

# MINIMAL SETS OF REIDEMEISTER MOVES

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ABSTRACT. It is well known that any two diagrams representing the same oriented link are related by a finite sequence of Reidemeister moves  $\Omega_1$ ,  $\Omega_2$  and  $\Omega_3$ . Depending on orientations of fragments involved in the moves, one may distinguish 4 different versions of each of the  $\Omega_1$  and  $\Omega_2$  moves, and 8 versions of the  $\Omega_3$  move. We introduce a minimal generating set of four oriented Reidemeister moves, which includes two moves of type  $\Omega_1$ , one move of type  $\Omega_2$  and one move of type  $\Omega_3$ . We then study other sets of moves, considering various sets with one move of type  $\Omega_3$ , and show that only few sets generate all Reidemeister moves. An unexpected non-equivalence of different  $\Omega_3$  moves is discussed.

## 1. INTRODUCTION

A standard way to describe a knot or a link is via its *diagram*, i.e. a generic plane projection of a link such that the only singularities are transversal double points, endowed with the over- undercrossing information at each double point. Two diagrams are equivalent if there is an orientation-preserving diffeomorphism of the plane that takes one diagram to the other diagram. A classical result of Reidemeister [Re] states that any two diagrams of isotopic links are related by a finite sequence of simple moves  $\Omega_1$ ,  $\Omega_2$ , and  $\Omega_3$ , shown in Figure 1.

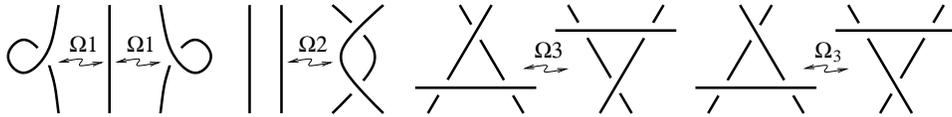


FIGURE 1. Reidemeister moves

Here we assume that two diagrams related by a move coincide outside a disk shown in the picture, called the *changing disk*. If a link is oriented, the diagram is also endowed with the orientation. Depending

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2000 *Mathematics Subject Classification.* 57M25, 57M27.

*Key words and phrases.* Reidemeister moves, knot and link diagrams.

on orientations of fragments involved in the moves, one may distinguish four different versions of each of the  $\Omega 1$  and  $\Omega 2$  moves, and eight versions of the  $\Omega 3$  move, see Figures 2, 3, and 4 respectively<sup>1</sup>.

