

Compute-and-Forward: Harnessing Interference through Structured Codes

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Abstract

Interference is usually viewed as an obstacle to communication in wireless networks. This paper proposes a new strategy, compute-and-forward, that exploits interference to obtain significantly higher rates between users in a network. The key idea is that relays should decode linear functions of transmitted messages according to their observed channel coefficients rather than ignoring the interference as noise. After decoding these linear equations, the relays simply send them towards the destinations, which given enough equations, can recover their desired messages. The underlying codes are based on nested lattices whose algebraic structure ensures that integer combinations of codewords can be decoded reliably. Encoders map messages from a finite field to a lattice and decoders recover equations of lattice points which are then mapped back to equations over the finite field. This scheme is applicable even if the transmitters lack channel state information. Its potential is demonstrated through examples drawn from cooperative communication including distributed MIMO and wireless network coding.

Index Terms

Relaying, cooperative communication, structured codes, nested lattice codes, reliable computation, AWGN networks, interference.

I. INTRODUCTION

In a wireless network, a transmission from a single node is heard not only by the intended receiver, but also by all other nearby nodes; by analogy, any receiver not only captures the signal from its designated transmitter, but from all other nearby transmitters. The resulting interference is usually viewed as highly undesirable and clever algorithms and protocols have been devised to avoid interference between transmitters. Collectively, these strategies transform the physical layer into a set of *reliable bit pipes*, i.e. each link can accommodate a certain number of bits per time unit. These bit pipes can then be used seamlessly by higher layers in the protocol stack.

Since wireless terminals must compete for the same fixed chunk of spectrum, interference avoidance results in diminishing rates as the network size increases. Recent work on cooperative communication has shown that this penalty can be overcome by adopting new strategies at the physical layer. The key idea is that users should help relay each other's messages by exploiting the broadcast and multiple-access properties of the wireless medium; properties that are usually viewed as a hindrance and are not captured by a bit pipe interface. To date, most proposed cooperative schemes have relied on one of the following three core relaying strategies:

- *Decode-and-Forward*: The relay decodes at least some part of the transmitted messages. The recovered bits are then re-encoded for collaborative transmission to the next relay. Although this strategy offers significant advantages, the relay is ultimately interference-limited as the number of transmitted messages increases [1]–[4].
- *Compress-and-Forward*: The signal observed at the relay is vector quantized and this information is passed towards the destination. If the destination receives information from multiple relays, it can treat the network as a multiple-input multiple-output (MIMO) channel. Unfortunately, since no decoding is performed at intermediate nodes, noise builds up as messages traverse the network [1], [3], [5]–[7].
- *Amplify-and-Forward*: The relay simply acts as a repeater and transmits a scaled version of its observation. Like compress-and-forward, this strategy converts the network into a large MIMO channel with the added possibility of a beamforming gain. However, noise also builds up with each retransmission. [2], [4], [8]–[10].

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In this paper, we propose a new strategy, *compute-and-forward*, that enables relays to decode linear equations of the transmitted messages using the noisy linear combinations provided by the channel. A destination, given sufficiently many linear combinations, can solve for its desired messages. Our strategy is based on the use of structured codes, particularly nested lattice codes. This ensures that integer combinations of codewords are themselves codewords. A relay is free to determine which linear combination to recover, but those closer to the channel's fading coefficients are available at higher rates.

This strategy simultaneously affords protection against noise and the opportunity to exploit interference for cooperative gains. Whereas compress-and-forward and amplify-and-forward convert a network into a set of noisy linear equations, compute-and-forward can convert it into a set of *reliable linear equations*. These equations can in turn be used for a digital implementation of cooperative schemes that could fit into a (slightly revised) network protocol stack. Classical relaying strategies seem to require a cross-layer design that dispenses with bit pipes and gives higher layers in the network stack direct access to the wireless medium. However, this would negate many of the advantages of a modular design [11]. Compute-and-forward provides a natural solution to this problem by permitting a slight revision of the interface from bits to equations of bits.

