

CONTRACTIONS, MATRIX PARAMATRIZATIONS, AND QUANTUM INFORMATION

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ABSTRACT. In this note, we discuss dilation-theoretic matrix parametrizations of contractions and positive matrices. These parametrizations are then applied to some problems in quantum information theory. First we establish some properties of positive maps, or entanglement witnesses. Two further applications, concerning concrete dilations of completely positive maps, in particular quantum operations, are given.

1. INTRODUCTION

It is well-known that positive operator-matrices, or more generally positive kernels, can be parametrized by contractions [5]. In this paper, we show that analogous results can be obtained for matrix contractions. A common feature of these parametrizations is that, while the explicit expressions may seem intricate, due to their combinatorial nature they can be easily understood by inspecting the so-called *lattice diagrams*. These diagrams will be used repeatedly to illustrate accompanying results.

The organization of this note is as follows. The structure of row and column contractions are discussed first. They already possess an elegant combinatorial structure and play a central role in our parametrizations. Next we consider matrix contractions. The 2×2 matrix contractions were already characterized in [2]. Here we extend the description to matrices of arbitrary size and point out the combinatorial aspect of this parametrization. Then the special case of unitary matrices is examined. We also review the parametrization of positive matrices. While the definitive treatment of the positive case is [5], our discussion differs slightly in some minor technical details. Turning to applications, we first obtain some properties of positive maps. The structure of positive maps and contractions is applied to show that general positive maps are more than merely positive when restricted to certain subsets of positive matrices. Results of this type were obtained in [4] and the parametrization of positive matrices allows one to explore their extensions in a non-ad hoc way. By the correspondence between positive maps and entanglement witnesses [7], we thus show that certain families of bipartite mixed states are separable. The last two applications concerns the unitary dilation of completely positive maps on matrix algebras. While a celebrated result by Stinespring [11] shows that such dilations always exist, the parametrization of contractions allows one to give a concrete constructive procedure for such dilations.

The results on positive maps are first stated in operator-theoretic terms before being placed in physical context. The last two applications are phrased more directly in the language of quantum information theory. For general background in quantum information, we refer the reader to [1] and [9].

Although we only consider matrices of finite size, all parametrizations described in this paper can be extended to (semi-)infinite matrices, where convergence is given by an appropriate operator topology.

2. ROW AND COLUMN CONTRACTIONS

The facts outlined below can be found in [5], in which the structure of such contractions plays a crucial role in obtaining a parametrization for positive kernels, the *Schur-Constantinescu parameters*. Our presentation here is different in that the notion of *defect spaces* is dispensed with. In [5], the defect spaces are used in forcing the uniqueness of certain operator angles. Rather, we identify the unique positive square roots explicitly via partial isometries.

In the following, \mathcal{H} and \mathcal{H}_i denote Hilbert spaces, and $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the bounded operators from \mathcal{H}_1 to \mathcal{H}_2 .

Lemma 1. *Let $X \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $Y \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be bounded operators between Hilbert spaces. $X^*X \leq Y^*Y$ iff there exist a contraction $\Gamma : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ such that $\Gamma X = Y$.*

Proof: Suppose $X^*X \leq Y^*Y$. Define $\Gamma' : \text{by } \Gamma'Xh = Yh$ on $\text{Ran}X$, the range of X . Extending by continuity to the closure of $\text{Ran}X$ and then by 0 to the orthogonal complement gives a contraction Γ . The expression $X^*X \leq Y^*Y$ implies $\text{Ker}Y \subset \text{Ker}X$. So Γ is a contraction satisfying $\Gamma X = Y$. The converse is also straightforward. \square

This leads to the following well-known fact regarding the freedom in the square-roots of a bounded operator.

Lemma 2. *If $X^*X = Y^*Y$, then there exist a partial isometry V such that $VX = Y$. Equivalently, $X^*V^* = Y^*$.*

Proof: It is clear that we can take V to be the Γ defined above. \square

The partial isometry V is unique if the condition $\text{Ker}V \subset (\text{Ran}L)^\perp$ is imposed. This uniqueness condition will be assumed throughout. The operator V is unitary if both $\text{Ran}X$ and $\text{Ran}Y$ are dense. When $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = \mathcal{H}$ is finite dimensional, V can also be assumed unitary. Given a positive operator $A \in \mathcal{L}(\mathcal{H})$ with unique positive square root $A^{\frac{1}{2}}$, every L such that $L^*L = A$ is related to $A^{\frac{1}{2}}$ by $A^{\frac{1}{2}} = VL$, or $A^{\frac{1}{2}} = L^*V^*$.

These two lemmas will be used to exhibit the structure of row contractions. Before doing so, we first introduce a bit of terminology. For a contraction $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, the

positive operator $(I - T^*T)^{\frac{1}{2}} \in \mathcal{L}(\mathcal{H}_1)$ is called the *defect operator* of T and denoted by D_T . Similarly, $D_{T^*} = (I - TT^*)^{\frac{1}{2}} \in \mathcal{L}(\mathcal{H}_2)$ is the defect operator of T^* . When $\mathcal{H}_1 = \mathcal{H}_2$, by the continuous functional calculus,

$$T(I - T^*T) = (I - TT^*)T$$

implies $TD_T = D_{T^*}T$. This relation shows that the operator

$$J(T) = \begin{bmatrix} T & D_{T^*} \\ D_T & -T^* \end{bmatrix}$$

, called the *Julia operator* of the contraction T , is unitary. The Julia operator can be viewed as the building blocks of our parametrizations. It is represented by figure 1.

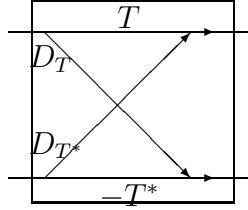


FIGURE 1. The Julia operator $J(T)$

Next we describe row contractions of length two.

Proposition 1. *Let $T = [T_1 \ T_2] \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H})$, then $\|T\| \leq 1$ iff there exist contractions $\Gamma_1 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $\Gamma_2 \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$ such that $T = [\Gamma_1 \ D_{\Gamma_1^*}\Gamma_2]$.*

Proof: (\Rightarrow) We can take Γ_1 to be T_1 . $\|T\| \leq 1$ implies

$$I - TT^* = I - \Gamma_1\Gamma_1^* - T_2T_2^* \geq 0$$

i.e. $D_{\Gamma_1^*}^2 \geq T_2T_2^*$. By lemma, $\Lambda D_{\Gamma_1^*} = T_2^*$. choosing $\Gamma_2 = \Lambda^*$ finishes the argument.

