A no-go theorem for string warped compactifications

S. Ivanov[1] and G. Papadopoulos[2]

1. Department of Mathematics
University of Sofia
"St. Kl. Ohridski"

2. CERN
Theory Division
1211 Geneva, 23
Switzerland

Abstract

We give necessary conditions for the existence of perturbative heterotic and type II string warped compactifications preserving eight and four supersymmetries to four spacetime dimensions, respectively. In particular, we find that the only compactifications of heterotic string with the spin connection embedded in the gauge connection and type II strings are those on Calabi-Yau manifolds with constant dilaton. We obtain similar results for compactifications to six and to two dimensions.

In past few years there has been interest in investigating perturbative [1] and non-perturbative string warped compactifications [2], [3], [4]. One of the attractive features of such compactifications is the appearance of a scalar potential in the effective low dimensional effective action depending on some of moduli fields. This leads to the lifting of some of the flat directions of the associated compactifications without a warp factor. The presence of the potential can be understood in various ways. One way is that in warped compactifications some ten- or eleven-dimensional supergravity form field strengths receive an expectation value. Therefore one can view the presence of the potentials as a consequence of a Scherk-Schwarz type of mechanism. Another feature of these warped compactifications is that they preserve a fraction of spacetime supersymmetry. Warped compactifications have also been related to Randall-Sundrum type of senario [5].

In perturbative string theory the warped compactifications have been investigated sometime ago by Strominger in [1]. This was achieved by allowing the dilaton to be a non-constant function of the compact space. Necessary conditions were given for the existence of such compactifications. In particular it was shown that the compact manifold of such warped compactification is a 2n-dimensional hermitian, but no Kähler, manifold equipped with a holomorphic (n, 0)-form. Therefore it is required that the Betti number $h^{n,0} = \dim H^{n,0}$ of the compact manifold is $h^{n,0} \geq 1$.

In this paper, we shall investigate the conditions for the existence of holomorphic $(n,0)$-forms on hermitian manifolds using the Gauduchon theorem [15]-[17]. We shall show that there are restrictions for the existence of such forms which manifest themselves as conditions on the string three-form field strength H and its exterior derivative dH. In particular, we shall find that heterotic warped compactifications with the spin connection embedded in the gauge connection and type II string warped compactifications are ruled out. So the only compactifications of these systems with eight and four remaining supersymmetries in four dimensions are the standard Calabi-Yau compactifications for which dilaton is constant and the string three-form field strength vanishes. Recently other no-go theorems have appeared in the literature for Randall-Sundrum [18, 19, 20] and warped compactifications [21].

We shall begin our analysis with a summary of the relevant results of [1]. Then after giving some definitions of various curvature tensors associated with hermitian geometry, we shall present our main result. This is an inequality involving the length of the torsion of the Chern connection and the scalar derived by taking twice the trace of dH with the complex structure of the hermitian manifold. We shall then explore this inequality in the context of both heterotic and type II strings. We shall conclude with some remarks regarding the relationship between the Killing spinor equation associated with the gravitino and that associated with the dilatino. In particular, we shall argue that the former does not always imply the latter.

Perturbative string compactifications involve supergravity solutions for which the non-vanishing bosonic fields are those of the NS⊗NS sector, ie the metric G, the dilaton ϕ and the string three-form field strength H. In type II strings, H is closed but in the heterotic string due to the anomaly cancellation mechanism, $dH \neq 0$. In what follows, we shall be mainly concerned with compactifications of the heterotic string though our results can be

easily adapted to the type II strings. The relevant heterotic Killing spinor equations in the string frame are

$$
\begin{aligned}
\hat{\nabla}\eta &= 0 \\
(\Gamma^M \partial_M \phi \; - \; \frac{1}{6}\Gamma^{MNR}H_{MNR})\eta &= 0 \; ,
\end{aligned}
\tag{0.1}
$$

where $M, N, R = 0, \ldots, 9$, Γ^M are the gamma matrices of spacetime and $\hat{\nabla}$ is a connection with torsion H, ie

$$
\hat{\nabla}_M Y^N = \nabla_M Y^N + \frac{1}{2}H_{MR}{}^N Y^R \; ;
\tag{0.2}
$$

