

Quantising the Foliation in History Quantum Field Theory

C.J. Isham¹

The Blackett Laboratory
Imperial College of Science, Technology & Medicine
South Kensington
London SW7 2BZ

and

K. Savvidou²

The Blackett Laboratory
Imperial College of Science, Technology & Medicine
South Kensington
London SW7 2BZ

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Abstract

As a preliminary to discussing the quantisation of the foliation in a history form of general relativity, we show how the discussion in [1] of a history version of free, scalar quantum field theory can be augmented in such a way as to include the quantisation of the unit-length, time-like vector that determines a Lorentzian foliation of Minkowski spacetime. We employ a Hilbert bundle construction that is motivated by: (i) discussing the role of the external Lorentz group in the existing history quantum field theory [1]; and (ii) considering a specific representation of the extended history algebra obtained from the multi-symplectic representation of scalar field theory.

¹email: c.isham@ic.ac.uk

²email: k.savvidou@ic.ac.uk

1 Introduction

The goal of the present paper is to extend the discussion in [1] of the construction of a history version of quantum scalar field theory in Minkowski spacetime. In particular, we shall show how the formalism can be developed to include the quantisation of the four-vector n that determines the spacetime foliation that plays a central role in the theory. The motivation for such a step, and the relevant background information, is as follows.

The ‘consistent-histories’ approach to quantum theory was originally introduced to provide a novel way of re-interpreting standard quantum theory, particularly in regard to the role played by measurement. However, because of the novel way in which time is handled, consistent-history theory also has the potential for providing new and powerful ways of studying quantum theories of gravity. Most recently, in [2] the formalism was applied to construct a history version of the canonical form of classical general relativity. The possibility also arises to use this formalism in the context of generalised ideas of time and space: for example, in models where spacetime is not represented by a differentiable manifold.

A first step in developing the framework with this goal in mind was taken in [3] where a new mathematical formalism—the ‘History Projection Operator’ (HPO) method—was introduced. This places emphasis on the idea of ‘quantum temporal logic’, and potentially allows substantial generalisations of the notion of time. The heart of this formalism is the idea that propositions about the *temporal history* of a system should be represented by projection operators on a ‘history’ Hilbert space³. In the case of simple, Newtonian time, and histories labelled by a finite set of discrete time points, the history Hilbert space is a tensor product of a copy of the standard canonical Hilbert space for each such time point.

The idea of representing history propositions by projection operators lead in turn to the notion of a ‘*history group*’. This is the history analogue of the Weyl group and its associated canonical commutation relations; in particular, the spectral projectors of the history operators in the Lie algebra of the history group represent propositions about the associated history quantities.

The introduction of a history group is particularly useful in the context of histories with a *continuous* time label, since it is by no means a trivial matter to define the continuous analogue of a tensor product. Instead, one finds the history Hilbert space by looking for representations of the appropriate history algebra.

For example, for the case of a point particle moving in one dimension, the history algebra for histories labelled with a continuous time parameter t is [4] [5]

$$[\hat{x}_t, \hat{x}_{t'}] = 0 \tag{1.1}$$

$$[\hat{p}_t, \hat{p}_{t'}] = 0 \tag{1.2}$$

³This is to be contrasted with the situation in standard quantum theory in which projection operators represent propositions about the system at a *single* time.

$$[\hat{x}_t, \hat{p}_{t'}] = i\hbar\tau\delta(t - t'), \quad (1.3)$$

and the basic history propositions in the theory refer to the value of time-averaged quantities such as $\frac{1}{\tau} \int dt x_t f(t)$ and $\frac{1}{\tau} \int dt p_t h(t)$ where f and h are smearing functions. Note that, in Eq. (1.3), τ is a new constant in the theory with the dimension of time.⁴

In equations (1.1)–(1.3), the label t on the operators \hat{x}_t and \hat{p}_t refers to the time at which propositions about the system are asserted—the time of ‘temporal logic’. It was to include in an explicit way such a time of temporal logic that the HPO formalism was originally developed. However, a clear notion of dynamics was not implemented for the, naturally time-averaged, physical quantities of the theory.

A major advance in the HPO formalism took place when time was introduced in a completely new way [6] [7]. It was realised that it is natural to consider time in a two-fold manner: the ‘time of being’—the time at which events ‘happen’ (and from this perspective, the time label t in Eqs. (1.1)–(1.3) and in Eq. (1.4) below can be regarded as such), and the ‘time of becoming’—the time of dynamical change, represented by a time label s . This *second* time appears in the history analogue $\hat{x}_t(s)$ of the Heisenberg picture, which is defined as

$$\hat{x}_t(s) := e^{is\hat{H}/\hbar} \hat{x}_t e^{-is\hat{H}/\hbar} \quad (1.4)$$

where $\hat{H} := \frac{1}{\tau} \int dt \hat{H}_t$ is the history quantity that represents the time average of the energy of the system. The notion of time evolution is now recovered for the time-averaged physical quantities, for example

$$\hat{x}_f(s) := e^{is\hat{H}/\hbar} \hat{x}_f e^{-is\hat{H}/\hbar} \quad (1.5)$$

where $f(t)$ is a smearing function.

Associated with these two manifestations of the concept of time are two types of time transformation: the ‘external’ translation

$$\hat{x}_t(s) \mapsto \hat{x}_{t+t'}(s), \quad (1.6)$$

and the ‘internal’ translation

$$\hat{x}_t(s) \mapsto \hat{x}_t(s + s'). \quad (1.7)$$

The external time translation is generated by the ‘Louvillie’ operator [6]

$$V := \int dt \hat{p}_t \frac{d\hat{x}_t}{dt} \quad (1.8)$$

whereas the internal time translation is generated by the time-averaged energy operator \hat{H} .

⁴In discussions involving the use of a history algebra with continuous time there is a tendency to choose units in which $\tau = 1$. However, this constant remains lurking in the background.

More importantly, it was shown in [6] that the generator of time translation in the HPO theory is the ‘action’ operator S defined as

$$S := \int dt \hat{p}_t \frac{d\hat{x}_t}{dt} - H = V - H. \quad (1.9)$$

Hence the action operator is the generator of *both* types of time translation

$$\hat{x}_t(s) \mapsto \hat{x}_{t+t'}(s + s'). \quad (1.10)$$

It is a very striking result that in the HPO theory the quantum analogue of the classical action functional is an actual operator in the formalism, and is the generator of time translations [6].

The idea of ‘two times’—and the associated two types of time translation—has recently been generalised to relativistic field theory [1] where, in particular, it is shown that the analogue of the two types of time translation is the existence of two *Poincaré* groups. The goal of the present paper is to develop the ideas in [1] in one particular respect: namely, the way in which spacetime foliations enter the theory.

That the idea of a Lorentzian⁵ spacetime foliation should play an important role in a history quantum field theory is understandable. Indeed, the obvious analogue of the history algebra Eqs. (1.1)–(1.3) for a quantum scalar field theory is (choosing units from now on such that $\tau = 1$)

$$[{}^n\hat{\phi}(t, \underline{x}), {}^n\hat{\phi}(t', \underline{x}')] = 0 \quad (1.11)$$

$$[{}^n\hat{\pi}(t, \underline{x}), {}^n\hat{\pi}(t', \underline{x}')] = 0 \quad (1.12)$$

$$[{}^n\hat{\phi}(t, \underline{x}), {}^n\hat{\pi}(t', \underline{x}')] = i\hbar\delta(t - t')\delta^3(\underline{x} - \underline{x}'), \quad (1.13)$$

where, for each $t \in \mathbb{R}$, the fields ${}^n\hat{\phi}(t, \underline{x})$ and ${}^n\hat{\pi}(t, \underline{x})$ are defined on the space-like hypersurface characterised by the unit length time-like vector n , and by the foliation parameter t . In particular, the three-vector \underline{x} in ${}^n\hat{\phi}(t, \underline{x})$ or in ${}^n\hat{\pi}(t, \underline{x})$, denotes a vector in this space. Note that the pair (t, \underline{x}) can be used to identify a unique point X in spacetime, and hence to write ${}^n\hat{\phi}(t, \underline{x})$ as ${}^n\hat{\phi}(X)$. The history algebra Eqs. (1.11)–(1.13) can then be written in the more covariant looking form

$$[\hat{\phi}(X), \hat{\phi}(X')] = 0 \quad (1.14)$$

$$[\hat{\pi}(X), \hat{\pi}(X')] = 0 \quad (1.15)$$

$$[\hat{\phi}(X), \hat{\pi}(X')] = i\hbar\delta^4(X - X') \quad (1.16)$$

where we have dropped the n superscript on the fields since the algebra itself is n -independent. Of course, this does not stop individual *representations* from depending on the foliation vector n ; indeed, as we shall see below, this is precisely what happens.

In what follows we shall denote by $H_+ := \{n \in M \mid n \cdot n = 1, n^0 > 0\}$ the set of all unit length, forward pointing time-like vectors on Minkowski spacetime M . We

⁵By ‘Lorentzian’ we mean that each leaf of the foliation is a hyperplane in the Minkowski spacetime.

are using a metric $\eta_{\mu\nu}$ on M with the signature $(+, -, -, -)$; also we use the notation $a \cdot b := a^\mu b^\nu \eta_{\mu\nu}$ for any four-vectors a and b in M .