(\Leftarrow) Direct computation. \square

The defect operators for the contraction $T = [T_1 \ T_2]$ can be directly calculated:

$$D_T^2 = \begin{bmatrix} D_{\Gamma_1} & 0 \\ -\Gamma_2^*\Gamma_1 & D_{\Gamma_1} \end{bmatrix} \begin{bmatrix} D_{\Gamma_1} & -\Gamma_1^*\Gamma_2 \\ 0 & D_{\Gamma_1} \end{bmatrix}$$

Invoking lemma, we have

$$D_T = \begin{bmatrix} D_{\Gamma_1} & 0 \\ -\Gamma_2^* \Gamma_1 & D_{\Gamma_1} \end{bmatrix} V$$

for some unique partial isometry V . Similarly,

$$D_{T^*}^2 = (D_{\Gamma_1^*} D_{\Gamma_2^*} D_{\Gamma_2^*}^* D_{\Gamma_1^*}^*).$$

The general structure of row contractions is described by:

Theorem 1. *Let $T = [T_1 \ T_2 \ \cdots] \in \mathcal{L}(\oplus \mathcal{H}_i, \mathcal{H})$, then $\|T\| \leq 1$ iff there exist contractions $\Gamma_i \in \mathcal{L}(\mathcal{H}_i, \mathcal{H})$ such that*

$$T = [\Gamma_1, D_{\Gamma_1^*} \Gamma_2, D_{\Gamma_1^*} D_{\Gamma_2^*} \Gamma_3, \cdots, D_{\Gamma_1^*} \cdots D_{\Gamma_{n-1}^*} \Gamma_n].$$

Furthermore, the defect operators are of the form $D_T^2 =$

$$\begin{bmatrix} D_{\Gamma_1} & 0 & \cdots & 0 \\ -\Gamma_2^* \Gamma_1 & D_{\Gamma_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\Gamma_n^* D_{\Gamma_{n-1}^*} \cdots D_{\Gamma_2^*} \Gamma_1 & -\Gamma_n^* D_{\Gamma_{n-1}^*} \cdots D_{\Gamma_2^*} \Gamma_2 & \cdots & D_{\Gamma_n} \end{bmatrix} \begin{bmatrix} D_{\Gamma_1} & -\Gamma_1^* \Gamma_2 & \cdots & -\Gamma_1^* D_{\Gamma_2^*} \cdots D_{\Gamma_{n-1}^*} \Gamma_n \\ 0 & D_{\Gamma_2} & \cdots & -\Gamma_2^* D_{\Gamma_3^*} \cdots D_{\Gamma_{n-1}^*} \Gamma_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_{\Gamma_n} \end{bmatrix}$$

and

$$D_{T^*}^2 = D_{\Gamma_1^*} \cdots D_{\Gamma_n^*} D_{\Gamma_n^*}^* \cdots D_{\Gamma_1^*}^*.$$

Proof: The argument is by induction. The length $n = 2$ case was shown above. Now suppose the claim holds for length $n - 1$. For a row contraction T of length n , put

$$T = [S, T_n]$$

where by inductive hypothesis,

$$S = [\Gamma_1, D_{\Gamma_1^*} \Gamma_2, D_{\Gamma_1^*} D_{\Gamma_2^*} \Gamma_3, \cdots, D_{\Gamma_1^*} \cdots \Gamma_{n-1}].$$

According to lemma, there exists a contraction Λ such that

$$T_n = D_{S^*} \Lambda.$$

But

$$D_{S^*} = D_{\Gamma_1^*} \cdots D_{\Gamma_n^*} V$$

for some partial isometry V . Choosing $\Gamma_n = V\Lambda$ shows T is of the desired form. Applying the defect operator identity proves the remaining claims. \square

The combinatorial content of the theorem can be depicted pictorially. Figure 2 below shows the parametrization of length 3 row contractions. The downward arrows indicate input ports and the upward arrows output ports. For example, for a matrix

(T_{ij}) , each path from input 3 to output 1 gives rise to a term in the expression of the T_{13} entry (in this particular case, T_{13} is the third entry of a row contraction T).

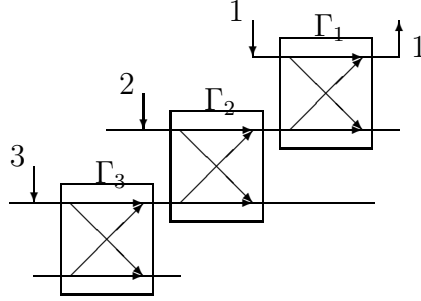


FIGURE 2. Structure of row contractions of length 3

Figures 3 and 4 describe the parametrizations of the defect operators (more precisely that of the natural square roots, or Cholesky factors). In the special case that $TT^* = I_{\mathcal{H}}$, i.e. T is a surjective partial isometry, the contraction Γ_n is in fact a partial isometry.

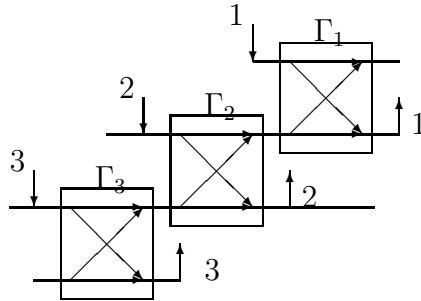


FIGURE 3. Structure of D_T where T is a row contraction of length 3

The structure of column contractions can be exhibited in similar fashion. For completeness, we state the corresponding result for column contractions.

Theorem 2. *An operator*

$$T = \begin{bmatrix} T_1 \\ \vdots \\ T_n \end{bmatrix} : \mathcal{H} \rightarrow \oplus_1^n \mathcal{H}_i$$

is a contraction if and only if

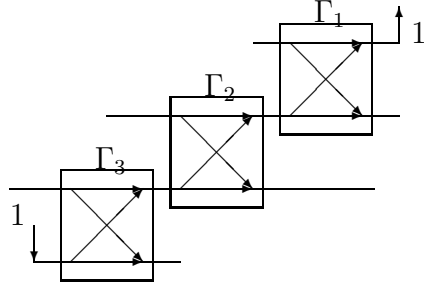


FIGURE 4. Structure of D_T^* where T is a row contraction of length 3

$$T_1 = \Gamma_1 \quad \text{and} \quad T_k = \Gamma_k D_{\Gamma_{k-1}} \cdots D_{\Gamma_1},$$

where Γ_i 's are contractions. Moreover, the defect operators of T and T^* take the form

$$D_T^2 = D_{\Gamma_1} \cdots D_{\Gamma_n} D_{\Gamma_n} \cdots D_{\Gamma_1},$$

and

$$D_{T^*}^2 = \begin{bmatrix} D_{\Gamma_1^*} & 0 & \cdots & 0 \\ -\Gamma_2 \Gamma_1^* & D_{\Gamma_2^*} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\Gamma_n D_{\Gamma_{n-1}} \cdots D_{\Gamma_2} \Gamma_1^* & -\Gamma_n D_{\Gamma_{n-1}} \cdots D_{\Gamma_2} \Gamma_2^* & \cdots & D_{\Gamma_n^*} \end{bmatrix} \begin{bmatrix} D_{\Gamma_1^*} & -\Gamma_1 \Gamma_2^* & \cdots & -\Gamma_1 D_{\Gamma_2} \cdots D_{\Gamma_n} \\ 0 & D_{\Gamma_2^*} & \cdots & -\Gamma_2 D_{\Gamma_3} \cdots D_{\Gamma_n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_{\Gamma_n^*} \end{bmatrix}$$

One can be easily convinced that there are lattice diagrams corresponding to the above description. If a column contraction T is such that $T^*T = I_{\mathcal{H}}$ (i.e. T is an isometry from \mathcal{H} to $\oplus_1^n \mathcal{H}_i$), Γ_n is a partial isometry.

