

THE SELF-FORCE OF A CHARGED PARTICLE IN CLASSICAL ELECTRODYNAMICS WITH A CUT-OFF

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ABSTRACT

We discuss, in the context of classical electrodynamics with a Lorentz invariant cut-off at short distances, the self-force acting on a point charged particle. It follows that the electromagnetic mass of the point charge occurs in the equation of motion in a form consistent with special relativity. We find that the exact equation of motion does not exhibit runaway solutions or non-causal behavior, when the cut-off is larger than half of the classical radius of the electron.

I - INTRODUCTION

The calculation of the self-force acting on a charged particle is a long outstanding problem in electrodynamics since the days of Abraham (1903) and Lorentz (1904), who derived for the first time the radiation reaction force on an extended electron¹. Assuming that the electron has a spherically symmetric rigid charge distribution of radius r_0 in its instantaneous rest frame, they were able to show that a particle subjected to an external force \mathbf{F}_{ext} obeys the following equation of motion:

$$\frac{4}{3} \frac{U}{c^2} \dot{\mathbf{v}} - \frac{2}{3} \frac{e^2}{c^3} \ddot{\mathbf{v}} + \frac{2e^2}{3c^3} \sum_{n=2}^{\infty} \frac{(-1)^n}{n!c^n} \gamma_n \frac{d^{n+1} \mathbf{v}}{dt^{n+1}} = \mathbf{F}_{\text{ext}} \quad (1)$$

where $\dot{\mathbf{v}}$ is the acceleration of the particle and U represents its electrostatic energy:

$$U = \frac{1}{2} \int d^3x \int d^3x' \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} . \quad (2)$$

The constants γ_n are proportional to r_0^{n-1} and characterize the way the charge is distributed within the particle.

The factor $4/3$ in front of the electromagnetic mass U/c^2 in equation (1) is just one of the several well-known difficulties involved in the Abraham-Lorentz theory of charged particles. To overcome this problem, Poincaré (1905) suggested that the charged particle could not be held together unless other attractive and nonelectromagnetic forces were present. These Poincaré stresses would add a mass m_0 to the electromagnetic mass of the particle, so that the requirements of special relativity would apply only to the physical, observed mass $m = m_0 + m_{\text{el}}$. However, there are two problems with that solution. Firstly, a charged particle endowed with Poincaré stresses would be unstable under deformations of shape which might occur when the particle is acted by external or self-electromagnetic forces¹. Secondly, classical electrodynamics is a covariant theory by itself, so one expects that a correct calculation should not violate the requirements of Lorentz covariance. The current point of view is that the electromagnetic energy-momentum used by Abraham,

Lorentz and Poincaré is not a covariant quantity, so that when the covariance condition is taken properly into account it should furnish the expected factor of unity.²⁻⁶

Anyway, the extended electron theory is not compatible with the experimental facts, which indicate that the electron may be considered as a point particle at least up to distances of order of 10^{-16} cm.⁷ In the point charge limit, all the structure-dependent constants γ_n in equation (1) go to zero, but then the electromagnetic mass m_{el} diverges as $1/r_0$ when $r_0 \rightarrow 0$, so that this limit is not meaningful in the Maxwell theory. One may assume that the terms involving the γ_n factors in equation (1) could be disregarded when r_0 is very small, provided the changes in the motion of the particle which occur during short time intervals of order r_0/c are negligible. One then obtains the Abraham-Lorentz equation of motion:

$$(m_0 + m_{\text{el}}) \dot{\mathbf{v}} - \frac{2}{3} \frac{e^2}{c^3} \ddot{\mathbf{v}} = \mathbf{F}_{\text{ext}} \quad (3)$$

where we have added a mechanical nonelectromagnetic mass m_0 . As remarked by Feynman⁸, one would be in trouble only if the energy changes were also infinite. Unfortunately, this is the case: even if we renormalize the mass, keeping $m = m_0 + m_{\text{el}}$ fixed as $r_0 \rightarrow 0$, the solution of equation (3) when $\mathbf{F} = 0$ would have an exponentially growing acceleration:

$$\dot{\mathbf{v}}(t) = \dot{\mathbf{v}}(0) \exp(t/\tau) \quad (4)$$

where $\tau = 2e^2/3mc^3$. This is called a runaway solution of the Abraham-Lorentz equation.^{3-6,8-10} When there is an external force acting on the charged particle, the runaway solution still persists:

$$\dot{\mathbf{v}}(t) = \left[\dot{\mathbf{v}}(0) - \frac{1}{m\tau} \int_0^t dt' \exp(-t'/\tau) \mathbf{F}_{\text{ext}}(t') \right] \exp(t/\tau) , \quad (5)$$

unless we impose, following Dirac⁹, the very peculiar initial condition:

$$\dot{\mathbf{v}}(0) = \frac{1}{m\tau} \int_0^\infty dt' \exp(-t'/\tau) \mathbf{F}_{\text{ext}}(t') . \quad (6)$$

But in this case, the acceleration $\dot{\mathbf{v}}(t)$ would depend on the force $\mathbf{F}_{\text{ext}}(t+t')$ at times greater than t . This non-causal effect, which is more pronounced during times t' of order τ is called preacceleration.

The above behavior indicates that the assumption about the neglect of the γ_n factors in this regime may be inconsistent, since during short time intervals of order τ the changes in the motion of the particle appear to be important. On the other hand it is well known that due to quantum effects, classical electrodynamics cannot remain valid at such small distances and time intervals when the runaways and the preacceleration effects are relevant. Thus, it is possible that a modification of the laws of electrodynamics at short distances might lead to a regularized, causal and runaway-free theory. In fact, because of the existence of a cut-off in such a theory, we shall show that the γ_n terms are nonvanishing in the point particle limit, being essential for the suppression of the unphysical runaway solutions.

Some time ago, Coleman¹¹, treating the electron as a point charge from the very beginning, introduced a cut-off in Maxwell's electrodynamics. This enabled him to derive unambiguously the relativistic equation of motion which reduces to equation (3) in the nonrelativistic limit, called the Lorentz-Dirac equation.⁹ In his work, the cut-off was merely a computational device, whose effects were disregarded at the end of the calculation. A few years later, Moniz and Sharp^{12,13,14} have shown in the context of a quantum theory of the electron, that the interaction of the point electron with his own electromagnetic field induces effectively a natural cut-off of order of the electron's Compton wavelength $\lambda = \hbar/mc$. This may arise in consequence of the creation of virtual electron-positron pairs in the neighbourhood of the point electron, which effectively spread-out its charge distribution. Subsequently, these and other related aspects have been further investigated by several authors.¹⁵⁻²⁰

Based on these facts, we believe that a possible way to remove the divergences, runaway solutions or noncausal behavior from classical electrodynamics is by the introduction of a gauge and Lorentz invariant cut-off at short distances in the Maxwell theory. We shall use such a cut-off at the threshold of the classical regime, which allows for the existence of a finite and well defined point particle limit. One of the authors²¹ has recently shown how calculating the electromagnetic mass in this framework solves the 4/3 problem of the classical theory. In the regularized classical electrodynamics one finds the correct factor of unity in the point charge limit as well as a finite electromagnetic mass $m_{\text{el}} = e^2/2\ell c^2$, where ℓ is the cut-off. When $\ell = \hbar/mc$ this becomes, apart from a logarithmic factor, of same order as the electromagnetic mass found in quantum electrodynamics.²²

