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Superstring Interactions in a pp-wave Background

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Abstract

We construct light-cone gauge superstring field theory in a pp-wave background with Ramond-Ramond flux. The leading term in the interaction Hamiltonian is determined up to an overall function of p^+ by requiring closure of the pp-wave superalgebra. The bosonic and fermionic Neumann matrices for this cubic vertex are derived, as is the interaction point operator. We comment on the development of a $1/\mu p^+$ expansion for these results.

Contents

1. Introduction

Recently Berenstein, Maldacena and Nastase [1] have argued that a sector of $\mathcal{N}=4$ SU(N) Yang-Mills theory containing operators with large R-charge J is dual to IIB superstring theory on a certain gravitational plane wave background (pp-wave) with Ramond-Ramond flux. The pp-wave solution of type IIB supergravity is [2]

$$ds^2 = -4dx^+dx^- - \mu^2 x_I x^I (dx^+)^2 + dx_I dx^I, \qquad F_{+1234} = F_{+5678} = 2\mu, \tag{1.1}$$

where $I = 1, \ldots, 8$. It has 32 supersymmetries and can be obtained as a Penrose limit from $AdS_5 \times S^5$ [3]. This application of the AdS/CFT correspondence [4] is particularly exciting because the string worldsheet theory in this background is exactly solvable, as shown by Metsaev and Tseytlin [5,6], whereas in the more familiar $AdS \times S$ dualities it is difficult to go beyond the supergravity approximation on the string theory side. Furthermore, in this case there are two expansion parameters $a = \frac{g_{YM}^2 N}{J^2} = \frac{1}{(\mu p^+ \alpha')^2}$ and $g = \frac{J^2}{N} = 4\pi g_s (\mu p^+ \alpha')^2$, so there exists the intriguing possibility that there is a regime in which both sides of the 'duality' are perturbative [7].

The authors of [1] obtained the light-cone gauge worldsheet action for a single free string in the pp-wave background by summing certain planar diagrams. It is natural to ask whether string interactions can be incorporated into this picture. In this paper we study the light-cone vertex corresponding to the splitting of one closed string into two closed strings (see Figure 1) in the pp-wave background. Our analysis follows closely the corresponding flat space calculation of Green, Schwarz and Brink [8] (see also [9]). We construct the light-cone gauge superstring field theory for this background and derive expressions for the Neumann matrices and interaction point operator. Note that our calculation is purely on the 'string side,' although it is our hope to eventually compare with results from the 'field theory side.'

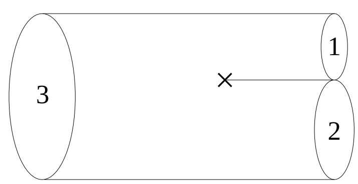

Fig. 1: One closed string splits into two via a local interaction X on the worldsheet.

Light-cone string field theory is an old subject, harking back to the work of Mandelstam [10,11] and extensively developed for the superstring by Green and Schwarz [12-15], and for the closed type IIB string by Green, Schwarz and Brink [8]. The light-cone formulation is especially appropriate for the pp-wave background because of the beautiful way in which light-cone 'string bits' naturally emerge in the field theory description [1] (see Figure 2). Although there are various equivalent approaches to studying string interactions, it is clear that any attempt to connect field theory results to string theory results benefits from consideration of light-cone gauge.

Furthermore, a significant advantage of the light-cone approach is that it is particularly clean (at least conceptually) in the pp-wave background[1]. In a general background, it is

[1] Modulo the difficulties associated with states having $p^+ = 0$, which are a perennial nuisance of light-cone theories.

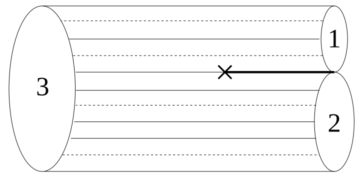

Fig. 2: A typical field theory diagram which corresponds to the string diagram of Figure 1 and represents the decay of of $\mathrm{Tr}[Z^{J_3}\phi^4]$ into $\mathrm{Tr}[Z^{J_2}\phi^2]\mathrm{Tr}[Z^{J_1}\phi^2]$. (We have been very schematic—see [1] for a detailed description of the identification between field theory operators and string states.)

unknown even how to determine what the observables of string theory are. In flat space we have the S-matrix and in AdS we have boundary correlation functions. In the pp-wave background, all string modes are massive in light-cone gauge, so a general state is labelled by p^+ and an infinite number of harmonic oscillator occupation numbers. In string perturbation theory, the observables of this theory are simply quantum mechanical transition amplitudes. This paper reports the first non-trivial term in the interaction Hamiltonian for this theory.

Having extolled the virtues of light-cone gauge, let us now be honest about some of its limitations. In flat space, the cubic string vertex is determined uniquely by the requirement that the nonlinearly realized super-Poincaré algebra closes to first order in the string coupling. The pp-wave superalgebra has the same number of supersymmetries as the flat space algebra, but it has fewer bosonic generators. In particular, there are no symmetries analogous to the J^{+-} or J^{-I} symmetries of flat space. Thus there are no symmetries which relate states of different values of p^+, so the pp-wave superalgebra is sufficient only to determine the cubic vertex up to a function $f(p^+_{(1)}, p^+_{(2)}, p^+_{(3)})$ of the values of p^+ for the three strings. (Of course p^+ is still conserved, so $\sum p^+_{(r)} = 0$.) Since this factor is independent of the particular string states in question, one could derive this factor once and for all from some other method—for example, by comparing with a supergravity calculation (which would be valid for $\mu p^+ \alpha' \ll 1$) or with the dual field theory (for $\mu p^+ \alpha' \gg 1$). Some other technical difficulties of working in light-cone gauge have been highlighted by Berkovits [16], who has constructed a quantizable, covariant, conformal worldsheet action for the superstring in a pp-wave background.

The plan of the paper is the following. In section 2 we review free string theory in the pp-wave background. In section 3 we explain how the interaction vertex is determined in light-cone string field theory à la Green, Schwarz and Brink. Then in section 4 we get our feet wet by studying the interaction just of the supergravity modes. In section 5 we derive the bosonic and fermionic Neumann matrices which couple all of the string modes, and in section 6 we construct the operators which supersymmetry requires to be inserted at the interaction point. Finally in section 7 we discuss the problem of developing a $1/\mu p^+$ expansion for comparison with calculations on the field theory side.

A number of recent papers have considered related issues, including obtaining pp-wave string theory from field theories with $\mathcal{N} = 1$ supersymmetry [17-19], orbifolded pp-waves [20-23], holography in the pp-wave background [24-26], and other aspects of Penrose limits and strings in plane wave backgrounds [27-34].

2. Free String Theory

In this section we review the Hilbert space and Hamiltonian for a free string on the pp-wave background (1.1). This has been discussed in [5,6], but it is useful to include a review here in order to fix notation and especially to display explicitly the nontrivial basis transformation required to write the fermionic part of the Hamiltonian in a canonical form.

We consider a string with fixed light-cone momentum $p^+ \neq 0$. We allow for p^+ to be negative, so that in the following sections when we consider string interactions, strings with $p^+ > 0$ ($p^+ < 0$) will be in outgoing (incoming) strings. Following [8], we set $\alpha' = 2$ and use the notation $\alpha = 2p^+$, $e(\alpha) = \text{sign}(\alpha)$.

2.1. Bosonic Sector

The light-cone action for the bosons is

$$S = \frac{e(\alpha)}{8\pi} \int d\tau \int_0^{2\pi|\alpha|} d\sigma \, [\partial_+ x^I \partial_- x^J - \mu^2 x^I x^J]\delta_{IJ}, \tag{2.1}$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$ and $I = 1, \ldots, 8$ labels the directions transverse to the light-cone. Since the bosonic sector of the theory is SO(8) invariant, we will frequently omit the I index in order to avoid unnecessary clutter.

Let us expand the string coordinate $x(\sigma)$ and momentum density $p(\sigma)$ in Fourier modes x_n and p_n according to the formula

$$\begin{aligned} x(\sigma) &= x_0 + \frac{1}{\sqrt{2}} \sum_{n\neq 0} (x_{|n|} - ie(n)x_{-|n|})e^{in\sigma/|\alpha|}, \\ p(\sigma) &= \frac{1}{2\pi|\alpha|} \left[p_0 + \frac{1}{\sqrt{2}} \sum_{n\neq 0} (p_{|n|} - ie(n)p_{-|n|})e^{in\sigma/|\alpha|} \right], \end{aligned} \tag{2.2}$$

where n ranges from $-\infty$ to $+\infty$. The complicated form of the coefficients has been chosen so that the modes x_n and p_n are associated to Hermitian operators $\hat{x}_n$ and $\hat{p}_n$. We will use capital letters $X(\sigma)$, $P(\sigma)$ to denote the operators given by (2.2) with x_n and p_n replaced by $\hat{x}_n$ and $\hat{p}_n$. The normalization is chosen so that the canonical commutation relation

$$[X(\sigma), P(\sigma')] = i\delta(\sigma - \sigma') \tag{2.3}$$

follows from imposing

$$[\hat{x}_m, \hat{p}_n] = i\delta_{mn}. \tag{2.4}$$

The coefficients p_n in the expansion (2.2) are related to the coefficients used in [8] by

$$p_0 \leftrightarrow p, \qquad p_n \leftrightarrow \sqrt{n} p_n^{\mathrm{I}}, \qquad p_{-n} \leftrightarrow \sqrt{n} p_n^{\mathrm{II}}, \tag{2.5}$$

for $n > 0$. This relation will help us transcribe some of the results of [8] with little difficulty. The basis (2.2) is more convenient for the pp-wave since there is no need to treat the zero mode separately. All of the string oscillations—left-movers, right-movers, and the zero modes—can be treated on an equal footing.

The Hamiltonian is

$$h = \frac{e(\alpha)}{2} \int_0^{2\pi|\alpha|} d\sigma \left[4\pi p^2 + \frac{1}{4\pi}((\partial_\sigma x)^2 + \mu^2 x^2) \right]. \tag{2.6}$$

Inserting $X(\sigma)$ and $P(\sigma)$ into (2.6) gives the operator version

$$H = \frac{1}{\alpha} \sum_{n=-\infty}^{\infty} \left[\hat{p}_n^2 + \frac{1}{4} \omega_n^2 \hat{x}_n^2 \right], \tag{2.7}$$

where $\omega_n = \sqrt{n^2 + (\alpha\mu)^2}$. We can write this in a canonical normal-ordered form by introducing the lowering operators

$$a_n = \frac{1}{\sqrt{\omega_n}} \hat{p}_n - \frac{i}{2} \sqrt{\omega_n} \hat{x}_n \tag{2.8}$$

which obey

$$[a_m, a_n^\dagger] = \delta_{mn}. \tag{2.9}$$

Then the Hamiltonian may be written as

$$H = \frac{1}{\alpha} \left[\sum_{n=-\infty}^{\infty} \omega_n a_n^\dagger a_n + A \right], \tag{2.10}$$

where

$$A = \frac{\delta^I{}_I}{2} \sum_{n=-\infty}^{\infty} \omega_n \tag{2.11}$$

is a normal-ordering constant which will cancel against the fermionic contribution in the next subsection.