It was shown in [1] that for each fixed n in H_+ it is possible to find a representation of the history algebra Eqs (1.11)–(1.13) on a Hilbert space \mathcal{H}_n with the property that the time-averaged energy exists as a well-defined self-adjoint operator ${}^n\hat{H}$ (this is the history analogue of an old theorem of Araki in the context of canonical quantum field theory [9]). This operator generates translations along the time-like direction n and, as such, is one of the generator of the *internal* Poincaré group that exists for each n : full expressions for all these generators are given in [1].

One of the key questions for our present purposes is how the external Poincaré group acts for each fixed choice of the foliation vector n . The translation part should obviously act in analogue to Eq. (1.6) by taking ${}^n\hat{\phi}(X)$ to ${}^n\hat{\phi}(X+a)$ for any four-vector n . Thus there should be an operator $U(a)$ such that

$$U(a){}^n\hat{\phi}(X)U(a)^{-1} = {}^n\hat{\phi}(X+a), \quad (1.17)$$

with a similar action on ${}^n\hat{\pi}(X)$.

The Lorentz subgroup of the Poincaré group is more interesting since as well as acting on the spacetime points, it might also be expected to act on the foliation vector n , and hence to take us out of the Hilbert space \mathcal{H}_n . In [1] this problem is solved by showing that even though the representations of the field algebra (1.11)–(1.13) are unitarily inequivalent for different choices of n , it is nevertheless possible to construct the fields for all n on a *common* Fock space \mathcal{F} (see Section 2 of the present paper for details). Hence it is meaningful to look for a unitary operator $U(\Lambda)$ such that, for all Lorentz transformations Λ ,

$$U(\Lambda){}^n\hat{\phi}(X)U(\Lambda)^{-1} = {}^{\Lambda n}\hat{\phi}(\Lambda X), \quad (1.18)$$

and similarly for ${}^n\hat{\pi}(X)$, where the operators are all defined on \mathcal{F} . Of course, the operators $U(\Lambda)$ are expected to form a unitary representation of the Lorentz group in the sense that

$$U(\Lambda')U(\Lambda) = U(\Lambda'\Lambda). \quad (1.19)$$

In the present paper we shall extend this formalism by quantising the foliation vector n itself. The main motivation for this step is our belief that, when constructing the quantum history theory of general relativity [2], it will be necessary to include the spacetime foliation⁶ as a genuine ‘history variable’, and which must therefore be represented by operators in the corresponding quantum theory. In the context of our present discussion, the vector n is the Minkowskian analogue of a foliation of a general spacetime: hence an investigation into what is meant by quantising n is a valuable precursor to the study of the quantisation of foliations of a general spacetime.

⁶In [8] also, the analogue of the foliation is a part of the postulated history group, this time in the context of the Bosonic string.

As an introduction to the quantisation of n , it is useful to return to the idea that, for each n , the theory is defined on a Hilbert space \mathcal{H}_n , and to ask again how the external Lorentz group might act. In these circumstances, Eq. (1.18) is not meaningful since the operators ${}^n\hat{\phi}(X)$ and ${}^{\Lambda n}\hat{\phi}(\Lambda X)$ are defined on different Hilbert spaces (\mathcal{H}_n and $\mathcal{H}_{\Lambda n}$ respectively). The natural thing instead is to seek a family of unitary intertwining operators $U(n; \Lambda) : \mathcal{H}_n \rightarrow \mathcal{H}_{\Lambda n}$ with the property that

$$U(n; \Lambda) {}^n\hat{\phi}(X) = {}^{\Lambda n}\hat{\phi}(\Lambda X) U(n; \Lambda) \quad (1.20)$$

$$U(n; \Lambda) {}^n\hat{\pi}(X) = {}^{\Lambda n}\hat{\pi}(\Lambda X) U(n; \Lambda) \quad (1.21)$$

which can usefully be represented by the commutative diagram

$$\begin{array}{ccc} \mathcal{H}_n & \xrightarrow{U(n; \Lambda)} & \mathcal{H}_{\Lambda n} \\ \downarrow {}^n\hat{\phi}(X) & & \downarrow {}^{\Lambda n}\hat{\phi}(\Lambda X) \\ \mathcal{H}_n & \xrightarrow{U(n; \Lambda)} & \mathcal{H}_{\Lambda n} \end{array} \quad (1.22)$$

and similarly for the operator ${}^n\hat{\pi}(X)$.

These operators $U(n; \Lambda) : \mathcal{H}_n \rightarrow \mathcal{H}_{\Lambda n}$ are expected to give a type of ‘representation’ of the external Lorentz group in the sense that, for all $n \in H_+$ and for all Lorentz transformations Λ , we have

$$U(\Lambda n; \Lambda') U(n; \Lambda) = U(n; \Lambda' \Lambda) \quad (1.23)$$

which is the appropriate analogue of the genuine representation Eq. (1.19). The specific form of Eq. (1.23) follows by considering the commutative diagram

$$\begin{array}{ccccc} \mathcal{H}_n & \xrightarrow{U(n; \Lambda)} & \mathcal{H}_{\Lambda n} & \xrightarrow{U(\Lambda n, \Lambda')} & \mathcal{H}_{\Lambda' \Lambda n} \\ \downarrow {}^n\hat{\phi}(X) & & \downarrow {}^{\Lambda n}\hat{\phi}(\Lambda X) & & \downarrow {}^{\Lambda' \Lambda n}\hat{\phi}(\Lambda' \Lambda X) \\ \mathcal{H}_n & \xrightarrow{U(n; \Lambda)} & \mathcal{H}_{\Lambda n} & \xrightarrow{U(\Lambda n, \Lambda')} & \mathcal{H}_{\Lambda' \Lambda n} \end{array} \quad (1.24)$$

whose outer square should equal the diagram

$$\begin{array}{ccc} \mathcal{H}_n & \xrightarrow{U(n; \Lambda' \Lambda)} & \mathcal{H}_{\Lambda' \Lambda n} \\ \downarrow {}^n\hat{\phi}(X) & & \downarrow {}^{\Lambda' \Lambda n}\hat{\phi}(\Lambda' \Lambda X) \\ \mathcal{H}_n & \xrightarrow{U(n; \Lambda' \Lambda)} & \mathcal{H}_{\Lambda' \Lambda n} \end{array} \quad (1.25)$$

We note that Eq. (1.23) is the type of relation that occurs naturally whenever we have a group G that acts on some G -set X , together with a family of operators $U_x(g)$, $x \in X$, defined on vector spaces V_x , $x \in X$, and satisfying the equation (cf. Eq. (1.23))

$$U(gx, g') U(x, g) = U(x, g'g) \quad (1.26)$$

for all $x \in X$ and $g, g' \in G$. There is also a version of Eq. (1.26) that uses a multiplier, but we shall not need that here.

Mathematically speaking, the appropriate picture (for the specific case of Eq. (1.23)) is a bundle of Hilbert spaces \mathcal{H}_n , $n \in H_+$, with base space H_+ , in which the action $n \mapsto \Lambda n$ of the external Lorentz group $SO(3,1)$ on H_+ is lifted to the bundle by the maps $U(n, \Lambda) : \mathcal{H}_n \rightarrow \mathcal{H}_{\Lambda n}$; note that Eq. (1.23) is precisely the statement that the operators $U(n, \Lambda)$ ‘cover’ the action of $SO(3,1)$ on the base space H_+ .

Under these circumstances, it is natural to consider the new Hilbert space formed by the *cross-sections* of this vector bundle. However this Hilbert space is quite different from the individual spaces \mathcal{H}_n , $n \in H_+$: in particular, the foliation vector itself becomes an operator under the natural action on a section Ψ as

$$\{\hat{n}^\mu \Psi\}(n) := n^\mu \Psi(n) \tag{1.27}$$

for all $n \in H_+$.

All this will be discussed properly in Section 3, but for the moment it suffices to summarise our remarks above by saying that the mathematical formalism for a history quantum field theory developed in [1] itself suggests a natural way in which the foliation vector could become a quantum operator.

Such a step is also understandable from a more conceptual perspective. For we should recall that the consistent history theory deals with ‘beables’ (albeit contextualised by the choice of a particular consistent set of histories) not observables. Thus, in quantising n , we are not saying that the foliation is something that is determined by nature—in particular, something that must be observed—but rather that the existing history QFT formalism depends on the choice of n in such an intrinsic way that it is natural to formulate propositions about things in the context⁷ of specifying n . And, as we have seen, one way of doing this in a form that is coherent with respect to the action of the external Lorentz group is to let n become a quantum operator.

The challenge now arises of finding a proper theory of a quantised foliation vector, and thereby justifying the rather heuristic ideas presented above. In particular, in the spirit of our approach to history theories, we must find the correct extension of the history field algebra to include an operator \hat{n}_μ and its conjugate variables.

We will address this issue in Section 3.1 by considering the *multi-symplectic* approach to a scalar field theory. There is a relation between multi-symplectic structures and history theory, and—as we shall see—in the case of a scalar field, attempting to quantise the corresponding multi-symplectic structure leads naturally to a quantised foliation vector in the context of a history group whose Lie algebra generators include n and an appropriate set of conjugate variables.

The plan of the paper is as follows. We start in Section 2 by summarising the results in [1] for constructing the quantum history theory of a free scalar field. Then in Section 3 we study the main problem of quantising the foliation vector. We base the first few steps in constructing the appropriate history algebra on the discussion in

⁷This also suggests that a topos approach could be useful: we shall make a few remarks about this later on in the paper.

