GROMOV-WITTEN INVARIANTS OF STABILIZATIONS OF SYMPLECTIC 4-MANIFOLDS

AHMET BEYAZ

ABSTRACT. We relate the Gromov-Witten invariants of $X \times S^2$ to the Seiberg-Witten invariants of X where X is a simply-connected symplectic 4-manifold. We also give examples that expose the similarity between the classification of smooth 4-manifolds and some classification problems regarding symplectic 6-manifolds.

0. INTRODUCTION

In this paper, we calculate some of the Gromov-Witten invariants of the product symplectic structure of $X \times S^2$ where X is a symplectic, simply-connected 4-manifold. $X \times S^2$ is called the stabilization of X in the sense that when two homeomorphic but non-diffeomorphic 4-manifolds are stabilized the difference between the smooth structures vanish and stabilized manifolds are diffeomorphic (Wall [12], Jupp [3]). Gromov-Witten invariants are calculated in terms of the Seiberg-Witten invariants of the base 4-manifold.

Our main theorem is the following:

Main Theorem. *Let X be a simply-connected, symplectic 4-manifold with $b_2^+(X) > 1$. Let A be second homology class of X and $g(A)$ be the genus of A which is assigned by the adjunction formula. Let $\alpha_1, \cdots, \alpha_k$ be cohomology classes of M. Then the following can be stated about the Gromov-Witten invariants of $X \times S^2$:*

(1) *If $g(A) = 0$ and A is nonzero, then $GW^{X \times S^2}_{A,g(A),k}(\alpha_1, \cdots, \alpha_k) = \pm 1$ only if $A = E$ is an exceptional sphere in X, α_j is $PD(E)$ for a unique j and α_i is $PD(E \times S^2)$ for $i \neq j$. Otherwise $GW^{X \times S^2}_{A,g(A),k}(\alpha_1, \cdots, \alpha_k)$ is zero.*

(2) *If $g(A) = 1$, then $GW^{X \times S^2}_{A,g(A),k}(\alpha_1, \cdots, \alpha_k)$ is non-zero only if the self-intersection A^2 of A is zero. When $A^2 = 0$ then $GW^{X \times S^2}_{A,g(A),0}()$ is equal to the sum of $SW^X(2A - K)$ and $1/12$ of the number of isolated fishtail curves in the class A in X.*

While the study of smooth and symplectic 4-dimensional manifolds has attracted much attention and has resulted in a richness of examples and constructions, there still appears to be little hope for a classification scheme. In particular, any classification scheme will require more than the standard algebraic topological invariants associated to a 4-manifold. In smooth topology in dimension 4, the main tool to distinguish the smooth types on a topological 4-manifold is gauge theoretical invariants that are called the Seiberg-Witten invariants.

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Gromov-Witten invariants are applied to get information about the symplectic structures on 6 dimensional manifolds ([**?**], [6]). In particular, they are used to distinguish the symplectic deformation types on a smooth 6-manifold.

Seiberg-Witten and Gromov-Witten for 4 dimensional symplectic manifolds are related to each other through Gromov-Taubes invariants. This relation is going to be revisited in Section 1.

The motivation for this paper is to study the symplectic structures on a given 6-manifold. We want to use smooth structures on a topological 4-manifold to study them. This kind of work was first done by Ruan ([8]). He also conjectures that given a topological 4-manifold X which admits symplectic structures, the classification of smooth structures on this 4-manifold is equivalent to the classification of symplectic deformation types of symplectic structures 6-manifolds which are products of symplectic structures on the 4-manifold and a 2-sphere. In [10], it is shown that this conjecture holds for elliptic surfaces.

Fintushel and Stern conjectures that two homeomorphic simply-connected smooth 4-manifolds are related with a sequence of surgeries on tori of self-intersection zero (Conjecture 6.1 of [2]). According to this conjecture, the second claim of this theorem implies that for different smooth structures X_1 and X_2 on a minimal 4-manifold, $X_1 \times S^2$ and $X_2 \times S^2$ are symplectically different. Nevertheless, next corollary and the examples expose that the study of symplectic structures on stabilized 4-manifolds as profound as the study on the smooth structures on a 4-manifold:

Corollary 0.1. *Let T be a self-intersection zero torus embedded in X such that $c_1(X)T$ is zero. Let X_T and X be homeomorphic but non-diffeomorphic symplectic 4-manifolds with X_T is obtained from X by surgery on T. Then $X_T \times S^2$ and $X \times S^2$ are diffeomorphic but not symplectic deformation equivalent.*

In Section 1, we provide brief background information on the Seiberg-Witten invariants. We also review the Gromov-Witten invariants and the relations between the Seiberg-Witten, Gromov-Taubes and the Gromov-Witten invariants. We prove the main theorem in Section 2. In Section 3 we give some examples of diffeomorphic manifolds $X_1 \times S^2$ and $X_2 \times S^2$, where X_1 and X_2 are homeomorphic symplectic 4-manifolds. Some of these stabilized 4-manifolds are diffeomorphic despite the fact that the base 4-manifolds are not. However, the symplectic structures on these 6-manifolds are not deformation equivalent.