The approach we follow involves adding a new term to the Maxwell Lagrangian, which leads to an effective Lagrangian for classical electrodynamics that takes into account the relevant effects from the quantum theory. The form of the new term can be restricted by a few reasonable and simple properties, which leave the Maxwell theory as unaltered as possible: (a) The Lagrangian must be gauge and Lorentz invariant. (b) It should yield local field equations which are still linear in the field quantities. The simplest possibility that includes a cut-off ℓ leads to a Lagrangian containing second order derivatives of the electromagnetic potentials $A_\alpha = (\mathbf{A}, i\phi)$:

$$\mathcal{L}(\ell) = -\frac{1}{16\pi} F_{\alpha\beta} F_{\alpha\beta} - \frac{\ell^2}{8\pi} \partial_\beta F_{\alpha\beta} \partial_\gamma F_{\alpha\gamma} + \frac{1}{c} j_\alpha A_\alpha, \quad (7)$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the usual electromagnetic field tensor and $j_\alpha = (\mathbf{j}, ic\rho)$ is the conserved four-current. At distances much larger than the cut-off, the fields described by equation (7) become essentially equivalent to the fields governed by the usual Maxwell theory. Such a modification of classical electrodynamics was proposed a long time ago by Podolsky and others.²³

In section II we present the calculation of the self-force that acts on a point charged

particle, within the framework of the generalized Maxwell theory described by the Lagrangian (7). We evaluate all the terms involving higher order derivatives of the velocity which appear in the exact equation of motion of a point charged particle. The contributions of the higher order terms can be summed in closed form, from which the absence of runaway behavior follows in the case when the cut-off is larger than half of the classical radius of the electron. Furthermore, we find that in this case the solutions are consistent with the principle of causality. Similar conclusions may be obtained from the exact equation of motion of a relativistic point charged particle, which is discussed in the last section.

II. EVALUATION OF THE SELF-FORCE OF A POINT CHARGED PARTICLE

The Lagrangian $\mathcal{L}(\ell)$ leads to the following linear partial differential equations:

$$(1 - \ell^2 \square) \square A_\alpha = -\frac{4\pi}{c} j_\alpha \quad (8)$$

where we used the Lorentz gauge $\partial_\alpha A_\alpha = 0$. To determine these potentials, it is useful to find the retarded Green function for the equation

$$(1 - \ell^2 \square) \square G(\mathbf{x} - \mathbf{x}', t - t', \ell) = -4\pi \delta(\mathbf{x} - \mathbf{x}') \delta(t - t') \quad (9)$$

which is subjected to the causality condition that $G = 0$ for $t < t'$. In that way the solution of equation (8) will be:

$$A_\alpha(\mathbf{x}, t, \ell) = \frac{1}{c} \int d^3x' dt' G(\mathbf{x} - \mathbf{x}', t - t', \ell) j_\alpha(\mathbf{x}', t') . \quad (10)$$

Following the procedure described in reference 21, we arrive at

$$G(R, T, \ell) = \frac{c \theta(T - R/c)}{\ell \sqrt{c^2 T^2 - R^2}} J_1 \left(\frac{\sqrt{c^2 T^2 - R^2}}{\ell} \right) , \quad (11)$$

where $R = |\mathbf{x} - \mathbf{x}'|$, $T = t - t'$ and J_1 is the Bessel function of order one. The self-force may be shown to be:

$$\mathbf{F}_s(t) = - \int d^3x \rho(\mathbf{x}, t) \left[\nabla \phi(\mathbf{x}, t) + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}(\mathbf{x}, t) \right]. \quad (12)$$

Now, instead of performing a series expansion in powers of R/c , for R/c small, of the retarded Green function $G(R, T, \ell)$, we will rather take the point particle limit in equation (11),

$$G(0, T, \ell) = \frac{\theta(T)}{\ell T} J_1(cT/\ell), \quad (13)$$

which allows us to express the self-force in a closed form given by the expression:

$$\mathbf{F}_s(t) = \frac{e^2}{c^2} \int_0^\infty dT \frac{dG(0, T, \ell)}{dT} \left[\frac{\mathbf{r}(t) - \mathbf{r}(t - T)}{T} - \mathbf{v}(t - T) \right], \quad (14)$$

where \mathbf{r} is the coordinate of the particle and $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$ is its velocity.