We will develop a general framework for compute-and-forward that can be used in any relay network with linear channels and additive white Gaussian noise (AWGN). Transmitters send out messages taking values in a prime-sized finite field and relays recover linear equations of the messages over the same field, making this an ideal physical-layer interface for network coding. We will compare compute-and-forward to classical relaying strategies in two network scenarios, one based on distributed MIMO and the other wireless network coding. Classical relaying strategies perform well in either low or high signal-to-noise ratio (SNR) regimes. As we will see, compute-and-forward offers advantages in moderate SNR regimes where both interference and noise are significant factors.

A. Related Work

There is a large body of work on lattice codes and their applications in communications. We cannot do justice to all this work here and point the interested reader to an excellent survey by Zamir [12]. The basic insight is that, for many AWGN networks of interest, nested lattice codes can approach the performance of standard random coding arguments. One key result by Erez and Zamir showed that nested lattice codes (combined with lattice decoding) can achieve the capacity of the point-to-point AWGN channel [13]. More generally, Zamir, Shamai, and Erez demonstrated how to use nested lattice codes for many classical AWGN multi-terminal problems in [14]. Subsequent work by El Gamal, Caire, and Damen showed that nested lattice codes achieve the diversity-multiplexing tradeoff of MIMO channels [15]. Note that, in general, structured codes are not sufficient to prove capacity results. For instance, group codes cannot approach the capacity of asymmetric discrete memoryless channels [16].

It is tempting to assume that requiring codes to have a certain algebraic structure diminishes their usefulness for proving capacity theorems. However, it has become clear that for certain network communication scenarios, structured codes can actually outperform standard random coding arguments [17]. The first example of such behavior was found by Körner and Marton in [18]. They considered a decoder that wants to reconstruct the parity of two dependent binary sources observed by separate encoders. They found the rate region by using the same linear code at each encoder. More recently, we showed that structured codes offer large gains for reliable computation over multiple-access channels [19]. Philosof *et al.* demonstrated that structured codes enable distributed dirty paper coding for Gaussian multiple-access channels in [20].

The celebrated paper of Ahlswede *et al.* on network coding showed that for wired networks, relays must send out functions of received data, rather than just routing it [21]. Subsequent work showed that linear codes [22], [23] and linear codes with random coefficients [24] are sufficient for multicasting. There has recently been a great deal of interest in exploiting the physical layer of the wireless medium for network coding [25], [26]. In [27], [28], we used structured codes for network coding over finite-field and Gaussian multiple-access networks. In a similar vein, Narayanan, Wilson, and Sprintson developed a nested lattice strategy for the two-way relay channel [29]. Nam, Chung, and Lee studied the two-way relay channel and Gaussian multiple-access networks with asymmetric power constraints in [30], [31]. Gündüz *et al.* used structured codes for multicasting over a compound multiple-access channel with the help of a relay [32].

Work on interference alignment by Maddah-Ali, Motahari, and Khandani [33] and Cadambe and Jafar [34] has shown that large gains are possible for interference channels at high SNR. The key is to have users transmit along

subspaces chosen such that all interference stacks up in the same dimensions at the receivers. Lattice codes can be used to realize these gains at finite SNR. Bresler, Parekh, and Tse used lattice codes to approximate the capacity of the many-to-one and one-to-many interference channels to within a constant number of bits [35]. This scheme was employed for bursty interference channels in [36]. For symmetric interference channels, Sridharan *et al.* developed a layered lattice strategy in [37]. Structured codes are also useful for ergodic alignment over fast fading interference channels [38] and multi-hop networks [39] as well as decentralized processing in cellular networks [40], [41].

Distributed source coding can also benefit from the use of structured codes. Krithivasan and Pradhan have employed nested lattice codes for distributed compression of linear functions of jointly Gaussian sources [42]. More recently, they have developed a framework for discrete memoryless sources using nested group codes [43]. Wagner derived an outer bound for the Gaussian case in [44].