3. MATRIX CONTRACTIONS

Let $T : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{K}_1 \oplus \mathcal{K}_2$,

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

be a contraction. Then $[A \ B] = P_{\mathcal{K}_1} T$, where $P_{\mathcal{K}_1}$ denotes projection onto \mathcal{K}_1 , is necessarily a row contraction, therefore of the form $[\Gamma_1 \ D_{\Gamma_1^*} \Gamma_2]$ where Γ_i are contractions. Similarly, we have

$$\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} \Gamma_1 \\ \Gamma_3 D_{\Gamma_1} \end{bmatrix}.$$

So we have

$$T = \begin{bmatrix} \Gamma_1 & D_{\Gamma_1^*} \Gamma_2 \\ \Gamma_3 D_{\Gamma_1} & D \end{bmatrix}.$$

We will show that the entry D , unspecified so far, can also be parametrized by contractions. To this end, view T as a row contraction $T = [S_1 \ S_2]$ with

$$S_1 = \begin{bmatrix} \Gamma_1 \\ \Gamma_3 D_{\Gamma_1} \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} D_{\Gamma_1^*} \Gamma_2 \\ D \end{bmatrix}.$$

Therefore $S_2 = D_{S_1^*} \Lambda$ for some (column) contraction Λ . S_1 is a column contraction and direct calculation gives

$$D_{S_1^*}^2 = \begin{bmatrix} D_{\Gamma_1^*} & 0 \\ -\Gamma_3 \Gamma_1^* & D_{\Gamma_3} \end{bmatrix} \begin{bmatrix} D_{\Gamma_1^*} & -\Gamma_1 \Gamma_3^* \\ 0 & D_{\Gamma_3} \end{bmatrix}.$$

Thus

$$S_2 = \begin{bmatrix} D_{\Gamma_1^*} & 0 \\ -\Gamma_3 \Gamma_1^* & D_{\Gamma_3} \end{bmatrix} \begin{bmatrix} \Lambda_1 \\ \Lambda_2 D_{\Lambda_1} \end{bmatrix}.$$

Comparing entries and invoking the uniqueness condition shows that $\Lambda_1 = \Gamma_2$. Rename Λ_2 as Γ_4 and we have

$$D = -\Gamma_3 \Gamma_1^* \Gamma_2 + D_{\Gamma_3} \Gamma_4 D_{\Gamma_2}.$$

The above can be summarized by [2]:

Theorem 3. *Every 2×2 contraction $T : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{K}_1 \oplus \mathcal{K}_2$ is of the form*

$$T = \begin{bmatrix} \Gamma_1 & D_{\Gamma_1^*} \Gamma_2 \\ \Gamma_3 D_{\Gamma_1} & -\Gamma_3 \Gamma_1^* \Gamma_2 + D_{\Gamma_3} \Gamma_4 D_{\Gamma_2} \end{bmatrix}$$

where Γ_i are contractions.

The general structure of a $n \times m$ matrix contraction can be obtained in a similar way.

Theorem 4. *Let*

$$T = \begin{bmatrix} T_{11} & \cdots & T_{1m} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nm} \end{bmatrix} : \oplus_1^m \mathcal{H}_i \rightarrow \oplus_1^n \mathcal{K}_i$$

be a contraction. Then the column contraction

$$C_1 = \begin{bmatrix} T_{11} \\ \vdots \\ T_{n1} \end{bmatrix} \text{ is of the form } \begin{bmatrix} \Gamma_1 \\ \Gamma_2 D_{\Gamma_1} \\ \vdots \\ \Gamma_n D_{\Gamma_{n-1}} \cdots D_{\Gamma_1} \end{bmatrix} \text{ where } \Gamma_i \text{ are contractions.}$$

For $1 < k \leq m$, the column

$$\begin{bmatrix} T_{1k} \\ \vdots \\ T_{nk} \end{bmatrix} \text{ is obtained inductively by } \begin{bmatrix} T_{1k} \\ \vdots \\ T_{nk} \end{bmatrix} = D_{C_1^*} D_{C_2^*} \cdots D_{C_{k-1}^*} C_k$$

where C_i is the column contraction parametrized by $\Gamma_{n(i-1)+1} \cdots \Gamma_{ni}$ with C_1 being as specified above.

The proof is immediate and omitted. The combinatorial structure of matrix contractions can also be nicely described via lattice paths. For example, the lattice diagram for the 2×2 case is figure 5.

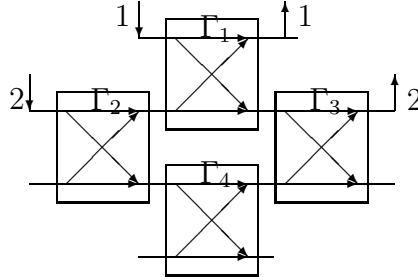


FIGURE 5. A 2×2 matrix contraction

The defect operators of matrix contractions can also be calculated. Due to the "two-layer" nature of its parametrization, the explicit formulae may seem complicated. It is helpful to first look at the lattice diagrams. From inspecting the above figure, we anticipate that the Cholesky factor of D_T and D_{T^*} to have the corresponding pictures given by figures 6 and 7 respectively.

In other words, one should have

$$D_T^2 = \begin{bmatrix} D_{\Gamma_3} D_{\Gamma_1} & 0 \\ -\Gamma_2^* \Gamma_1 D_{\Gamma_3} - D_{\Gamma_2} \Gamma_4^* \Gamma_3 & D_{\Gamma_2} D_{\Gamma_4} \end{bmatrix} \begin{bmatrix} D_{\Gamma_3} D_{\Gamma_1} & -D_{\Gamma_3} \Gamma_1^* \Gamma_2 - \Gamma_3^* \Gamma_4 D_{\Gamma_2} \\ 0 & D_{\Gamma_4} D_{\Gamma_2} \end{bmatrix},$$

and

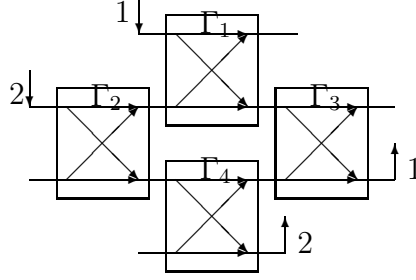


FIGURE 6. The natural Cholesky factor of D_T where T is a 2×2 matrix contraction

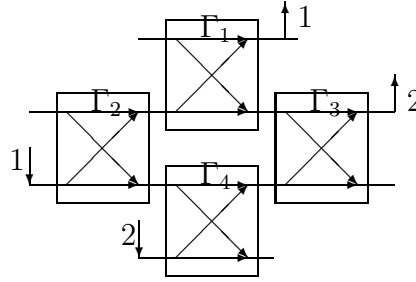


FIGURE 7. The Cholesky factor for D_{T^*}

$$D_{T^*}^2 = \begin{bmatrix} D_{\Gamma_1^*} D_{\Gamma_2^*} & 0 \\ -\Gamma_2 \Gamma_4^* D_{\Gamma_3^*} - D_{\Gamma_2^*} \Gamma_1^* \Gamma_3^* & D_{\Gamma_4^*} D_{\Gamma_2^*} \end{bmatrix} \begin{bmatrix} D_{\Gamma_2^*} D_{\Gamma_1^*} & -D_{\Gamma_3^*} \Gamma_4 \Gamma_2^* - \Gamma_3 \Gamma_1 D_{\Gamma_2^*} \\ 0 & D_{\Gamma_2^*} D_{\Gamma_4^*} \end{bmatrix}.$$

This can be confirmed by a straightforward but perhaps tedious calculation, which we shall not bore the reader with. The defect operators for matrix contraction of any finite size can be obtained in similar fashion.