2.2. Fermionic Sector

The action is

$$S = \frac{1}{8\pi} \int d\tau \int_0^{2\pi|\alpha|} d\sigma \ [i(\bar{\theta}\partial_\tau\theta + \theta\partial_\tau\bar{\theta}) - \theta\partial_\sigma\theta + \bar{\theta}\partial_\sigma\bar{\theta} - 2\mu\bar{\theta}\Pi\theta]. \tag{2.12}$$

Here θ^a is a complex positive chirality SO(8) spinor (although we will frequently omit the spinor index $a = 1, \ldots, 8$) and $\Pi = \Gamma^1\Gamma^2\Gamma^3\Gamma^4$. We expand θ and its conjugate momentum[2] as $\lambda \equiv \frac{1}{4\pi}\bar{\theta}$ as[3]

$$\begin{aligned} \theta(\sigma) &= \vartheta_0 + \frac{1}{\sqrt{2}} \sum_{n\neq 0} (\vartheta_{|n|} - ie(n)\vartheta_{-|n|}) e^{in\sigma/|\alpha|}, \\ \lambda(\sigma) &= \frac{1}{2\pi|\alpha|} \left[\lambda_0 + \frac{1}{\sqrt{2}} \sum_{n\neq 0} (\lambda_{|n|} - ie(n)\lambda_{-|n|}) e^{in\sigma/|\alpha|} \right], \end{aligned} \tag{2.13}$$

with the reality condition $\lambda_n^* = \frac{|\alpha|}{2}\vartheta_n$. The anticommutation relation

$$\{\Theta^a(\sigma), \Lambda^b(\sigma')\} = \delta^{ab}\delta(\sigma - \sigma') \tag{2.14}$$

follows from

$$\{\hat{\vartheta}_m^a, \hat{\lambda}_n^b\} = \delta^{ab}\delta_{mn}. \tag{2.15}$$

The Hamiltonian is

$$h = \frac{1}{2} \int_0^{2\pi|\alpha|} d\sigma \ \left[-4\pi\lambda\partial_\sigma\lambda + \frac{1}{4\pi}\theta\partial_\sigma\theta + 2\mu(\lambda\Pi\theta) \right]. \tag{2.16}$$

In terms of the mode operators, we have

$$H = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left[\frac{n}{2} \left(\frac{4}{\alpha^2} \hat{\lambda}_{-n}\hat{\lambda}_n - \hat{\vartheta}_{-n}\hat{\vartheta}_n \right) + 2\mu\hat{\lambda}_n\Pi\hat{\vartheta}_n \right]. \tag{2.17}$$

We would like to write the Hamiltonian in canonical form in terms of fermionic lowering operators b_n, $n = -\infty, \ldots, \infty$, satisfying

$$\{b_m, b_n^\dagger\} = \delta_{mn} \tag{2.18}$$

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[2] Actually, λ is $-i$ times the conjugate momentum to θ.

[3] The coefficients are related to those of [8] by $\lambda_0 = \lambda,\ \lambda_n = \lambda_n^{\mathrm{I}},\ \lambda_{-n} = \lambda_n^{\mathrm{II}}$ for $n > 0$.

(with spinor indices a, b suppressed). Because of the last term in (2.17), the required change of basis is nontrivial. For $n > 0$ we define

$$\begin{aligned}
\hat{\vartheta}_n &= \frac{1}{\sqrt{|\alpha|}} c_n \left[(1 + \rho_n \Pi) b_n + e(\alpha)(1 - \rho_n \Pi) b_{-n}^\dagger \right], \\
\hat{\vartheta}_{-n} &= \frac{1}{\sqrt{|\alpha|}} c_n \left[(1 + \rho_n \Pi) b_{-n} - e(\alpha)(1 - \rho_n \Pi) b_n^\dagger \right], \\
\hat{\lambda}_n &= \frac{\sqrt{|\alpha|}}{2} c_n \left[e(\alpha)(1 - \rho_n \Pi) b_{-n} + (1 + \rho_n \Pi) b_n^\dagger \right], \\
\hat{\lambda}_{-n} &= \frac{\sqrt{|\alpha|}}{2} c_n \left[-e(\alpha)(1 - \rho_n \Pi) b_n + (1 + \rho_n \Pi) b_{-n}^\dagger \right],
\end{aligned} \tag{2.19}$$

where

$$\rho_n = \frac{\omega_n - n}{\alpha\mu}, \qquad c_n = \frac{1}{\sqrt{1 + \rho_n^2}}, \tag{2.20}$$

and for the zero-modes we take

$$\begin{aligned}
\hat{\vartheta}_0 &= \frac{1}{\sqrt{2|\alpha|}} \left[(1 + e(\alpha)\Pi) b_0 + (1 - e(\alpha)\Pi) b_0^\dagger \right], \\
\hat{\lambda}_0 &= \frac{1}{2} \sqrt{\frac{|\alpha|}{2}} \left[(1 + e(\alpha)\Pi) b_0^\dagger + (1 - e(\alpha)\Pi) b_0 \right].
\end{aligned} \tag{2.21}$$

In terms of the b_n, the Hamiltonian takes the desired simple form

$$H = \frac{1}{\alpha} \left[\sum_{n=-\infty}^{\infty} \omega_n b_n^\dagger b_n + B \right], \tag{2.22}$$

where the normal-ordering constant

$$B = -\frac{\delta^a{}_a}{2} \sum_{n=-\infty}^{\infty} \omega_n \tag{2.23}$$

cancels (2.11) as advertised.

2.3. The Hilbert Space

We have expressed the normal-ordered free string Hamiltonian

$$H = \frac{1}{\alpha} \sum_{n=-\infty}^{\infty} \omega_n (\delta_{IJ} a_n^{I\dagger} a_n^J + \delta_{ab} b_n^{a\dagger} b_n^b), \qquad \omega_n = \sqrt{n^2 + (\alpha\mu)^2} \tag{2.24}$$

in terms of an infinite number of canonically normalized raising and lowering operators. The vacuum is defined by

$$a_n|\mathrm{vac}\rangle = b_n|\mathrm{vac}\rangle = 0, \qquad \forall n, \tag{2.25}$$

and a general state is constructed by acting on $|\mathrm{vac}\rangle$ with the creation operators $a_n^\dagger$ and $b_n^\dagger$. The subspace $\mathcal{H}_1$ of physical states is obtained by imposing the constraint

$$\sum_{n=-\infty}^{\infty} n(\delta_{IJ} a_n^{I\dagger} a_n^J + \delta_{ab} b_n^{a\dagger} b_n^b)|\Psi\rangle = 0, \tag{2.26}$$

which expresses the fact that there is no significance to the choice of origin for the σ coordinate on the string worldsheet.

The state $|\mathrm{vac}\rangle$ defined by (2.25) is not the "ground state" that we will be using for the rest of this paper. A more convenient choice for $|0\rangle$ [6] is a state which is the ground state for all the bosonic oscillators as well as the non-zero fermionic oscillators, but satisfies

$$\theta_0|0\rangle = 0. \tag{2.27}$$

The energy of this state is $H|0\rangle = 4\mu|0\rangle$, so the zero should be thought of signifying the occupation number, and not the energy. The energy can be raised (for $\alpha > 0$) by applying to $|0\rangle$ any of the four components of $(1+\Pi)\hat{\lambda}_0$, or lowered by applying any of the four components of $(1-\Pi)\hat{\lambda}_0$. The Hilbert spaces built on $|0\rangle$ and $|\mathrm{vac}\rangle$ are of course isomorphic and differ just by a relabeling of the states.

2.4. Symmetries of the Free Theory

The pp-wave superalgebra is a contraction of the $AdS_5 \times S^5$ superalgebra [36,37]. The isometries are generated by H, P^+, J^{+I}, J^{ij} and $J^{i'j'}$, where the indices run over $I = 1, \ldots, 8$, $i = 1, \ldots, 4$ and $i' = 5, \ldots, 8$. There are also 32 supercharges Q^+, $\bar{Q}^+$, Q^- and $\bar{Q}^-$. The interesting (anti)-commutation relations (i.e., those which are not the same as in flat space) are

$$[H, P^I] = i\mu^2 J^{+I}, \qquad [P^I, Q^-_{\dot{a}}] = \mu(\Pi\gamma^I)_{\dot{a}b} Q^+_\beta, \qquad [H, Q^+_a] = \mu \Pi_{ab} Q^+_b, \tag{2.28}$$

where $\Pi = \Gamma^1\Gamma^2\Gamma^3\Gamma^4$, and

$$\{Q^-_{\dot{a}}, \bar{Q}^-_{\dot{b}}\} = 2\delta_{\dot{a}\dot{b}} H + i\mu(\gamma_{ij}\Pi)_{\dot{a}\dot{b}} J^{ij} + i\mu(\gamma_{i'j'}\Pi)_{\dot{a}\dot{b}} J^{i'j'}, \tag{2.29}$$

with similarly complicated formulas for $\{Q^+,\bar{Q}^-\}$ and $\{Q^-,\bar{Q}^+\}$ (see for example [6]). We do not write the rest of the algebra since we will have no need for the precise form of these relations. We have only written (2.28) and (2.29) to highlight a very important fact about the pp-wave superalgebra, which is that although the algebra satisfied by the free generators is complicated, we will see that the algebra satisfied by the interaction terms in H, Q^- and $\bar{Q}^-$ (when they are promoted to operators which act non-linearly on the full string theory Hilbert space) satisfy essentially the same algebra as in flat space, in a sense made precise in the following section.

We will however make extensive use of the supercurrents[4]

$$q^+ = \int_0^{2\pi|\alpha|} d\sigma\ \sqrt{2}\lambda, \qquad \bar{q}^+ = \int_0^{2\pi|\alpha|} d\sigma\ \frac{\alpha}{\sqrt{2}}\theta \tag{2.30}$$

and

$$\begin{aligned} q^- &= \int_0^{2\pi|\alpha|} d\sigma\ \left[4\pi e(\alpha)p^I\gamma_I\lambda - \frac{i}{4\pi}\partial_\sigma x^I\gamma_I\theta - i\mu x^I\gamma_I\Pi\lambda\right], \\ \bar{q}^- &= \int_0^{2\pi|\alpha|} d\sigma\ \left[p^I\gamma_I\theta + ie(\alpha)\partial_\sigma x^I\gamma_I\lambda + \frac{i}{4\pi}e(\alpha)\mu x^I\gamma_I\Pi\theta\right]. \end{aligned} \tag{2.31}$$

The latter can be expressed in terms of string mode operators as

$$\begin{aligned} Q^- &= \frac{2}{\alpha}\left[\hat{p}_0^I\gamma_I\hat{\lambda}_0 - i\frac{\alpha}{2}\mu\hat{x}_0^I\gamma_I\Pi\hat{\lambda}_0\right] + \frac{1}{\sqrt{|\alpha|}}\sum_{n=-\infty}^{\infty}\sqrt{\omega_n}Q_n^-, \\ \bar{Q}^- &= \hat{p}_0^I\gamma_I\hat{\vartheta}_0 + i\frac{\alpha}{2}\mu\hat{x}_0^I\gamma_I\Pi\hat{\theta}_0 + \frac{e(\alpha)}{\sqrt{|\alpha|}}\sum_{n=-\infty}^{\infty}\sqrt{\omega_n}(Q_n^-)^\dagger, \end{aligned} \tag{2.32}$$

where the sum excludes $n = 0$, and

$$Q_n^- = c_n\left[e(\alpha)a_n^I\gamma_I(1+\rho_n\Pi)b_n^\dagger + e(n)a_n^{I\dagger}\gamma_I(1-\rho_n\Pi)b_{-n}\right]. \tag{2.33}$$

The fact that the expressions (2.31) differ so slightly (by the addition of a single term proportional to μ) from the flat space supercurrents will lead to considerable simplification below.

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[4] Note that $(q^\pm)^\dagger = e(\alpha)\bar{q}^\pm$. We must allow for this sign if we want $\{Q^-,\bar{Q}^-\} = 2H$, $\{Q^+,\bar{Q}^+\} = 2P^+$, where in both cases the right-hand side has sign $e(\alpha)$.