Section 3.1 of the multi-symplectic formalism as applied to the relativistic scalar field. The quantisation of this formalism is discussed in Section 3.2, and this is completed in Section 3.3 where we apply group-theoretical quantisation techniques to a classical system whose configuration space is the set H_+ of all foliation vectors. Then in Section 4 we discuss the representations of this algebra, and show how a particularly simple one reproduces the heuristic ideas of a Hilbert bundle sketched above.

2 The Quantum History Theory of a Scalar Field

2.1 The field operators

The starting point for the construction in [1] of a quantum history version of a free, scalar field is a Fock space \mathcal{F} defined via annihilation and creation operators, $\hat{b}(X)$ and $\hat{b}^\dagger(X)$ respectively, that satisfy the commutation relations:

$$[\hat{b}(X), \hat{b}(X')] = 0 \quad (2.1)$$

$$[\hat{b}^\dagger(X), \hat{b}^\dagger(X')] = 0 \quad (2.2)$$

$$[\hat{b}(X), \hat{b}^\dagger(X')] = \hbar\delta^4(X - X'). \quad (2.3)$$

This bosonic Fock space has (generalised) basis vectors $|X_1, X_2, \dots, X_k\rangle$ defined by

$$|X_1, X_2, \dots, X_k\rangle := \hat{b}^\dagger(X_1)\hat{b}^\dagger(X_2)\dots\hat{b}^\dagger(X_k)|0\rangle \quad (2.4)$$

where $|0\rangle$ is the cyclic ‘vacuum’ state of the Fock space.

On this Fock space \mathcal{F} , field operators for each $n \in H_+$ can be defined as

$${}^n\hat{\phi}(X) := \frac{1}{\sqrt{2}} {}^n\Gamma^{-1/4}(\hat{b}(X) + \hat{b}^\dagger(X)) \quad (2.5)$$

$${}^n\hat{\pi}(X) := \frac{1}{i\sqrt{2}} {}^n\Gamma^{1/4}(\hat{b}(X) - \hat{b}^\dagger(X)) \quad (2.6)$$

where ${}^n\Gamma$ is the elliptic, partial differential operator on $L^2(\mathbb{R}^4)$ defined as

$${}^n\Gamma := (\eta^{\mu\nu} - n^\mu n^\nu)\partial_\mu\partial_\nu + m^2 \quad (2.7)$$

where m is the mass parameter in the theory. It is easy to check that, for each foliation vector $n \in H_+$, the fields ${}^n\hat{\phi}(X)$ and ${}^n\hat{\pi}(X)$ defined in Eqs. (2.5)–(2.6) satisfy the history algebra Eqs. (1.14)–(1.16). We note that, as promised, these fields are all defined on the *same* space \mathcal{F} even though the associated representations of the history algebra are unitarily inequivalent for different choices of $n \in H_+$.

The time-averaged energy for each n is represented by the operator

$${}^n\hat{H} = \frac{1}{2} : \int d^4X \{ {}^n\hat{\pi}(X)^2 + {}^n\hat{\phi}(X) {}^n\Gamma {}^n\hat{\phi}(X) \} : \quad (2.8)$$

$$= \int d^4X \hat{b}^\dagger(X) \sqrt{{}^n\Gamma} \hat{b}(X) \quad (2.9)$$

which is a well-defined self-adjoint operator. It is to guarantee the existence of these operators for all $n \in H_+$ that the basic fields ${}^n\hat{\phi}(X)$ and ${}^n\hat{\pi}(X)$ are defined as they are in Eqs. (2.5)–(2.6).

2.2 The external Poincaré group

There is a natural unitary representation of the ‘external’ Poincaré group on the Fock space \mathcal{F} . This is defined in the obvious way on the basic vectors $|X_1, X_2, \dots, X_k\rangle$ as

$$U(\Lambda)|0\rangle := |0\rangle \quad (2.10)$$

$$U(\Lambda)|X_1, X_2, \dots, X_k\rangle := |\Lambda X_1, \Lambda X_2, \dots, \Lambda X_k\rangle \quad (2.11)$$

$$U(a)|0\rangle := |0\rangle \quad (2.12)$$

$$U(a)|X_1, X_2, \dots, X_k\rangle := |X_1 + a, X_2 + a, \dots, X_k + a\rangle. \quad (2.13)$$

This induces the action on the annihilation operators of

$$U(\Lambda)\hat{b}(X)U(\Lambda)^{-1} = \hat{b}(\Lambda X) \quad (2.14)$$

$$U(a)\hat{b}(X)U(a)^{-1} = \hat{b}(X + a), \quad (2.15)$$

and similarly for the creation operators $\hat{b}^\dagger(X)$. It is straightforward to show that, as anticipated, the basic field operators ${}^n\hat{\phi}(X)$ and ${}^n\hat{\pi}(X)$ transform as

$$U(\Lambda){}^n\hat{\phi}(X)U(\Lambda)^{-1} = \Lambda^n\hat{\phi}(\Lambda X) \quad (2.16)$$

$$U(\Lambda){}^n\hat{\pi}(X)U(\Lambda)^{-1} = \Lambda^n\hat{\pi}(\Lambda X) \quad (2.17)$$

$$U(a){}^n\hat{\phi}(X)U(a)^{-1} = {}^n\hat{\phi}(X + a) \quad (2.18)$$

$$U(a){}^n\hat{\pi}(X)U(a)^{-1} = {}^n\hat{\pi}(X + a). \quad (2.19)$$

We note that it is possible to define another set of fields by

$$\hat{\Phi}(X) := \frac{1}{\sqrt{2}}(\hat{b}(X) + \hat{b}^\dagger(X)) \quad (2.20)$$

$$\hat{\Pi}(X) := \frac{1}{i\sqrt{2}}(\hat{b}(X) - \hat{b}^\dagger(X)) \quad (2.21)$$

which satisfy the basic history field algebra Eqs. (1.14)–(1.16). Under the action of the external Poincaré group, we have

$$U(\Lambda)\hat{\Phi}(X)U(\Lambda)^{-1} = \hat{\Phi}(\Lambda X) \quad (2.22)$$

$$U(a)\hat{\Phi}(X)U(a)^{-1} = \hat{\Phi}(X + a), \quad (2.23)$$

and similarly for the conjugate variable $\hat{\Pi}(X)$.

The role of these ‘covariant’ fields in the theory is intriguing. The relation of $\hat{\Phi}(X)$ to the fields ${}^n\hat{\phi}(X)$ suggests strongly that the former should be thought of as the

history analogue of the *Newton-Wigner* field (which, in standard quantum field theory, creates and annihilates localised *particle* states). However we note that—in the history theory— $\hat{\Phi}(X)$ is a genuine scalar field, whereas in standard quantum field theory the Newton-Wigner field transforms in a non-covariant way.

The formal explanation of this difference lies in the way the internal and external times interface with each other in the history theory. In particular, the history field $\hat{\Phi}(X)$ is a ‘Schrödinger picture’ object in the sense that it does not carry any dynamical information. On the other hand, the remarks above about the standard Newton-Wigner field apply in the Heisenberg picture: in the history case, this would involve invoking the second, internal time.

3 Quantising the Foliation Vector

3.1 The multi-symplectic formalism for a scalar field

One might be tempted to construct the classical history theory for a scalar field by positing the Poisson bracket algebra (cf. Eqs. (1.14)–(1.16))

$$\{ \phi(X), \phi(X') \} = 0 \tag{3.1}$$

$$\{ \pi(X), \pi(X') \} = 0 \tag{3.2}$$

$$\{ \phi(X), \pi(X') \} = \delta^4(X - X') \tag{3.3}$$

which has the advantage of appearing to be manifestly covariant under the action of the external Poincaré group (on the assumption that ϕ and π are scalar fields). However, this covariance is deceptive in the sense that the conjugate variable $\pi(X)$ has no clear *physical* meaning; not least because the actual field momentum for a physical system is manifestly foliation dependent: *i.e.* it means the momentum along some specified time-like direction n .

This problem is circumscribed in the approach summarised in Section 2 since the *representations* of the quantum algebra Eqs. (1.14)–(1.16) are manifestly n -dependent, and indeed an explicit n -label becomes attached to both the scalar field and its conjugate momentum via Eqs. (2.5)–(2.6). However, these explicit forms are chosen so that the *quantum* average-energy operator ${}^n\hat{H}$ exists, and to some extent therefore this leaves open the question of the structure of the underlying *classical* history theory. We shall now address this issue with the aid of some ideas drawn from an, apparently, quite different scheme: namely, the multi-symplectic formalism.

The multi-symplectic formalism arose from attempts to modify the standard classical *canonical* formalism so that it would be manifestly covariant under the appropriate group of spacetime transformations [10]. In the case of a scalar field on Minkowski spacetime M , the idea is to introduce a pair of space-time fields $\phi(X)$ and $\pi_\mu(X)$ where, physically, for any vector V , $V^\mu\pi_\mu(X)$ can be interpreted as the field momentum along the space-time direction V^μ . Then, for each choice of foliation vector n ,

there is defined the Poisson bracket

$$\{F, G\}_n(\phi, \pi) := \int_M d^4X \left(\frac{\delta F}{\delta \phi(X)} \frac{\delta G}{\pi_\mu(X)} - \frac{\delta G}{\delta \phi(X)} \frac{\delta F}{\pi_\mu(X)} \right) n_\mu \quad (3.4)$$

where F and G are functionals of ϕ and π . By this means, a family of symplectic structures is introduced, and the whole system is manifestly covariant under an action of the Poincaré group in which (i) ϕ and π_μ transform as genuine space-time objects in the obvious way; and (ii) the symplectic structure labeled by a foliation vector n is transformed into that labeled by Λn for all Lorentz transformations Λ .