Unitary Matrices The unitary operators are the extreme points of contractions, thus a special case. If

$$T = \begin{bmatrix} \Gamma_1 & D_{\Gamma_1^*} \Gamma_2 \\ \Gamma_3 D_{\Gamma_1} & -\Gamma_3 \Gamma_1^* \Gamma_2 + D_{\Gamma_3} \Gamma_4 D_{\Gamma_2} \end{bmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{K}_1 \oplus \mathcal{K}_2$$

is unitary, then $[\Gamma_1 \ D_{\Gamma_1^*} \Gamma_2]$ is a partial isometry therefore so is Γ_2 ; same goes for Γ_3 . When all spaces are finite dimensional and, in the expression

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

we have B and C being square matrices, Γ_2 and Γ_3 are unitary and the description of T becomes very simple:

$$T = \begin{bmatrix} \Gamma_1 & D_{\Gamma_1^*} \Gamma_2 \\ \Gamma_3 D_{\Gamma_1} & -\Gamma_3 \Gamma_1^* \Gamma_2 \end{bmatrix}.$$

In other words, all unitary matrices are related to the Julia operator via

$$T = \begin{bmatrix} 1 & 0 \\ 0 & \Gamma_3 \end{bmatrix} \begin{bmatrix} \Gamma_1 & D_{\Gamma_1^*} \\ D_{\Gamma_1} & -\Gamma_1^* \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \Gamma_2 \end{bmatrix}.$$

This will be applied in the sequel in calculating dilations of completely positive maps/quantum operations.

4. POSITIVE MATRICES

Similar dilation-theoretic techniques can be applied to positive matrices. As stated in the introduction, we outline basic results for completeness. See [5] for a comprehensive discussion. As for contractions, one can start by examining the 2×2 case then apply induction. Let

$$A = \begin{bmatrix} L_1^* L_1 & A_{12} \\ A_{12}^* & L_2^* L_2 \end{bmatrix} \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$$

be a positive semidefinite operator matrix whose entries are bounded operators, that is

$$\left\langle \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, A \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \right\rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2} \geq 0$$

for all $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$.

Theorem 5. *There exists a unique contraction $\Gamma \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ such that $A_{12} = L_1^* \Gamma L_2$.*

Proof: : Assume for the moment that both A_{11} and A_{22} are invertible. Then a *Frobenius-Schur identity* holds:

$$A = \begin{bmatrix} L_1^* L_1 & A_{12} \\ A_{12}^* & L_2^* L_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ A_{12}^* (L_1^* L_1)^{-1} & I \end{bmatrix} \cdot \begin{bmatrix} L_1^* L_1 & 0 \\ 0 & L_2^* L_2 - A_{12}^* (L_1^* L_1)^{-1} A_{12} \end{bmatrix} \cdot \begin{bmatrix} I & (L_1^* L_1)^{-1} A_{12} \\ 0 & I \end{bmatrix}.$$

It follows that A is positive if and only if its *Schur complement*

$$L_2^* L_2 - A_{12}^* (L_1^* L_1)^{-1} A_{12} \geq 0$$

i.e.

$$L_2^* L_2 \geq A_{12}^* (L_1^*)^{-1} (L_1)^{-1} A_{12}.$$

By lemma 1, there exist a contraction Γ such that $\Gamma L_2 = (L_1^{-1})^* A_{12}$, where we impose the condition required for uniqueness. Thus $A_{12} = L_1^* \Gamma L_2$.

For the general case where $A_{11} = L_1^* L_1$ and $A_{22} = L_2^* L_2$ need not be invertible, consider the sequences $\{\alpha_n^* \alpha_n = A_{11} + \frac{1}{n}\}$ and $\{\beta_n^* \beta_n = A_{22} + \frac{1}{n}\}$. By the spectral mapping theorem for self adjoint operators, $\alpha_n^* \alpha_n$ and $\beta_n^* \beta_n$ are invertible for all n . Therefore there exist contractions $\{\Gamma_n\}$ with $A_{12} = \alpha_n^* \Gamma_n \beta_n$. Since the unit ball in $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ is compact in the weak operator topology, Γ_n converges to some contraction Γ weakly. We can compute directly, for all $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$,

$$\langle h_1, L_1^* \Gamma L_2 h_2 \rangle_{\mathcal{H}_1} = \lim_n \langle h_1, L_1^* \Gamma_n L_2 h_2 \rangle_{\mathcal{H}_1} = \lim_n \langle h_1, \alpha_n^* \Gamma_n \beta_n h_2 \rangle_{\mathcal{H}_1} = \langle h_1, A_{12} h_2 \rangle_{\mathcal{H}_1}.$$

This proves the claim. \square

This can be generalized to positive operator matrices of arbitrary size in the obvious way. We present the finite case as an algorithm

Algorithm 1. [5] *Let $A = [A_{ij}]_{ij} \in \mathcal{L}(\oplus_{i=1}^n \mathcal{H}_i)$ be positive. The **Schur-Constantinescu**, or **SC**, parametrization of A can be calculated recursively as follows:*

i) $\begin{bmatrix} A_{n-1,n-1} & A_{n-1,n} \\ A_{n,n-1} & A_{n,n} \end{bmatrix}$ is positive and can be parametrized according to the 2×2 case.

ii) For $1 \leq k \leq n-2$, the SC parametrization of $\begin{bmatrix} A_{k,k} & A_{k,k+1} & \cdots & A_{k,n} \\ A_{k+1,k} & A_{k+1,k+1} & \cdots & A_{k+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,k} & A_{n,k+1} & \cdots & A_{n,n} \end{bmatrix}$

is calculated by first considering $\begin{bmatrix} A_{k+1,k+1} & \cdots & A_{k+1,n} \\ \vdots & \ddots & \vdots \\ A_{n,k+1} & \cdots & A_{n,n} \end{bmatrix} = L_{k+1,k+1}^* L_{k+1,k+1}$,

where L_{k+1} is the Cholesky factor calculated in the previous step then put

$$[A_{k,k+1} \ \cdots \ A_{k,n}] = A_{k,k}^{\frac{1}{2}} R_k L_{k+1}$$

with R_k being the corresponding row contraction.

The lattice diagram for a 4×4 positive matrix is given below.

Tensor product of positive matrices If $M = (m_{ij}) \in \mathbf{C}^{n \times n}$ is a positive matrix with complex entries and $A = B^* B \in \mathcal{L}(\mathcal{H})$ where \mathcal{H} is a Hilbert space. Suppose M is SC-parametrized by $\{\gamma_i\} \subset \mathbf{C}$, $i = 1, \dots, \frac{1}{2}m(m-1)$ with $|\gamma_i| \leq 1$. The parametrization of the positive matrix

$$M \otimes A = (m_{ij} L^* L) \in \mathbf{C}^{n \times n} \otimes \mathcal{L}(\mathcal{H})$$

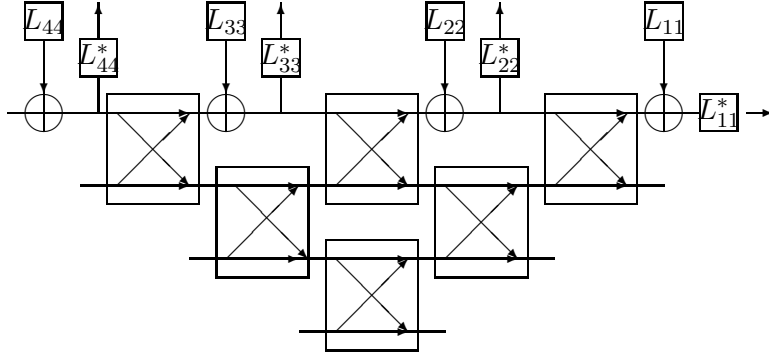


FIGURE 8. Lattice structure for 4×4 positive matrices

can be described easily [6]. Namely, let $\Gamma_i = \gamma_i I_{\mathcal{H}}$ and $L_i = \sqrt{m_{ii}} B$; it is clear that they parametrize $M \otimes A$ in the sense of Schur-Constantinescu.