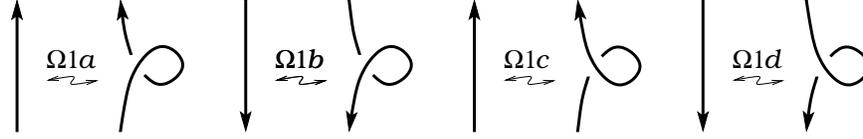


FIGURE 2. Oriented Reidemeister moves of type 1

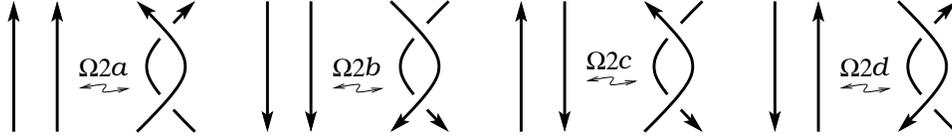


FIGURE 3. Oriented Reidemeister moves of type 2

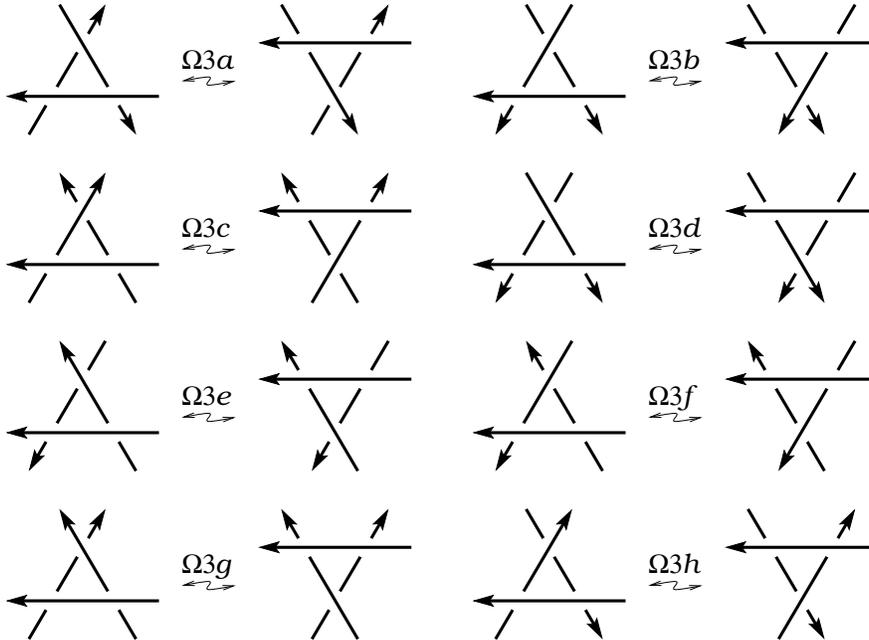


FIGURE 4. Oriented Reidemeister moves of type 3

<sup>1</sup>We will be interested mainly in two  $\Omega 3$  moves:  $\Omega 3a$  and  $\Omega 3b$ . Enumeration of other  $\Omega 3$  moves is somewhat arbitrary and was chosen only for a technical convenience.

When one checks that a certain function of knot or link diagrams defines a link invariant, it is important to minimize the number of moves. We will call a collection  $S$  of oriented Reidemeister moves a *generating set*, if any oriented Reidemeister move  $\Omega$  may be obtained by a finite sequence of isotopies and moves from the set  $S$  inside the changing disk of  $\Omega$ .

While some dependencies between oriented Reidemeister moves are well-known, the standard generating sets of moves usually include six different  $\Omega_3$  moves, see e.g. Kauffman [Ka]. For sets with a smaller number of  $\Omega_3$  moves there seems to be a number of different, often contradictory results. In particular, Turaev [Tu, proof of Theorem 5.4] introduces a set of five oriented Reidemeister moves with only one  $\Omega_3$  move. There is no proof (and in fact we will see in Section 3 that this particular set is not generating), with the only comment being a reference to a figure where, unfortunately, a move which does not belong to the set is used. Wu [Wu] uses the same set of moves citing [Tu], but puts the total number of oriented  $\Omega_3$  moves at 12 (instead of 8). Kaufmann [Ka, page 90] includes as an exercise a set of all  $\Omega_1$  and  $\Omega_2$  moves together with two  $\Omega_3$  moves. Meyer [Me] uses a set with four  $\Omega_1$ , two  $\Omega_2$ , and two  $\Omega_3$  moves and states (again without a proof) that the minimal number of needed  $\Omega_3$  moves is two. The number of  $\Omega_3$  moves used by Östlund [Oe] is also two, but his classification works only for knots and is non-local (depending on the cyclic order of the fragments along the knot). Series of exercises in Chmutov et al. [CDM] (unfortunately without proofs) suggest that only one  $\Omega_3$  suffices, but this involves all  $\Omega_2$  moves. These discrepancies are most probably caused by the fact that while many people needed some statement of this kind, it was only an auxiliary technical statement, a proof of which would be too long and would take the reader away from the main subject, so only a brief comment was usually made. We decided that it was time for a careful treatment. In this note we introduce a simple generating set of four Reidemeister moves, which includes two  $\Omega_1$  moves, one  $\Omega_2$  move and one  $\Omega_3$  move:

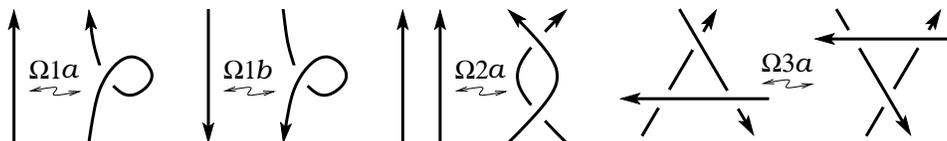


FIGURE 5. A minimal set of Reidemeister moves

**Theorem 1.1.** *Let  $D$  and  $D'$  be two diagrams, representing the same oriented link. Then one may pass from  $D$  to  $D'$  by isotopy and a finite sequence of four oriented Reidemeister moves  $\Omega 1a$ ,  $\Omega 1b$ ,  $\Omega 2a$ , and  $\Omega 3a$ , shown in Figure 5.*

This set of moves is minimal in the following sense. It is easy to show that any generating set should contain at least one move of each of the types two and three; Lemma 2.2 in Section 3 implies that there should be at least two moves of type one. Thus any generating set of Reidemeister moves should contain at least four moves.