3. Interacting String Theory

The primary purpose of this section is to explain in detail what it is that we are trying to calculate in this paper. We review the light-cone string field theory formulation used by Green, Schwarz and Brink [8] to calculate the cubic string interaction in flat space. The only essential conceptual differences in the pp-wave case are that the superalgebra is smaller, and that in this case even the zero modes of the string have nonzero energy.[5]

3.1. Light-cone String Field Theory

The m-string Hilbert space $\mathcal{H}_m$ is the product of m copies of the single string Hilbert space $\mathcal{H}_1$ described in the previous section. The Hilbert space of the full string theory is the sum $\mathcal{H} = |\text{vacuum}\rangle \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$. The basic object in string field theory is the field operator Φ which can create or destroy complete strings. In a momentum space representation, Φ is a function of x^+ (the light-cone time), α, and the worldsheet momentum densities $p^I(\sigma)$ and $\lambda^a(\sigma)$. To be specific, we can expand Φ in the number-basis representation as (we suppress the fermionic degrees of freedom and the transverse index I for simplicity)

$$\Phi[p(\sigma)] = \sum_{\{n_k\}} \varphi_{\{n_k\}} \prod_{k=-\infty}^{\infty} \psi_{n_k}(p_k). \tag{3.1}$$

Here p_k is the k-th Fourier mode of $p(\sigma)$, as in the expansion (2.2). The sum is over all possible sets of harmonic oscillator occupation numbers $\{n_k\}$ which satisfy the physical state condition (2.26) and $\psi_n(p)$ is the harmonic oscillator wavefunction for occupation number n. Finally $\varphi_{\{n_k\}}$ is an operator $\mathcal{H}_m \to \mathcal{H}_{m\pm 1}$ that creates (if $\alpha > 0$) or destroys (if $\alpha < 0$) a string in the state $|\{n_k\}\rangle$ at time $\tau = 0$.

The Hamiltonian as well as the other generators of the pp-wave superalgebra from the previous section are promoted to operators on the full Hilbert space by expressing them in terms of the string field Φ. For example, the free Hamiltonian is

$$H_2 = \frac{1}{2} \int d\alpha D^8 p(\sigma) D^8 \lambda(\sigma)\ \Phi^\dagger \left(p^2 - \frac{1}{4}(\alpha\mu)^2 \frac{\partial^2}{\partial p^2} + \alpha\mu\lambda\Pi \frac{\partial}{\partial\lambda} \right) \Phi. \tag{3.2}$$

The subscript 2 signifies that this term is quadratic in string fields. The leading interaction H_3 which we seek to determine has an integrand which is cubic in Φ.

[5] The term "zero modes" is nevertheless not a misnomer, since we mean the modes which are the zeroth Fourier coefficient in the σ coordinate on the worldsheet.

3.2. Symmetries of the Interaction

There are two essentially different kinds of symmetries. The kinematical generators

$$P^+, P^I, J^+, J^{ij}, J^{i'j'}, Q^+, \bar{Q}^+ \tag{3.3}$$

do not involve $\frac{\partial}{\partial x^+}$ and therefore act at fixed light-cone time. These generators are quadratic in the string field, as in (3.2), and receive no corrections in the interacting theory. In other words, they act diagonally on the Hilbert space, mapping $\mathcal{H}_{(r)} \to \mathcal{H}_{(r)}$, where $\mathcal{H}_{(r)}$ is the Hilbert space of the r-th string. On the other hand, the dynamical generators

$$H, Q^-, \bar{Q}^- \tag{3.4}$$

are corrected by interactions, so that they can create or annihilate strings. The full Hamiltonian has the form

$$H = H_2 + \kappa H_3 + \cdots, \tag{3.5}$$

with similar expansions for Q^- and $\bar{Q}^-$, where κ is the coupling constant.[6] Our goal in this paper will be to construct the leading terms H_3, Q_3^- and $\bar{Q}_3^-$ by requiring that the generators (3.3) and (3.4) satisfy the pp-wave superalgebra to order $\mathcal{O}(\kappa)$. Presumably one could fix the higher order terms in a similar fashion, although we do not consider them in this paper.

Let us make a couple of general observations about the cubic interaction. First, we note that although transverse momentum is not conserved in the pp-wave background (1.1) because $\partial/\partial x^I$ is not a Killing vector, the $\mathcal{O}(\kappa)$ terms of the first relation in (2.28) imply that

$$[H_3, P^I] = 0. \tag{3.6}$$

(Indeed the same relation holds for any H_k with $k > 2$.) Therefore, the interaction Hamiltonian is translationally invariant—it is only the free Hamiltonian H_2 which breaks this symmetry (by confining particles in a harmonic oscillator potential). The cubic interaction must therefore contain a Δ-functional which implements conservation of transverse momentum locally on the worldsheet:

$$\Delta^8\left[\sum_{r=1}^{3} p^I_{(r)}(\sigma)\right], \tag{3.7}$$

[6] The question of what κ actually *is* in terms of other parameters, such as g_s, μ, p^+, α', π, or 2, is irrelevant to our analysis. We simply use κ as an auxiliary parameter to keep track of the perturbative expansion.

where r labels the three strings. Similarly, we see from (2.28) that although Q^+ does not commute with the free Hamiltonian H_2, it does commute with the interaction H_3, so the latter must also contain the Δ-functional

$$\Delta^8 \left[\sum_{r=1}^{3} \lambda^a_{(r)}(\sigma) \right]. \tag{3.8}$$

Finally, P^+ is a good quantum number in the pp-wave background, implying the delta-function

$$\delta(\alpha_{(1)} + \alpha_{(2)} + \alpha_{(3)}). \tag{3.9}$$

To summarize, we have gotten a lot of mileage out of the fact that although the pp-wave superalgebra (2.28) looks quite different from the flat space super-Poincaré algebra, those differences only affect the leading ($\kappa = 0$) terms in the dynamical generators. The interaction terms all (anti)-commute with the kinematical generators (3.3). Much of the machinery of [8] therefore carries over to the present case.

3.3. Number Basis Vertex

The conservation laws (3.7), (3.8) and (3.9) imply that the cubic interaction can be written in the form

$$H_3 = \int d\mu_3 \; h_3(\alpha_{(r)}, p_{(r)}(\sigma), x'_{(r)}(\sigma), \lambda_{(r)}(\sigma))\Phi(1)\Phi(2)\Phi(3), \tag{3.10}$$

where $\Phi(r) = \Phi[x^+, \alpha_{(r)}, p_{(r)}(\sigma), \lambda_{(r)}(\sigma)]$ is the string field for string r, h_3 is a factor to be determined, and the measure is

$$d\mu_3 = \left(\prod_{r=1}^{3} d\alpha_{(r)} D^8 p_{(r)}(\sigma) D^8 \lambda_{(r)}(\sigma) \right) \delta(\textstyle\sum \alpha_{(r)}) \Delta^8 \left[\sum p_{(r)}(\sigma) \right] \Delta^8 \left[\sum \lambda_{(r)}(\sigma) \right]. \tag{3.11}$$

There will be similar expressions for Q_3^- and $\bar{Q}_3^-$, involving different factors q_3 and $\bar{q}_3$ but the same measure (3.11). Therefore it is fair that an entire section of this paper is devoted exclusively to the study of this measure, which is determined just by consideration of the kinematical symmetries (3.3), while we postpone discussion of the prefactors h_3, q_3 and $\bar{q}_3$, which are determined by more complicated dynamical considerations, until section 6.

The delta function of $\sum \alpha_{(r)}$ guarantees that H_3 creates or annihilates at most a single string. Therefore it is sufficient to consider the action of H_3 from $\mathcal{H}_1 \to \mathcal{H}_2$ (or the adjoint of this process), since any other strings simply go along for the ride. It is convenient to express H_3 not as an operator $\mathcal{H}_1 \to \mathcal{H}_2$ but rather as a state $|H\rangle$ in $\mathcal{H}_3$ via the identification

$$\langle \Phi_1 | \langle \Phi_2 | \langle \Phi_3 | H \rangle = \langle \Phi_1 | \langle \Phi_2 | H_3 | \Phi_3 \rangle. \tag{3.12}$$

Similarly we identify Q_3^- and $\bar{Q}_3^-$ with states $|Q^-\rangle$, $|\bar{Q}^-\rangle$ in $\mathcal{H}_3$.

Using an identity which is roughly of the form

$$\sum_{n=0}^{\infty} |n\rangle \psi_n(p) \sim \exp\left(-\frac{1}{4}p^2 + pa^\dagger - \frac{1}{2}a^\dagger a^\dagger\right)|0\rangle \equiv \psi(p)|0\rangle \tag{3.13}$$

(this particular $\psi(p)$ is for an oscillator with $\omega = 4$, as an example) to relate the momentum wavefunctions $\psi_n(p)$ which appear in the string field expansion (3.1) to number-basis states (and a similar relation for fermionic oscillators), it is straightforward [13] to show that the operator (3.10) can be expressed as a state in the three string Hilbert space by the formula (again we suppress the fermions and transverse indices)

$$|H\rangle = \left[\int d\mu_3 \; h_3 \prod_{r=1}^{3} \prod_{k=-\infty}^{\infty} \psi(p_{k(r)})\right]|0\rangle. \tag{3.14}$$

Since we postpone discussion of the prefactor h_3, we will actually calculate first the three-string vertex just with the kinematical delta-functions (3.11),

$$|V\rangle \equiv \left[\int d\mu_3 \prod_{r=1}^{3} \prod_{k=-\infty}^{\infty} \psi(p_{k(r)})\right]|0\rangle. \tag{3.15}$$

These delta-functions are common to the three dynamical generators, which can therefore be represented as

$$|H\rangle = \hat{h}_3|V\rangle, \qquad |Q^-\rangle = \hat{q}_3|V\rangle, \qquad |\bar{Q}^-\rangle = \hat{\bar{q}}_3|V\rangle \tag{3.16}$$

for some operators $\hat{h}_3$, $\hat{q}_3$ and $\hat{\bar{q}}_3$ which will be determined in section 6.

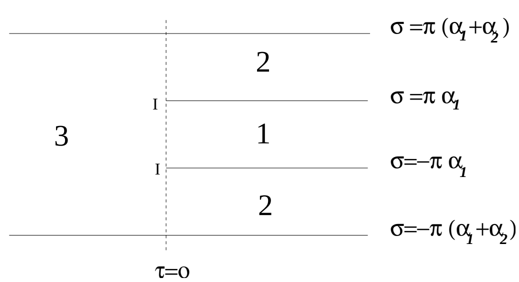

Fig. 3: Parameterization of the σ coordinate for the interaction of Figure 1. The interaction occurs at the points marked I, which are identified.

3.4. Parametrization of the Interaction

We consider three strings joining as in figure 2. The strings are labelled by $r = 1, 2, 3$ and in light-cone gauge their widths are $2\pi|\alpha_{(r)}|$, with $\alpha_{(1)} + \alpha_{(2)} + \alpha_{(3)} = 0$. We will take $\alpha_{(1)}$ and $\alpha_{(2)}$ positive for purposes of calculation, although the final vertex will be symmetric under the interchange of any two strings. The coordinates of the three strings may be parametrized by

$$\begin{aligned}
\sigma_{(1)} &= \sigma \qquad\qquad -\pi\alpha_{(1)} \leq \sigma \leq \pi\alpha_{(1)}, \\
\sigma_{(2)} &= \begin{cases} \sigma - \pi\alpha_{(1)} & \pi\alpha_{(1)} \leq \sigma \leq \pi(\alpha_{(1)} + \alpha_{(2)}), \\ \sigma + \pi\alpha_{(1)} & -\pi(\alpha_{(1)} + \alpha_{(2)}) \leq \sigma \leq -\pi\alpha_{(1)}, \end{cases} \\
\sigma_{(3)} &= -\sigma \qquad\quad -\pi(\alpha_{(1)} + \alpha_{(2)}) \leq \sigma \leq \pi(\alpha_{(1)} + \alpha_{(2)}).
\end{aligned} \tag{3.17}$$

In general, when we write a function of σ with a subscript (r), it is to be understood that the function has support only for σ within the range which coincides with string r. For example, $p_{(r)}(\sigma)$ will denote $p_{(r)}(\sigma) = \Theta_{(r)}(\sigma) p_{(r)}(\sigma_{(r)})$, where

$$\Theta_{(1)} = \theta(\pi\alpha_{(1)} - |\sigma|), \qquad \Theta_{(2)} = \theta(|\sigma| - \pi\alpha_{(1)}), \qquad \Theta_{(3)} = 1. \tag{3.18}$$

4. Supergravity Vertex

In this section we consider the cubic coupling of just the supergravity modes. This is interesting for its own sake, but more importantly it serves to clarify the procedure with a minimum of technical complication. The generators of the pp-wave superalgebra at $\kappa = 0$ are

$$\begin{aligned}
H &= \frac{1}{\alpha}\left(p^2 + \frac{1}{4}(\mu\alpha)^2 x^2\right) - \mu\theta\Pi\lambda, \\
P^I &= p^I, \\
J^{+I} &= \frac{1}{2}\alpha x^I, \\
Q_a^+ &= \sqrt{2}\lambda_a, \\
\bar{Q}_a^+ &= \frac{\alpha}{\sqrt{2}}\theta_a, \\
Q_{\dot{a}}^- &= \frac{2}{\alpha}\left[p^I(\gamma_I\lambda)_{\dot{a}} - i\frac{\alpha}{2}\mu x^I(\gamma_I\Pi\lambda)_{\dot{a}}\right], \\
\bar{Q}_{\dot{a}}^- &= p^I(\gamma_I\theta)_{\dot{a}} + i\frac{\alpha}{2}\mu x^I(\gamma_I\Pi\theta)_{\dot{a}}.
\end{aligned} \tag{4.1}$$

Here p, x, λ and θ are really the zero-modes p_0, x_0, λ_0 and θ_0 from section 2, but we will omit the subscript throughout this section.

