aligned} P^m &= \tfrac{1}{2}\psi^\alpha\gamma^m_{\alpha\beta}\psi^\beta \\ \omega_\alpha &= X_m\gamma^m_{\alpha\beta}\psi^\beta \end{aligned} \tag{1}$$

which implies that the conformal spinor $Z^A=[\psi^\alpha,\omega_\alpha]^t$ satisfies seven constraints, generating S^7 (a covariantly conformal form of these constraints is given in ref.4).

Consider now a set of phase-space variables $\Xi^M\in 56$ of E_7. When $E_7\rightarrow Sp(2)\times SO(2,10)$ the the branching rule[5] is $56\rightarrow(2,12)+(1,32)$ and for the adjoint $133\rightarrow(3,1)+(2,32')+(1,66)$, so that $\Xi^M=(S^{a\mu},Z^A)$ and $133\ni T^{\mathcal{A}}=(T^{\{ab\}},T^a_{A'},T^{[\mu\nu]})$. If $\{\Xi^M,\Xi^N\}=g^{MN}$, E_7 is generated by $T^{\mathcal{A}}=\frac{1}{2}\Xi^M\Omega^{\mathcal{A}}_{MN}\Xi^N$, $\Omega^{\mathcal{A}}_{MN}$ being Clebsh-Gordan coefficients for $56\times 56\times 133\rightarrow 1$. $S^{a\mu}=(X^\mu,P^\mu)^t$ is to be interpreted as the conformal space-time vectors and Z^A as the twistor variables. The set of constraints is chosen to be

$$T^{\mathcal{A}}\approx 0 \tag{2}$$

which is obviously first class. The content of eq.(2) will now be analyzed.

The pure $Sp(2)$ part $T^{ab} \approx 0$ contains exactly the constraints stated above for the space-time picture. I make the same gauge choices, with $c = 1/\sqrt{2}$ (this reduces manifest covariance to $SO(1,9)$), solve for $X^{\ominus}, P^{\ominus}$ and insert in the remaining constraints. The resulting set of equations is highly reducible. With some help from formulas in the appendix, it is easy to verify that $T^{\oplus m} \approx 0$ and the second component of $T^{1}_{A'} \approx 0$ are exactly the twistor transform equations (1). Once these are fulfilled, all other identities in eq.(2) hold. The system described by eq.(2) is thus equivalent to a massless particle in $D = 10$, and may be gauge-fixed to either the space-time or the twistor picture.

In the dimensionalities $D = 3, 4$ and 6, the constructions are analogous to the one above (in $D = 4$ and 6 the twistor variables obey bilinear constraints generating S^1 and S^3 that also are part of the symmetry). The resulting algebras are $Sp(6)$, $SU(6)$ and SO(12), respectively. The algebras can be collectively described as $Sp(6;\mathbf{K}_\nu)$ in D=ν+2, where $\mathbf{K}_\nu$, ν=1,2,4,8 is the division algebra of dimension ν. They are conformal algebras of the Jordan algebras of 3×3 hermitean matrices with entries in $\mathbf{K}_\nu$[6,7]. The conformal algebra in $D = \nu+2$ is $SO(2,\nu+2) \approx Sp(4;\mathbf{K}_\nu)$, so one sees how it is contained in the present larger structure.

The real forms of the algebras are defined by their maximal compact subalgebras, which are the symmetric subalgebras $SU(3)\times U(1) \subset Sp(6)$, $SU(3)^2\times U(1) \subset SU(6)$, $SU(6)\times U(1) \subset SO(12)$ and $E_6 \times U(1) \subset E_7$ respectively. It is worth mentioning that the real form of $SO(12)$ does not belong to the class $SO(n, 12-n)$, and that it, in contrast to the $D = 10$ conformal algebra $SO(2,10)$, does posess a superextension[7].

It is probably motivated to regard the extended conformal symmetries considered in this paper not only as convenient algebraic constructions, but as fundamental geometric properties of combined space-time/twistor spaces underlying $D = \nu + 2$ Minkowski space.

APPENDIX

$Sp(2)$ spinor indices are raised and lowered with ϵ_{ab} as $y_a = \epsilon_{ab} y^b$. The representation 3 consists of symmetric matrices M^{ab}.

$SO(1,9)$ gamma-matrices are denoted $\gamma^m_{\dot\alpha\alpha}$, $\tilde\gamma^{m\alpha\dot\alpha}$ and obey

$$\gamma^{\{m}\tilde\gamma^{n\}} = \eta^{mn}\mathbf{1}, \quad \tilde\gamma^{\{m}\gamma^{n\}} = \eta^{mn}\mathbf{1}$$

with $\eta^{mn} = diag(-1,1,\ldots,1)$. From these, $SO(2,10)$ gamma-matrices $\Gamma^\mu_{A'A}$ are constructed:

$$\Gamma^\oplus = \begin{bmatrix} \sqrt{2}\,\mathbf{1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \Gamma^\ominus = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2}\,\mathbf{1} \end{bmatrix}, \quad \Gamma^m = \begin{bmatrix} 0 & \tilde\gamma^m \\ \gamma^m & 0 \end{bmatrix}.$$

Then,

$$\Gamma^{\{\mu A}_{A'}\Gamma^{\nu\}A'}_{B} = \eta^{\mu\nu}\delta^A_B \ , \quad \Gamma^{\{\mu A}_{A'}\Gamma^{\nu\}B'}_{A} = \eta^{\mu\nu}\delta^{B'}_{A'}$$

with $\eta^{\oplus\ominus} = -1$, η^{mn} as above. $\Gamma^{\mu\nu}_{AB}$ is defined as $\Gamma^{[\mu A'}_{A}\Gamma^{\nu]}_{A'B}$. The invariant spinor metrics

$$g_{AB} = \begin{bmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{bmatrix} = g_{A'B'}$$

are used to raise and lower spinor indices as $Y_A = g_{AB} Y^B$ and analogously for primed indices.

The non-zero Clebsh-Gordan coefficients for $56 \times 56 \to 133$ of E_7 are, with indices according to the maximal subalgebra $Sp(2) \times SO(2,10)$ (no attention paid to normalization of the generators):

$$\Omega^{ab}_{c\mu,d\nu} = \tfrac{1}{2}\eta_{\mu\nu}\delta^{\{a}_c\delta^{b\}}_d \ , \quad \Omega^{aA'}_{b\mu,A} = \Omega^{aA'}_{A,b\mu} = \delta^a_b\Gamma^{A'}_{\mu A} \ , \quad \Omega^{\mu\nu}_{a\kappa,b\lambda} = \epsilon_{ab}\delta^{\mu\nu}_{\kappa\lambda} \ , \quad \Omega^{\mu\nu}_{A,B} = -\tfrac{1}{4}\Gamma^{\mu\nu}_{AB} \ .$$

The invariant metric for $56 \times 56 \to 1$ is g_{MN}, with $g_{a\mu,b\nu} = \eta_{\mu\nu}\epsilon_{ab}$ and g_{AB} as above.

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