The nature of this covariance is particularly clear if we look at the basic Poisson brackets that follow from Eq. (3.4):

$$\{\phi(X), \phi(X')\}_n = 0 \quad (3.5)$$

$$\{\pi_\mu(X), \pi_\nu(X')\}_n = 0 \quad (3.6)$$

$$\{\phi(X), \pi_\mu(X')\}_n = n_\mu \delta^4(X - X') \quad (3.7)$$

where X and X' are points in Minkowski spacetime M .

As remarked above, the multi-symplectic formalism was developed in the context of standard canonical theory. However—in so far as they are space-time objects—we could clearly think of ϕ and π_μ as classical *history* fields, and try to develop a history theory based on Eqs. (3.5)–(3.7) instead of Eqs. (3.1)–(3.3). As a mathematical possibility, this makes good sense. However, we should emphasise that, physically speaking, the history interpretation of the multi-symplectic formalism is quite different from the standard one.

For example, a frequent comment in the literature on the multi-symplectic formalism is that the basic Poisson brackets Eqs. (3.5)–(3.7) are not compatible with the equations of motion. But viewed as a history theory, this is no longer the case since the equations of motion are now to be associated with the introduction of the ‘internal’ time label. This is closely related to the fact that, from a history perspective, the fields ϕ and π_μ are the classical analogue of *Schrödinger* picture objects, and are used in a temporal logic sense as the carriers of propositions about the history of the system; they are *not* dynamical fields.

3.2 First steps to the quantum history algebra

We must now address the question of the quantum analogue of the parametrised (by n) family of Poisson brackets given in Eqs. (3.5)–(3.7). It is noteworthy that very little has been said in the literature on the multi-symplectic formalism about quantising such Poisson bracket relations, and by hindsight we can understand why: it is only in the context of a quantum *history* theory—for example, the consistent history theory—that the quantisation makes any physical sense.

If we approach quantisation in the traditional way of replacing Poisson brackets with operator commutators, then the first issue is how to handle the n -subscript that appears on the left hand side of the equations (3.5)–(3.7). Attaching a subscript to an operator commutator does have any *a priori* meaning other than, perhaps, to indicate different representations of an algebra, and one is tempted therefore to postulate the simple algebra (from now on we set $\hbar = 1$)

$$[\hat{\phi}(X), \hat{\phi}(X')] = 0 \quad (3.8)$$

$$[\hat{\pi}_\mu(X), \hat{\pi}_\nu(X')] = 0 \quad (3.9)$$

$$[\hat{\phi}(X), \hat{\pi}_\mu(X')] = in_\mu \delta^4(X - X') \quad (3.10)$$

with the understanding that the physically appropriate representation may depend on n .

However, the quantity n_μ now appears as a fixed c -number, and the manifest Poincaré covariance is lost. For example, one would like to postulate an action of the external Lorentz group of the form

$$U(\Lambda)\hat{\phi}(X)U(\Lambda)^{-1} = \hat{\phi}(\Lambda X) \quad (3.11)$$

$$U(\Lambda)\hat{\pi}_\mu(X)U(\Lambda)^{-1} = \Lambda_\mu{}^\nu \hat{\pi}_\nu(\Lambda X) \quad (3.12)$$

but this is incompatible with the right hand side of Eq. (3.10) because, since n_μ is a c -number, we have $U(\Lambda)n_\mu U(\Lambda)^{-1} = n_\mu$. The obvious resolution of this problem is to make n_μ itself into an *operator*, with the algebra

$$[\hat{\phi}(X), \hat{\phi}(X')] = 0 \quad (3.13)$$

$$[\hat{\pi}_\mu(X), \hat{\pi}_\nu(X')] = 0 \quad (3.14)$$

$$[\hat{\phi}(X), \hat{\pi}_\mu(X')] = i\hat{n}_\mu \delta^4(X - X') \quad (3.15)$$

and to augment the transformations Eqs. (3.11)–(3.12) with

$$U(\Lambda)\hat{n}_\mu U(\Lambda)^{-1} = \Lambda_\mu{}^\nu \hat{n}_\nu \quad (3.16)$$

so that the whole set is now compatible.

We now have four main tasks:

1. Extend Eqs. (3.13)–(3.15) to a complete history theory; in particular we must discuss the form of the conjugate variables to the quantised foliation vector \hat{n}_μ .
2. Find a physically appropriate representation of the extended algebra.
3. Show how dynamics is implemented in this scheme. In particular, how the idea arises of a second, ‘internal’ time and associated internal Poincaré group.
4. Give a physical interpretation of the algebra.

Of course, these different issues are closely related. For example, the average-energy operator for the system could be anticipated to be

$$\hat{H} = \frac{1}{2} : \int d^4 X \left\{ (\hat{n}^\mu \hat{\pi}_\mu(X))^2 + (\hat{n}^\mu \hat{n}^\nu - \eta^{\mu\nu}) \partial_\mu \hat{\phi}(X) \partial_\nu \hat{\phi}(X) + m^2 \hat{\phi}(X)^2 \right\} : \quad (3.17)$$

which should be compared with the expression in Eq. (2.8) for a fixed n -vector. Note that there is no longer an n -superscript on \hat{H} : there is now just a single operator. It is natural, therefore, to seek to fix the representation of the final history algebra by requiring that the expression in Eq. (3.17) exists as a genuine (essentially) self-adjoint operator.

3.3 Completing the history algebra

The conjugate variables to n . The next step is to consider the conjugate variables for the foliation vector. The key observation in this context is that, before quantisation, the vector n is of unit length in the sense that

$$n \cdot n := n^\mu n^\nu \eta_{\mu\nu} = 1, \quad (3.18)$$

and time-like. It seems appropriate that the quantisation of n should preserve these constraints, but this requirement is incompatible with the obvious commutator algebra

$$[\hat{p}^\mu, \hat{n}_\nu] = -i\delta_\nu^\mu \quad (3.19)$$

since the conjugate \hat{p}^μ would then generate translations in \hat{n}_μ , and these do not preserve the constraints.

What we are faced with is the quantisation of a system whose classical configuration space is not a vector space but rather the hyperboloid $H_+ := \{n \in \mathbb{R}^4 \mid n^\mu n_\mu = 1, n^0 > 0\}$ in \mathbb{R}^4 , which can be viewed as a non-compact version of the three-sphere S^3 . The quantisation of systems whose configuration spaces are not vector spaces was discussed at length in [11] which, in particular, contains a detailed description of the quantisation of a system whose classical configuration space is an n -sphere. The conclusion was that the appropriate canonical group is not the standard Weyl group (that is associated with the normal canonical commutation relations) but rather the euclidean group $SO(n+1) \circledast \mathbb{R}^{n+1}$ where \circledast denotes the semi-direct product.

The same general discussion applies in the present case with the hyperboloid H_+ as configuration space. The result is that the appropriate history group for the foliation variable is the semi-direct product $SO(3,1) \circledast \mathbb{R}^4$, with the Lie algebra relations

$$[\hat{n}_\alpha, \hat{n}_\beta] = 0 \quad (3.20)$$

$$[\hat{p}^{\alpha\beta}, \hat{p}^{\gamma\delta}] = i(\eta^{\alpha\gamma} \hat{p}^{\beta\delta} - \eta^{\beta\gamma} \hat{p}^{\alpha\delta} + \eta^{\beta\delta} \hat{p}^{\alpha\gamma} - \eta^{\alpha\delta} \hat{p}^{\beta\gamma}) \quad (3.21)$$

$$[\hat{n}_\alpha, \hat{p}^{\beta\gamma}] = i(\delta_\alpha^\beta \hat{n}^\gamma - \delta_\alpha^\gamma \hat{n}^\beta) \quad (3.22)$$

where $\hat{p}^{\alpha\beta} = -\hat{p}^{\beta\alpha}$.

We note that:

- i) Eq. (3.20) shows that the variables \hat{n}_α span the Lie algebra of the abelian group \mathbb{R}^4 .
- ii) Eq. (3.21) shows that the conjugate variables $\hat{p}^{\alpha\beta}$ satisfy the Lie algebra of the Lorentz group $SO(3, 1)$.
- iii) Eq. (3.22) reflects the semi-direct structure given by the action of $SO(3, 1)$ on \mathbb{R}^4 .

This group-theoretic scheme works because $\eta^{\mu\nu}\hat{n}_\mu\hat{n}_\nu$ is a Casimir operator for the algebra in Eqs. (3.20)–(3.22). Hence it is meaningful to look for a representation in which $\eta^{\mu\nu}\hat{n}_\mu\hat{n}_\nu$ has the constant value 1, thus maintaining compatibility with the classical constraint in Eq. (3.18).

Of course $SO(3, 1)\circledast\mathbb{R}^4$ is nothing but the familiar Poincaré group. But this should not be confused with either the internal or the external Poincaré groups to which we have referred earlier: the present group has arisen as a direct result of quantising the foliation vector n_μ .