Since H_3, Q_3^- and $\bar{Q}_3^-$ commute with the kinematical generators P^I, J^{+I}, Q^+ and $\bar{Q}^+$, we start by constructing a state $|V\rangle$ which satisfies

$$(\hat{p}_{(1)} + \hat{p}_{(2)} + \hat{p}_{(3)})|V\rangle = 0, \tag{4.2}$$

$$(\alpha_{(1)}\hat{x}_{(1)} + \alpha_{(2)}\hat{x}_{(2)} + \alpha_{(3)}\hat{x}_{(3)})|V\rangle = 0, \tag{4.3}$$

$$(\hat{\lambda}_{(1)} + \hat{\lambda}_{(2)} + \hat{\lambda}_{(3)})|V\rangle = 0, \tag{4.4}$$

$$(\alpha_{(1)}\hat{\theta}_{(1)} + \alpha_{(2)}\hat{\theta}_{(2)} + \alpha_{(3)}\hat{\theta}_{(3)})|V\rangle = 0. \tag{4.5}$$

The vertex $|V\rangle$ enforces conservation of P^I, J^{+I}, Q^+ and $\bar{Q}^+$ and forms the basic building block for constructing the interaction terms in the dynamical generators, which will take the form

$$|H\rangle = \hat{h}_3|V\rangle, \qquad |Q^{-\dot{a}}\rangle = \hat{q}_3^{-\dot{a}}|V\rangle, \qquad |\bar{Q}^{-\dot{a}}\rangle = \hat{\bar{q}}_3^{-\dot{a}}|V\rangle \tag{4.6}$$

for some operators $\hat{h}$, $\hat{q}^{-\dot{a}}$ and $\hat{\bar{q}}^{-\dot{a}}$. Of course these operators should (anti)-commute with the operators appearing in (4.2)-(4.5) so as not to ruin the conservation laws. We will then determine $\hat{H}$, $\hat{Q}^{-\dot{a}}$ and $\hat{\bar{Q}}^{-\dot{a}}$ by requiring the relations

$$\{Q^{-\dot{a}}, \bar{Q}^{-\dot{b}}\} = 2\delta^{\dot{a}\dot{b}}H + \mathcal{O}(\kappa), \qquad \{Q^{-\dot{a}}, Q^{-\dot{b}}\} = 0, \qquad \{\bar{Q}^{-\dot{a}}, \bar{Q}^{-\dot{b}}\} = 0 \tag{4.7}$$

to hold to first order in κ.

4.1. Bosonic Modes

In this subsection we construct the operator E_a such that $E_a|0\rangle$ is annihilated by $\sum \hat{p}_{(r)}$ and $\sum \alpha_{(r)}\hat{x}_{(r)}$. To construct such a state we use (3.14). The momentum eigenstate $|p\rangle$ may be expressed as $\psi(p)|0\rangle$, where

$$\psi(p) \sim \exp\left[-\frac{1}{2}a^\dagger a^\dagger + \frac{2}{\sqrt{|\alpha|\mu}}pa^\dagger - \frac{1}{|\alpha|\mu}p^2\right]. \tag{4.8}$$

The symbol $\sim$ denotes that we are ignoring overall factors. The bosonic part of the integral (3.14) is then

$$E_a^0 \equiv \int dp_{(1)}dp_{(2)}dp_{(3)}\psi(p_{(1)})\psi(p_{(2)})\psi(p_{(3)})\delta(p_{(1)} + p_{(2)} + p_{(3)}). \tag{4.9}$$

This is a simple Gaussian integral, and yields the result

$$E_a^0 \sim \exp\left[\frac{1}{2}\sum_{r,s=1}^{3} a_{(r)}^\dagger M^{rs} a_{(s)}^\dagger\right], \tag{4.10}$$

where

$$M^{rs} = \begin{pmatrix} \beta+1 & -\sqrt{-\beta(1+\beta)} & -\sqrt{-\beta} \\ -\sqrt{-\beta(1+\beta)} & -\beta & -\sqrt{1+\beta} \\ -\sqrt{-\beta} & -\sqrt{1+\beta} & 0 \end{pmatrix} \tag{4.11}$$

and $\beta = \alpha_{(1)}/\alpha_{(3)}$. It is trivial to check directly that the state $E_a^0|0\rangle$ satisfies the desired identities (4.2), (4.3). Additionally, we have the useful fact that

$$(\hat{x}_{(r)} - \hat{x}_{(s)})E_a^0|0\rangle = 0, \qquad \forall r, s. \tag{4.12}$$

Now let us ask the question: is $E_a^0|0\rangle$ the unique state which satisfies (4.2),(4.3)? Indeed, no: it is easy to see that any state of the form $f(\mathbb{P})E_a^0|0\rangle$, where f is an arbitrary function of

$$\mathbb{P} = \alpha_{(1)}\hat{p}_{(2)} - \alpha_{(2)}\hat{p}_{(1)}, \tag{4.13}$$

will satisfy the same identities by virtue of the fact that $\mathbb{P}$ commutes with $\sum \hat{p}_{(r)}$ and $\sum \alpha_{(r)}\hat{x}_{(r)}$. Note that we could have defined $\mathbb{P}$ with operators from any two of the three strings (instead of strings 1 and 2 as we have chosen), but it is easy to see from (4.2) and $\sum \alpha_{(r)} = 0$ that they are all equivalent when acting on $E_a^0|0\rangle$.

4.2. Fermionic Modes

Conservation of Q_a^+ and $\bar{Q}_a^+$ implies that the state $|V\rangle$ must include an operator E_b^0 which satisfies the identities $\sum \hat{\lambda}_{(r)}E_b^0|0\rangle = 0$ and $\sum \alpha_{(r)}\hat{\theta}_{(r)}E_b^0|0\rangle = 0$. These requirements almost uniquely determine

$$E_b^0 \sim \frac{1}{8!}\epsilon_{a_1\cdots a_8}\hat{\lambda}^{a_1}\cdots\hat{\lambda}^{a_8}|0\rangle, \qquad \text{where } \hat{\lambda} = \hat{\lambda}_{(1)} + \hat{\lambda}_{(2)} + \hat{\lambda}_{(3)}. \tag{4.14}$$

This choice also satisfies the useful relation

$$(\hat{\theta}_{(r)} - \hat{\theta}_{(s)})E_b^0|0\rangle = 0, \qquad \forall r, s. \tag{4.15}$$

The meaning of 'almost uniquely' is that we could include an arbitrary function $f(\Lambda)$ of

$$\Lambda = \alpha_{(1)}\hat{\lambda}_{(2)} - \alpha_{(2)}\hat{\lambda}_{(1)}. \tag{4.16}$$

This does not spoil (4.4), (4.5) since Λ anticommutes with $\sum \hat{\lambda}_{(r)}$ and $\sum \alpha_{(r)}\hat{\theta}_{(r)}$. Again the choice of which two strings to single out is irrelevant.

4.3. Prefactor

We have determined that the kinematical constraints (4.2)-(4.5) are satisfied by the state

$$|V\rangle \equiv E_a^0 E_b^0 |0\rangle, \tag{4.17}$$

and more generally by any state of the form $f(\mathbb{P},\Lambda)|V\rangle$ for any function f. Motivated by the expectation (explained in detail in section 6) that the prefactor will take essentially the same form as in flat space, we make the ansatz

$$\begin{aligned} |H\rangle &= \mathbb{P}^I \mathbb{P}^J v_{IJ}(\Lambda)|V\rangle, \\ |Q^-_{\dot a}\rangle &= \mathbb{P}^I s_{I\dot a}(\Lambda)|V\rangle, \\ |\bar Q^-_{\dot a}\rangle &= \mathbb{P}^I t_{I\dot a}(\Lambda)|V\rangle, \end{aligned} \tag{4.18}$$

where v, s and t are functions to be determined. Because of the terms proportional to μx in the supercharges (4.1), one might be tempted to include terms with $\mu\mathbb{R}^I$, where

$$\mathbb{R} \equiv \frac{1}{\alpha_{(3)}}(x_{(1)} - x_{(2)}), \tag{4.19}$$

but $\mathbb{R}|V\rangle = 0$ by virtue of (4.12). The $\mathcal{O}(\kappa)$ terms in the relations (4.7) require

$$\begin{aligned} \sum_{r=1}^3 Q^{-\dot a}_{(r)}|\bar Q^{-\dot b}\rangle + \sum_{r=1}^3 \bar Q^{-\dot b}_{(r)}|Q^{-\dot a}\rangle &= 2\delta^{\dot a\dot b}|H\rangle, \\ \sum_{r=1}^3 Q^{-\dot a}_{(r)}|Q^{-\dot b}\rangle + (\dot a \leftrightarrow \dot b) &= 0, \\ \sum_{r=1}^3 \bar Q^{-\dot a}_{(r)}|\bar Q^{-\dot b}\rangle + (\dot a \leftrightarrow \dot b) &= 0. \end{aligned} \tag{4.20}$$

Let us note the useful relations

$$\begin{aligned} \sum_{r=1}^3 [Q^{-\dot a}_{(r)}, \mathbb{P}^I] &= \mu(\gamma^I \Pi \Lambda)_{\dot a}, \\ \sum_{r=1}^3 [\bar Q^{-\dot a}_{(r)}, \mathbb{P}^I] &= \frac{1}{2}\alpha\mu(\gamma^I \Pi \Theta)_{\dot a}, \\ \sum_{r=1}^3 \{Q^{-\dot a}_{(r)}, \Lambda^b\} &= 0, \\ \sum_{r=1}^3 \{\bar Q^{-\dot a}_{(r)}, \Lambda^b\} &= \mathbb{P}^I(\gamma_I)_{\dot a b} - \frac{i}{2}\alpha\mu\mathbb{R}^I(\gamma_I\Pi)_{\dot a b}, \end{aligned} \tag{4.21}$$

where $\alpha \equiv \alpha_{(1)}\alpha_{(2)}\alpha_{(3)}$ and

$$\Theta = \frac{1}{\alpha_{(3)}}(\theta_{(1)} - \theta_{(2)}). \tag{4.22}$$

We also have

$$\sum_{r=1}^{3} \bar{Q}_{(r)}^{-\dot{a}}|V\rangle = 0, \qquad \sum_{r=1}^{3} Q_{(r)}^{-\dot{a}}|V\rangle = -\frac{2}{\alpha}\mathbb{P}^I(\gamma_I\Lambda)_{\dot{a}}|V\rangle. \tag{4.23}$$

Note that $\Theta|V\rangle = 0$ as a consequence of (4.15). Using these relations, we can substitute the ansatz (4.18) into (4.20) to obtain equations on v, s and t.