Our choice of the move  $\Omega 3a$  may look unusual, since this move (called a cyclic  $\Omega 3$  move, see e.g. [Ka]) is rarely included, contrary to a more common move  $\Omega 3b$ , which is the standard choice motivated by the braid theory<sup>2</sup> The reason is that, unexpectedly, these moves have different properties, as we discuss in detail in Section 3. Indeed, Theorem 1.2 below implies that a generating set of four Reidemeister moves which includes  $\Omega 3b$  simply does not exist. If we consider sets of five Reidemeister moves which contain  $\Omega 3b$ , then it turns out that out of 36 possible combinations of pairs of  $\Omega 1$  and  $\Omega 2$  moves, only 4 sets generate all Reidemeister moves. The only freedom is in the choice of  $\Omega 1$  moves, while  $\Omega 2$  moves are uniquely determined:

**Theorem 1.2.** *Let  $S$  be a generating set of at most five Reidemeister moves which contains only one move  $\Omega 3b$  of type three. Then  $S$  contains  $\Omega 2c$  and  $\Omega 2d$ . Also,  $S$  contains one of the pairs  $(\Omega 1a, \Omega 1b)$ ,  $(\Omega 1a, \Omega 1c)$ ,  $(\Omega 1b, \Omega 1d)$ , or  $(\Omega 1c, \Omega 1d)$ .*

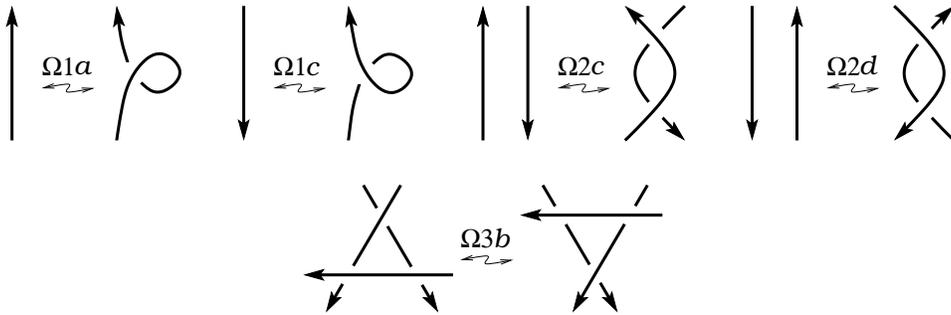


FIGURE 6. A minimal set of Reidemeister moves with  $\Omega 3b$

One of these generating sets is shown in Figure 6. It is interesting to note that while (by Markov theorem) the set  $\Omega 1a$ ,  $\Omega 1c$ ,  $\Omega 2a$ ,  $\Omega 2b$  and  $\Omega 3b$  allows one to pass between any two braids whose closure gives

<sup>2</sup>This is the only  $\Omega 3$  move with all three positive crossings.

the same link, this set is not sufficient to connect any pair of general diagrams representing the same link. Even more unexpected is the fact that all type one moves together with  $\Omega 2a$ ,  $\Omega 2c$  (or  $\Omega 2d$ ) and  $\Omega 3b$  are also insufficient (c.f. [Tu, Wu]).

All our considerations are local, and no global realization restrictions are involved. Therefore all our results hold also for virtual links.

Section 2 is dedicated to the proof of Theorem 1.1. In Section 3 we discuss various generating sets which contain  $\Omega 3b$  and prove Theorem 1.2

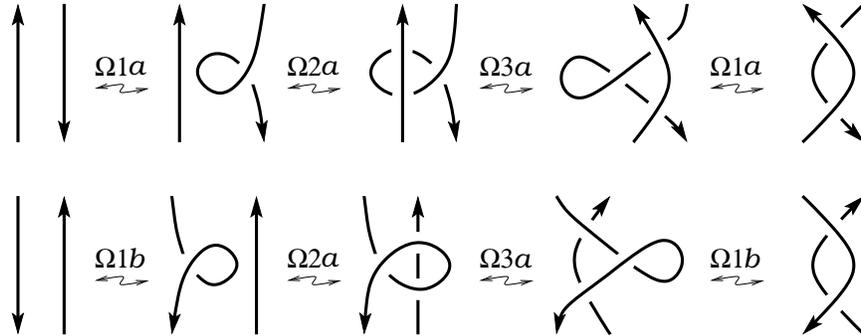
We are grateful to O. Viro for posing this question and to S. Chmutov for valuable discussions. The author was supported by an ISF grant 1261/05.