Large gains are also possible in multi-user source-channel coding [45]–[47]. For Gaussian settings, the modulo-lattice modulation scheme of Kochman and Zamir is particularly useful [48]. Finally, recent work by He and Yener has shown that lattices are useful for information theoretic secrecy [49]. See also [50].

B. Summary of Paper Results

Our basic strategy is to take messages from a finite field, map them onto lattice points, and transmit (dithered versions of) these across the channel. Each relay observes a linear combination of these lattice points and attempts to decode an integer combination of them. This equation of lattice points is finally mapped back to a linear equation over a finite field. Our main theorems are summarized below:

- Theorems 1 and 2 give our achievable rates for sending equations over a finite field from transmitters to relays. The strategy relies on a nested lattice code which is developed in Theorem 3.
- Theorems 4, 5, and 6 give sufficient conditions on the equation coefficients so that a destination can recover one or more of the original messages.
- In Theorems 7 and 8 we generalize our compute-and-forward scheme to include successive cancellation and superposition coding.

Finally, we give two applications of compute-and-forward and compare the performance to that of classical relaying strategies. In Section VIII, we consider a distributed MIMO problem and, in Section IX, we consider a wireless network coding problem.

II. PROBLEM STATEMENT

Our relaying strategy is applicable to any configuration of sources, relays, and destinations that are linked through linear AWGN channels. We will refer to such configurations as AWGN networks. To simplify the description of the scheme, we will first focus on how to deliver equations to a single set of relays. We will then show how a destination, given sufficiently many equations, can recover the intended messages. These two components are sufficient to completely describe an achievable rate region for any AWGN network. We begin with the necessary definitions for L transmitters to send equations to M relays over a wireless channel. Then, we will provide a framework for recovering the original messages given a set of equations.

Remark 1: In fact, one can apply this strategy to non-linear and non-Gaussian channels as well by using the modulo-lattice transformation developed by Erez and Zamir in [51].

A. Decoding Equations

We are primarily interested in narrowband wireless channel models so we will specify our encoding and decoding schemes for complex baseband and assume that modulation is handled separately. In Section II-C, we will restate a few definitions for real-valued channel models so that we can give corollaries that specialize our results to real values.

Let \mathbb{C} denote the complex field and \mathbb{F}_p denote the finite field of size p where p is always assumed to be prime. Let $+$ denote addition over the complex field and \oplus addition over the finite field. Furthermore, let \sum denote summation over the complex field and \bigoplus denote summation over the finite field. We also set $j = \sqrt{-1}$ and assume that the log operation is with respect to base 2. It will be useful to map between the finite field and a corresponding subset of the integers. We let $g : \mathbb{F}_p \rightarrow \{0, 1, 2, \dots, p-1\}$ be this one-to-one map. This is essentially an identity map except for the change of alphabet. If g or its inverse g^{-1} are applied to a vector we assume they operate element-wise.

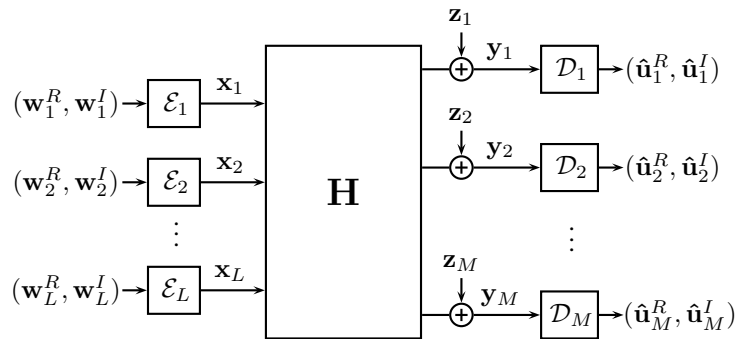


Fig. 1. L transmitters reliably communicate linear functions to M relays.