Three of the equations are identical to the analogous equations in flat space:

$$\begin{aligned} 2\delta_{\dot{a}\dot{b}}v_{IJ} &= \left[\frac{1}{2}(\gamma_I)_{\dot{a}b}\frac{\partial s_{J\dot{b}}}{\partial\Lambda^b} + \frac{1}{\alpha}t_{I\dot{a}}(\gamma_J\Lambda)_{\dot{b}}\right] + (I \leftrightarrow J), \\ 0 &= \left[(\gamma_I\Lambda)_{\dot{a}}s_{J\dot{b}} + (\dot{a} \leftrightarrow \dot{b})\right] + (I \leftrightarrow J), \\ 0 &= \left[(\gamma_I)_{\dot{a}a}\frac{\partial t_{J\dot{b}}}{\partial\Lambda^a} + (\dot{a} \leftrightarrow \dot{b})\right] + (I \leftrightarrow J). \end{aligned} \tag{4.24}$$

These equations arise as coefficients of $\mathbb{P}^I\mathbb{P}^J$ terms in (4.20) (hence the symmetrization on the I, J indices). But there are additional equations which come from the terms in (4.21) proportional to μ. These terms do not $\mathbb{P}^I\mathbb{P}^J$ so they must vanish separately:

$$\begin{aligned} 0 &= (\gamma^I\Pi\Lambda)_{\dot{a}}t_{I\dot{b}} + \frac{1}{2}\alpha(\gamma^I\Pi)_{\dot{b}b}\frac{\partial s_{I\dot{a}}}{\partial\Lambda^b}, \\ 0 &= (\gamma^I\Pi\Lambda)_{\dot{a}}s_{I\dot{b}} + (\dot{a} \leftrightarrow \dot{b}), \\ 0 &= (\gamma^I\Pi)_{\dot{a}a}\frac{\partial t_{I\dot{b}}}{\partial\Lambda^b} + (\dot{a} \leftrightarrow \dot{b}). \end{aligned} \tag{4.25}$$

The solution to the flat-space equations (4.24) is [8]

$$\begin{aligned} v_{IJ} &= \delta^{IJ} + \frac{1}{6\alpha^2}\gamma^{IK}_{ab}\gamma^{JK}_{cd}\Lambda^a\Lambda^b\Lambda^c\Lambda^d + \frac{16}{8!\alpha^4}\delta^{IJ}\epsilon^{abcdefgh}\Lambda^a\Lambda^b\Lambda^c\Lambda^d\Lambda^e\Lambda^f\Lambda^g\Lambda^h, \\ s_{I\dot{a}} &= 2\gamma^I_{\dot{a}a}\Lambda^a - \frac{8}{6!\alpha^2}\gamma^{IJ}_{ab}\gamma^J_{c\dot{a}}\epsilon^{abcdefgh}\Lambda^d\Lambda^e\Lambda^f\Lambda^g\Lambda^h, \\ t_{I\dot{a}} &= -\frac{2}{3\alpha}\gamma^{IJ}_{ab}\gamma^J_{c\dot{a}}\Lambda^a\Lambda^b\Lambda^c - \frac{16}{7!\alpha^3}\gamma^I_{a\dot{a}}\epsilon^{abcdefgh}\Lambda^b\Lambda^c\Lambda^d\Lambda^e\Lambda^f\Lambda^g\Lambda^h. \end{aligned} \tag{4.26}$$

It is straightforward, though tedious, to verify that functions satisfy the additional constraints (4.25).

This might seem like a small miracle. How can the flat space expressions (4.26), which are determined uniquely by super-Poincaré invariance, handle the additional burden of satisfying the extra constraints (4.25)? It turns out that (4.26) would satisfy (4.25) for any matrix Π which is symmetric and traceless. Together with $\Pi^2 = 1$, which is required to satisfy $\{Q^-, \bar{Q}^-\} = 2H$ at the level of free fields, we see that this miracle works *only* for the matrix $\Pi = \mathrm{diag}(1_4, -1_4)$ (in some basis), which is fortunately the Π that we happen to be interested in. Once again, we see how the pp-wave superalgebra miraculously makes our calculation almost as easy as in flat space.

In flat space supergravity, one is free to drop all but the quartic term in v, the quintic term in s, and the cubic term in t when the three external states are on-shell (see [12]). To see why this is so for v, note that the first and last terms are proportional to δ^{IJ} and hence give a contribution like

$$|H\rangle \sim \mathbb{P}^2 |V\rangle. \tag{4.27}$$

It is easy to check that

$$\mathbb{P}^2 = -\alpha \sum_{r=1}^{3} H_{(r)}, \tag{4.28}$$

so in fact the matrix element of (4.27) with any three on-shell states satisfying energy conservation will vanish. In fact the same is true for the states we have constructed in the pp-wave background, since here

$$\mathbb{P}^2 = -\alpha \sum_{r=1}^{3} \left[H_{(r)} - \frac{1}{4}\mu^2 \alpha_{(r)} \hat{x}_{(r)}^2 - \mu \hat{\lambda}_{(r)} \Pi \hat{\theta}_{(r)} \right]. \tag{4.29}$$

The second term vanishes when acting on $|V\rangle$ as a consequence of (4.12) and (4.3), and the third term vanishes when acting on $|V\rangle$ as a consequence of (4.4) and (4.15). We conclude that $\mathbb{P}^2|V\rangle$ vanishes when applied to three on-shell states.

5. String Theory Vertex

In this section we determine the operators E_a and E_b, constructed out of all the string modes, such that the vertex $|V\rangle = E_a E_b |0\rangle$ satisfies the kinematic constraints

$$\sum_{r=1}^{3} P_{(r)}(\sigma)|V\rangle = \sum_{r=1}^{3} e(\alpha_{(r)}) X_{(r)}(\sigma)|V\rangle = \sum_{r=1}^{3} \Lambda_{(r)}(\sigma)|V\rangle = \sum_{r=1}^{3} e(\alpha_{(r)}) \Theta_{(r)}(\sigma)|V\rangle = 0. \tag{5.1}$$

5.1. Bosonic Modes

The three-string vertex contains a Δ-functional of momentum conservation on the worldsheet:

$$\Delta\left[\sum_{r=1}^{3} p_{(r)}(\sigma)\right]. \tag{5.2}$$

Delta-functionals of this form can be defined as an infinite product of delta functions for the individual Fourier modes of the argument:

$$\Delta[f(\sigma)] = \prod_{m=-\infty}^{\infty} \delta\left(\int_{-\pi|\alpha_{(3)}|}^{\pi|\alpha_{(3)}|} e^{im\sigma/|\alpha_{(3)}|} f(\sigma)\, d\sigma\right) \tag{5.3}$$

The Fourier modes of (5.2) were worked out in [8], and it is an easy matter to use (2.5) to transcribe their result into the basis (2.2) for $p(\sigma)$ used in this paper. The result is

$$\Delta\left[\sum_{r=1}^{3} p_{(r)}(\sigma)\right] = \prod_{m=-\infty}^{\infty} \delta\left(p_{m(3)} + \sum_{n=-\infty}^{\infty} (X^{(1)}_{mn} p_{n(1)} + X^{(2)}_{mn} p_{n(2)})\right). \tag{5.4}$$

The matrices $X^{(1)}$ and $X^{(2)}$ are written in appendix B (we define $X^{(3)} = \mathbf{1}$). The Δ-functional (5.4) determines the Fourier modes of the third string in terms of the modes of the other two strings.

The properly normalized wavefunctions for momentum eigenstates in terms of the raising operators are

$$\psi(p_{n(r)}) = \left(\frac{2}{\pi\omega_{n(r)}}\right)^{1/4} \exp\left[-\frac{1}{\omega_{n(r)}} p_{n(r)}^2 + \frac{2}{\sqrt{\omega_{n(r)}}} p_{n(r)} a^\dagger_{n(r)} - \frac{1}{2} a^\dagger_{n(r)} a^\dagger_{n(r)}\right] \tag{5.5}$$

Therefore we have to do the integral

$$E_a = \int \left(\prod_{r=1}^{3} \prod_{n=-\infty}^{\infty} dp_{n(r)} \psi(p_{n(r)})\right) \Delta\left[\sum_{r=1}^{3} p_{(r)}(\sigma)\right]. \tag{5.6}$$

This is a fairly straightforward Gaussian integral, and in appendix C we show that the result is (up to an overall factor which is computed in the appendix)

$$E_a \sim \exp\left[\frac{1}{2} \sum_{r,s=1}^{3} \sum_{m,n=-\infty}^{\infty} a^\dagger_{m(r)} \overline{N}^{(rs)}_{mn} a^\dagger_{n(s)}\right] |0\rangle, \tag{5.7}$$

where the Neumann matrices $\overline{N}^{(rs)}_{mn}$ are given by

$$\overline{N}^{(rs)}_{mn} = \delta^{rs}\delta_{mn} - 2\sqrt{\omega_{m(r)}\omega_{n(s)}} (X^{(r)\mathrm{T}} \Gamma_a^{-1} X^{(s)})_{mn}, \tag{5.8}$$

in terms of the matrix

$$(\Gamma_a)_{mn} = \sum_{r=1}^{3} \sum_{p=-\infty}^{\infty} \omega_{p(r)} X^{(r)}_{mp} X^{(r)}_{np}. \tag{5.9}$$

It is straightforward to check that the result (5.8) reduces to the flat space Neumann matrices as $\mu \to 0$, although the zero modes must be separated and treated more carefully to find their contribution. In flat space, the worldsheet theory is conformal in light-cone gauge, so that it is natural to use a basis of left- and right-moving oscillators, $\alpha_n = \sqrt{n}(a_n + a_{-n})$ and $\tilde{\alpha}_n = \sqrt{n}(a_n - a_{-n})$. The fact that these decouple in a conformal theory requires an identity which in our notation is

$$\overline{N}^{(rs)}_{mn} = \overline{N}^{(rs)}_{-m,-n}. \tag{5.10}$$

In the notation of [8] the corresponding identity looks less trivial, but it is not hard to show that it is true. However, it is easy to check that the relation (5.10) no longer holds when $\mu \neq 0$, as expected for a worldsheet theory that is not conformal.

5.2. Fermionic Modes

The three-string vertex contains a Δ-functional of fermionic momentum conservation on the worldsheet:

$$\Delta\left[\sum_{r=1}^{3} \lambda_{(r)}(\sigma)\right] = \prod_{m=-\infty}^{\infty} \delta\left(\sum_{r=1}^{3} \sum_{n=-\infty}^{\infty} X^{(r)}_{mn} \lambda_{n(r)}\right). \tag{5.11}$$

It is natural to separate the zero mode, since the Grassmann wavefunction will couple the $n > 0$ and $n < 0$ modes. The zero modes give a contribution of E^0_b given by (4.14).