## 2. A MINIMAL SET OF ORIENTED REIDEMEISTER MOVES

In this section we prove Theorem 1.1 in several easy steps. The first step is to obtain  $\Omega 2c$ ,  $\Omega 2d$ :

**Lemma 2.1.** *Reidemeister move  $\Omega 2c$  may be realized by a sequence of  $\Omega 1a$ ,  $\Omega 2a$  and  $\Omega 3a$  moves. Reidemeister move  $\Omega 2d$  may be realized by a sequence of  $\Omega 1b$ ,  $\Omega 2a$  and  $\Omega 3a$  moves.*

*Proof.*

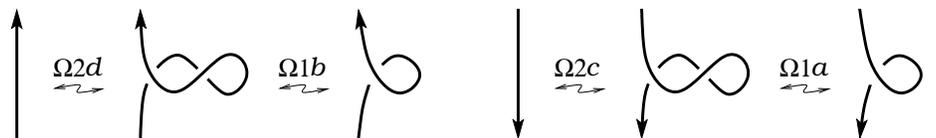


□

Now the remaining moves of type one may be obtained as in [Oe]:

**Lemma 2.2** ([Oe]). *Reidemeister move  $\Omega 1c$  may be realized by a sequence of  $\Omega 1b$  and  $\Omega 2d$  moves. Reidemeister move  $\Omega 1d$  may be realized by a sequence of  $\Omega 1a$  and  $\Omega 2c$  moves.*

*Proof.*

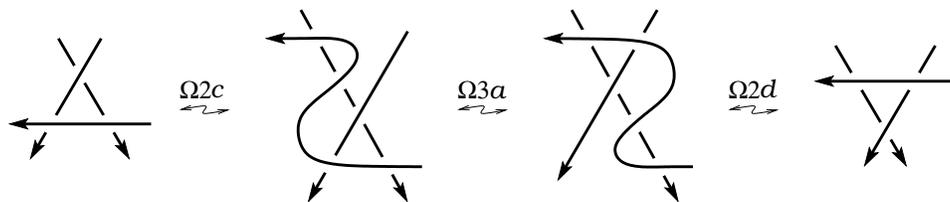


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This concludes the treatment of all  $\Omega 1$  and  $\Omega 2$  moves, except for  $\Omega 2b$ ; we will take care of it later. Having in mind Section 3, where we will deal with  $\Omega 3b$  instead of  $\Omega 3a$ , we will first consider  $\Omega 3b$ :

**Lemma 2.3.** *Reidemeister move  $\Omega 3b$  may be realized by a sequence of  $\Omega 2c$ ,  $\Omega 2d$ , and  $\Omega 3a$  moves.*

*Proof.*

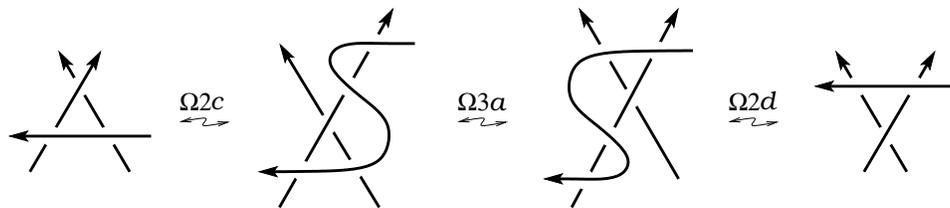


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To deal with  $\Omega 2b$  we will need another move of type three:

**Lemma 2.4.** *Reidemeister move  $\Omega 3c$  may be realized by a sequence of  $\Omega 2c$ ,  $\Omega 2d$ , and  $\Omega 3a$  moves.*

*Proof.*

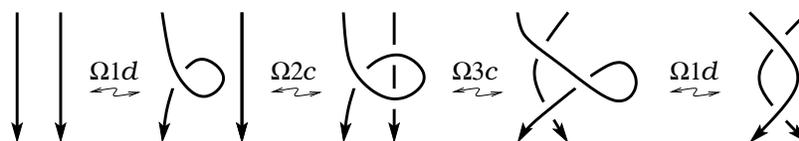


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At this stage we can obtain the remaining move  $\Omega 2b$  of type two:

**Lemma 2.5.** *Reidemeister move  $\Omega 2b$  may be realized by a sequence of  $\Omega 1d$ ,  $\Omega 2c$  and  $\Omega 3c$  moves.*

*Proof.*

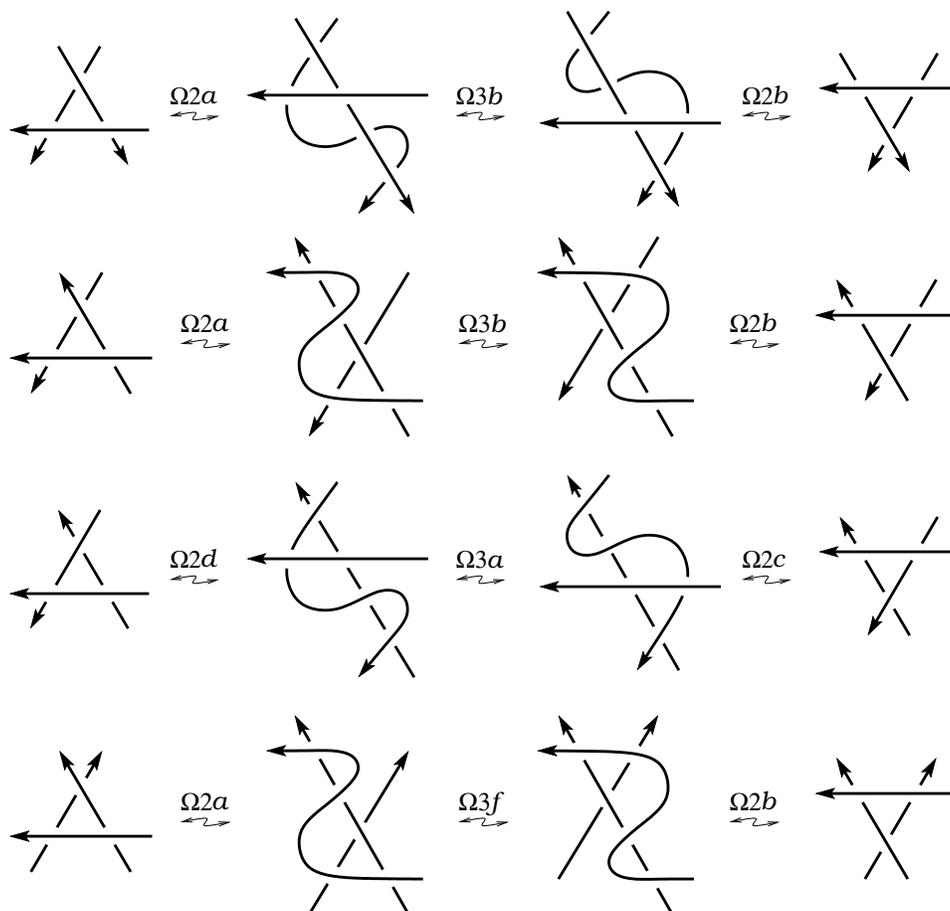


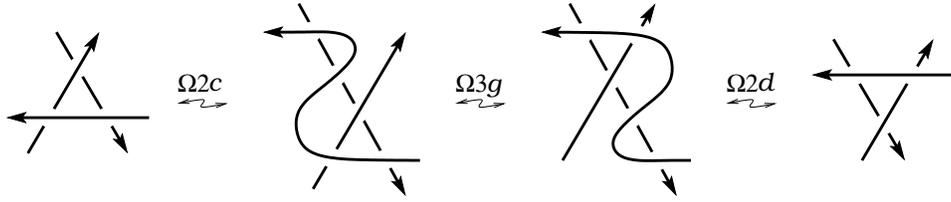
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To conclude the proof of Theorem 1.1, it remains to obtain  $\Omega 3d - \Omega 3h$ . Since by now we have in our disposal all moves of type two, this becomes an easy exercise:

**Lemma 2.6.** *Reidemeister moves  $\Omega 3d - \Omega 3h$  of type three may be realized by a sequence of type two moves,  $\Omega 3a$ , and  $\Omega 3b$ .*

*Proof.* We consider the moves in the alphabetic order, using moves obtained in previous steps:





□

**Remark 2.7.** There are other generating sets which include  $\Omega 3a$ . In particular,  $\Omega 1a$ ,  $\Omega 1b$ ,  $\Omega 2b$  and  $\Omega 3a$  also give a generating set. To adapt the proof of Theorem 1.1 to this case, one needs only a slight modification of Lemma 2.1. All other lemmas do not change.