Unlike the Abraham-Lorentz equation, expression (14) does not involve any explicit second order derivatives of the velocity with respect to time. For that reason, the exact equation of motion of the particle,

$$m_0 \dot{\mathbf{v}} - \mathbf{F}_s(t) = \mathbf{F}_{\text{ext}} \quad (15)$$

has substantially different properties from those of the Abraham-Lorentz equation. In particular, the homogeneous solutions of equation (15) do not display runaway behavior when m_0 is non-negative.

To see this we assume an ansatz of the form $\mathbf{r} = \mathbf{r}_0 \exp(\eta t)$. There will be no runaway solutions if the real part of η is negative or vanishes. Then, the possible solutions of equation (15) in the absence of external forces are determined by the condition that

$$m_0 \eta^2 + \frac{e^2}{c^2 \ell} \int_0^\infty dT [\exp(-\eta T) - 1 + \eta T \exp(-\eta T)] \frac{1}{T} \frac{d}{dT} \left[\frac{J_1(cT/\ell)}{T} \right] = 0. \quad (16)$$

The T -integration may be performed,²⁴ giving

$$\left(\eta^2 + \frac{c^2}{\ell^2} \right)^{1/2} \left(2\eta^2 - \frac{c^2}{\ell^2} \right) = 2\eta^3 - \frac{c^3}{\ell^3} - \frac{3m_0 c^3}{e^2} \eta^2. \quad (17)$$

Squaring both sides and noticing that $\eta = 0$ is a doubly degenerate solution, we may rewrite (17) as the cubic equation:

$$m_0 \eta^3 - \frac{3}{4} \frac{m_0^2 c^3}{e^2} \eta^2 + \frac{1}{3} \frac{e^2}{\ell^3} \eta - \frac{1}{4} \frac{c e^2}{\ell^4} - \frac{1}{2} \frac{m_0 c^3}{\ell^3} = 0 . \quad (18)$$

The solutions of (17) are a subset of those determined by the cubic equation (18). The solutions of such an equation are well known and it can be verified that its complex conjugate roots do not satisfy the original equation (17). Therefore, we are allowed to write the solutions of (17) as $\eta = cx/\ell$, where x is a real quantity satisfying equation (17) in the form

$$(1 + x^2)^{1/2} (2x^2 - 1) = 2x^3 - 1 - px^2 , \quad (19)$$

where $p = 3m_0 \ell c^2 / e^2$ is a dimensionless real parameter. Note that the sign of the mechanical mass m_0 determines that of p .

Apart from the trivial solution $x = 0$, we must distinguish three cases in order to find the other solutions:

- (i) $p = 0$. A very simple analysis shows that the left hand side of (19) is always larger than the right hand side, except when $x = 0$. Hence, there are no additional solutions when $p = 0$.
- (ii) $p > 0$. The above argument holds still stronger in this case. Thus, we cannot get extra solutions in this case either.
- (iii) $p < 0$. There is a continuous set of solutions $x = x(p)$. To see that, consider the inverse relation $p = p(x)$ which, according to (19), is given by

$$p = 2x - \frac{1}{x^2} + (1 + x^2)^{1/2} \left(\frac{1}{x^2} - 2 \right) . \quad (20)$$

When $x \gg 1$, p approaches zero as $-1/x^2$, while for $x \ll 1$, p behaves approximately as $-3/2 + 2x$. A plot of the graph of p versus x helps us to grasp these features (see figure 1).