For the rest of this section, we always take $n > 0$ and write $-n$ to be explicit when we mean a negative component. In order to write (5.11) in an oscillator basis, we need to write the fermionic analogue of the wavefunction (5.5). Consider a state of the form

$$|\lambda_n \lambda_{-n}\rangle \sim \exp\left[\frac{2}{\alpha}\left(\lambda_{-n}\tilde{b}^\dagger_n - \lambda_n \tilde{b}^\dagger_{-n} + \frac{n}{2\omega_n}\tilde{b}^\dagger_n \tilde{b}^\dagger_{-n}\right)\right]|0\rangle, \tag{5.12}$$

where we define

$$\tilde{b}^\dagger_{\pm n} = \sqrt{|\alpha|}\frac{(1+\rho_n \Pi)}{c_n(1-\rho_n^2)} b^\dagger_{\pm n}. \tag{5.13}$$

It is easy to check that this state satisfies the eigenvalue equations

$$\begin{aligned} \frac{\sqrt{|\alpha|}}{2} c_n \left[e(\alpha)(1-\rho_n\Pi)b_{-n} + (1+\rho_n\Pi)b^\dagger_n\right]|\lambda_n\lambda_{-n}\rangle &= \lambda_n|\lambda_n\lambda_{-n}\rangle, \\ \frac{\sqrt{|\alpha|}}{2} c_n \left[-e(\alpha)(1-\rho_n\Pi)b_n + (1+\rho_n\Pi)b^\dagger_{-n}\right]|\lambda_n\lambda_{-n}\rangle &= \lambda_{-n}|\lambda_n\lambda_{-n}\rangle. \end{aligned} \tag{5.14}$$

Therefore, (5.12) is the desired expression, up to a term without $b^\dagger$ operators. That term is determined by checking that $\langle 0|\lambda_n\lambda_{-n}\rangle$ is the ground state wavefunction for the b_n and b_{-n} harmonic oscillators. To determine this wavefunction we invert the transformation (2.19) to find

$$b_n = \frac{\sqrt{|\alpha|}}{2} c_n \left[(1+\rho_n \Pi) \frac{\partial}{\partial \lambda_n} - \frac{2}{\alpha}(1-\rho_n\Pi)\lambda_{-n} \right], \tag{5.15}$$

where we have represented $\hat{\vartheta}_n = \frac{\partial}{\partial \lambda_n}$. The ground state wavefunction should be annihilated by b_n, which determines

$$\langle 0|\lambda_n \lambda_{-n}\rangle = \exp\left[\frac{2}{\alpha}\lambda_n P_n^2 \lambda_{-n}\right], \qquad P_n = \frac{(1-\rho_n\Pi)}{\sqrt{1-\rho_n^2}}. \tag{5.16}$$

We have omitted the overall normalization. Putting everything together gives the desired expression $|\lambda_n\lambda_{-n}\rangle = \chi(\lambda_n, \lambda_{-n})|0\rangle$, where

$$\chi(\lambda_n, \lambda_{-n}) = \exp\left[\frac{2}{\alpha}\left(\lambda_n P_n^2 \lambda_{-n} + \lambda_{-n}\tilde{b}_n^\dagger - \lambda_n \tilde{b}_{-n}^\dagger + \frac{n}{2\omega_n}\tilde{b}_n^\dagger \tilde{b}_{-n}^\dagger\right)\right]. \tag{5.17}$$

The integral we are interested in,

$$E_b = \int \left(\prod_{r=1}^{3}\prod_{n=1}^{\infty} d\lambda_{n(r)} d\lambda_{-n(r)}\ \chi(\lambda_{n(r)}, \lambda_{-n(r)})\right) \Delta\left[\sum_{r=1}^{3}\lambda_{(r)}(\sigma)\right], \tag{5.18}$$

can be performed in much the same way as the bosonic integral in appendix A. To express the result we introduce the matrices

$$Y_{mn}^{(r)} = P_{|m|}^{(3)} X_{mn}^{(r)} P_{|n|}^{(r)-1}, \qquad \overline{Y}_{mn}^{(r)} = P_{|m|}^{(3)} X_{-m,-n}^{(r)} P_{|n|}^{(r)-1} \tag{5.19}$$

with the understanding that $P_0 = 1$, and

$$\Gamma_b = \sum_{r=1}^{3} \overline{Y}^{(r)} Y^{(r)\mathrm{T}} \tag{5.20}$$

Then the vertex takes the form

$$E_b = \exp\left[\sum_{r,s=1}^{3}\sum_{m,n=1}^{\infty} b_{-m(r)}^\dagger Q_{mn}^{(rs)} b_{n(s)}^\dagger\right] E_b^0 |0\rangle, \tag{5.21}$$

where we have restored the zero mode contribution from (4.14), and where

$$Q_{mn}^{(rs)} = U_m^{(r)}\left[\frac{1}{\alpha_{(r)}}\delta^{rs}\delta_{mn}\left(1 + \frac{\mu\alpha_{(r)}}{\omega_{n(r)}}\Pi\right) - 2(\overline{Y}^{(r)\mathrm{T}}\Gamma_b^{-1}Y^{(s)})_{mn}\right] U_n^{(s)}, \tag{5.22}$$

with

$$U_n^{(r)} = \sqrt{|\alpha_{(r)}|\frac{(1+\rho_n\Pi)^2}{c_n(1-\rho_n^2)^{3/2}}}. \tag{5.23}$$

6. Interaction Point Operator

In this section we derive the operators $\hat{h}_3$, $\hat{q}_3$ and $\hat{\tilde{q}}_3$ which must be inserted in the three-string vertex in order to preserve supersymmetry. The necessity of introducing these operators arises from a short-distance effect on the worldsheet. Since this short-distance behaviour is unaffected by the addition of a mass term μ to the worldsheet action, the interaction point operators should be the same for the pp-wave as in flat space. This expectation has also been noted in [16].

6.1. The Necessity of a Prefactor

Suppose[7] we tried to take $\hat{h}_3 = 1$ so that $|H_3\rangle = |V\rangle$. Then the $\mathcal{O}(\kappa)$ terms in the relation $\{Q^-, H\} = 0$ tell us that

$$0 = \sum_{r=1}^{3} H_{(r)}|Q^-\rangle + \sum_{r=1}^{3} Q^-_{(r)}|V\rangle. \tag{6.1}$$

The first term vanishes when applied against an on-shell external state with $\sum_r H_r = 0$, so we need $\sum Q^-_{(r)}|V\rangle = 0$. From (2.32), this means we would need

$$\sum_{r=1}^{3} \int d\sigma_{(r)} \left[4\pi e(\alpha_{(r)}) p^I_{(r)} \gamma_I \lambda_{(r)} - \frac{i}{4\pi} \partial_\sigma x^I_{(r)} \gamma_I \theta_{(r)} - i\mu x^I_{(r)} \gamma_I \Pi \lambda_{(r)} \right] |V\rangle = 0. \tag{6.2}$$

Naively, the kinematical constraints (4.2)-(4.5) indeed guarantee that this is zero. The problem is that the operators in (6.2) applied to $|V\rangle$ are singular near the interaction point $\sigma = \pi\alpha_{(1)}$. For example, we have $p(\sigma)\lambda(\sigma)|V\rangle \sim \partial_\sigma x(\sigma)\theta(\sigma)|V\rangle \sim \epsilon^{-1}|V\rangle$ for $\sigma = \pi\alpha_{(1)} - \epsilon$. Therefore, although (6.2) vanishes pointwise for all $\sigma \neq \pi\alpha_{(1)}$, the singular operators nevertheless give a finite contribution when integrated over σ. This contribution can be calculated by deforming the contour in an appropriate way and reading off the residues of the poles [9].

However, the last term in (6.2) is not singular: $x(\sigma)\lambda(\sigma)|V\rangle \sim \epsilon^0|V\rangle$. Therefore (6.2) gives precisely the same constraint as in flat space, which suggests that the form of the prefactor is the same as it is in flat space. This is a manifestation of the aforementioned fact that μ should not affect the short-distance behaviour of the worldsheet theory. Armed with this knowledge, all that remains is to find stringy generalizations of the operators $\mathbb{P}$

[7] This argument appears in [9].

and Λ which we used in section 4 to write the prefactor in supergravity. These should be chosen so that when we make an ansatz analogous to (4.18), the form of the equations (4.24) and (4.25) on the functions s, t and v remains unchanged. Then the solutions s, t and v will also be unchanged.[8]

Now the derivation of the equations (4.24) and (4.25) follows quite trivially from the identities (4.21) and (4.23). Therefore, if we want to preserve these equations, we should look for stringy generalizations of $\mathbb{P}$ and Λ which preserve the relations (4.21) and (4.23) as closely as possible. For example, from the second equation in (4.23), we see that we can read off $\mathbb{P}$ and Λ by considering the action of $\sum Q^-_{(r)}$ on $|V\rangle$. But we have already considered this in (6.2), and concluded that the result is the same as in flat space! Therefore, we conclude that the stringy generalizations of $\mathbb{P}$ and Λ are *identical* to the corresponding flat space expressions in the continuum basis. These factors are written in the next subsection.

6.2. *Full Vertex with Prefactor*

A careful analysis of the singular terms in (6.2), which are the same as in flat space, necessitate consideration of the operators defined by [8]

$$
\begin{aligned}
P|V\rangle &= \lim_{\sigma\to\pi\alpha_{(1)}} -2\pi\sqrt{-\alpha}(\pi\alpha_{(1)}-\sigma)^{1/2}\left(P_{(1)}(\sigma)+P_{(1)}(-\sigma)\right)|V\rangle,\\
\partial X|V\rangle &= \lim_{\sigma\to\pi\alpha_{(1)}} -2\pi\sqrt{-\alpha}(\pi\alpha_{(1)}-\sigma)^{1/2}\left(\partial_\sigma X_{(1)}(\sigma)+\partial_\sigma X_{(1)}(-\sigma)\right)|V\rangle,\\
\Lambda|V\rangle &= \lim_{\sigma\to\pi\alpha_{(1)}} -2\pi\sqrt{-\alpha}(\pi\alpha_{(1)}-\sigma)^{1/2}\left(\Lambda_{(1)}(\sigma)+\Lambda_{(1)}(-\sigma)\right)|V\rangle.
\end{aligned}
\tag{6.3}
$$

Although we have singled out string 1, the final result turns out to be independent of this choice.

In terms of these, the operators which appear in the cubic string vertex in flat space are

$$
\begin{aligned}
\hat{h}_3 &= \left(P^I+\frac{1}{4\pi}\partial X^I\right)\left(P^J-\frac{1}{4\pi}\partial X^J\right)v_{IJ}(\Lambda),\\
\hat{q}_3^{-\dot{a}} &= P^I s_{I\dot{a}}(\Lambda)+\frac{i}{4\pi}\partial X^I t_{I\dot{a}}(\Lambda),\\
\hat{\tilde{q}}_3^{-\dot{a}} &= P^I t_{I\dot{a}}(\Lambda)-\frac{i}{4\pi}\partial X^I s_{I\dot{a}}(\Lambda).
\end{aligned}
\tag{6.4}
$$

[8] They will change slightly when we go to string theory because we will need terms which are not symmetric in I, J. By "unchanged," we mean unchanged relative to flat space.