We will use boldface lowercase letters to denote column vectors and boldface uppercase letters to denote matrices. For example, $\mathbf{h} \in \mathbb{C}^L$ and $\mathbf{H} \in \mathbb{C}^{M \times L}$. Let $\|\mathbf{h}\| \triangleq \sqrt{\sum_{i=1}^L |h[i]|^2}$ denote the ℓ^2 -norm of \mathbf{h} . Also, let \mathbf{h}^* and \mathbf{h}^T denote the Hermitian (or conjugate) transpose and the regular transpose of \mathbf{h} , respectively. Finally, let $\mathbf{0}$ denote the zero vector, δ_ℓ denote the unit vector with 1 in the ℓ^{th} entry and 0 elsewhere, and $\mathbf{I}^{M \times M}$ denote the identity matrix of size M .

Our main goal is to reliably transmit messages across the network at the highest possible rates. However, intermediate nodes in the network may only need to decode a function of the messages. These functions can be decoded at high rates if we choose their algebraic structure to match the structure of the channel. Thus, we will split our messages over the finite field into two parts: one for the real part of the channel and the other for the imaginary part.

Definition 1 (Messages): Each transmitter (indexed by $\ell = 1, 2, \dots, L$) has two length- k_ℓ vectors over a prime-size finite field, $\mathbf{w}_\ell^R, \mathbf{w}_\ell^I \in \mathbb{F}_p^{k_\ell}$. The superscript denotes whether the vector is intended for the real part or the imaginary part of the channel. Together these vectors are the *message* of transmitter ℓ , $\mathbf{w}_\ell = (\mathbf{w}_\ell^R, \mathbf{w}_\ell^I)$. Without loss of generality, we assume that the transmitters are numbered by increasing message length. Since we are interested in functions of these messages vectors, we zero-pad them to a common length $k \triangleq \max_\ell k_\ell$ prior to encoding.

Remark 2: We are interested in average probability of error results so we assume that all messages are drawn independently and uniformly from the set of all possible values.

Definition 2 (Encoders): Each transmitter is equipped with an *encoder*, \mathcal{E}_ℓ , that maps length- k messages over the finite field to length- n codewords over the complex field:

$$\mathcal{E}_\ell : \mathbb{F}_p^k \times \mathbb{F}_p^k \rightarrow \mathbb{C}^n \quad (1)$$

for $\ell = 1, 2, \dots, L$.

Definition 3 (Message Rate): The *message rate* r_ℓ of each transmitter is:

$$r_\ell = \frac{2k_\ell}{n} \log p \quad (2)$$

Definition 4 (Power Constraint): Each transmitter's length- n channel input, $\mathbf{x}_\ell = \mathcal{E}_\ell(\mathbf{w}_\ell)$, is subject to the usual *power constraint*:

$$\frac{1}{n} \|\mathbf{x}_\ell\|^2 \leq \text{SNR} \quad (3)$$

for $\text{SNR} \geq 0$ and $\ell = 1, 2, \dots, L$.

Remark 3: Note that asymmetric power constraints can be easily modeled by scaling the channel coefficients appropriately.

Definition 5 (Channel Model): Each relay observes a noisy linear combination of the transmitted signals through the *channel*:

$$\mathbf{y}_m = \sum_{\ell=1}^L h_{m\ell} \mathbf{x}_\ell + \mathbf{z}_m \quad (4)$$

for $m = 1, 2, \dots, M$ where $h_{m\ell} \in \mathbb{C}$ are the channel coefficients and \mathbf{z} is i.i.d. circularly symmetric Gaussian noise, $\mathbf{z} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}^{n \times n})$. Let $\mathbf{h}_m = [h_{m1} \cdots h_{mL}]^T$ denote the vector of channel coefficients to relay m and let $\mathbf{H} = \{\mathbf{h}_m\}$ denote the entire channel matrix. Note that by this convention the m^{th} row of \mathbf{H} is \mathbf{h}_m^T .

Remark 4: For our initial analysis, we will assume that the channel coefficients are fixed for all time. However, these results can easily be extended to the slow fading case under an outage formulation which we develop in Section VIII.