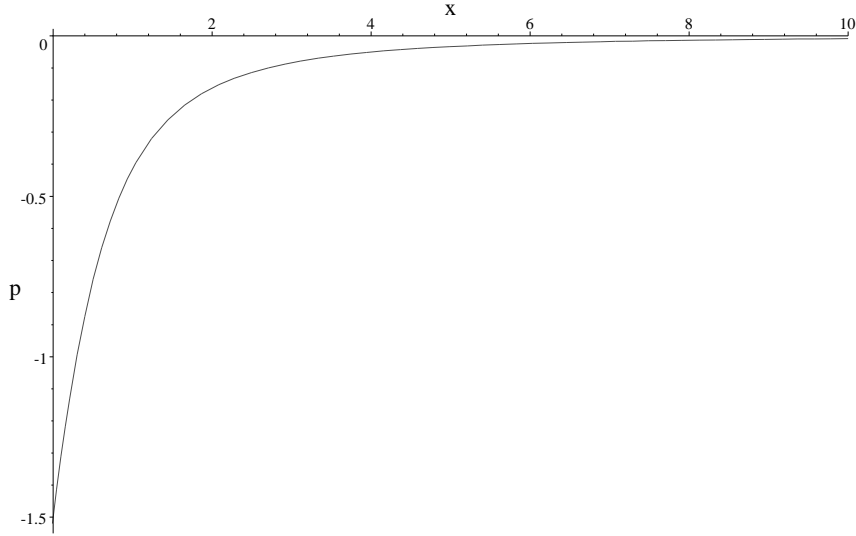


Figure 1: Behavior of the parameter p as a function of the root x .

Since $m = m_0 + m_{\text{el}}$, and the electromagnetic mass m_{el} is given in the regularized electrodynamics for a point particle by²¹

$$m_{\text{el}} = \frac{e^2}{2\ell c^2} , \quad (21)$$

it is possible to express p in terms of the cut-off ℓ and the classical radius of the electron $r_0 = e^2/mc^2$ as:

$$p = \frac{3m_0\ell c^2}{e^2} = -\frac{3}{2} + \frac{3\ell}{r_0} . \quad (22)$$

This shows that p is necessarily larger than $-3/2$, for both ℓ and r_0 are positive constants. Therefore, we are led to the conclusion that x must be restricted to positive values. Consequently, if $p < 0$, and hence $m_0 < 0$, η is real and positive and runaway motion takes place.

We see that runaway solutions can be presented if, and only if, $m_0 < 0$. Expression (22) shows that this is possible provided ℓ is smaller than half the classical radius of the electron. It is interesting to examine these solutions in the limit $\ell \rightarrow 0$. In this case, we would have $x \simeq 3\ell/2r_0$. Thus, the homogeneous solution of the exact equation

of motion (15) may be written in the limit $\ell \rightarrow 0$ as

$$\dot{\mathbf{v}}(t) = \dot{\mathbf{v}}_0 \exp(\eta t) = \dot{\mathbf{v}}_0 \exp[cx(p)t/\ell] \simeq \dot{\mathbf{v}}_0 \exp(3ct/2r_0) \quad (23)$$

which is identical to the homogeneous solution (4) of the Abraham-Lorentz equation of motion (3).

Let us finally examine the inhomogeneous solution of the regularized equation of motion (15), corresponding to a nonrelativistic motion of the particle, subject to a time-dependent external force. This solution is easily obtained after introducing the Fourier transforms

$$\mathbf{r}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \exp(-i\omega t) \mathbf{r}(\omega) ; \quad \mathbf{F}_{\text{ext}}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \exp(-i\omega t) \mathbf{F}_{\text{ext}}(\omega) . \quad (24)$$

Using the expression (14) we arrive at the relation

$$\begin{aligned} \mathbf{r}(\omega) = & \left\{ m_0(-i\omega)^2 + \frac{e^2}{c^2\ell} \int_0^\infty dT [\exp(i\omega T) - 1 + \right. \\ & \left. + (-i\omega T) \exp(i\omega T)] \frac{1}{T} \frac{d}{dT} \left(\frac{J_1(cT/\ell)}{T} \right) \right\}^{-1} \mathbf{F}_{\text{ext}}(\omega) . \end{aligned} \quad (25)$$