The functions s and t are as given in (4.26), but the function v has some new terms which are not symmetric in I, J and therefore were ignored when we were working just with the supergravity modes. The full expression is

$$\begin{aligned} v^{IJ} = \delta^{IJ} - \frac{i}{\alpha}\gamma^{IJ}_{ab}\Lambda^a\Lambda^b + \frac{1}{6\alpha^2}\gamma^{IK}_{ab}\gamma^{JK}_{cd}\Lambda^a\Lambda^b\Lambda^c\Lambda^d \\ - \frac{4i}{6!\alpha^3}\gamma^{IJ}_{ab}\epsilon_{abcdefgh}\Lambda^c\Lambda^d\Lambda^e\Lambda^f\Lambda^g\Lambda^h + \frac{16}{8!\alpha^4}\delta^{IJ}\epsilon_{abcdefgh}\Lambda^a\Lambda^b\Lambda^c\Lambda^d\Lambda^e\Lambda^f\Lambda^g\Lambda^h. \end{aligned} \tag{6.5}$$

Since the result (6.4) automatically satisfies the superalgebra at $\mu = 0$, we need only check the $\mathcal{O}(\mu)$ terms in (4.20). The generalization of the relations (4.21) to include string modes is

$$\begin{aligned} [Q^{-\dot{a}}_{(r)}, P^I] &= -\frac{1}{4\pi}(\gamma^I\partial\Theta)_{\dot{a}} + \mu(\gamma^I\Pi\Lambda)_{\dot{a}}, \\ \{Q^{-\dot{a}}_{(r)}, \Lambda_b\} &= -\frac{i}{4\pi}\partial X^I(\gamma_I)_{\dot{a}b}, \\ [Q^{-\dot{a}}_{(r)}, \partial X^I] &= -4\pi i(\gamma^I\partial\Lambda)_{\dot{a}}, \\ [\bar{Q}^{-\dot{a}}_{(r)}, P^I] &= (\gamma^I\partial\Lambda)_{\dot{a}} - \frac{1}{4\pi}\mu(\gamma^I\Pi\Theta)_{\dot{a}}, \\ \{\bar{Q}^{-\dot{a}}_{(r)}, \Lambda_b\} &= P^I(\gamma_I)_{\dot{a}b} + \frac{i}{4\pi}\mu X^I(\gamma_I\Pi)_{\dot{a}b}, \\ [\bar{Q}^{-\dot{a}}_{(r)}, \partial X^I] &= -i(\gamma^I\partial\Theta)_{\dot{a}}, \end{aligned} \tag{6.6}$$

while the generalization of (4.23) is

$$\begin{aligned} Q^{-\dot{a}}|V\rangle &= -\frac{2}{\alpha}P^I(\gamma_I\Lambda)_{\dot{a}}|V\rangle, \\ \bar{Q}^{-\dot{a}}|V\rangle &= -\frac{i}{2\pi\alpha}\partial X^I(\gamma_I\Lambda)_{\dot{a}}|V\rangle. \end{aligned} \tag{6.7}$$

The operators X, Θ, $\partial\Lambda$ and $\partial\Theta$ are defined via the same limiting process of (6.3). Then $X|V\rangle = \Theta|V\rangle = 0$, while the resulting operators $\partial\Lambda$ are $\partial\Theta$ are singular when acting on $|V\rangle$. To render all of the relations (6.6) finite one should always work with the left- or right-moving linear combinations $Q^- \pm i\bar{Q}^-$ and $P \pm \frac{1}{4\pi}\partial X$, in which case all singular terms cancel. However we do not need to worry about this since we are only interested in the $\mathcal{O}(\mu)$ terms in (6.6)—we know from [8] that the terms independent of μ work out properly.

Using (6.6) and (6.7), it is trivial to check that the $\mathcal{O}(\mu)$ constraints in (4.20) reduce to the equations (4.25) which we have already shown to be satisfied by s and t. This completes the proof that the prefactors (6.4) are the same as in flat space, when written in a continuum basis. It would be straightforward to express these operators in the number basis by acting with the operators (6.3) on the bosonic and fermionic Neumann matrices in $|V\rangle$.

7. String Bits and a $1/\mu p^+$ Expansion

One of the motivations for this work is the hope to compare string interactions in the pp-wave background with predictions from the field theory of [1], which is expected to be perturbative in $1/\mu p^+$ [7] (we still have $\alpha' = 2$). Since we have obtained the Neumann matrices for arbitrary μp^+, it is therefore desirable to develop an expansion in $1/\mu p^+$ for large μ. In this section we outline how this might be accomplished. We hope to address this in more detail in future work.

In the limit of large μp^+, all Fourier modes of the string have the same energy, $\omega = |\alpha|\mu$, so the Fourier basis is no longer particularly natural. Instead of having a harmonic oscillator for each Fourier mode of the string, it is more natural to have a harmonic oscillator for each point on the worldsheet. This is because in this limit, the worldsheet action for the string becomes ultra-local, so that there is no coupling between $x(\sigma)$ and $x(\sigma')$ for $\sigma \neq \sigma'$.[9] Consideration of the dual field theory also suggests that it is natural to work in a local basis on the worldsheet, with one harmonic oscillator for each string 'bit' of [1].

With this point of view in mind we consider the case when $\alpha \neq 0$ is an integer, and we define $J = |\alpha| = 2|p^+|$. Discretizing the string worldsheet into J bits, we write the fields $x(\sigma)$ and $p(\sigma)$ (note that we continue to suppress the transverse index $I = 1, \dots, 8$) as

$$x(\sigma) = \sqrt{J} \sum_{a=0}^{J-1} \mathtt{x}_a \chi_a(\sigma), \qquad p(\sigma) = \frac{1}{2\pi\sqrt{J}} \sum_{a=0}^{J-1} \mathtt{p}_a \chi_a(\sigma), \tag{7.1}$$

where $\chi_a(\sigma)$ is a step function with support in the interval $[2\pi a, 2\pi(a+1)]$. The coordinate σ is identified modulo $2\pi J$. The commutation relations $[X(\sigma), P(\sigma')] = i\delta(\sigma - \sigma')$ are a consequence of

$$[\hat{\mathtt{x}}_a, \hat{\mathtt{p}}_b] = i\delta_{ab}. \tag{7.2}$$

Substituting (7.1) into the Hamiltonian (2.6) and ignoring the $\partial_\sigma X$ terms gives

$$H = \frac{e(\alpha)}{J} \sum_{a=0}^{J-1} \left[\hat{\mathtt{p}}_a^2 + \frac{1}{4}(\mu J)^2 \hat{\mathtt{x}}_a^2 \right]. \tag{7.3}$$

In terms of

$$\mathtt{a}_a = \frac{1}{\sqrt{\mu J}} \hat{\mathtt{p}}_a - \frac{i}{2}\sqrt{\mu J}\hat{\mathtt{x}}_a, \tag{7.4}$$

[9] We are grateful to L. Motl for emphasizing this point to us.

we have

$$H = \mu e(\alpha) \sum_{a=0}^{J-1} \mathtt{a}_a^\dagger \mathtt{a}_a, \tag{7.5}$$

where we have dropped the normal-ordering constant.

Now consider the interaction of three strings with $J_3 = J_1 + J_2 \equiv J$. We define

$$x_{(r)}(\sigma) = \sqrt{J_r} \sum_{a=0}^{J-1} \mathtt{x}_{a(r)} \chi_{a-\frac{1}{2}J_1}(\sigma), \qquad p_{(r)}(\sigma) = \frac{1}{2\pi\sqrt{J_r}} \sum_{a=0}^{J-1} \mathtt{p}_{a(r)} \chi_{a-\frac{1}{2}J_1}(\sigma). \tag{7.6}$$

The step functions have been shifted so that $a = 0$ coincides with $\sigma = -\pi J_1 = -\pi\alpha_{(1)}$. It is to be understood that $\mathtt{x}_{a(1)}$ and $\mathtt{p}_{a(1)}$ are zero for $a \geq J_1$, while $\mathtt{x}_{a(2)}$ and $\mathtt{p}_{a(2)}$ are zero for $a < J_1$.

Now we wish to write down the vertex $|V\rangle$ which expresses the constraint

$$\Delta \left[x_{(1)}(\sigma) + x_{(2)}(\sigma) - x_{(3)}(\sigma) \right]. \tag{7.7}$$

We should do an integral analogous to (5.6), with wavefunctions $\psi(\mathtt{p}_a)$ appropriate for the Hamiltonian (7.3), but it is trivial to check directly without having to do the integrals that the solution is

$$|V\rangle = \exp\left[\sum_{a=0}^{J_1-1} \mathtt{a}_{a(1)}^\dagger \mathtt{a}_{a(3)}^\dagger + \sum_{a=J_1}^{J-1} \mathtt{a}_{a(2)}^\dagger \mathtt{a}_{a(3)}^\dagger \right] |0\rangle. \tag{7.8}$$

Writing (7.8) it in the form

$$\exp\left[\frac{1}{2} \sum_{r,s=1}^{3} \sum_{a,b=0}^{J-1} \mathtt{a}_{a(r)}^\dagger \overline{N}_{ab}^{(rs)} \mathtt{a}_{b(s)}^\dagger \right], \tag{7.9}$$

shows that the Neumann matrices $\overline{N}_{ab}^{(rs)}$ are diagonal in ab and are given by

$$\overline{N}_{aa}^{(rs)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}_{rs}, \qquad \overline{N}_{aa}^{(rs)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}_{rs}, \tag{7.10}$$

respectively for $a < J_1$ and $a \geq J_1$.

A systematic expansion in $1/\mu p^+$ can be developed by retaining sub-leading terms in the Hamiltonian (7.3). These would modify the wave functions $\psi(\mathtt{p}_a)$ used to calculate the Neumann matrices through the integral (5.6). The main subtlety in deriving a $1/\mu p^+$ expansion is the fact that the limit $\mu p^+ \to \infty$ does not seem continuous. Although the modes are degenerate at $\mu p^+ = \infty$, for any finite value of μp^+, no matter how large, there are always string modes with $n \gg \alpha' \mu p^+$ which have much higher energy than the low-n modes. Even if one restricts attention to states which are occupied for small n, the calculation of the Neumann matrices requires the inversion of an infinite matrix Γ which is sensitive to arbitrarily large n.

It is also interesting to consider the change of basis between $x_{n(r)}$ and $\mathtt{x}_{a(r)}$, which takes the form

$$x_{n(r)} = \sum_{a=0}^{J-1} T^{(r)}_{na} \mathtt{x}_{a(r)} \tag{7.11}$$

for some easily calculated matrices $T^{(r)}_{na}$. Since an identical formula relates $p_{n(r)}$ to $\mathtt{p}_{n(r)}$, we can write the ladder operators in the number basis as

$$a_{n(r)} = \sum_{a=0}^{J-1} T^{(r)}_{na} \left[\frac{1}{\sqrt{\omega_{n(r)}}} \mathtt{p}_{a(r)} + \frac{i}{2} \sqrt{\omega_{n(r)}} \mathtt{x}_{a(r)} \right], \qquad \omega_{n(r)} = \sqrt{n^2 + (\mu J_r)^2}. \tag{7.12}$$

Comparison with (7.4) shows that in the strict $\mu J = \infty$ limit, the Bogolyubov transformation (7.12) does not mix creation and annihilation operators. However for finite μ the Bogolyubov transformation is nontrivial. This greatly complicates the calculation of factors like $\exp[(a^\dagger)^2] \to \exp[(\mathtt{a} + \mathtt{a}^\dagger)^2]$ which now have exponentials of non-commuting operators.

8. Conclusion

The bulk of this paper contains detailed calculations of various pieces of the three-string vertex. We summarize here some of the more interesting qualitative features of the result. The normal modes of a string in the pp-wave background have frequency $\omega_n = \sqrt{n^2 + (2p^+\mu)^2}$. Since even the zero-mode has nonzero energy, there is no need to separate the modes into left-movers, right-movers, and the zero modes. We simply work with all $n = -\infty, \ldots, \infty$ at once. The cubic string vertex written in a number oscillator basis contains a factor

$$\exp\left[\frac{1}{2} \sum_{r,s=1}^{3} \sum_{m,n=-\infty}^{\infty} a^\dagger_{m(r)} \overline{N}^{(rs)}_{mn} a^\dagger_{n(s)} \right] |0\rangle, \tag{8.1}$$

where r, s label the three strings (note that in our conventions, $a^\dagger_m$ is a raising operator regardless of the sign of m) and a formula for the Neumann matrices $\overline{N}^{(rs)}_{mn}$ is presented in (5.8). The slightly more complicated formula for the fermionic oscillators is given in

(5.21). In flat space, the left- and right-movers on the string worldsheet decouple in the corresponding expression for (8.1). This is of course no longer true in the present case since the worldsheet action is not conformal in light-cone gauge. Note that in this paper we have derived the 'un-amputated' Neumann coefficients, the analogue of $\overline{N}$ in [9]. It would be interesting to derive formulas for the 'amputated' Neumann matrices N, which might be more useful in studying on-shell processes. The vertex also contains a prefactor, given by the operator (6.4) evaluated at the interaction point. It is easy to check that the form of this operator (written in the continuum basis as in (6.3)) is identical to the corresponding expression in flat space [8].