Matrices given by a strict inequality. The natural square roots given by the SC-parametrization are upper(or lower)-triangular, i.e. they are Cholesky factors. Owing to this fact, if $A^*A \geq B^*B$, and the SC parameters of A^*A are known, B is readily described. Take for instance the 2×2 case. $B = \Gamma A$, where Γ is a contraction. Let

$$A = \begin{bmatrix} L_{11} & \Lambda L_{22} \\ 0 & D_{\Lambda} L_{22} \end{bmatrix} \quad \text{and} \quad \Gamma = \begin{bmatrix} \Gamma_1 & D_{\Gamma_1^*} \Gamma_2 \\ \Gamma_3 D_{\Gamma_1} & -\Gamma_3 \Gamma_1^* \Gamma_2 + D_{\Gamma_3} \Gamma_4 D_{\Gamma_2} \end{bmatrix},$$

then ΓA corresponds to the following figure:

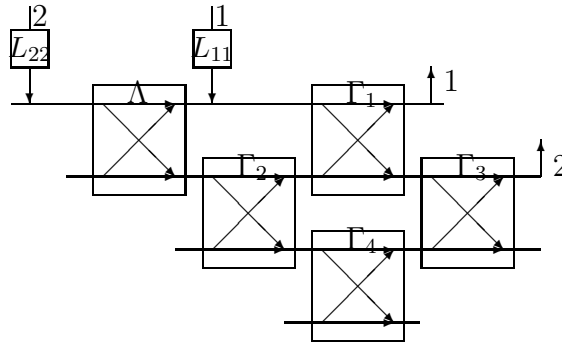


FIGURE 9. The 2×2 matrix ΓA

5. APPLICATIONS

Due to the ubiquity of positive matrices, the Schur-Constantinescu parametrization of positive matrices has numerous applications [5]. More recently, it has been applied in the context of quantum information theory. For example, it was used to parametrize completely positive maps (quantum channels) in [6]. A cylinder-like condition, called

the *Bloch cylinder*, was obtained for positive matrices of trace 1 (quantum states) of any finite dimension. This provides an alternative to the well-known Bloch sphere. In [12] it was applied to show that every positive map is completely positive to a certain extent, thus establishing the separability of certain families of quantum states in arbitrary finite dimensions. In a similar vein, in this section we obtain more results in this direction, in a sense extending what was found in [4]. Also, we consider two further applications that are matrix completion problems in disguise and can be solved by utilizing parametrization of matrix contractions.

5.1. Positive Maps. In this section, the structure of contractions and positive matrices are applied to extend properties of positive maps.

Definition 1. Let \mathcal{H} and \mathcal{K} be Hilbert spaces. A linear map $\Phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{K})$ is said to be **positive** if it preserves the cone of positive elements, i.e. $A \geq 0$ implies $\Phi(A) \geq 0$. Let $\mathbf{C}^{n \times n}$ denote the $n \times n$ matrices of complex numbers and I_n the identity map on $\mathbf{C}^{n \times n}$, then a map Φ is said to be **n-positive** if the induced map

$$I_d \otimes \Phi : \mathbf{C}^{n \times n} \otimes \mathcal{L}(\mathcal{H}) \rightarrow \mathbf{C}^{n \times n} \otimes \mathcal{L}(\mathcal{K})$$

is positive, and Φ is **completely positive**, or **CP**, if it is n -positive for all n .

We state the following result without proof [10].

Theorem 6. (Russo-Dye) Let Φ be a positive map between unital C^* -algebras, then $\|\Phi\| \leq \|\Phi(I)\|$.

In particular, if Φ is unital and Γ a contraction, then

$$\Phi(\Gamma^*)\Phi(\Gamma) = \|\Phi(\Gamma)\|^2 \leq \Gamma^*\Gamma \leq I.$$

Similarly, $\Phi(\Gamma)\Phi(\Gamma^*) \leq I$. Making use of this, one has [4]:

Theorem 7. Let $\Phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{K})$ be a positive map, then for all positive

$$\rho = \begin{bmatrix} T & S \\ S^* & T \end{bmatrix} \in \mathbf{C}^{2 \times 2} \otimes \mathcal{L}(\mathcal{H}), \quad \text{we have} \quad (I_2 \otimes \Phi)(\rho) = \begin{bmatrix} \Phi(T) & \Phi(S) \\ \Phi(S^*) & \Phi(T) \end{bmatrix} \geq 0.$$

We recast the proof so that the role played by contractions is made more apparent.

Proof: Assume for the moment that T^{-1} exists and $\Phi(I)$ is invertible, therefore so is $\Phi(T)$. According to theorem 5, $S = T^{\frac{1}{2}}\Gamma T^{\frac{1}{2}}$ for some contraction Γ . It is equivalent to show that the Schur complement

$$\Phi(T) - \Phi(S^*)\Phi(T)^{-1}\Phi(S) \geq 0$$

i.e.

$$I \geq \Phi(T)^{-\frac{1}{2}}\Phi(T^{\frac{1}{2}}\Gamma^*T^{\frac{1}{2}})\Phi(T)^{-\frac{1}{2}} \cdot \Phi(T)^{-\frac{1}{2}}\Phi(T^{\frac{1}{2}}\Gamma T^{\frac{1}{2}})\Phi(T)^{-\frac{1}{2}}.$$

This suggests that we define $\Psi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{K})$ by

$$\Psi(A) = \Phi(T)^{-\frac{1}{2}} \Phi(T^{\frac{1}{2}} A T^{\frac{1}{2}}) \Phi(T)^{-\frac{1}{2}}.$$

Ψ is an unital positive map. By Russo-Dye,

$$I \leq \Psi(\Gamma) \Psi(\Gamma^*) = \Phi(T)^{-\frac{1}{2}} \Phi(T^{\frac{1}{2}} \Gamma^* T^{\frac{1}{2}}) \Phi(T)^{-\frac{1}{2}} \cdot \Phi(T)^{-\frac{1}{2}} \Phi(T^{\frac{1}{2}} \Gamma T^{\frac{1}{2}}) \Phi(T)^{-\frac{1}{2}},$$

which is what we want.