Standard techniques, such as the convolution theorem, allows us to write the inhomogeneous solution in the form

$$\dot{\mathbf{v}}(t) = \int_{-\infty}^{+\infty} dt' \mathcal{G}(t-t') \mathbf{F}_{\text{ext}}(t') , \quad (26)$$

where the Green function $\mathcal{G}(t-t')$ is given by:

$$\begin{aligned} \mathcal{G}(t-t') = & \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-i\omega(t-t')) (-i\omega)^2 \left\{ m_0(-i\omega)^2 + \frac{e^2}{c^2\ell} \int_0^\infty dT \times \right. \\ & \left. \times [\exp(i\omega T) - 1 + (-i\omega T) \exp(i\omega T)] \frac{1}{T} \frac{d}{dT} \left(\frac{J_1(cT/\ell)}{T} \right) \right\}^{-1} d\omega . \end{aligned} \quad (27)$$

The charge will move in a causal way if the acceleration at time t depends only upon the force field at times earlier than t . That such a behavior can be ensured by the retarded

Green function, which is characterized by the fact that its singularities lie in the lower half of the complex ω -plane,^{4,20} is well known. Substituting η for $(-i\omega)$, this property requires that all zeros of the expression in curly brackets in equation (27) must be situated in the left half of the complex η -plane. But this condition is identical to that given by equation (16) in connection with the absence of runaway solutions. It then follows from our previous analysis that no improper solution is present, when the cut-off is larger than half the classical radius of the electron.

One important aspect to note is that the removal of runaways and of preacceleration is intimately related to keeping the higher order terms in the expansion of the self-force.²¹ In our framework, these higher order terms do not vanish in the point particle limit because of the existence of the cut-off ℓ . Then, the self-force may be written as

$$\mathbf{F}_s(t) = \sum_{n=0}^{\infty} \mathbf{F}_s^n(t) , \quad (28)$$

where

$$\mathbf{F}_s^n = \frac{b_n e^2}{n! c^{n+2}} \ell^{n-1} \frac{d^{n+1} \mathbf{v}}{dt^{n+1}} , \quad (29)$$

and the constants b_n may be determined using the techniques described in 21. We only cite the results. One finds that $b_0 = 1/2$, $b_1 = -2/3$ and

$$b_n = \frac{(-1)^{n/2} (n+1)}{(n-1)(n+2)} [(n-1)!!]^2 \quad (30)$$

when $n \geq 2$ is even and $b_n = 0$ otherwise.

We see from the above equations that the factor $2b_0$, which multiplies the electromagnetic mass $e^2/2c^2\ell$, has the correct value of unity which is consistent with special relativity.

III. DISCUSSION

A relativistic generalization of equation (15) must have the form

$$m_0 \frac{dv_\mu}{d\tau} - F_\mu = F_\mu^{\text{ext}} \quad (31)$$

where τ is the particle's proper time and v_μ is its four-velocity:

$$v_\mu = (1 - v^2/c^2)^{-1/2} [\mathbf{v}, ic] = [\gamma\mathbf{v}, i\gamma c] . \quad (32)$$

Here, F_μ represents the covariant generalization of the self-force \mathbf{F}_s , which acts in the instantaneous rest frame of the charged particle. Using the Lorentz transformation properties of the four-vector F_μ , together with the general constraint that $F_\mu v_\mu = 0$, we obtain for F_μ the expression

$$F_\mu = \left[\mathbf{F}_s + (\gamma - 1) \frac{\mathbf{v} \cdot \mathbf{F}_s}{v^2} \mathbf{v}, i\gamma \frac{\mathbf{v} \cdot \mathbf{F}_s}{c} \right] . \quad (33)$$

The self-force F_μ can be expanded in a power series involving higher-order derivatives of the four-velocity with respect to the proper time. This may be done conveniently using the relation

$$F_\mu = \frac{e}{c} F_{\mu\nu} v_\nu = \frac{e}{c} (\partial_\mu A_\nu - \partial_\nu A_\mu) v_\nu , \quad (34)$$

where the self-potentials A_α given by equation (10) must be evaluated at the position of the point charge. The actual calculation is rather involved and can be carried out along the lines indicated in reference 11. The result is