Y is a vector field. The gamma matrices have been chosen to be hermitian along the space directions and antihermitian along the time direction. The spinor η is in the Majorana-Weyl representation of the spin group $\mathrm{Spin}(1,9)$. In the heterotic string, there is an additional Killing spinor equation associated with the gaugino. However this will not enter in our investigation and it will be only very briefly mentioned later. In the case of type II strings there are two additional Killing spinor equations to (0.1) but again do not enter in our analysis. It has been shown in [1] that for warped compactifications to $\mathbb{R}^{2(5-n)}$, the solution for the background in the string frame can be written as

$$
\begin{aligned}
ds^2 &= ds^2(\mathbb{R}^{2(5-n)}) + ds^2(M) \\
\phi &= \phi(y) \\
H &= \frac{1}{3!}H(y)_{ijk}dy^i \wedge dy^j \wedge dy^k \; ,
\end{aligned}
\tag{0.3}
$$

where $\{y^i; i = 1, \ldots, 2n\}$ are the coordinates of the 2n-dimensional compact space M and $ds^2(M)$ is the metric on M. The manifold M is even-dimensional and this will be explained shortly. The warp factor is not apparent in the string frame but in the Einstein frame because the dilaton is not taken to be constant, the spacetime metric becomes

$$
ds_E^2 = e^{-\frac{1}{2}\phi}ds^2 \; ,
\tag{0.4}
$$

where ds_E^2 is the Einstein metric.

For the first Killing spinor equation in (0.1) to have solutions, the holonomy of the connection $\hat{\nabla}$ has to be an appropriate subgroup of $SO(1,9)$. For the compactifications (0.3) we are considering, the non-trivial part of the connection $\hat{\nabla}$ is given by a connection on the compact manifold M which we shall denote with the same letter. Therefore for the first Killing spinor equation in (0.1) to have solutions, we take the holonomy of $\hat{\nabla}$ to be in $SU(n)$. This in particular implies that for compactifications to four dimensions ($n = 3$), four supersymmetries will be preserved. The solutions of this Killing spinor equation are $\hat{\nabla}$-parallel spinors which can be identified with the singlets in the decomposition of the Majorana-Weyl spinor representation of $SO(1,9)$ under $SU(n)$. In addition it was shown in [1] that M is a hermitian manifold, i.e. M admits a complex structure and it is equipped

with a compatible hermitian metric. The complex structure is given in terms of a killing spinor η_+ as

$$J^i{}_j = -i\eta_+^\dagger \Gamma^i{}_j \eta_+ \; , \tag{0.5}$$

satisfying

$$\hat{\nabla}_k J^i{}_j = 0 \; , \tag{0.6}$$

i.e. it is parallel with respect to the $\hat{\nabla}$ connection. The torsion of the connection $\hat{\nabla}$ is determined in terms of the metric and the complex structure on M as

$$H_{ijk} = -3J^m{}_{[i}d\Omega_{|m|jk]} \; , \tag{0.7}$$

where $\Omega_{ij} = g_{ik}J^k{}_j$ is the Kähler form of M and $d\Omega$ is its exterior derivative, $(d\Omega)_{ijk} = 3\partial_{[i}\Omega_{jk]}$ and $ds^2(M) = g_{ij}dy^idy^j$ is the metric on M. In mathematics $\hat{\nabla}$ has appeared sometime ago in the context of hermitian manifolds [10] and recently it is called Bismut connection [11], and in physics has appeared in the context of supersymmetric sigma models [12, 13, 14]. Hermitian manifolds equipped with the Bismut connections are also called KT (Kähler with torsion) manifolds [14].

Let us now investigate the second Killing spinor equation associated with the dilatino. For the compactifications we are considering this Killing spinor equation for η_+ becomes

$$(\Gamma^i\partial_i\phi - \frac{1}{6}\Gamma^{ijk}H_{ijk})\eta_+ = 0 \; . \tag{0.8}$$

Next we multiply this equation from the left and its conjugate from the right with Γ^m, respectively. After contracting the two equations with $\eta_+^\dagger$ and η_+ appropriately and then substracting them, we get schematically

$$\eta_+^\dagger([\Gamma^m, d\phi] - \frac{1}{6}\{\Gamma^m, H\})\eta_+ = 0 \; . \tag{0.9}$$

After some gamma matrix algebra, this equation gives

$$\theta_i = 2\partial_i\phi \; , \tag{0.10}$$

where θ_i is the so called Lie form of J defined as

$$\theta_i = -\nabla^k\Omega_{km}J^m{}_i = \frac{1}{2}J^m{}_iH_{mkn}\Omega^{kn} \; . \tag{0.11}$$