If T is not invertible, consider the sequence $\{T_n = T + \frac{1}{n}\}$. T_n tends to T uniformly and positive maps are bounded. So

$$\Phi(S) = \lim_n \Phi(T_n)^{\frac{1}{2}} \cdot \Lambda_n \cdot \lim_n \Phi(T_n)^{\frac{1}{2}}$$

for contractions Λ_n . Let $\Lambda \in \mathcal{L}(\mathcal{K})$ be a weak operatorial limit of Λ_n , then

$$\Phi(S) = \Phi(\lim_n T_n)^{\frac{1}{2}} \cdot \Lambda \cdot \Phi(\lim_n T_n)^{\frac{1}{2}} = \Phi(T)^{\frac{1}{2}} \Lambda \Phi(T)^{\frac{1}{2}}.$$

So the claim holds.

If, in addition, $\Phi(I)$ is not invertible, take a positive functional ϕ with $\phi(I) = 1$. Define $\Phi_n(A) = \Phi(A) + \frac{1}{n}\phi(A)$. We have $\Phi_n \rightarrow \Phi$ in the operator norm of linear maps, and

$$\Phi(S) = \lim_n \Phi_n(S) = \lim_n \Phi_n(T)^{\frac{1}{2}} \cdot \Lambda'_n \cdot \lim_n \Phi_n(T)^{\frac{1}{2}}.$$

The same weak limit argument shows that $\Phi(S) = \Phi(T)^{\frac{1}{2}} \Lambda' \Phi(T)^{\frac{1}{2}}$ for some contraction Λ' . This proves the theorem. \square

In other words, any positive map is 2-positive on the 2×2 Toeplitz matrices. Now we extend this to a sub-family of 3×3 positive matrices. Recall that an operator $A \in \mathcal{L}(\mathcal{H})$ is said to be **subnormal** if it is the compression of a 2×2 normal upper-triangular N , i.e. if there exist some Hilbert space \mathcal{K} and a normal $N \in \mathcal{L}(\mathcal{K})$ such that N is of the form

$$N = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}.$$

The following fact, which we state without proof, will be used [4] :

Lemma 3. *For any unital positive map $\Phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{K})$ and any normal $A \in \mathcal{L}(\mathcal{H})$,*

$$\Phi(A^* A) \geq \Phi(A^*) \Phi(A) \quad \text{and} \quad \Phi(A^* A) \geq \Phi(A) \Phi(A^*).$$

What is known as *Kadison's inequality* will also be needed: for every unital positive map Φ and every self-adjoint S , $\Phi(S^2) \geq \Phi(S)^2$ [8]. What we will show that is essentially every positive map is 3-positive in a certain limited sense. We first notice that subnormal contractions enjoy a property stronger than that prescribed by Russo-Dye.

Lemma 4. *If Φ is a unital positive map and Γ a subnormal contraction, then*

$$I - \Phi(\Gamma^*\Gamma) - \Phi(D_{\Gamma^*})^2 \geq 0 \quad \text{and} \quad I - \Phi(\Gamma^*\Gamma) - \Phi(D_\Gamma)^2 \geq 0.$$

Proof: We directly compute

$$\begin{aligned} I - \Phi(\Gamma^*\Gamma) - \Phi(D_{\Gamma^*})^2 &\geq I - \Phi(\Gamma^*\Gamma) - \Phi(D_{\Gamma^*}^2) \\ &= I - \Phi(\Gamma^*\Gamma) - \Phi(I) + \Phi(\Gamma^*\Gamma) = \Phi(\Gamma^*\Gamma) - \Phi(\Gamma^*\Gamma), \end{aligned}$$

which is positive, by the preceding lemma. The second inequality is similar. \square

Theorem 8. *i) Consider positive matrices in $(A_{ij}) \in \mathbf{C}^{3 \times 3} \otimes \mathcal{L}(\mathcal{H})$ that are SC-parametrized in the following way: for $i = 1, 2, 3$, choose $A_{ii} = T \geq 0$. Of the three contractions, choose Γ_{12} to be subnormal, $\Gamma_{23} = 0$, and $\Gamma_{13} = I$. In other words, we consider 3×3 positive matrices of the form*

$$(A_{ij}) = \begin{bmatrix} T & T^{\frac{1}{2}}\Gamma T^{\frac{1}{2}} & T^{\frac{1}{2}}D_{\Gamma^*}T^{\frac{1}{2}} \\ T^{\frac{1}{2}}\Gamma^*T^{\frac{1}{2}} & T & 0 \\ T^{\frac{1}{2}}D_{\Gamma^*}T^{\frac{1}{2}} & 0 & T \end{bmatrix}.$$

Then

$$(I_3 \otimes \Phi)(A_{ij}) \geq 0,$$

for any positive map Φ acting on $\mathcal{L}(\mathcal{H})$.

ii) The same is true if Γ_{23} is subnormal, $\Gamma_{12} = 0$, $\Gamma_{13} = I$, i.e. if

$$(A_{ij}) = \begin{bmatrix} T & 0 & T^{\frac{1}{2}}D_\Gamma T^{\frac{1}{2}} \\ 0 & T & T^{\frac{1}{2}}\Gamma T^{\frac{1}{2}} \\ T^{\frac{1}{2}}D_\Gamma T^{\frac{1}{2}} & T^{\frac{1}{2}}\Gamma^*T^{\frac{1}{2}} & T \end{bmatrix}.$$

We do not completely recover Choi's result by considering the 2×2 leading minor in part i). The requirement that Γ be subnormal is particular to the 3×3 case, due to the presence of D_{Γ^*} in the parametrization.

Proof: i) Assume for the moment that $\Phi(I)$ and T are invertible. Invoking again theorem 5 on the 2×2 case, it is equivalent to show that

$$\Phi(T) \geq \begin{bmatrix} \Phi(T^{\frac{1}{2}}\Gamma T^{\frac{1}{2}}) & \Phi(T^{\frac{1}{2}}D_{\Gamma^*}T^{\frac{1}{2}}) \end{bmatrix} \begin{bmatrix} \Phi(T)^{-1} & 0 \\ 0 & \Phi(T)^{-1} \end{bmatrix} \begin{bmatrix} \Phi(T^{\frac{1}{2}}\Gamma^*T^{\frac{1}{2}}) \\ \Phi(T^{\frac{1}{2}}D_{\Gamma^*}T^{\frac{1}{2}}) \end{bmatrix}.$$

The right hand side is

$$\Phi(T^{\frac{1}{2}}\Gamma T^{\frac{1}{2}})\Phi(T)^{-1}\Phi(T^{\frac{1}{2}}\Gamma^*T^{\frac{1}{2}}) + \Phi(T^{\frac{1}{2}}D_{\Gamma^*}T^{\frac{1}{2}})\Phi(T)^{-1}\Phi(T^{\frac{1}{2}}D_{\Gamma^*}T^{\frac{1}{2}}).$$

Again we define an unital positive map Ψ by

$$\Psi(B) = \Phi(T)^{-\frac{1}{2}}\Phi(T^{\frac{1}{2}}BT^{\frac{1}{2}})\Phi(T)^{-\frac{1}{2}}.$$

By the subnormality of Γ and lemma 4,

$$I \geq \Psi(\Gamma)\Psi(\Gamma^*) + \Psi(D_{\Gamma^*})^2$$

which proves the claim. The general case can be shown using argument similar in that of theorem 7.

ii) The argument is analogous to i) and omitted. \square

The following result is of similar nature. The special 2×2 case, proven in [4], says that any positive map is positive on 2×2 Hankel matrices.