Remark 5: Our coding scheme only requires that each relay knows the channel coefficients from each transmitter to itself. Specifically, relay m only needs to know \mathbf{h}_m . Each transmitter only needs to know its target message rate, not the realization of the channel.

Definition 6 (Decoders): Each relay is equipped with a decoder that maps the observed channel output from the complex field back to two equations of messages over the finite field:

$$\mathcal{D}_m : \mathbb{C}^n \rightarrow \mathbb{F}_p^k \times \mathbb{F}_p^k \quad (5)$$

for $m = 1, 2, \dots, M$.

Our choice of equation structure is inspired by the real-valued decomposition of a complex-valued channel. Recall that for any $\mathbf{H} \in \mathbb{C}^{M \times L}$, $\mathbf{x} \in \mathbb{C}^L$, and $\mathbf{z} \in \mathbb{C}^M$, the channel output $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}$ can be written as:

$$\begin{bmatrix} \text{Re}(\mathbf{y}) \\ \text{Im}(\mathbf{y}) \end{bmatrix} = \begin{bmatrix} \text{Re}(\mathbf{H}) & -\text{Im}(\mathbf{H}) \\ \text{Im}(\mathbf{H}) & \text{Re}(\mathbf{H}) \end{bmatrix} \begin{bmatrix} \text{Re}(\mathbf{x}) \\ \text{Im}(\mathbf{x}) \end{bmatrix} + \begin{bmatrix} \text{Re}(\mathbf{z}) \\ \text{Im}(\mathbf{z}) \end{bmatrix}$$

where $\text{Re}(\cdot)$ and $\text{Im}(\cdot)$ represent the real and imaginary parts.

Note that the structure of the equations at an intermediate relay is not important, so long as they are useful for conveying messages to the destination at the highest possible rate.

Definition 7 (Desired Equations): The goal of each relay is to reliably recover a *linear combination* of the transmitted messages. Relay m selects coefficients $q_{m\ell}^R, q_{m\ell}^I \in \mathbb{F}_p$ for $\ell = 1, 2, \dots, L$ and attempts to decode two equations:

$$\mathbf{u}_m^R = \bigoplus_{\ell=1}^L q_{m\ell}^R \mathbf{w}_\ell^R \oplus (-q_{m\ell}^I) \mathbf{w}_\ell^I \quad (6)$$

$$\mathbf{u}_m^I = \bigoplus_{\ell=1}^L q_{m\ell}^I \mathbf{w}_\ell^R \oplus q_{m\ell}^R \mathbf{w}_\ell^I \quad (7)$$

where $(-q_{m\ell})$ denotes the additive inverse of $q_{m\ell}$.

Although our desired equations are evaluated over the finite field \mathbb{F}_p , the channel operates over the complex field \mathbb{C} . Our coding scheme will allow us to efficiently exploit the channel for reliable computation if the desired equation coefficients are close to the channel coefficients in an appropriate sense. The definition below provides an embedding from the finite field to the complex field that will be useful in quantifying this closeness.

Definition 8 (Coefficient Vector): The equation with coefficient vector $\mathbf{a}_m = [a_{m1} \ a_{m2} \ \cdots \ a_{mL}]^T \in \{\mathbb{Z} + j\mathbb{Z}\}^L$ is the linear combination of the transmitted messages with coefficients given by

$$q_{m\ell}^R = g^{-1}([\text{Re}(a_{m\ell})] \bmod p) \quad (8)$$

$$q_{m\ell}^I = g^{-1}([\text{Im}(a_{m\ell})] \bmod p). \quad (9)$$

Definition 9 (Probability of Error): We say that the equations with coefficient vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_M \in \{\mathbb{Z} + j\mathbb{Z}\}^L$ are decoded with *average probability of error* ϵ if:

$$(\hat{\mathbf{u}}_m^R, \hat{\mathbf{u}}_m^I) \triangleq \mathcal{D}_m(\mathbf{y}_m) \quad (10)$$

$$P \left(\bigcup_{m=1}^M \{(\hat{\mathbf{u}}_m^R, \hat{\mathbf{u}}_m^I) \neq (\mathbf{u}_m^R, \mathbf{u}_m^I)\} \right) < \epsilon. \quad (11)$$