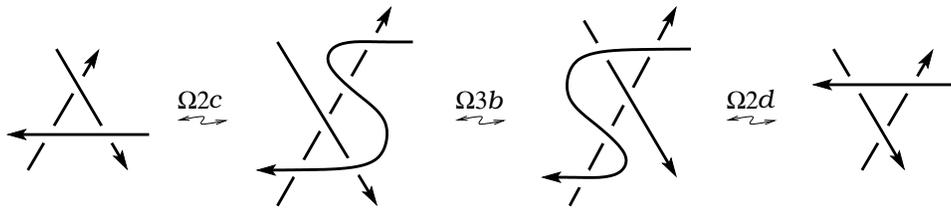
### 3. OTHER SETS OF REIDEMEISTER MOVES

In this section we discuss other generating sets and prove Theorem 1.2. Unexpectedly, different  $\Omega 3$  moves have different properties as far as minimal sets of Reidemeister moves are concerned. Let us study the case of  $\Omega 3b$  in more details.

In a striking contrast to Theorem 1.1 which involves  $\Omega 3a$ , Theorem 1.2 implies that there does not exist a generating set of four moves which includes  $\Omega 3b$ . It is natural to ask where does the proof in Section 2 break down, if we attempt to replace  $\Omega 3a$  with  $\Omega 3b$ .

The only difference between  $\Omega 3a$  and  $\Omega 3b$  may be pinpointed to Lemma 2.1: it does not have an analogue with  $\Omega 3b$  replacing  $\Omega 3a$ , as we will see in the proof of Lemma 3.2 below.

An analogue of Lemma 2.3 is readily shown to exist. Indeed,  $\Omega 3a$  may be realized by a sequence of  $\Omega 2c$ ,  $\Omega 2d$  and  $\Omega 3b$  moves, as illustrated below:



Using this fact instead of Lemma 2.3, together with the rest of Lemmas 2.2-2.6, implies that  $\Omega 1a$  and  $\Omega 1b$ , taken together with  $\Omega 2c$ ,  $\Omega 2d$ , and  $\Omega 3b$ , indeed provide a generating set. Moreover, a slight modification of Lemma 2.2 shows that any of the other three pairs of  $\Omega 1$  moves in the statement of Theorem 1.1 may be used instead of  $\Omega 1a$  and  $\Omega 1b$ . Thus we obtain the positive part of Theorem 1.2. It remains to show that the remaining pairs of  $\Omega 1$  and  $\Omega 2$  moves, taken together with  $\Omega 3b$ , do not

result in generating sets. The first step is to eliminate two remaining pairs  $(\Omega1a, \Omega1d)$  and  $(\Omega1b, \Omega1c)$  of  $\Omega1$  moves.

To show that a certain set of Reidemeister moves is not generating, we will construct an invariant of these moves which, however, is not preserved under the set of all Reidemeister moves. The simplest classical invariants of this type are the *writhe*  $w$  and the *winding number*  $rot$  of the diagram. The winding number of the diagram grows (respectively drops) by one under  $\Omega1b$  and  $\Omega1d$  (respectively  $\Omega1a$  and  $\Omega1c$ ). The writhe of the diagram grows (respectively drops) by one under  $\Omega1a$  and  $\Omega1b$  (respectively  $\Omega1c$  and  $\Omega1d$ ). Moves  $\Omega2$  and  $\Omega3$  do not change  $w$  and  $rot$ . These simple invariants suffice to deal with moves of type one (see e.g. [Oe]):

**Lemma 3.1** ([Oe]). *None of the two pairs  $(\Omega1a, \Omega1d)$  or  $(\Omega1b, \Omega1c)$ , taken together with all  $\Omega2$  and  $\Omega3$  moves, gives a generating set.*

*Proof.* Indeed, both  $\Omega1a$  and  $\Omega1d$  preserve  $w + rot$ , so this pair together with  $\Omega2$  and  $\Omega3$  moves cannot generate all Reidemeister moves. The case of  $\Omega1b$  and  $\Omega1c$  is obtained by the reversal of an orientation (of all components) of the link.  $\square$

The situation with  $\Omega2$  moves is more cumbersome. We are to show that except for  $(\Omega2c, \Omega2d)$ , no other pair of  $\Omega2$  moves, taken together with two  $\Omega1$  moves and  $\Omega3b$ , gives a generating set. The case of a pair  $(\Omega2a, \Omega2b)$  requires a separate consideration.