Theorem 9. *If A is a positive matrix of the form*

$$A = \begin{bmatrix} T & 0 & \cdots & 0 & S_1 \\ 0 & T & \cdots & 0 & S_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & T & \vdots \\ S_1 & S_2 & \cdots & 0 & R \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} T & S_1 & \cdots & \cdots & S_{m-1} \\ S_1 & R & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & R & 0 \\ S_{m-1} & 0 & \cdots & 0 & R \end{bmatrix} \in C^{m \times m} \otimes \mathcal{L}(\mathcal{H}),$$

where m is arbitrary, then

$$(I_m \otimes \Phi)(A) \geq 0$$

for any positive map Φ .

Proof: Suppose A is of the first form in the claim. The argument uses only the structure of 2×2 matrices and thus we consider first the case $m = 2$. By virtue of previous arguments, it can be assumed without loss of generality that T and $\Phi(I)$ are both invertible. To show

$$\begin{bmatrix} \Phi(T) & \Phi(S) \\ \Phi(S) & \Phi(R) \end{bmatrix} \geq 0$$

, it suffices to obtain $\Phi(ST^{-1}S) \geq \Phi(S)\Phi(T)^{-1}\Phi(S)$ because the Schur complement $R - ST^{-1}S$ is positive. This is equivalent to

$$\Phi(T)^{-\frac{1}{2}}\Phi(ST^{-1}S)\Phi(T)^{-\frac{1}{2}} \geq \Phi(T)^{-\frac{1}{2}}\Phi(S)\Phi(T)^{-\frac{1}{2}} \cdot \Phi(T)^{-\frac{1}{2}}\Phi(S)\Phi(T)^{-\frac{1}{2}},$$

which suggests we define a unital positive map by

$$\Psi(B) = \Phi(T)^{-\frac{1}{2}}\Phi(B)\Phi(T)^{-\frac{1}{2}}.$$

Invoking Kadison's inequality then proves the 2×2 case. For $m > 2$, it is enough to show that

$$\begin{aligned} \Phi\left(\begin{bmatrix} S_1 & \cdots & S_{m-1} \end{bmatrix} \begin{bmatrix} T^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & T^{-1} \end{bmatrix} \begin{bmatrix} S_1 \\ \cdots \\ S_{m-1} \end{bmatrix}\right) \\ \geq \sum_{i=1}^{m-1} \Phi(S_i)\Phi(T)^{-1}\Phi(S_i), \end{aligned}$$

i.e.

$$\Phi\left(\sum_{i=1}^{m-1} S_i T^{-1} S_i\right) \geq \sum_{i=1}^{m-1} \Phi(S_i)\Phi(T)^{-1}\Phi(S_i).$$

But A is positive only if the principal 2×2 minors

$$\begin{bmatrix} T & S_i \\ S_i & R \end{bmatrix}$$

are positive. Thus the desired inequality holds by linearity of Φ and the 2×2 argument. \square

Remark What we have show above is that a positive map is 3-positive and CP (in the case of theorems 8 and 9 respectively) to a certain extent. Results in the similar vein were obtained in [12] that are also applications of SC parameters. Namely positive maps were shown to be CP if restricted to certain families, of arbitrary finite size, which can be SC-parametrized by two real parameters. In that approach, Choi's result on 2×2 Hankel matrices also become a special case. For comparison, a result from [12] for matrices with entries in $\mathbf{C}^{3 \times 3}$ is stated below.

Theorem 10. *Let \mathcal{S} be the linear span of*

$$\left\{A = \begin{bmatrix} a & a & b \\ a & a & b \\ b & b & c \end{bmatrix} \mid a, b, c \in \mathbf{C}\right\} \subset \mathbf{C}^{3 \times 3}$$

and where $m \in \mathbf{N}$ be arbitrary. Then for all positive $\rho \in \mathbf{C}^{m \times m} \otimes \mathcal{S}$,

$$(I_m \otimes \Phi)(\rho) \geq 0$$

where Φ is any positive map acting on $\mathbf{C}^{3 \times 3}$. Furthermore, the claim holds if A is replaced by

$$\begin{bmatrix} a & c & a \\ c & b & c \\ a & c & a \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a & c & c \\ c & b & b \\ c & b & b \end{bmatrix}.$$

Separable Quantum States In physical language, trace-class positive matrices are un-normalized mixed states [1].

Definition 2. [13] *Let the state space of a bipartite quantum system be the tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where \mathcal{H}_i are Hilbert spaces. A state $\sigma \in \mathcal{L}(\mathcal{H})$ is **separable** if it lies in the closure, in the trace norm, of states of the form*

$$\rho = \sum_{i=1}^k \rho_i^1 \otimes \rho_i^2,$$

where ρ_i^j are states in \mathcal{H}_j .

The membership problem for separable states is sometimes called the *separability problem*. The following theorem establishes the correspondence between the classification of positive maps and the membership problem for separable states[7]:

Theorem 11. *If a mixed state $\sigma \in \mathcal{L}(\mathcal{H}_A) \otimes \mathcal{L}(\mathcal{H}_B)$ is such that for every positive map Φ from $\mathcal{L}(\mathcal{H}_B)$ to $\mathcal{L}(\mathcal{H}_A)$, the operator $(I_A \otimes \Phi)(\sigma)$ is positive, then σ is separable.*

The above result is of geometric nature and a consequence of the hyperplane-separation variant of Hahn-Banach. Thus in quantum information theory, positive but not CP maps are called *entanglement witnesses*, for they detect the entanglement of some state. If a family \mathcal{S} of positive matrices is such that any positive map behaves as a CP map when restricted to \mathcal{S} , then \mathcal{S} must consist of separable states, due to lack to entanglement witnesses. Thus what was shown in the previous discussion translate to that all positive matrices of the forms specified in theorems 8 and 9 are separable states. In particular, $2 \times m$ block-Toeplitz and block-Hankel states are separable, where m need not be finite.

5.2. POVM's. We first give a few relevant definitions and basic results; the reader is referred to [1] for more background information. In the von Neumann measurement scheme, the *effects* of a quantum measurement are assumed to satisfy the *projective hypothesis*, i.e. they are self-adjoint projections and form the resolution of identity of a self adjoint operator. A resolution of the identity is sometimes called *projection-valued measure*, or *PVM*. A more general formulation of measurement replaces these projections by positive operators:

Definition 3. [10] *Let X be a compact Hausdorff space, \mathcal{B} be the Borel σ -algebra on X , and \mathcal{H} a Hilbert space. A **positive operator-valued measure**, or **POVM** is a map $E : \mathcal{B} \rightarrow \mathcal{L}(\mathcal{H})$ such that $E(B) \geq 0$ for all B , and E is countably additive in the weak topology on \mathcal{H} , i.e. for any pairwise disjoint collection $\{B_i\}_{i \geq 1} \subset \mathcal{B}$,*

$$\langle E(\cup_i B_i)x, y \rangle = \sum_i \langle E(B_i)x, y \rangle$$

for all $x, y \in \mathcal{H}$.