Definition 10 (Computation Rate): We say that the *computation rate* $R(\mathbf{h}, \mathbf{a})$ is achievable if for any $\epsilon > 0$ and n large enough, there exist encoders and decoders, $\mathcal{E}_1, \dots, \mathcal{E}_L, \mathcal{D}_1, \dots, \mathcal{D}_M$, such that for any set of channels,

$\mathbf{h}_1, \dots, \mathbf{h}_M \in \mathbb{C}^L$, and coefficient vectors, $\mathbf{a}_1, \dots, \mathbf{a}_M \in \{\mathbb{Z} + j\mathbb{Z}\}^L$, all relays can recover their desired equations with average probability of error ϵ so long as the underlying message rates r_1, \dots, r_L satisfy:

$$r_\ell < \min_{m: a_{m\ell} \neq 0} R(\mathbf{h}_m, \mathbf{a}_m) \quad (12)$$

Remark 6: Note that the above definition means that each relay is free to choose which equation to recover. The only constraint is that the message rates be lower than the computation rate for that choice of coefficients. In many cases, it will be useful to have each relay recover the equation that is available at the highest computation rate.

Remark 7: The minimization in (12) is taken over non-zero equation coefficients since only the information in these messages is required for the desired equation.

B. Recovering Messages

After the relays decode their equations, they forward them towards the appropriate destinations which may only be interested in recovering the original messages. If the equation coefficients satisfy appropriate conditions, then the destinations can solve for the original messages. Although our scheme can be employed in any AWGN network, we will omit formal definitions for such networks and assume that equations of messages arrive at a destination. This may occur through a single layer of channels as described above or through multiple layers.

Definition 11 (Recovery): We say that message $\mathbf{w}_\ell \in \mathbb{F}_p^{k_\ell}$ can be *recovered* at rate r_ℓ from the equations with coefficient vectors $\mathbf{a}_1, \dots, \mathbf{a}_M \in \{\mathbb{Z} + j\mathbb{Z}\}^L$ if for any $\epsilon > 0$ and n large enough, there exists a decoder $\mathcal{D} : \{\mathbb{F}_p^k \times \mathbb{F}_p^k\}^M \rightarrow \mathbb{F}_p^{k_\ell} \times \mathbb{F}_p^{k_\ell}$ such that:

$$\hat{\mathbf{w}}_\ell = \mathcal{D}((\mathbf{u}_1^R, \mathbf{u}_1^I), \dots, (\mathbf{u}_M^R, \mathbf{u}_M^I)) \quad (13)$$

$$P(\hat{\mathbf{w}}_\ell \neq \mathbf{w}_\ell) < \epsilon \quad (14)$$

where $(\mathbf{u}_m^R, \mathbf{u}_m^I)$ represents the equations from relay m as in Definition 7.

C. Real-Valued Channels

We will now restate some of the above definitions for real-valued channel models. Each transmitter ℓ has a length- k_ℓ message that takes values over a prime-size finite field, $\mathbf{w}_\ell \in \mathbb{F}_p^{k_\ell}$. As before, we zero-pad all messages to a common length $k = \max_\ell k_\ell$ prior to encoding. The message rate is given by $r_\ell = \frac{k_\ell}{n} \log p$. The channel model is as in (4) except all inputs and outputs are over the reals instead of the complex field and the noise is i.i.d. Gaussian with mean zero and variance one.