1. Background

In this section, we are going to review Seiberg-Witten and Gromov-Taubes invariants in dimension 4, and Gromov-Witten invariants.

As a convention we denote the homology class of a manifold M represented by a submanifold A as A. Another convention is that we will denote the image of a submanifold A of X under the inclusion map into another manifold M by A as well.

1.1. **Seiberg-Witten Invariants.**

Detailed definition of Seiberg-Witten invariants can be found in [7]. We consider Seiberg-Witten as a function from the set of characteristic second cohomology classes of X to the set of integers. If we fix a characteristic class L, then all characteristic classes can be written as $L + 2\alpha$ for some $\alpha \in H^2(X, \mathbb{Z})$. A class on which the Seiberg-Witten function is nonzero is called a basic class. The set of all basic classes is denoted by Bas_X. This is a finite subset of $H_2(X;\mathbb{Z})$. For almost-complex manifolds the canonical class $K = -c_1(X)$ is a basic class.

The Seiberg-Witten function is an invariant for the smooth type of a 4-manifold. It carries information on the embedded surfaces in a 4-manifold. Following lemma is one of the structure theorems for Seiberg-Witten invariants established by Taubes.

Lemma 1.1. *(Adjunction Inequality) Let X be a simply-connected 4-manifold of simple type with $b_2^+(X) > 1$. Assume that $A \subset X$ is an embedded, oriented, connected surface of genus $g(A) > 0$. Then for every basic class L of X, we have $2g(A) - 2 \geq A^2 + |L \cdot A|$. If $b_2^+(X) = 1$ then we have $2g(A) - 2 \geq A^2 - L \cdot A$.*

This lemma is valid for genus zero surfaces if we add a trivial handle to it.

1.2. **Gromov-Witten Invariants.** Next we review the Gromov-Witten invariants. Gromov-Witten invariants of symplectic manifolds count J-holomorphic curves in a particular homology class which pass through of a number of points. In practice, for a given homology class this count is done by tracing the oriented intersection points of a certain moduli space and a number of cohomology classes. Gromov-Witten invariants are invariants of the deformation type of symplectic structures on a given smooth manifold.

Given a symplectic manifold M, let J be a compatible almost-complex structure on M. A J-holomorphic curve in a symplectic manifold M is an embedding f of a complex curve into M such that $J \circ df = df \circ i$ where i is the usual complex structure on the curve. Given a homology class $A \in H_2(M;\mathbb{Z})$, consider the moduli space $\mathcal{M}_{A,g}$ of all genus g J-holomorphic curves in M where the homology class of the image is A. If one chooses distinct marked points of the curve, say $x_1, \cdots, x_k$, then the moduli space $\mathcal{M}_{A,g,k}$ will consist of the tuples $(f, x_1, \cdots, x_k)$. At this point, $\mathcal{M}_{A,g,k}$ needs to be compactified. Kontsevich and Manin suggested that it is more convenient to compactify the moduli space of stable J-holomorphic maps rather than the space of J-holomorphic maps ([4]). We will adopt this approach here as did Ruan-Tian and Zinger. We denote the space of the stable maps also by $\mathcal{M}_{A,g,k}$. The dimension of $\mathcal{M}_{A,g,k}$ is $(1-g)(2n-6) + 2c_1(M)A + 2k$ where $2n$ is the real dimension of the manifold. When the dimension of the underlying manifold is 6 or less, $\mathcal{M}_{A,g,k}$ can be assumed to be compact ([6] Chapter 7). When $g = 0$, the image of $\mathcal{M}_{A,g,k}$ under the evaluation map $ev : \mathcal{M}_{A,g,k} \to M \times \cdots \times M$ is a (pseudo)cycle in the Cartesian product of k copies of M. The count of the J-holomorphic curves in our manifolds reduces to the intersections of homology classes. The genus zero Gromov-Witten invariants are defined as

$$GW_{A,g,k}(\alpha_1, \cdots, \alpha_k) = ev(\mathcal{M}_{A,g,k}) \cdot ev^*(\alpha_1) \cdots ev^*(\alpha_k)$$

(1) *or*

$$GW_{A,g,k}(\alpha_1, \cdots, \alpha_k) = ev(\mathcal{M}_{A,g,k}) \cdot PD(\alpha_1) \cdots PD(\alpha_k)$$

where $\alpha_1, \cdots, \alpha_k$ are cohomology classes of M. Nonzero invariants can be obtained only when the sum of degrees of these cohomology classes is equal to the dimension of $\mathcal{M}_{A,g,k}$, which is given above.