If $E(B)$ is self adjoint for all B and $E(B_1 \cap B_2) = E(B_1)E(B_2)$, then each $E(B)$ is a self adjoint projection. Thus E is a PVM and we recover the von Neumann Scheme. A natural question is whether a POVM can be "lifted" to a larger space where it is a PVM. The general answer to this dilation-theoretic question is given by:

Theorem 12. (Naimark) *Let X be a compact Hausdorff space. Suppose E is a POVM on the σ -algebra generated by the Borel sets of X and E takes values in $\mathcal{L}(\mathcal{H})$. There exist a Hilbert space \mathcal{K} containing \mathcal{H} as a subspace and a PVM F on X with values in $\mathcal{L}(\mathcal{K})$ such that*

$$E(B) = P_{\mathcal{H}}F(B)P_{\mathcal{H}}$$

,for all Borel $B \subset X$.

The proof follows from the fact that $C(X)$ is a commutative C^* -algebra and therefore the induced map

$$f \in C(X) \xrightarrow{\Phi} \int_X f(x)dE(x)$$

is completely positive, rather than merely positive. Stinespring's theorem on CP maps [11] then says Φ can be dilated to a homomorphism Φ' . The PVM corresponding to Φ' is the desired F . For a complete proof, see [10]. Stinespring's theorem is a generalization of the Gelfand-Naimark representation theorem of positive functionals.

In quantum information theory, Of particular interest is the case when X is finite, with the discrete topology. In that case, one would like and can indeed find solutions of more concrete nature. Let $X = \{1, 2, \dots, n\}$. Without losing generality, we can consider only POVM's whose elements are rank-1 projections that may not be mutually orthogonal. Suppose a POVM on X is given by $E(i) = v_i v_i^*$, $i = 1, \dots, n$ with $\sum_i v_i v_i^* = I_m$ where $m \leq n$ and I_m is the identity in $\mathbf{C}^{m \times m}$. In other words,

$$M = [v_1 \quad \dots \quad v_n] \in \mathbf{C}^{m \times n}$$

is an isometry, i.e. $M^*M = I$. We want to specify a PVM F taking value in some $\mathcal{L}(\mathcal{K})$ whose restriction to \mathbf{C}^m is E . This is a trivial completion problem: given a (rectangular) isometry M , find a suitable rectangular N s.t. $\begin{bmatrix} M \\ N \end{bmatrix}$ is unitary. It is an elementary fact from linear algebra that such an N can always be found.

However, in physical considerations, a suitable POVM is often obtained by coupling to the original system an *ancilla*. This amounts to finding appropriate operators A , B , and C such that

$$U = \begin{bmatrix} M & A \\ B & C \end{bmatrix} \in \mathbf{C}^{k \times k}$$

is unitary. The columns of U , $\{u_1, \dots, u_k\}$, then gives a PVM on $\mathbf{C}^k = \mathbf{C}^m \oplus \mathbf{C}^{k-m}$ with the desired properties.

The completion of 2×2 unitary matrices offers an immediate solution to this problem. Since M is an isometry, it is trivially a contraction. For example, consider the special case of Julia operator. So the PVM are the projections onto the column vectors of

$$J(M) = \begin{bmatrix} M & D_{M^*} \\ D_M & -M^* \end{bmatrix}$$

$M^*M = I$ means $D_M = 0$ and similarly $D_{M^*} = (I - MM^*)^{\frac{1}{2}} = (I - MM^*)$ because $I - MM^*$ is a projection. Thus

$$J(M) = \begin{bmatrix} M & I - MM^* \\ 0 & -M^* \end{bmatrix}$$

and a suitable PVM F is obtained without doing any calculation whatsoever. If M is $m \times n$ with $m \leq n$, the Hilbert space \mathcal{K} is of dimension $m + n$. From the discussion on unitary operators, we see that the freedom in obtaining a suitable PVM is described by

$$J'(M) = \begin{bmatrix} I & \\ 0 & U_1 \end{bmatrix} \begin{bmatrix} M & I - MM^* \\ 0 & -M^* \end{bmatrix} \begin{bmatrix} I & \\ 0 & U_2 \end{bmatrix}$$

where U_i are unitary matrices of suitable size. An effect of the POVM can be obtained by first performing the corresponding yes-no measurement from the dilated PVM then taking the partial trace with respect to the ancilla variables.

Note Physical reasons require that the dimension of $J'(M)$ be $(m \times k) \times (m \times k)$ where the ancilla has state space of dimension k . This can always be achieved via direct sum with an identity matrix of appropriate size.

5.3. Mocking Up a Quantum Operation. We now extend our discussion to general quantum operations, of which measurement is a special case. We are interested in the finite dimensional case.

Definition 4. *A quantum operation is a completely positive map $\Phi : \mathbf{C}^{n \times n} \rightarrow \mathbf{C}^{m \times m}$ between density matrices that does not increase the trace.*

It is well known that Φ must take the form $\Phi(\rho) = \sum_{i=1}^{nm} E_i \rho E_i^*$ [3]. By *mocking up* [9] we mean, similar to the POVM case, coupling the system to an ancilla and find an unitary evolution on the combined system such that the reduced state, obtained via the partial trace, is $\sum_{i=1}^{nm} E_i \rho E_i^*$. In other words, given a CP map Φ , we wish to find an unitary dilation. Again Stinespring's theorem ensures the existence of such a dilation. But in practice perhaps one would like to obtain a less abstract solution. The parametrization of matrix contractions provides one such explicit and easy procedure.

Choose the ancilla to be \mathbf{C}^{nm} . Any quantum state can be purified [9], and we can assume the ancilla is in a pure state, a rank-1 projection, of the special form $e_0 e_0^*$, where

$$e_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The state of the composite system is

$$e_0 e_0^* \otimes \rho = \begin{bmatrix} \rho & 0 \\ 0 & 0 \\ & \ddots \end{bmatrix}.$$

If U is the proposed unitary evolution, then

$$\begin{aligned} U \rho U^* &= \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \\ & \ddots \end{bmatrix} \begin{bmatrix} \rho & 0 \\ 0 & 0 \\ & \ddots \end{bmatrix} \begin{bmatrix} U_{11}^* & U_{21}^* \\ U_{12}^* & U_{22}^* \\ & \ddots \end{bmatrix} \\ &= \begin{bmatrix} U_{11} \rho U_{11}^* & U_{11} \rho U_{21}^* \\ U_{21} \rho U_{11}^* & U_{21} \rho U_{21}^* \\ & \ddots \end{bmatrix}. \end{aligned}$$

Tracing out the ancilla, the first system, gives the reduced density matrix $\sum_1^{nm} U_{i1} \rho U_{i1}^*$. Therefore to specify U is to find appropriate operators A , B , and C such that

$$U = \begin{bmatrix} T & A \\ B & C \end{bmatrix} \in \mathbf{C}^{k \times k}$$

is unitary, where T is the contraction

$$T = \begin{bmatrix} E_1 \\ \vdots \\ E_{nm} \end{bmatrix}.$$

As before, this can be achieved by forming the operator

$$J'(T) = \begin{bmatrix} I & \\ 0 & U_1 \end{bmatrix} \begin{bmatrix} T & D_{T^*} \\ D_T & -T^* \end{bmatrix} \begin{bmatrix} I & \\ 0 & U_2 \end{bmatrix}.$$

Notice purification of mixed states was applied only to the ancilla. There is another approach that relies on purification more heavily. Namely, one treats the matrix

$(\Phi(E_{ij}))_{ij}$, where E_{ij} are standard matrix units, as a state and consider its purification. The differences are, first, in the latter only the range of Φ is coupled to an ancilla and, second, the lifted map still may not be an unitary evolution.

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