For real-valued channels, the desired linear combination of messages at relay m is:

$$\mathbf{u}_m = \bigoplus_{\ell=1}^L q_{m\ell} \mathbf{w}_\ell \quad (15)$$

for coefficients $q_{m\ell} \in \mathbb{F}_p$. These finite field coefficients can be embedded in the reals as well. The equation with coefficient vector $\mathbf{a}_m \in \mathbb{Z}^L$ is the linear combination of the transmitted messages with coefficients given by $q_{m\ell} = [g^{-1}(a_{m\ell})] \bmod p$. The rest of the definitions map in a straightforward fashion to the reals.

III. NESTED LATTICE CODES

In order to allow relays to decode integer combinations of codewords, we need codebooks with a linear structure. We will use lattice codes that have both good statistical and good algebraic properties. Erez and Zamir showed that nested lattice codes can approach the capacity of point-to-point AWGN channels in [13]. These codes operate under a modulo arithmetic that is well-suited for mapping operations over a finite field to the complex field.

First, we will provide some necessary definitions from [13] on nested lattice codes. Note that all of these definitions are given over \mathbb{R}^n . Our scheme will use the same lattice code over the real and imaginary parts of the channel input (albeit with different messages). Next, we will describe how to construct nested lattice codes which have certain desirable properties at sufficiently high dimension.

A. Lattice Definitions

Definition 12 (Lattice): An n -dimensional lattice, Λ , is a set of points in \mathbb{R}^n such that if $\mathbf{x}, \mathbf{y} \in \Lambda$, then $\mathbf{x} + \mathbf{y} \in \Lambda$, and if $\mathbf{x} \in \Lambda$, then $-\mathbf{x} \in \Lambda$. A lattice can always be written in terms of a lattice generator matrix $\mathbf{L} \in \mathbb{R}^{n \times n}$:

$$\Lambda = \{\mathbf{x} = \mathbf{L}\mathbf{w} : \mathbf{w} \in \mathbb{Z}^n\}. \quad (16)$$

Note that the origin is always a point in the lattice.

Definition 13 (Nested Lattices): A lattice Λ is said to be *nested* in a lattice Λ_1 if $\Lambda \subseteq \Lambda_1$. We will sometimes refer to Λ as the coarse lattice and Λ_1 as the fine lattice. More generally, a sequence of lattices $\Lambda, \Lambda_1, \dots, \Lambda_L$ is nested if $\Lambda \subseteq \Lambda_1 \subseteq \dots \subseteq \Lambda_L$.

Definition 14 (Quantizer): A lattice quantizer is a map, $Q_\Lambda : \mathbb{R}^n \rightarrow \Lambda$, that sends a point, \mathbf{x} , to the nearest lattice point in Euclidean distance:

$$Q_\Lambda(\mathbf{x}) = \arg \min_{\lambda \in \Lambda} \|\mathbf{x} - \lambda\|. \quad (17)$$

Definition 15 (Voronoi Region): The *fundamental Voronoi region*, \mathcal{V} , of a lattice, is the set of all points in \mathbb{R}^n that are closest to the zero vector: $\mathcal{V} = \{\mathbf{x} : Q_\Lambda(\mathbf{x}) = \mathbf{0}\}$. Let $\text{Vol}(\mathcal{V})$ denote the volume of \mathcal{V} .

Definition 16 (Modulus): Let $[\mathbf{x}] \bmod \Lambda$ denote the quantization error of $\mathbf{x} \in \mathbb{R}^n$ with respect to the lattice Λ :

$$[\mathbf{x}] \bmod \Lambda = \mathbf{x} - Q_\Lambda(\mathbf{x}) \quad (18)$$

For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\Lambda \subseteq \Lambda_1$, the mod Λ operation satisfies:

$$[\mathbf{x} + \mathbf{y}] \bmod \Lambda = [[\mathbf{x}] \bmod \Lambda + \mathbf{y}] \bmod \Lambda \quad (19)$$

$$[Q_{\Lambda_1}(\mathbf{x})] \bmod \Lambda = [Q_{\Lambda_1}([\mathbf{x}] \bmod \Lambda)] \bmod \Lambda \quad (20)$$

$$[a\mathbf{x}] \bmod \Lambda = [a[\mathbf{x}] \bmod \Lambda] \bmod \Lambda \quad \forall a \in \mathbb{Z} \quad (21)$$

$$[\beta\mathbf{x}] \bmod \Lambda = [\beta[\mathbf{x}] \bmod \beta\Lambda] \bmod \beta\Lambda \quad \forall \beta \in \mathbb{R} \quad (22)$$