Genus zero Gromov-Witten invariants are explained in full detail by McDuff and Salamon in [6].

The calculations for genus one Gromov-Witten invariants are harder than the genus zero case. Genus one Gromov-Witten invariants are studied by Zinger. We use the following result of Zinger to calculate the invariants [13].

Theorem 1.2. *(Zinger) Let M be a symplectic manifold and $A \in H_2(M;\mathbb{Z})$. Then*

$$GW^M_{A,1,k}(\alpha_1,\cdots,\alpha_k) - GW^{M,0}_{A,1,k}(\alpha_1,\cdots,\alpha_k) = \begin{cases} 0 & ,\dim M = 4 \\ \frac{2-c_1(X)A}{24} GW^M_{A,0,k}(\alpha_1,\cdots,\alpha_k) & ,\dim M = 6. \end{cases}$$

where $GW^{M,0}_{A,1,k}$ is the genus one GW invariants where the domain is a smooth torus.

A fact about Gromov-Witten invariants which we use in the proof of the main theorem is that if one of the cohomology classes is the Poincare dual of the fundamental class of the manifold then the invariant vanishes ([6] p191).

Lemma 1.3. *Let M be a manifold with dimension 6 or less, A be a nonzero second homology class and $k \geq 1$. Then $GW_{A,g,k}(\alpha_1,\cdots,\alpha_{k-1},PD([M]))$ is zero. In other words there can not be a degree 0 cohomology class among α_i's.*

1.3. **Gromov-Taubes Invariants.** Gromov-Taubes invariant of symplectic 4-manifold X counts the number of J-holomorphic curves in a particular homology class A which pass through $k_A = A^2 + 1 - g(A)$ points. Taubes' work ([11]) on Seiberg-Witten and Gromov-Taubes invariants for 4-manifolds shows that these invariants are equivalent with the following explicit relationship which is taken from [1]:

Lemma 1.4. *Let X be a symplectic 4-manifold with $K = PD(-c_1(X))$. After choosing appropriate orientations, for $b_2^+(X) > 1$ and for all $A \in H_2(X;\mathbb{Z})$, $GT^X(A) = SW^X(2A-K)$. For $b_2^+(X) = 1$, $GT^X(A) = SW^{X,-}(2A-K)$ for all $A \in H_2(X;\mathbb{Z})$ such that $A \cdot E \geq -1$ for each E represented by an embedded symplectic 2-sphere of self-intersection -1.*

For a 4-manifold X with $b_2^+(X) = 1$, we write SW^X instead of $SW^{X,-}$. Gromov-Taubes and Gromov-Witten invariants are related as the next lemma shows. This relationship is mentioned in the MR review of [4], but no proof is provided. A proof is presented here.

Lemma 1.5. *Let X be a simply-connected, symplectic 4-manifold. Then the following equality holds:*

$$GT^X(A) = GW^X_{A,g(A),k_A}(PD(point),\cdots,PD(point))$$

where $k_A = A^2 + 1 - g(A)$ and $PD(point)$ is repeated k_A times.

Proof. Since X is simply-connected, $GT^X(A) = GW^X_{A,g(A),k}(\alpha_1,\cdots,\alpha_k))$ for some k and for some cohomology classes α_j where $1 < j < k$ ([11]). The dimension of the moduli space $\mathcal{M}_{A,g(A),k}$ is $(1-g(A))(2n-6) + 2c_1(X)A + 2k$. Dimension of X is $2n = 4$. By the adjunction formula, $c_1(X) = A^2 + 2 - 2g(A)$. On the other hand, the number of points for Gromov-Taubes invariants is given by Taubes as