We remark that the field and Killing spinor equations of supergravity theory are well-defined if we take the dilaton to be locally defined on M and therefore θ to be closed but not exact. However in the supergravity action, ϕ appears without a derivative acting on it and therefore in order the Lagrangian to be a scalar, ϕ is required to be a globally defined scalar on M. In many backgrounds that arise in the worldsheet-conformal field theory approach to strings, ϕ is only locally defined function on the background. For example, the background associated with the Wess-Zumino-Witten model on $S^1 \times S^3$ is supersymmetric,

in the supergravity sense, if one introduces a dilaton which depends linearly on the angular coordinate of S^1 factor. Such a dilaton is *not* globally defined on $S^1 \times S^3$; though is well-defined in the universal cover of $S^1 \times S^3$ which is the near horizon geometry of the NS5-brane. As we shall see in conformal field theory, a more general situation can arise.

The key observation in [1] is that the existence of a solution to the supergravity Killing spinor equations implies the presence of a holomorphic $(n,0)$-form on M. To see this observe that M admits a $\hat{\nabla}$-parallel $(n,0)$-form ϵ because of the requirement that the holonomy of $\hat{\nabla}$ is in $SU(n)$. Next we assume that ϕ is a *globally-defined* scalar on M and write the form

$$\tilde{\epsilon} = e^{-2\phi}\epsilon \ . \tag{0.12}$$

Then using $\hat{\nabla}\epsilon = 0$,

$$\hat{\Gamma}^{\gamma}_{\bar{\beta}\gamma} = \theta_{\bar{\beta}} \ ,$$

as it can be verified with an explicit computation, and (0.10), we find

$$\begin{aligned} \partial_{\bar{\beta}}\tilde{\epsilon}_{\alpha_1...\alpha_n} &= -2\partial_{\bar{\beta}}\phi\tilde{\epsilon}_{\alpha_1...\alpha_n} + n\hat{\Gamma}^{\gamma}_{\bar{\beta}[\alpha_1}\tilde{\epsilon}_{|\gamma|\alpha_2...\alpha_n]} \\ &= -2\partial_{\bar{\beta}}\phi\tilde{\epsilon}_{\alpha_1...\alpha_n} + \hat{\Gamma}^{\gamma}_{\bar{\beta}\gamma}\tilde{\epsilon}_{\alpha_1...\alpha_n} \\ &= -2\partial_{\bar{\beta}}\phi\tilde{\epsilon}_{\alpha_1...\alpha_n} + \theta_{\bar{\beta}}\tilde{\epsilon}_{\alpha_1...\alpha_n} = 0 \ , \end{aligned} \tag{0.13}$$

where $\alpha_1, \alpha_2, \ldots \alpha_n, \gamma, \delta = 1, \ldots, n$ are holomorphic indices with respect to J. So $\tilde{\epsilon}$ is holomorphic. This concludes our summary of the warped string compactifications.

There are conditions for the existence of holomorphic $(n,0)$-forms on hermitian manifolds. In particular, we shall show there does not exist such a holomorphic (n,0)-form for a large class of the compactifications we have described. For this, we first define the Chern connection $\tilde{\nabla}$ on M as

$$\tilde{\nabla}_i Y^j = \nabla_i Y^j + \frac{1}{2}J^m{}_i(d\Omega)_{mkn}g^{nj}Y^k \ , \tag{0.14}$$

where Y is a vector field of M. This connection is compatible with the metric ($\tilde{\nabla}g = 0$), the complex structure ($\tilde{\nabla}J = 0$) and the holomorphic structure of the tanget bundle of M (ie the curvature of the Chern connection is a $(1,1)$-form on M). The torsion of the Chern connection is

$$C_{ijk} = \frac{1}{2}(J^m{}_i d\Omega_{mjk} + J^m{}_j d\Omega_{imk}) \ , \tag{0.15}$$

which can be expressed in terms of the string three-form H as

$$C_{ijk} = \frac{1}{4}(J^m{}_i J^n{}_j H_{mnk} - H_{ijk}) \ . \tag{0.16}$$

Observe that if the torsion of the Chern connection vanishes, then the manifold M is Kähler and consequently H also vanishes. Next we define

$$b = -\frac{1}{2}\hat{R}_{ijkl}\Omega^{ij}\Omega^{kl} \ , \tag{0.17}$$

and

$$u = -\frac{1}{4}\tilde{R}_{ijkl}\Omega^{ij}\Omega^{kl} , \tag{0.18}$$

where $\hat{R}$ and $\tilde{R}$ are the curvature tensors of the Bismut and Chern connections on M, respectively. Moreover a computation reveals that

$$2u = b + C_{ijk}C^{ijk} + \frac{1}{4}dH_{ijkl}\Omega^{ij}\Omega^{kl} . \tag{0.19}$$

(For the derivation of this see [9].) Of course if the holonomy of the Bismut connection is contained in $SU(n)$, then $b = 0$.