Completing the history algebra. We must now try to complete the history algebra by considering the cross commutators between the pair $\hat{\phi}(X), \hat{\pi}_\mu(X)$, and the pair $\hat{n}_\mu, \hat{p}^{\alpha\beta}$. As a first step we take the commutator of both sides of Eq. (3.15) with $\hat{p}^{\alpha\beta}$, then use the Jacobi identity on the left hand side, and the commutator Eq. (3.22) on the right hand side, to give

$$[\hat{\phi}(X), [\hat{p}^{\alpha\beta}, \hat{\pi}_\mu(X')]] + [\hat{\pi}_\mu(X'), [\hat{\phi}(X), \hat{p}^{\alpha\beta}]] = (\delta_\mu^\alpha \hat{n}^\beta - \delta_\mu^\beta \hat{n}^\alpha) \delta^4(X - X'). \quad (3.23)$$

It is natural to think of $\hat{\phi}(X)$ and \hat{n}_μ as disjoint configuration variables, which suggests that

$$[\hat{\phi}(X), \hat{n}_\mu] = 0 = [\hat{\pi}_\mu(X), \hat{n}_\mu], \quad (3.24)$$

and for this reason it is arguably also natural to assume that $[\hat{\phi}(X), \hat{p}^{\alpha\beta}] = 0$. We note that a more general possibility is

$$[\hat{\phi}(X), \hat{p}^{\alpha\beta}] = ia(\hat{n}^\alpha \hat{\pi}^\beta(X) - \hat{n}^\beta \hat{\pi}^\alpha(X)) \quad (3.25)$$

for some real constant a . However, this is rather ugly in the sense that the right hand side of Eq. (3.25) is a non-linear function of our basic fields, and from now on we shall assume that $a = 0$.

We note that, by virtue of Eq. (3.14) and the assumption in Eq. (3.24), even if the commutator in Eq. (3.25) is non-zero, it does not contribute to the left hand side of Eq. (3.23). Thus, even if $a \neq 0$, the obvious choice for the commutator $[\hat{p}^{\alpha\beta}, \hat{\pi}_\mu(X)]$ is

$$[\hat{p}^{\alpha\beta}, \hat{\pi}_\mu(X)] = -i(\delta_\mu^\alpha \hat{\pi}^\beta(X) - \delta_\mu^\beta \hat{\pi}^\alpha(X)). \quad (3.26)$$

In summary, the entire history algebra is postulated to be as follows:

$$[\hat{\phi}(X), \hat{\phi}(X')] = 0 \quad (3.27)$$

$$[\hat{\pi}_\mu(X), \hat{\pi}_\nu(X')] = 0 \quad (3.28)$$

$$[\hat{\phi}(X), \hat{\pi}_\mu(X')] = i\hat{n}_\mu\delta^4(X - X') \quad (3.29)$$

$$[\hat{n}_\alpha, \hat{n}_\beta] = 0 \quad (3.30)$$

$$[\hat{p}^{\alpha\beta}, \hat{p}^{\gamma\delta}] = i(\eta^{\alpha\gamma}\hat{p}^{\beta\delta} - \eta^{\beta\gamma}\hat{p}^{\alpha\delta} + \eta^{\beta\delta}\hat{p}^{\alpha\gamma} - \eta^{\alpha\delta}\hat{p}^{\beta\gamma}) \quad (3.31)$$

$$[\hat{n}_\alpha, \hat{p}^{\beta\gamma}] = i(\delta_\alpha^\beta\hat{n}^\gamma - \delta_\alpha^\gamma\hat{n}^\beta) \quad (3.32)$$

$$[\hat{\phi}(X), \hat{n}_\alpha] = 0 \quad (3.33)$$

$$[\hat{\pi}_\mu(X), \hat{n}_\alpha] = 0 \quad (3.34)$$

$$[\hat{\phi}(X), \hat{p}^{\alpha\beta}] = 0 \quad (3.35)$$

$$[\hat{\pi}_\mu(X), \hat{p}^{\alpha\beta}] = i(\delta_\mu^\alpha\hat{\pi}^\beta(X) - \delta_\mu^\beta\hat{\pi}^\alpha(X)). \quad (3.36)$$

It is easy to check that the Jacobi identities are satisfied for this algebra.

An ansatz for the operator $\hat{\pi}_\mu(X)$. At this point we note that Eqs. (3.29) and (3.33) imply that

$$[\hat{\phi}(X), \hat{\pi}_\mu(X') - \hat{n}_\mu \hat{n}_\nu \hat{\pi}^\nu(X')] = 0 \quad (3.37)$$

in a representation in which $\hat{n} \cdot \hat{n} = 1$. For an arbitrary value of this Casimir operator we have instead

$$[\hat{\phi}(X), \hat{n} \cdot \hat{n} \hat{\pi}_\mu(X') - \hat{n}_\mu \hat{n}_\nu \hat{\pi}^\nu(X')] = 0. \quad (3.38)$$

Equation (3.37) suggests that $\hat{\pi}_\mu(X) - \hat{n}_\mu \hat{n}_\nu \hat{\pi}^\nu(X)$ might be a function of $\hat{\phi}(X)$: indeed, this statement is necessarily true if the algebra generated by the spacetime fields $\hat{\phi}(X)$ is assumed to be a maximal commutative subalgebra of the history algebra. One natural possibility is to set

$$\hat{\pi}_\mu(X) - \hat{n}_\mu \hat{n}_\nu \hat{\pi}^\nu(X) = 0, \quad (3.39)$$

which suggests that $\hat{\pi}_\mu(X)$ can be defined using a single ‘master’ field $\hat{\pi}(X)$ as

$$\hat{\pi}_\mu(X) := \hat{n}_\mu \hat{\pi}(X). \quad (3.40)$$

We shall discuss this option at some length below. Note that it is compatible with the supposed commutator $[\hat{\pi}_\mu(X), \hat{\pi}_\nu(X')] = 0$ if we postulate that $[\hat{\pi}(X), \hat{\pi}(X')] = 0$. It is also compatible with the remaining commutators in Eqs. (3.27)–(3.36) that involve $\hat{\pi}_\mu(X)$.

A natural generalisation of the definition Eq. (3.40) of the operator $\hat{\pi}_\mu(X)$ in terms of a single $\hat{\pi}(X)$ is

$$\hat{\pi}_\mu(X) := \hat{n}_\mu \hat{\pi}(X) + b(\partial_\mu \hat{\phi}(X) - \hat{n}_\mu \hat{n} \cdot \partial \hat{\phi}(X)) \quad (3.41)$$

for some real constant b . Bearing in mind that (assuming that $\hat{n} \cdot \hat{n} = 1$)

$$\hat{n}^\mu (\partial_\mu \hat{\phi}(X) - \hat{n}_\mu \hat{n} \cdot \partial \hat{\phi}(X)) \equiv 0 \quad (3.42)$$

we see that Eq. (3.41) can be viewed as the decomposition of $\hat{\pi}_\mu(X)$ into a ‘longitudinal’ part $\hat{n}_\mu \hat{\pi}(X)$ and a ‘transverse part’ $\partial_\mu \hat{\phi}(X) - \hat{n}_\mu \hat{n} \cdot \partial \hat{\phi}(X)$, with the implication in particular that the transverse part is essentially the spatial derivatives of the field $\hat{\phi}(X)$. There are several attractive features to assuming Eq. (3.41). However, it does have the implication that

$$[\hat{\pi}_\mu(X), \hat{\pi}_\nu(X')] = 2ib(\hat{n}_{(\nu} \partial_{\mu)} - \hat{n}_\mu \hat{n}_\nu \hat{n} \cdot \partial) \delta^4(X - X') \quad (3.43)$$

where the partial derivatives on the right hand side are with respect to the X label. This would mean making a change in the postulated commutator in Eq. (3.28).

3.4 The external and internal Poincaré groups

The action of the external Poincaré group. There is a natural automorphism of the complete history algebra Eqs. (3.27)–(3.36) by the external Poincaré group, which we might hope could be unitarily implemented as an extension of Eqs. (3.11)–(3.12) and Eq. (3.16):

$$U(\Lambda) \hat{\phi}(X) U(\Lambda)^{-1} = \hat{\phi}(\Lambda X) \quad (3.44)$$

$$U(\Lambda) \hat{\pi}_\mu(X) U(\Lambda)^{-1} = \Lambda_\mu^\nu \hat{\pi}_\nu(\Lambda X) \quad (3.45)$$

$$U(\Lambda) \hat{n}_\mu U(\Lambda)^{-1} = \Lambda_\mu^\nu \hat{n}_\nu \quad (3.46)$$

$$U(\Lambda) \hat{p}^{\alpha\beta} U(\Lambda)^{-1} = \Lambda^\alpha_\mu \Lambda^\beta_\nu \hat{p}^{\mu\nu} \quad (3.47)$$

and with the translations acting as

$$U(a) \hat{\phi}(X) U(a)^{-1} = \hat{\phi}(X + a) \quad (3.48)$$

$$U(a) \hat{\pi}_\mu(X) U(a)^{-1} = \hat{\pi}_\mu(X + a) \quad (3.49)$$

$$U(a) \hat{n}_\mu U(a)^{-1} = \hat{n}_\mu \quad (3.50)$$

$$U(a) \hat{p}^{\alpha\beta} U(a)^{-1} = \hat{p}^{\alpha\beta}. \quad (3.51)$$

The internal Poincaré group. The situation with the internal Poincaré group is interesting. As remarked above, we expect the average-energy operator of the system to be

$$\hat{H} = \frac{1}{2} : \int d^4 X \left\{ (\hat{n}^\mu \hat{\pi}_\mu(X))^2 + (\hat{n}^\mu \hat{n}^\nu - \eta^{\mu\nu}) \partial_\mu \hat{\phi}(X) \partial_\nu \hat{\phi}(X) + m^2 \hat{\phi}(X)^2 \right\} : \quad (3.52)$$

However, in this situation, \hat{H} is not defined with respect to any *particular* foliation (unlike, for example, the time-averaged energy in Eq. (2.8)), and hence it cannot be identified as the time-like component of a four-vector \hat{P}_μ in any obvious way.