$\frac{1}{2}(A^2+c_1(X)A)$. By the adjunction formula this is equal to $A^2+1-g(A)$. The sum of the degrees of α_j must be equal to the dimension of the moduli space $\mathcal{M}_{A,g(A),k}$ which is $4A^2+4-4g(A)$ or $4k_A$. None of these classes is degree 0 by Lemma 1.3. If there is a cohomology class of degree 2, then to get the sum $4k_A$ there must be another class of degree 0. This reveals that $GT^X(A)$ is nonzero only if the degree of each cohomology class is 4. A point is the Poincare dual of the fourth cohomology class which is the generator of the fourth cohomology group. □

2. Proof of the Main Theorem

Proof. (1) By the adjunction inequality, the self-intersection of A must be less than zero. If it is less than -1, Theorem 1.2 of [5] says that A is a multiple cover of an exceptional sphere. Then this multiple cover is not regular with respect to any generic regular almost-complex J and it does not contribute to the Gromov-Witten.

Assume that the self-intersection of A is -1. By the dimension condition, the sum of degrees of α_i should be equal to the dimension of the moduli space $\mathcal{M}^{X\times S^2}_{A,g,k}$ which is $(1-g(A))(2n-6)+2c_1(M)A+2k = 2+2k$. By Lemma 1.3, in order to get a nonzero invariant, there must be one class of degree 4, and the remaining ones are of degree 2.

By positivity of intersections of J-holomorphic curves in an almost-complex manifold, there is only one curve presenting E in X which will be denoted by E as well. For each point of S^2, we have the curve E in X. If k is one, i. e. we put the condition of passing through a generic point, this adds up two real dimensions to the moduli space for the freedom of choosing a point on the sphere E. The moduli space will be $E\times S^2$. The evaluation map ev sends it to into $X\times S^2$. For any $k>0$, the image of the moduli space under ev is $E\times S^2$ times $(k-1)$ copies of E. The evaluation map gives an embedding as below.

$$ev : (E\times S^2)\times E\times\cdots\times E \hookrightarrow (X\times S^2)\times\cdots\times(X\times S^2)$$

In this picture, each E component is embedded in the corresponding copy of $(X\times S^2)$. Now let's turn to the cohomology classes α_j, $1<j<k$. We want to apply Formula 1 given above for the calculation, therefore to get nonzero invariants, the fourth degree class corresponds to $E\times S^2$. Let's call it α_1. It is Poincare dual to a second homology class of $X\times S^2$ which is either (the image of X under the inclusion map in $X\times S^2$) of a second cohomology class of X or $\cdot\times S^2$. The transversal intersection of $PD(\alpha_1)$ with $E\times S^2$ should be nonzero. If we consider only connected and not multiply covered representations, we see that α_1 is $PD(E)$.

Each of the other cohomology classes, α_j is of degree two, so it corresponds to one of the E components of the moduli space. It is Poincare dual to X or $C\times S^2$, where C is a second homology class of X. To get a nonzero invariant it must be $E\times S^2$.

We see that $GW^{X\times S^2}_{E,0,1}(PD(E)) = ev(\mathcal{M}^{X\times S^2}_{E,0,1})\cdot E = E\times S^2\cdot E = -1$. Inductively, we conclude that $GW^{X\times S^2}_{E,0,k}(PD(E), E\cdots E)$ is one upto a sign. If any other cohomology class exists among the α_i then because of the zero intersection, Gromov-Witten invariant vanishes.

(2) Adjunction inequality shows that the self-intersection of A is at most zero. The number of embedded genus one curves in X is given by Lemmas 1.4 and 1.5 as

$SW^X(2A-K)$. However if the self-intersection of A is less than zero then number k_A of points in Lemma 1.5 would be less than zero so we would have our invariant equal to zero. Therefore A^2 should be zero.

Theorem 1.2 points out that we need a count of certain curves of genus zero and genus one in X in order to calculate the genus one Gromov-Witten invariants of the 6-manifolds $X \times S^2$. Once we know the counts, by the inclusion map, we know that these numbers does not change when we move from X to $X \times S^2$. The genus zero curves in the homology class A are either the fishtail curves or the cusp curves in that class. Each cusp curve can be perturbed to form two distinct fishatail curves. Therefore, the Gromov-Witten invariant $GW^{X\times S^2}_{A,1,0}$ is equal to the sum of $SW^X(2A-K)$ and the number of isolated fishtail curves divided by 12 in the homology class of A by Theorem 1.2.

□

For any symplectic manifold M, if $A = 0$ and $g(A) = 0$, then $GW^M_{A,g(A),k}$ is zero unless k is 3. When k is 3, $GW^M_{0,0,3}(\alpha_1, \alpha_2, \alpha_3)$ is the cup product of the three cohomology classes evaluated on the fundamental class of M.