Necessary conditions on the hermitian manifold M for the existence of a holomorphic $(n,0)$-form can be derived by the application of the Gauduchon plurigenera theorem; for applications of other vanishing theorems [8] see [9]. For this, we first observe that given a hermitian manifold M with complex structure J and metric g, one can find another hermitian structure on M with the same complex structure but with metric which is related to g by a conformal rescaling, i.e. metric $e^f g$ where f is a function on M. It has been shown that it is possible to always find a metric h within the conformal class of g such that

$$(\nabla^{(h)})^i\theta_i^{(h)} = 0 ,$$

where $\nabla^{(h)}$ is the Levi-Civita connection of the metric h and $\theta^{(h)}$ is the Lie form with respect to the metric h. The metric h that fixes the conformal gauge is called Gauduchon metric. If the string metric g on M is related to h by $g = e^f h$, then we have

$$2e^f u = 2u^{(h)} + n(n-1)h^{ij}(\theta^{(h)})_i\partial_j f - \Delta^{(h)}f , \tag{0.20}$$

where the tensors label with (h) are taken with respect to the Gauduchon metric h and $\Delta^{(h)}$ is the Laplacian of h.

As an application of the Gauduchon plurigenera theorem, we have that $h^{n,0} \in \{0,1\}$, if

$$\int_M \sqrt{h}d^{2n}y \, u^{(h)} \geq 0 . \tag{0.21}$$

Moreover if the inequality is strict, then $h^{n,0} = 0$. Using (0.20) and (0.19), and after integrating by parts, we find that

$$
\begin{aligned}
\int_M \sqrt{h}d^{2n}y \;\; u^{(h)} &= \int_M \sqrt{h}d^{2n}y \;\; e^f u \\
&= \frac{1}{2}\int_M \sqrt{g}d^{2n}y \;\; e^{-(n-1)f} \;\; (C_{ijk}C^{ijk} + \frac{1}{4}dH_{ijkl}\Omega^{ij}\Omega^{kl}) .
\end{aligned} \tag{0.22}
$$

This is our main equation. Further, the torsion C of the Chern connection can be expressed in terms of the string three-form H using (0.16) but the above expression will suffice.

Now consider warped compactifications of the heterotic string with compact space the hermitian, but no Kähler manifold, M and with spin connection embedded in the gauge one. If this is the case, then $dH = 0$ and so

$$\int_M \sqrt{h}d^{2n}y \;\; u^{(h)} = \frac{1}{2}\int_M \sqrt{g}d^{2n}y \;\; e^{-(n-1)f} \;\; C_{ijk}C^{ijk} > 0 . \tag{0.23}$$

From the Gauduchon theorem stated above we conclude that $h^{n,0} = 0$ and a holomorphic $(n, 0)$ form on M cannot exist. Thus there are no warped compactifications of the heterotic string with the spin connection embedded in the gauge one. The same applies for warped compactifications of type II strings for which H is closed , $dH = 0$. From these, one concludes that the only compactifications of type II strings and heterotic string for which the spin connection is embedded in the gauge connection are those on Calabi-Yau manifolds [7]. In particular the dilaton is constant and so there is no warp factor in the metric.

The only remaining possibility for existence of warped perturbative string compactifications is that of heterotic strings for which the spin connection is *not* embedded in the gauge connection and so $dH \neq 0$. A necessary condition for having $h^{n,0} = 1$ as required is that

$$\int_M \sqrt{g} d^{2n} y \ e^{-(n-1)f} \ [C_{ijk} C^{ijk} + \frac{1}{4} dH_{ijkl} \Omega^{ij} \Omega^{kl}] \leq 0 \ . \tag{0.24}$$