The resolution of this issue is as follows. In the original form of history quantum field theory, summarised in Section 2, there is a four-vector operator ${}^n\hat{P}_\mu$ that is related to the associated time-averaged energy operator ${}^n\hat{H}$ by

$${}^n\hat{P}_\mu := n_\mu {}^n\hat{H} + \int d^4X \left(\partial_\mu {}^n\hat{\phi}(X) - n_\mu n \cdot \partial {}^n\hat{\phi}(X) \right) {}^n\hat{\pi}(X), \quad (3.53)$$

and where we note that

$$n^\mu \int d^4X \left(\partial_\mu {}^n\hat{\phi}(X) - n_\mu n \cdot \partial {}^n\hat{\phi}(X) \right) {}^n\hat{\pi}(X) \equiv 0. \quad (3.54)$$

Thus the ‘ n -longitudinal’ part of ${}^n\hat{P}_\mu$ is ${}^n\hat{H} \equiv n^\mu {}^n\hat{P}_\mu$, whereas the ‘ n -transverse’ part is $\int d^4X \left(\partial_\mu \hat{\phi}(X) - n_\mu n \cdot \partial \hat{\phi}(X) \right) {}^n\hat{\pi}(X)$.

In the present case, where n is quantised, the expression in Eq. (3.53) suggests that we define the translation generators of the internal Poincaré group by

$$\text{int}\hat{P}_\mu := \hat{n}_\mu \hat{H} + \int d^4X \left(\partial_\mu \hat{\phi}(X) - \hat{n}_\mu \hat{n} \cdot \partial \hat{\phi}(X) \right) {}^n\hat{\pi}(X) \quad (3.55)$$

where \hat{H} is defined in Eq. (3.52). The remaining generators of the internal Poincaré group can be defined in a similar way using the expressions given in [1] where the foliation vector n is fixed.

We note that, according to Eqs. (3.44)–(3.45), under the action of the external Lorentz group, the generators of the translations of the internal Poincaré group transform as

$$U(\Lambda) \text{int}\hat{P}_\mu U(\Lambda)^{-1} = \Lambda_\mu{}^\nu \text{int}\hat{P}_\nu \quad (3.56)$$

whereas, for a fixed n we have

$$U(\Lambda) {}^n\hat{P}_\mu U(\Lambda)^{-1} = \Lambda_\mu{}^\nu \Lambda^n \hat{P}_\nu. \quad (3.57)$$

The internal time. In the context of the discussion above of the internal Poincaré group, it is clear that one way in which a second, internal time variable s could enter the formalism is by the definition of a ‘Heisenberg picture’ field $\hat{\phi}(X; s)$ as

$$\hat{\phi}(X; s) := e^{is\hat{H}} \hat{\phi}(X) e^{-is\hat{H}}. \quad (3.58)$$

We see that, in this approach, there is now a *single* extra time variable s —for each choice of a foliation vector n —and this is not associated with any particular foliation vector. However, it is still true that the interpretation of the formalism must be such that s automatically has the correct meaning in the correct context.

However, this is not the only option. For example, it is arguably more natural to have a separate internal time variable $s(n)$ for each value of $n \in H_+$, and such that

$s(n) \geq 0$ for all n . In the quantum case, an operator $s(\hat{n})$ can be defined using the spectral theorem for the self-adjoint operator \hat{n} , and then we can define (c.f. Eq. (3.52))

$$\hat{H}[s] := \frac{1}{2} : \int d^4X s(\hat{n}) \left\{ (\hat{n}^\mu \hat{\pi}_\mu(X))^2 + (\hat{n}^\mu \hat{n}^\nu - \eta^{\mu\nu}) \partial_\mu \hat{\phi}(X) \partial_\nu \hat{\phi}(X) + m^2 \hat{\phi}(X)^2 \right\} : \quad (3.59)$$

This suggests defining an associated ‘generalised Heisenberg picture’ object $\hat{\phi}(X; s]$ (c.f. Eq. (3.58)) as

$$\hat{\phi}(X; s] := e^{i\hat{H}[s]} \hat{\phi}(X) e^{-i\hat{H}[s]} \quad (3.60)$$

where the brackets in $\hat{\phi}(X; s]$ serve to remind us that $\hat{\phi}$ is a function of the spacetime point X , but a *functional* of the function $s : H_+ \rightarrow \{0\} \cup \mathbb{R}_+$.

4 Representations of the History Algebra

4.1 The Hilbert bundle construction

From what has been said above, it is clear that one way of satisfying the history algebra Eqs. (3.27)–(3.36) would be to have a single ‘master’ momentum field $\hat{\omega}(X)$, and then to define

$$\hat{\pi}_\mu(X) := \hat{n}_\mu \hat{\omega}(X) \quad (4.1)$$

with the assumption that $\hat{n}_\mu(X)$ commutes with $\hat{\omega}(X)$ so that there are no operator-ordering problems. This gives us the simpler algebra

$$[\hat{\phi}(X), \hat{\phi}(X')] = 0 \quad (4.2)$$

$$[\hat{\omega}(X), \hat{\omega}(X')] = 0 \quad (4.3)$$

$$[\hat{\phi}(X), \hat{\omega}(X')] = i\delta^4(X - X') \quad (4.4)$$

$$[\hat{n}_\alpha, \hat{n}_\beta] = 0 \quad (4.5)$$

$$[\hat{p}^{\alpha\beta}, \hat{p}^{\gamma\delta}] = i(\eta^{\alpha\gamma} \hat{p}^{\beta\delta} - \eta^{\beta\gamma} \hat{p}^{\alpha\delta} + \eta^{\beta\delta} \hat{p}^{\alpha\gamma} - \eta^{\alpha\delta} \hat{p}^{\beta\gamma}) \quad (4.6)$$

$$[\hat{n}_\alpha, \hat{p}^{\beta\gamma}] = i(\delta_\alpha^\beta \hat{n}^\gamma - \delta_\alpha^\gamma \hat{n}^\beta) \quad (4.7)$$

with all other commutators vanishing. Of course, this is just the direct sum of the field algebra Eqs. (4.2)–(4.4) with the algebra Eqs. (4.5)–(4.7). Note that the commutator $[\hat{\pi}_\mu(X), \hat{p}^{\alpha\beta}] = i(\delta_\mu^\alpha \hat{\pi}^\beta(X) - \delta_\mu^\beta \hat{\pi}^\alpha(X))$ in Eq. (3.36) need no longer be assumed since it is implied now by the commutation relation $[\hat{n}_\alpha, \hat{p}^{\beta\gamma}] = i(\delta_\alpha^\beta \hat{n}^\gamma - \delta_\alpha^\gamma \hat{n}^\beta)$ in Eq. (4.7).

We anticipate that the key average-energy operator (that will eventually be associated with translations along the internal time direction) is (cf. Eq. (3.52))

$$\hat{H} := \frac{1}{2} : \int d^4X \left\{ \hat{\omega}(X)^2 + (\hat{n}^\mu \hat{n}^\nu - \eta^{\mu\nu}) \partial_\mu \hat{\phi}(X) \partial_\nu \hat{\phi}(X) + m^2 \hat{\phi}(X)^2 \right\} : \quad (4.8)$$

in which case the main task is to find a representation of the history algebra Eqs. (4.2)–(4.7) in which Eq. (4.8) exists as a genuine self-adjoint operator.

We proceed as follows. For each $n \in H_+$ we construct the Hilbert space \mathcal{H}_n that carries operators ${}^n\hat{\phi}(X)$, ${}^n\hat{\pi}(X)$ that satisfy the history algebra Eqs. (1.14)–(1.16):

$$[{}^n\hat{\phi}(X), {}^n\hat{\phi}(X')] = 0 \quad (4.9)$$

$$[{}^n\hat{\pi}(X), {}^n\hat{\pi}(X)] = 0 \quad (4.10)$$

$$[{}^n\hat{\phi}(X), {}^n\hat{\pi}(X')] = i\delta^4(X - X') \quad (4.11)$$

and with the property that the average-energy operator in Eq. (2.8)

$${}^n\hat{H} := \frac{1}{2} : \int d^4X \left\{ {}^n\hat{\pi}(X)^2 + (n^\mu n^\nu - \eta^{\mu\nu}) \partial_\mu {}^n\hat{\phi}(X) \partial_\nu {}^n\hat{\phi}(X) + m^2 {}^n\hat{\phi}(X)^2 \right\} : \quad (4.12)$$

exists as a genuine self-adjoint operator. The n -superscripts on the fields in Eqs. (4.9)–(4.11) serve to indicate that we have chosen the representation of the abstract history algebra Eqs. (1.14)–(1.16) in which this operator ${}^n\hat{H}$ exists. In fact, as we know from the work in [1], for all $n \in H_+$ these fields can be constructed on the same abstract Fock space even though the corresponding representations of the history field algebra are unitarily inequivalent for different n . However, for our present purposes, it is clearer if we continue to refer to the Hilbert space on which ${}^n\hat{H}$ exists as \mathcal{H}_n .