$$F_\mu = -\frac{e^2}{2\ell c^2} \frac{dv_\mu}{d\tau} + \frac{2e^2}{3c^3} \left[\frac{d^2 v_\mu}{d\tau^2} - \frac{1}{c^2} \frac{dv_\nu}{d\tau} \frac{dv_\nu}{d\tau} v_\mu \right] + \sum_{n=2}^{\infty} \frac{b_n e^2}{n! c^{n+2}} \ell^{n-1} V_\mu^n , \quad (35)$$

where $e^2/2\ell c^2$ is the electromagnetic mass of a point particle and b_n are the constants given in eq. (30). The four-vector V_μ^n may be expressed in terms of the proper time derivatives of the four-velocity, $v_\mu^{(n)} = d^n v_\mu / d\tau^n$, as follows:

$$\begin{aligned}
V_\mu^n &= \frac{1}{(n+1)^2} \left[v_\mu^{(n+1)} + \frac{v_\nu v_\nu^{(n+1)}}{c^2} v_\mu \right] + \\
&+ \frac{n+2}{n+1} \frac{v_\nu}{c^2} \sum_{k=1}^n \binom{n}{k} \frac{1}{n+2-k} \left[v_\nu^{(n+2-k)} v_\mu^{(k-1)} - (\mu \leftrightarrow \nu) \right]. \quad (36)
\end{aligned}$$

In the nonrelativistic regime the three-vector part of V_μ^n , namely \mathbf{V}^n , is practically equal to $d^{n+1}\mathbf{v}/dt^{n+1}$, in accordance with the result given by equation (29).

In the limit $\ell \rightarrow 0$ the self-force four-vector becomes

$$F_\mu = -m_{\text{el}} \frac{dv_\mu}{d\tau} + \frac{2e^2}{3c^3} \left[\frac{d^2 v_\mu}{d\tau^2} - \frac{1}{c^2} \frac{dv_\nu}{d\tau} \frac{dv_\nu}{d\tau} v_\mu \right] \quad (37)$$

with a diverging electromagnetic mass $m_{\text{el}} = e^2/2\ell c^2$. In this limit, the exact equation of motion (31) reduces to the Lorentz-Dirac equation with the physical mass $m = m_0 + e^2/2\ell c^2$,

$$m \frac{dv_\mu}{d\tau} - \frac{2e^2}{3c^3} \left[\frac{d^2 v_\mu}{d\tau^2} - \frac{1}{c^2} \frac{dv_\nu}{d\tau} \frac{dv_\nu}{d\tau} v_\mu \right] = F_\mu^{\text{ext}}. \quad (38)$$

The Lorentz-Dirac equation is known to exhibit the familiar maladies of runaway solutions and noncausal behavior. These problems may be ascribed to the appearance of a negative bare mass m_0 to counterbalance the diverging electromagnetic mass, in order to furnish the observed, finite mass m to the point charged particle.

The relativistic equation of motion (31), with F_μ given by the exact expression (35), predicts in general the same kind of behavior as that described by its nonrelativistic counterpart (15). To understand this feature, we remark that when $\ell < r_0/2$, the mechanical mass m_0 must be negative in order to ensure the observed value of the physical mass m . Then, a runaway behavior is consistent with the conservation of energy, which is the sum of the particle kinetic energy $(\gamma - 1)m_0 c^2$ and the positive electromagnetic field energy. The kinetic energy of a negative mechanical mass, which is negative and decreasing with the increasing velocity of the particle, can compensate the increase of the field energy, maintaining an overall constant energy.

On the other hand, when ℓ is larger than half the classical radius of the electron, m_0 is positive and the particle cannot undergo a runaway motion. Such a motion would violate the conservation of energy, since it would increase the particle positive mechanical energy as well as the positive electromagnetic field energy. Therefore, we conclude that if the quantum processes induce in the classical regime an effective cut-off of order of the Compton wavelength of the electron, then the exact equation of motion of a point charged particle will admit only physical solutions.

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FIGURE CAPTION

FIG. 1. Behavior of the parameter p as a function of the root x .