From the heterotic string anomaly cancellation formula

$$dH = \lambda[\text{Tr}(R' \wedge R') - \text{Tr}(F \wedge F)] \ , \tag{0.25}$$

we have

$$(dH)_{ijkl} \Omega^{ij} \Omega^{kl} = -\frac{\lambda}{2} (\text{Tr} R'_{ij} R'^{ij} - \text{Tr} F_{ij} F^{ij}) \ , \tag{0.26}$$

where λ is a constant that depends on the string tension, R' is the curvature of the connection with torsion $-H$, F is the curvature of a vector bundle on M and the trace is taken in the gauge indices which have been suppressed. Observe that only the zeroth order term of H in λ contributes in the one-loop anomaly and that $dH = 0 + \mathcal{O}(\lambda)$. To derive (0.26), we have used the fact that $\hat{R}_{ij,kl} = R'_{kl,ij}$ and therefore both R' and F are (1,1)-forms on M which in addition satisfy $\Omega^{ij} F_{ija}{}^b = \Omega^{ij} R'_{ijk}{}^l = 0$. These conditions on F are derived by solving the Killing spinor equations associated with the gaugino. (Global anomaly cancellation requires that dH is exact.) Now whether or not (0.24) is satisfied depends on the details of the choice of the gauge connection and that of the hermitian structure.

For compactifications to six dimensions, if $dH = 0$ then the only compactifications are those on K_3 with constant dilaton [6]. On the other hand if $dH \neq 0$, then the string metric g is in the same conformal class as the K_3 metric. The latter of course can be identified with the Gauduchon metric. Therefore $u^{(h)} = 0$ and there is a holomorphic (2,0)- form which is that of K_3.

We shall conclude with a remark regarding the relationship between the first and second Killing spinor equations in (0.1). In particular we show that the first Killing spinor equation does *not* always imply the second. To derive this, we shall construct an example of a manifold that solves the first Killing spinor equation but not the second. For this we take $M = SU(2) \times SU(2) = S^3 \times S^3$; This is a group manifold model as those in [22, 23]. Moreover let as denote the left-invariant one-forms on M with $\{(\sigma^r, \bar{\sigma}^r); r = 0, 1, 2\}$, where σ^r are associated with the first $SU(2)$ in M and $\bar{\sigma}^r$ with the second. In particular, we have

$$d\sigma^r = -\frac{1}{2} \epsilon^r{}_{st} \sigma^s \wedge \sigma^t \tag{0.27}$$

7

and similarly for $\bar{\sigma}^r$. The metric on M can be chosen to be

$$ds^2 = \delta_{rs}\sigma^r\sigma^s + \delta_{rs}\bar{\sigma}^r\bar{\sigma}^s \ . \qquad (0.28)$$

(There is a two parameter family of bi-invariant metrics on M but this choice will suffice for our purpose.) The Kähler form of the metric that we shall consider is

$$\Omega = \sigma^0 \wedge \bar{\sigma}^0 + \sigma^1 \wedge \sigma^2 + \bar{\sigma}^1 \wedge \bar{\sigma}^2 \qquad (0.29)$$

In particular we find that the string form field strength in this case is

$$H = -\sigma^0 \wedge \sigma^1 \wedge \sigma^2 - \bar{\sigma}^0 \wedge \bar{\sigma}^1 \wedge \bar{\sigma}^2 \ . \qquad (0.30)$$

The Bismut connection is the left-invariant connection on the group $SU(2) \times SU(2)$ and so it has holonomy group the identity. Thus its holonomy is contained in $SU(3)$ and the first Killing spinor equation has the required solutions. It remains to see whether the second Killing spinor equation can be solved as well. For this we shall attempt to verify the equation $\theta_i = -2\partial_i\phi$. To do this we compute the Lie form θ and find

$$\theta = \sigma^0 - \bar{\sigma}^0 \ . \qquad (0.31)$$

Thus $d\theta \neq 0$ and so the second Killing spinor equation is not satisfied. The manifold $M = SU(2) \times SU(2)$ is associated with a two-dimensional $N = 2$ superconformal theory, for example that of a Wess-Zumino-Witten model with target space M. Of course such conformal theories do not have the appropriate central charge to serve as string backgrounds, i.e. the dilaton field equation is not satisfied. Nevertheless, it is surprising that there is no choice of a dilaton for the above complex structure, even after adding for example background charges, for which the associated supergravity background preserves some of the spacetime supersymmetry. However if the complex structure J is chosen in a different way, then by adding two appropriate background charges, the resulting background can be thought of as having geometry $S^1 \times S^3 \times S^1 \times S^3$ which preserves some spacetime supersymmetry.

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