We close this section with a remark in the first part of the main theorem. In general, when $g(A) = 0$ and $k > 3$, the order of the cohomology classes matter. However since all calculations in the first part of the theorem include intersection with a copy of $PD(E)$ in them, the order is not important here.

3. Examples

The first example applies to manifolds with $b_2^+ > 1$ as well.

Example 3.1. *(Ruan [8]) Let Y be a simply-connected symplectic 4-manifold and X be its blowup at one point, i.e $X = Y\#\overline{\mathbb{C}P^2}$. Let's find the Gromov-Witten invariant of X and $X \times S^2$ for the homology class of the exceptional sphere E and its image in $X \times S^2$, respectively. We are going to denote this image as E as well. The number of points in X is determined by Taubes as $k_E = E^2 + 1 - g(E) = 0$. The moduli space $\mathcal{M}^X_{E,g(E),k_E} = \mathcal{M}^X_{E,0,0}$ is a finite set of finite points with orientation. By positivity of intersections of J-holomorphic curves in an almost-complex manifold, there is only curve presenting E in X. So $GW^X_{E,0,0} = 1$. That means $GW^X_{E,0,0} = GT^X(E) = SW^X(2E-K)$ where the last equality is justified by the blowup formula. Remark that this is in accordance with the results of Taubes ([11]). For the 6-manifold $X \times S^2$, the dimension of $\mathcal{M}^{X\times S^2}_{E,g(E),k}$ is $c_1(X \times S^2)E + 2k = c_1(X)E+2k$. By the adjunction formula, $c_1(X)E$ is equal to $E^2+2-2g(E)$. Therefore the dimension of the moduli space is $2+2k$. Let's calculate $GW^{X\times S^2}_{E,0,1}(PD(E))$. For each point of the S^2 we have the curve representing E in X. If we put the condition of passing through a generic point, this adds two real dimensions to the moduli space for the freedom of choosing a point on the sphere. The moduli space will be $E \times S^2$. So $GW^{X\times S^2}_{E,0,1}(PD(E)) = ev(\mathcal{M}^{X\times S^2}_{E,0,1}) \cdot E = E \times S^2 \cdot E = -1$.*

Let's give some examples on elliptic surfaces.

Example 3.2. *Let $X = E(2)$ and $A = 0 \in H_2(X;\mathbb{Z})$. Then $SW^X(2A-K) = SW^X(0) = 1$. By Lemma 1.4, $GT^X(A) = SW^X(2A-K) = 1$. By Theorem 1.2, $GW^{X,0}_{A,1,0} = GW^X_{A,1,0} = GT^X(A) = 1$. If we send X into $X \times S^2$ by the inclusion* *map, we have* $GW_{A,1,0}^{X\times S^2,0} = GW_{A,1,0}^{X,0} = 1$. *Since* $c_1(X\times S^2)A$ *is equal to zero, we also have* $\frac{2-c_1(X\times S^2)A}{24} GW_{A,0,0}^{X\times S^2} = 0$. *Thus we end up with* $GW_{A,1,0}^{X\times S^2,0} = 1$. *If we apply the knot surgery ([2]) on* $E(2)$ *with a nontrivial fibred knot* K, *and get a manifold* $E(2)_K$ *which is not diffeomorphic to* $E(2)$, *then the third part of the main theorem gives that* $E(2)\times S^2$ *and* $E(2)_K\times S^2$ *are not deformation equivalent as symplectic manifolds.*

Another example on elliptic surfaces is given below.

Example 3.3. *Let* $X = E(n)$, *for* $n > 1$ *and* $A = nF \in H_2(X)$ *where* F *is the homology class of the fiber for the elliptic fibration on* X *(which is also a Lefschetz fibration). the canonical class* K *of* X *is* $PD((n-2)F)$. *By Lemma 1.4,* $GT^X(A) = SW^X(2A-K) = 0$ *since* $2A-K$ *is not a basic class. The inclusion of* X *into* $X\times S^2$ *gives that* $GW_{A,1,0}^{X,0} = GW_{A,1,0}^{X} = GT^X(A) = 0$. $GW_{A,0,0}^{M}$ *is count of the* 12 *because* n *fishtail fibers of the fibration on* $E(n)$ *represents* A. $\frac{2-c_1(X\times S^2)A}{24} GW_{A,0,0}^{M} = 1$. *So* $GW_{A,1,0}^{X\times S^2,0} = 1$.

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Department of Mathematics, Middle East Technical University, Ankara 06531 Turkey
E-mail address: beyaz@metu.edu.tr