We now link up with the heuristic ideas in the Introduction by constructing a Hilbert bundle whose base space is the hyperboloid H_+ , and in which the fiber over each $n \in H_+$ is defined to be the Hilbert space \mathcal{H}_n .⁸ The Hilbert space of our history theory is then defined to be the direct integral

$$\mathcal{H} := \int_{H_+}^{\oplus} \mathcal{H}_n d\mu(n). \quad (4.14)$$

Here $d\mu(n)$ is the usual $SO(3,1)$ -invariant measure on the hyperboloid H_+ : it is just the standard measure used in normal quantum field theory, but now applied to n -space rather than momentum space.

The vectors in this direct-integral Hilbert space \mathcal{H} are defined to be the cross-sections of the Hilbert bundle: *i.e.*, maps $\Psi : H_+ \rightarrow \bigcup_{n \in H_+} \mathcal{H}_n$ with the property that $\Psi(n) \in \mathcal{H}_n$ for all $n \in H_+$. The inner product between a pair of such cross-sections Ψ_1 and Ψ_2 is defined as

$$\langle \Psi_1, \Psi_2 \rangle := \int_{H_+} d\mu(n) \langle \Psi_1(n), \Psi_2(n) \rangle_{\mathcal{H}_n} \quad (4.15)$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}_n}$ denotes the inner product in the Hilbert space fiber \mathcal{H}_n .

⁸We note that the $SO(3,1)$ subgroup of the history group (*i.e.*, the part associated with the \hat{n}_μ variables) acts transitively on H_+ with stability group $SO(3)$, so that $H_+ \simeq SO(3,1)/SO(3)$. Thus we have the principle bundle

$$SO(3) \longrightarrow SO(3,1) \longrightarrow SO(3,1)/SO(3) \simeq H_+ := \{n \in \mathbb{R}^4 \mid n \cdot n = 1, n^0 > 0\}. \quad (4.13)$$

This suggests that we could try using a non-trivial representation of $SO(3)$ to ‘twist’ the fibers of the Hilbert bundle. However, we shall not explore that option here.

Of course, if we make the specific identification of each Hilbert space \mathcal{H}_n with the Fock space \mathcal{F} , as summarised in Section 2, then the new Hilbert space \mathcal{H} can be viewed as the vector space of all measurable functions $\Psi : H_+ \rightarrow \mathcal{F}$ with the inner product

$$\langle \Psi_1, \Psi_2 \rangle := \int_{H_+} d\mu(n) \langle \Psi_1(n), \Psi_2(n) \rangle_{\mathcal{F}}. \quad (4.16)$$

Note that there is a natural cyclic ‘ground’ state which is defined to be the cross-section Ω such that, for all $n \in H_+$,

$$\Omega(n) := |0\rangle_n \quad (4.17)$$

where $|0\rangle_n$ is the ground state of the average-energy operator ${}^n\hat{H}$ in the Hilbert-space \mathcal{H}_n .

4.2 The field and foliation operators

The field operators. The next step is to define the history field operators $\hat{\phi}(X)$ and $\hat{\omega}(X)$ on $\mathcal{H} := \int_{H_+}^{\oplus} \mathcal{H}_n d\mu(n)$ as follows:

$$\{\hat{\phi}(X)\Psi\}(n) := {}^n\hat{\phi}(X)\Psi(n) \quad (4.18)$$

$$\{\hat{\omega}(X)\Psi\}(n) := {}^n\hat{\omega}(X)\Psi(n) \quad (4.19)$$

for all $n \in H_+$. These equations are meaningful since the vectors $\Psi(n)$, $n \in H_+$, in the right hand sides belong to the Hilbert space \mathcal{H}_n on which the field operators ${}^n\hat{\phi}(X)$ and ${}^n\hat{\omega}(X)$ are defined. In other words, the maps $n \mapsto {}^n\hat{\phi}(X)$ and $n \mapsto {}^n\hat{\omega}(X)$ define fields of operators over the base space H_+ , and are hence well-defined⁹ operators on the direct integral $\int_{H_+}^{\oplus} \mathcal{H}_n d\mu(n)$.

It is clear that the operators defined by Eqs. (4.18) and (4.19) satisfy the history algebra Eqs. (4.2)–(4.4). For example,

$$\{\hat{\phi}(X)\hat{\omega}(X')\Psi\}(n) = {}^n\hat{\phi}(X)\{\hat{\omega}(X')\Psi\}(n) = {}^n\hat{\phi}(X){}^n\hat{\omega}(X')\Psi(n) \quad (4.20)$$

and similarly

$$\{\hat{\omega}(X')\hat{\phi}(X)\Psi\}(n) = {}^n\hat{\omega}(X')\{\hat{\phi}(X)\Psi\}(n) = {}^n\hat{\omega}(X'){}^n\hat{\phi}(X)\Psi(n) \quad (4.21)$$

so that, for all $n \in H_+$,

$$\begin{aligned} \{[\hat{\phi}(X), \hat{\omega}(X')]\Psi\}(n) &= [{}^n\hat{\phi}(X), {}^n\hat{\omega}(X')]\Psi(n) \\ &= i\delta^4(X - X')\Psi(n) \end{aligned} \quad (4.22)$$

which means that (modulo subtleties about domains) we have the basic history field commutator $[\hat{\phi}(X), \hat{\omega}(X')] = i\delta^4(X - X')$.

⁹Of course, to do this rigorously one needs to discuss the domains of the various operators concerned, but we shall not dwell on such niceties here.

Note that, if we exploit the fact that the Hilbert spaces can all be identified with the same Fock space \mathcal{F} , then using the definitions in Eqs. (2.5)–(2.6), we can further write

$$\{\hat{\phi}(X)\Psi\}(n) := {}^n\hat{\phi}(X)\Psi(n) = \frac{1}{\sqrt{2}} {}^n\Gamma^{-1/4}(\hat{b}(X) + \hat{b}^\dagger(X))\Psi(n) \quad (4.23)$$

$$\{\hat{\omega}(X)\Psi\}(n) := {}^n\hat{\omega}(X)\Psi(n) = \frac{1}{i\sqrt{2}} {}^n\Gamma^{1/4}(\hat{b}(X) - \hat{b}^\dagger(X))\Psi(n). \quad (4.24)$$

Here the operator $\hat{b}(X)$ (and similarly for $\hat{b}^\dagger(X)$) is defined as the constant field of operators over H_+ obtained by identifying each fiber of the Hilbert bundle with the Fock space \mathcal{F} .

The foliation operators. The operators that represent the foliation vector are easy to define in the Hilbert space $\int_{H_+}^{\oplus} \mathcal{H}_n d\mu(n)$. Specifically:

$$\{\hat{n}_\mu\Psi\}(n) := n_\mu\Psi(n) \quad (4.25)$$

and

$$\{\hat{p}^{\alpha\beta}\Psi\}(n) := i \left\{ n_\alpha \frac{\partial}{\partial n_\beta} - n_\beta \frac{\partial}{\partial n_\alpha} \right\} \Psi(n). \quad (4.26)$$

Note that, strictly speaking, if the history states are considered as sections of the Hilbert bundle, then the right hand side of Eq. (4.26) involves taking the difference between vectors belonging to different Hilbert-space fibers, and hence it is only meaningful if there is a *connection* in the bundle. However, this is not a problem in our case since the fibers \mathcal{H}_n , $n \in H_+$, can all be identified with the basic Fock space \mathcal{F} , and this is assumed to have been done when writing Eq. (4.26).

The time-averaged energy operator. The natural way of defining a time-averaged energy operator is to exploit the fact that, on each Hilbert space fiber \mathcal{H}_n , $n \in H_+$, the operator ${}^n\hat{H}$ defined in Eq. (4.12) exists, and represents the time-averaged value of the energy for that particular foliation. Thus we can define¹⁰ an operator \hat{H} by

$$\{\hat{H}\Psi\}(n) := {}^n\hat{H}\Psi(n) \quad (4.27)$$

for all $n \in H_+$. Note that, as anticipated in Eq. (4.8), the operator thus defined can be written in terms of the basic history fields as

$$\hat{H} := \frac{1}{2} : \int d^4X \left\{ \hat{\omega}(X)^2 + (\hat{n}^\mu \hat{n}^\nu - \eta^{\mu\nu}) \partial_\mu \hat{\phi}(X) \partial_\nu \hat{\phi}(X) + m^2 \hat{\phi}(X)^2 \right\} : \quad (4.28)$$

The remaining generators of the internal Poincaré group can be defined in an analogous way.

¹⁰As usual, to be fully rigorous we should worry about domains of essential self-adjointness for these operators.

The internal time function. If an internal time function is introduced as in Eq. (3.59), then the action of the operator $\hat{H}[s]$ on a section Ψ of the Hilbert bundle is

$$\{\hat{H}[s]\Psi\}(n) := s(n) {}^n\hat{H}\Psi(n), \quad (4.29)$$

which shows clearly the sense in which $s(n)$ is the internal time associated with the foliation vector n .