We have not obtained an analytic expression for the matrix elements $\overline{N}^{(rs)}_{mn}$ for arbitrary μ, because (5.7) requires the inversion of a complicated infinite matrix (5.9). In practice, we have resorted to numerical calculation using a level-truncation scheme. Of particular interest is the limit $\mu p^+ \to \infty$, since this is the weak-coupling limit in the dual Yang-Mills theory. In this limit it is more convenient to use a basis where one has one harmonic oscillator for each point on the string worldsheet. The Neumann matrices in this limit are presented in section 7, and we have discussed how one might develop a $1/\mu p^+$ expansion in order to compare with transition amplitudes calculated from the field theory.

A more ambitious project would be to compare one-loop corrections to the masses of string states in the pp-wave background. The corresponding field theory calculation has been achieved in [7] for some states. In order to reproduce this calculation on the string theory side we would need to know the quartic term H_4 in the interaction Hamiltonian, since the masses receive corrections from H_4 as well as H_3^2. Finally, it would be interesting to study the interactions of open strings. D-branes in the pp-wave background have recently been considered in [38-43], and open strings have been constructed in [44,45]. Finally, it would be very interesting if the $\mu p^+ = \infty$ theory is sufficiently simple to allow an exact solution—meaning a construction of the interaction Hamiltonian to all orders.

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Appendix A. Some Matrices

In this appendix we write down the matrices $X^{(r)}$ which appear in the Δ-functional (5.4). We define $\beta = \alpha_{(1)}/\alpha_{(3)}$. Then we consider for $m, n > 0$ the matrices of [13],

$$\begin{aligned} A^{(1)}_{mn} &= (-1)^m \sqrt{\frac{n}{m}} \frac{2}{\pi\alpha_{(1)}} \int_0^{\pi\alpha_{(1)}} d\sigma \cos\frac{m\sigma}{\alpha_{(3)}} \cos\frac{n\sigma}{\alpha_{(1)}} \\ &= (-1)^{m+n+1} \frac{2\sqrt{mn}}{\pi} \frac{\beta \sin m\pi\beta}{n^2 - m^2\beta^2}, \\ A^{(2)}_{mn} &= (-1)^m \sqrt{\frac{n}{m}} \frac{2}{\pi\alpha_{(2)}} \int_{\pi\alpha_{(1)}}^{\pi(\alpha_{(1)}+\alpha_{(2)})} d\sigma \ \cos\frac{m\sigma}{\alpha_{(3)}} \cos\frac{n(\sigma - \pi\alpha_{(1)})}{\alpha_{(2)}} \\ &= (-1)^{m+1} \frac{2\sqrt{mn}}{\pi} \frac{(\beta+1)\sin m\pi\beta}{n^2 - m^2(\beta+1)^2} \\ C_{mn} &= m\delta_{mn}, \end{aligned} \tag{A.1}$$

and the vector

$$B_m = -\frac{(-1)^m}{\sqrt{m}} \frac{2}{\pi\alpha_{(1)}\alpha_{(2)}} \int_0^{\pi\alpha_{(1)}} d\sigma \ \cos\frac{m\sigma}{\alpha_{(3)}} = (-1)^{m+1} \frac{2}{\pi} \frac{\alpha_{(3)}}{\alpha_{(1)}\alpha_{(2)}} m^{-3/2} \sin m\pi\beta. \tag{A.2}$$

We define $X^{(3)}_{mn} = \delta_{mn}$, while for $r = 1, 2$ we can express the matrix $X^{(r)}$ as

$$\begin{aligned} X^{(r)}_{mn} &= (C^{1/2} A^{(r)} C^{-1/2})_{mn} && \text{if } m, n > 0, \\ &= \frac{\alpha_{(3)}}{\alpha_{(r)}} (C^{-1/2} A^{(r)} C^{1/2})_{-m,-n}, && \text{if } m, n < 0, \\ &= -\frac{1}{\sqrt{2}} \epsilon^{rs} \alpha_{(s)} (C^{1/2} B)_m && \text{if } n = 0 \text{ and } m > 0, \\ &= 1 && \text{if } m = n = 0, \\ &= 0 && \text{otherwise.} \end{aligned} \tag{A.3}$$

Schematically, the matrix $X^{(r)}_{mn}$ is given for $r = 1, 2$ by

$$\begin{pmatrix} \begin{pmatrix} & \vdots & \\ \cdots & \frac{\alpha_{(3)}}{\alpha_{(r)}} (C^{-1/2} A^{(r)} C^{1/2})_{-m,-n} & \cdots \\ & \vdots & \end{pmatrix} & & \\ & 1 & \\ & \begin{pmatrix} \vdots \\ -\frac{1}{\sqrt{2}} \epsilon^{rs} \alpha_{(s)} (C^{1/2} B)_m \\ \vdots \end{pmatrix} & \begin{pmatrix} & \vdots & \\ \cdots & (C^{1/2} A^{(r)} C^{-1/2})_{mn} & \cdots \\ & \vdots & \end{pmatrix} \end{pmatrix}. \tag{A.4}$$

Appendix B. Harmonic Oscillator Integrations

In this appendix we evaluate the integral

$$E_a = \int \left(\prod_{r=1}^{3} \prod_{n=-\infty}^{\infty} dp_{n(r)} \psi(p_{n(r)}) \right) \Delta \left[\sum_{r=1}^{3} p_{(r)}(\sigma) \right] \tag{B.1}$$

using (5.4) and (5.5). Overall factors of 2 and π will be dropped. Let us define the matrix

$$(C_{(r)})_{mn} = \omega_{n(r)} \delta_{mn} = \sqrt{n^2 + (\alpha_{(r)}\mu)^2} \delta_{mn}. \tag{B.2}$$

These matrices become singular in the flat-space limit $\mu \to 0$. The quantity in (B.1) in parentheses may be written as

$$\prod_{r=1}^{3} dp_{(r)} \ \left(\det C_{(r)}\right)^{-1/4} \exp\left[-p_{(r)}^{\mathrm{T}} C_{(r)}^{-1} p_{(r)} + 2a_{(r)}^{\dagger\mathrm{T}} C_{(r)}^{-1/2} p_{(r)} - \frac{1}{2} a_{(r)}^{\dagger\mathrm{T}} a_{(r)}^{\dagger} \right], \tag{B.3}$$

where we have introduced an obvious vector notation in order to suppress the n index which is summed from $-\infty$ to $+\infty$. Let us make the change of variable

$$C_{(r)}^{-1/2} p_{(r)} \to p_{(r)}. \tag{B.4}$$

Then (B.3) becomes

$$\prod_{r=1}^{3} dp_{(r)} \ \left(\det C_{(r)}\right)^{1/4} \exp\left[-p_{(r)}^{\mathrm{T}} p_{(r)} + 2a_{(r)}^{\dagger\mathrm{T}} p_{(r)} - \frac{1}{2} a_{(r)}^{\dagger\mathrm{T}} a_{(r)}^{\dagger} \right]. \tag{B.5}$$

Meanwhile, the Δ-functional (5.4) becomes

$$\delta\left(\sum_{r=1}^{3} X^{(r)} p_{(r)} \right) \to \delta\left(\sum_{r=1}^{3} X^{(r)} C_{(r)}^{1/2} p_{(r)} \right) = \left(\det C_{(3)}\right)^{-1/2} \delta\left(\sum_{r=1}^{3} \widetilde{X}^{(r)} p_{(r)} \right), \tag{B.6}$$

where

$$\widetilde{X}^{(r)} = C_{(3)}^{-1/2} X^{(r)} C_{(r)}^{1/2}. \tag{B.7}$$

Using (B.6) to eliminate $p_{(3)}$ in the integral (B.5) leaves an integral of the form

$$E_a = \mu_0 \int dp_{(1)} dp_{(2)} \ \exp\left[-\begin{pmatrix} p_{(1)}^{\mathrm{T}} & p_{(2)}^{\mathrm{T}} \end{pmatrix} M \begin{pmatrix} p_{(1)} \\ p_{(2)} \end{pmatrix} + 2H^{\mathrm{T}} \begin{pmatrix} p_{(1)} \\ p_{(2)} \end{pmatrix} - \frac{1}{2} \sum_{r=1}^{3} a_{(r)}^{\dagger\mathrm{T}} a_{(r)}^{\dagger} \right], \tag{B.8}$$

where we have introduced the determinant

$$\mu_0 = \left(\frac{\det C_{(1)} \det C_{(2)}}{\det C_{(3)}} \right)^{1/4}, \tag{B.9}$$

the matrix

$$M = \begin{pmatrix} 1 + \widetilde{X}^{(1)\mathrm{T}} \widetilde{X}^{(1)} & \widetilde{X}^{(1)\mathrm{T}} \widetilde{X}^{(2)} \\ \widetilde{X}^{(2)\mathrm{T}} \widetilde{X}^{(1)} & 1 + \widetilde{X}^{(2)\mathrm{T}} \widetilde{X}^{(2)} \end{pmatrix} \tag{B.10}$$

and the vector

$$H = \begin{pmatrix} a^\dagger_{(1)} - \widetilde{X}^{(1)\mathrm{T}} a^\dagger_{(3)} \\ a^\dagger_{(2)} - \widetilde{X}^{(2)\mathrm{T}} a^\dagger_{(3)} \end{pmatrix}. \tag{B.11}$$

Performing the Gaussian integral in (B.8) gives

$$E_a = \mu_0 (\det M)^{-1/2} \exp \left[H^{\mathrm{T}} M^{-1} H - \frac{1}{2} \sum_{r=1}^{3} a^{\dagger\mathrm{T}}_{(r)} a^\dagger_{(r)} \right]. \tag{B.12}$$

Fortunately it is not to difficult to invert the matrix M. We have

$$M^{-1} = \begin{pmatrix} 1 - \widetilde{X}^{(1)\mathrm{T}} \widetilde{\Gamma}_a^{-1} \widetilde{X}^{(1)} & -\widetilde{X}^{(1)\mathrm{T}} \widetilde{\Gamma}_a^{-1} \widetilde{X}^{(2)} \\ -\widetilde{X}^{(2)\mathrm{T}} \widetilde{\Gamma}_a^{-1} \widetilde{X}^{(1)} & 1 - \widetilde{X}^{(2)\mathrm{T}} \widetilde{\Gamma}_a^{-1} \widetilde{X}^{(2)} \end{pmatrix} \tag{B.13}$$

in terms of the matrix

$$\widetilde{\Gamma}_a = \sum_{r=1}^{3} \widetilde{X}^{(r)} \widetilde{X}^{(r)\mathrm{T}}. \tag{B.14}$$

Note that $\det \widetilde{\Gamma}_a = \det M$. Combining (B.11), (B.12) and (B.13) gives the final result

$$E_a = \mu_0 (\det M)^{-1/2} \exp \left[\frac{1}{2} \sum_{r,s=1}^{3} a^{\dagger\mathrm{T}}_{(r)} N^{(rs)} a^\dagger_{(s)} \right], \tag{B.15}$$

where the Neumann matrices are given by

$$N^{(rs)} = \delta^{rs} \mathbf{1} - 2 \widetilde{X}^{(r)\mathrm{T}} \widetilde{\Gamma}_a^{-1} \widetilde{X}^{(s)}. \tag{B.16}$$

One might be concerned that (B.7) singles out string number 3, since the vertex should be symmetric under interchange of the three strings. However, if we plug (B.7) into (B.16), we find that the factor of $C_{(3)}^{-1/2}$ drops out, leaving

$$N^{(rs)} = \delta^{rs} 1 - 2 C_{(r)}^{1/2} X^{(r)\mathrm{T}} \Gamma_a^{-1} X^{(s)} C_{(s)}^{1/2}, \qquad \Gamma_a = \sum_{r=1}^{3} X^{(r)} C_{(r)} X^{(r)\mathrm{T}}. \tag{B.17}$$

Then the determinant which appears in (B.15) may be written as

$$\mu_a = \mu_0 (\det M)^{-1/2} = \left(\det C_{(1)} \det C_{(2)} \det C_{(3)} \right)^{1/4} (\det \Gamma_a)^{-1/2}. \tag{B.18}$$

Actually the vertex contains eight powers of μ_a, one for each transverse direction.

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