**Lemma 3.2.** *Let  $S$  be a set which consists of two Reidemeister moves of type one,  $\Omega2a, \Omega2b$ , and  $\Omega3b$ . Then  $S$  is not generating.*

*Proof.* Given a link diagram, smooth all double points of the diagram respecting the orientation, as illustrated in Figure 7.



FIGURE 7. Smoothing the diagram respecting the orientation

Count the numbers  $C^-$  and  $C^+$  of clockwise and counter-clockwise oriented circles of the smoothed diagram, respectively. Note that  $\Omega2a$ ,  $\Omega2b$ , and  $\Omega3b$  preserve an isotopy class of the smoothed diagram, thus preserve both  $C^+$  and  $C^-$ . On the other hand,  $\Omega1b$  and  $\Omega1d$  add one to  $C^+$ , and  $\Omega1a$ ,  $\Omega1c$  add one to  $C^-$ . Thus if  $S$  contains  $\Omega1a$  and  $\Omega1c$ , all moves of  $S$  preserve  $C^+$ . The case of  $\Omega1b$  and  $\Omega1d$  is obtained by the

reversal of an orientation (of all components) of the link. If  $S$  contains  $\Omega 1a$  and  $\Omega 1b$ , all moves of  $S$  preserve  $C^+ + C^- - w$ . Similarly, if  $S$  contains  $\Omega 1c$  and  $\Omega 1d$ , all moves of  $S$  preserve  $C^+ + C^- + w$ . In all the above cases, moves from  $S$  can not generate  $\Omega 2c$ ,  $\Omega 2d$ , since each of  $\Omega 2c$  and  $\Omega 2d$  may change  $C^+$  as well as  $C^+ + C^- \pm w$  (while preserving  $w$  and  $C^+ - C^- = rot$ ).  $\square$

The remaining four cases are more delicate, since here the standard algebraic/topological invariants, reasonably well behaved under compositions, can not be applied. The reason can be explained on a simple example: suppose that we want to show that  $\Omega 2d$  cannot be obtained by a sequence of Reidemeister moves which includes  $\Omega 2c$ . Then our invariant should be preserved under  $\Omega 2c$  and distinguish two tangles shown in Figure 8a. However, if we compose them with a crossing, as shown in Figure 8b, we may pass from one to another by  $\Omega 2c$ . Thus the invariant should not survive composition of tangles.

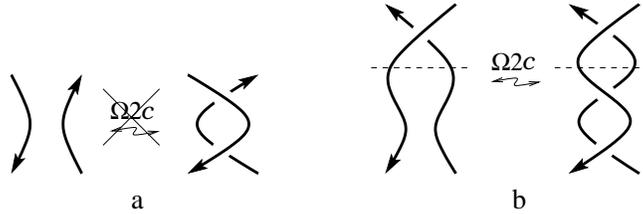


FIGURE 8. Composition destroys inequivalence

Instead, we will use a certain notion of positivity, which is indeed destroyed by such compositions. It is defined as follows. Let  $D$  be a  $(2, 2)$ -tangle diagram with two oriented ordered components  $D_1$ ,  $D_2$ . Decorate all arcs of both components of with an integer weight by the following rule. Start walking on  $D_1$  along the orientation. Assign zero to the initial arc. Each time when we pass an overcrossing (we don't count undercrossings) with  $D_2$ , we add a sign (the local writhe) of this overcrossing to the weight of the previous arc. Now, start walking on  $D_2$  along the orientation. Again, assign zero to the initial arc. Each time when we pass an undercrossing (now we don't count overcrossings) with  $D_1$ , we add a sign of this undercrossing to the weight of the previous arc. See Figure 9a. Two simple examples are shown in Figure 9b,c.