This suggests an interesting application of the ideas discussed in [12] of possible uses of topos theory in quantum gravity and quantum theory. In particular, one might try to view the construction above as being, rather than of a bundle, instead of a sheaf of Hilbert spaces over the base space H_+ , which is now construed as the category of ‘contexts’ in which assertions about the history system are to be made.

By viewing our construction as an object in the topos of sheaves over H_+ , we can exploit the existence in any topos of both external and internal views: ‘external’ in the sense of how things look from the perspective of normal mathematics; and ‘internal’ in the sense of how things look from the perspective of the mathematical structure based on the topos itself. In particular, when viewed externally, the time function $n \mapsto s(n)$ appears precisely as that: *i.e.*, a function. On the other hand, when viewed internally it corresponds to a real number in the topos of sheaves over H_+ . Thus what we have called the ‘internal time *function*’ is just a *real number* when viewed internally in the topos. We intend to devote a future paper to the general question of the ways in which topos ideas can be productively applied to history theory.

4.3 The external Poincaré group

The key step in constructing a representation of the external Lorentz group in the Hilbert space \mathcal{H} of cross-sections is to have a family of intertwining operators $U(n; \Lambda) : \mathcal{H}_n \rightarrow \mathcal{H}_{\Lambda n}$ that satisfy the conditions given in Eq. (1.23):

$$U(\Lambda n; \Lambda')U(n; \Lambda) = U(n; \Lambda'\Lambda). \quad (4.30)$$

Indeed, the conditions in Eq. (4.30) mean precisely that the intertwining operators $U(n; \Lambda)$ ‘cover’ (*i.e.*, act coherently with respect to) the action of $SO(3, 1)$ on the base space H_+ of the Hilbert bundle.

In these circumstances, for each $\Lambda \in SO(3, 1)$, we can define an operator $W(\Lambda) : \mathcal{H} \rightarrow \mathcal{H}$ by

$$\{W(\Lambda)\Psi\}(n) := U(\Lambda^{-1}n; \Lambda)\Psi(\Lambda^{-1}n) \quad (4.31)$$

for all $n \in H_+$. This is clearly unitary since

$$\begin{aligned} \langle W(\Lambda)\Psi, W(\Lambda)\Psi \rangle_{\mathcal{H}} &= \int_{H_+} d\mu(n) \langle \{W(\Lambda)\Psi\}(n), \{W(\Lambda)\Psi\}(n) \rangle_{\mathcal{H}_n} \\ &= \int_{H_+} d\mu(n) \langle U(\Lambda^{-1}n; \Lambda)\Psi(\Lambda^{-1}n), U(\Lambda^{-1}n; \Lambda)\Psi(\Lambda^{-1}n) \rangle_{\mathcal{H}_n} \end{aligned}$$

$$\begin{aligned}
&= \int_{H_+} d\mu(n) \langle \Psi(\Lambda^{-1}n), \Psi(\Lambda^{-1}n) \rangle_{\mathcal{H}_{\Lambda^{-1}n}} \\
&= \int_{H_+} d\mu(n) \langle \Psi(n), \Psi(n) \rangle_{\mathcal{H}_n} \\
&= \langle \Psi, \Psi \rangle_{\mathcal{H}}
\end{aligned} \tag{4.32}$$

where we have used (i) the assumed unitarity of the intertwining operators $U(n; \Lambda) : \mathcal{H}_n \rightarrow \mathcal{H}_{\Lambda n}$; and (ii) the invariance of the measure $d\mu$ on H_+ under the action of $SO(3, 1)$.

To see that $W(\Lambda)$ defined in Eq. (4.31) satisfies the group law we compute as follows:

$$\begin{aligned}
\{W(\Lambda_2)W(\Lambda_1)\Psi\}(n) &= U(\Lambda_2^{-1}n; \Lambda_2) \left(\{W(\Lambda_1)\Psi\}(\Lambda_2^{-1}n) \right) \\
&= U(\Lambda_2^{-1}n; \Lambda_2)U(\Lambda_1^{-1}\Lambda_2^{-1}n; \Lambda_1)\Psi(\Lambda_1^{-1}\Lambda_2^{-1}n).
\end{aligned} \tag{4.33}$$

But, from Eq. (4.30) we have

$$U(\Lambda_2^{-1}n; \Lambda_2)U(\Lambda_1^{-1}\Lambda_2^{-1}n; \Lambda_1) = U(\Lambda_1^{-1}\Lambda_2^{-1}n; \Lambda_2\Lambda_1) = U((\Lambda_2\Lambda_1)^{-1}n; \Lambda_2\Lambda_1) \tag{4.34}$$

and hence, for all $n \in H_+$,

$$\begin{aligned}
\{W(\Lambda_2)W(\Lambda_1)\Psi\}(n) &= U((\Lambda_2\Lambda_1)^{-1}n; \Lambda_2\Lambda_1)\Psi((\Lambda_2\Lambda_1)^{-1}n) \\
&= \{W(\Lambda_2\Lambda_1)\Psi\}(n)
\end{aligned} \tag{4.35}$$

as is required to give a representation of the Lorentz group.

As was mentioned earlier, in our particular case, the existence of intertwining operators $U(n; \Lambda) : \mathcal{H}_n \rightarrow \mathcal{H}_{\Lambda n}$ satisfying Eq. (4.30) is demonstrated rather easily by exploiting the fact that the Hilbert spaces \mathcal{H}_n , $n \in H_+$, can all be identified naturally with the same Fock space generated by creation and annihilation operators $\hat{b}(X)^\dagger$ and $\hat{b}(X)$. Indeed, as discussed earlier, we simply get operators $U(\Lambda) : \mathcal{F} \rightarrow \mathcal{F}$ which in themselves give a representation of the external Lorentz group, and which satisfy Eqs. (2.16)–(2.17).

The translation subgroup of the external Poincaré group is easier to define since the translations do not act on H_+ . Thus we have the simple definition

$$\{W(a)\Psi\}(n) := {}^nU(a)\Psi(n) \quad \forall n \in H_+, \tag{4.36}$$

where ${}^nU(a)$ denotes the operators of the translation subgroup of the external Poincaré group in \mathcal{H}_n .

5 Conclusions

We have shown how the discussion in [1] of a history version of scalar quantum field theory can be augmented in such a way as to include the quantisation of the unit-length, time-like vector n that determines the Lorentzian foliation of Minkowski spacetime.

The Hilbert bundle construction that we employed was motivated by: (i) a heuristic discussion of the role of the external Lorentz group in the existing history quantum field theory [1]; and (ii) a more technical discussion of a specific representation of the extended history algebra obtained from the multi-symplectic representation of classical scalar field theory. In the latter context it should be remarked that there exist representations of this algebra other than the simple one given here—the significance of such representations is a subject for future research.

The construction of a Hilbert bundle over $H_+ := \{n \in M \mid n \cdot n = 1, n^0 > 0\}$ is a natural idea at a technical level, but it is also interesting from a conceptual perspective. For example, the direct integral representation of the history Hilbert space—together with the postulated non-dependence of the average energy operator on the variables conjugate to \hat{n}^μ —suggests that we have a type of history analogue of what, in ordinary quantum theory, would be regarded as a system with continuous super-selection sectors labelled by $n \in H_+$. But in a ‘neo-realist’ theory such as consistent histories, the role of super-selection sectors is somewhat different from that which arises in an instrumentalist theory such as standard quantum mechanics.

However, the main motivation behind the present paper is to present certain mathematical techniques that can be proved useful when quantising the spacetime foliations that are expected to arise in a history version of general relativity. This important issue in the history approach to quantum gravity, is something to which we shall return in a later paper.

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References

- [1] K. Savvidou Poincare invariance for continuous time histories. gr-qc/0104053 (2001).
- [2] K. Savvidou General relativity histories theory. *Class. Quant. Grav.* 18 (2001) 3611-3628.
- [3] C.J. Isham. Quantum logic and the histories approach to quantum theory. *J. Math. Phys.* 35:2157 (1994).
- [4] C.J. Isham and N. Linden. Continuous histories and the history group in generalised quantum theory. *J. Math. Phys.* 36: 5392 (1995).
- [5] C. Isham, N. Linden, K. Savvidou and S. Schreckenberg. Continuous time and consistent histories. *J. Math. Phys.* 37:2261 (1998).

- [6] K. Savvidou. The action operator in continuous time histories. *J. Math. Phys.* 40: 5657 (1999).
- [7] K. Savvidou. Continuous Time in Consistent Histories. PhD Thesis in gr-qc/9912076 (1999).
- [8] I. Kouletsis and K.V. Kuchar. Diffeomorphisms as symplectomorphisms in history phase space: Bosonic string model. gr-qc/0108022 (2001).
- [9] H. Araki Hamiltonian formalism and the canonical commutation relations in quantum field theory. *J. Math. Phys.* 1: 492 (1960).
- [10] J.E. Marsden, R. Montgomery, P.J. Morrison and W.B. Thompson Covariant Poisson brackets for classical fields. *Ann. Phys.* 169:29 (1986).
- [11] C.J. Isham Global and topological aspects of quantum theory In “Relativity, Groups and Topology II”, 1059–1290 eds. B. S. DeWitt and R. Stora, North-Holland, Amsterdam (1984).
- [12] J. Butterfield and C.J. Isham Some possible roles for topos theory in quantum theory and quantum gravity *Found. Phys.* 37:1707 (2000).