We call a component *positively weighted*, if weights of all its arcs are non-negative. E.g., both components of the (trivial) tangle in Figure 9b are positively weighted. None of the components of a diagram in Figure 9c are positively weighted (since the weights of the middle arcs on both

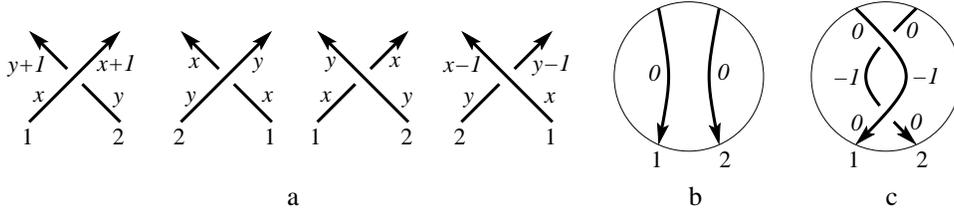


FIGURE 9. Weights of diagrams

components are  $-1$ ). Behavior of positivity under Reidemeister moves is considered in the next lemmas.

Denote by  $S^+$  the set which consists of all Reidemeister moves of type one,  $\Omega 2a$ , and  $\Omega 3b$ .

**Lemma 3.3.** *Let  $D$  be a  $(2, 2)$ -tangle diagram with positively weighted components. Then any diagram obtained from it by a sequence of moves from  $S^+$  also has positively weighted components.*

*Proof.* Indeed, an application of a first Reidemeister move does not change this property since we count only intersections of two different components. An application of  $\Omega 2a$  adds (or removes) two crossings on each component in such a way, that walking along a component we first meet a positive crossing and then the negative one, so the weights of the middle arcs are either the same or larger than on the surrounding arcs, see Figure 10a. An application of  $\Omega 3b$  preserves the weights since  $\Omega 3b$  involves only positive crossings.  $\square$

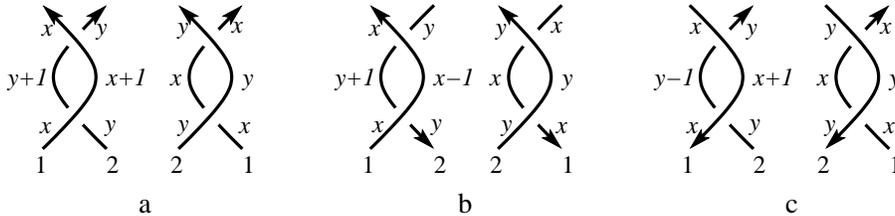


FIGURE 10. Weights and Reidemeister moves of type two

**Lemma 3.4.** *Let  $D$  be a  $(2, 2)$ -tangle diagram with a positively weighted second component. Then any diagram obtained from it by  $\Omega 2c$  also has a positively weighted second component.*

*Proof.* An application of  $\Omega 2c$  may add (or remove) two undercrossings on  $D_2$ , but in such a way that we first meet a positive undercrossing and then the negative one, so the weight of a middle arc is larger than on the surrounding arcs, see Figure 10b.  $\square$

**Lemma 3.5.** *Let  $D$  be a  $(2, 2)$ -tangle diagram with a positively weighted first component. Then any diagram obtained from it by  $\Omega 2d$  also has a positively weighted first component.*

*Proof.* An application of  $\Omega 2d$  may add (or remove) two overcrossings on  $D_1$ , but in such a way that we first meet a positive overcrossing and then the negative one, so the weight of a middle arc is larger than on the surrounding arcs, see Figure 10c.  $\square$

Comparing Figures 9b and 9c we conclude

**Corollary 3.6.** *None of the two sets  $S^+ \cup \Omega 2c$  and  $S^+ \cup \Omega 2d$  generates  $\Omega 2b$ .*

**Remark 3.7.** In [Tu, Theorem 5.4] (and later [Wu]) the set  $S^+ \cup \Omega 2c$  is considered as a generating set. Fortunately (V. Turaev, personal communication), an addition of  $\Omega 2d$  does not change the proof of the invariance in [Tu, Theorem 5.4].

The remaining cases of pairs  $(\Omega 2b, \Omega 2c)$  and  $(\Omega 2b, \Omega 2d)$  are obtained by the reversal of orientations (of both components) of the tangle in the above construction. This concludes the proof of Theorem 1.2.

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