Definition 17 (Nested Lattice Codes): A *nested lattice code* \mathcal{L} is the set of all points of a fine lattice Λ_1 that are within the fundamental Voronoi region \mathcal{V} of a coarse lattice Λ :

$$\mathcal{L} = \Lambda_1 \cap \mathcal{V} = \{\mathbf{x} : \mathbf{x} = \lambda \bmod \Lambda, \lambda \in \Lambda_1\}. \quad (23)$$

The rate of a nested lattice code is:

$$R = \frac{1}{n} \log |\mathcal{L}| = \frac{1}{n} \log \frac{\text{Vol}(\mathcal{V})}{\text{Vol}(\mathcal{V}_1)}. \quad (24)$$

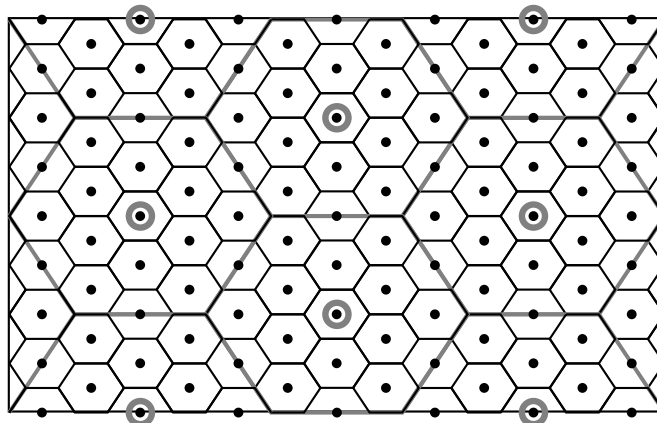


Fig. 2. Part of a nested lattice $\Lambda \subset \Lambda_1 \subset \mathbb{R}^2$. Black points are elements of the fine lattice Λ_1 and gray circles are elements of the coarse lattice Λ . The Voronoi regions for the fine and coarse lattice are drawn in black and gray respectively. A nested lattice code is the set of all fine lattice points within the Voronoi region of the coarse lattice centered at zero.

Our scheme relies on mapping messages from a finite field to codewords from a nested lattice code. The relay will first decode an integer combination of lattice codewords and then convert this into an equation of the messages.

Definition 18 (Lattice Equation): A lattice equation $\mathbf{v} \in \mathcal{L}$ is an integer combination of lattice codewords $\mathbf{t}_\ell \in \mathcal{L}$ modulo the coarse lattice:

$$\mathbf{v} = \left[\sum_{\ell=1}^L a_\ell \mathbf{t}_\ell \right] \bmod \Lambda \quad (25)$$

for some coefficients $a_\ell \in \mathbb{Z}$.

Let $\mathcal{B}(r)$ denote an n -dimensional ball of radius r :

$$\mathcal{B}(r) \triangleq \{\mathbf{x} : \|\mathbf{x}\| \leq r, \mathbf{x} \in \mathbb{R}^n\} \quad (26)$$

and let $\text{Vol}(\mathcal{B}(r))$ denote its volume.

Definition 19 (Covering Radius): The covering radius of a lattice Λ is the smallest real number r_{cov} such that $\mathbb{R}^n \subseteq \Lambda + \mathcal{B}(r_{\text{cov}})$.

Definition 20 (Effective Radius): The effective radius of a lattice with Voronoi region \mathcal{V} is the real number r_{EFFEC} that satisfies $\text{Vol}(\mathcal{B}(r_{\text{EFFEC}})) = \text{Vol}(\mathcal{V})$.

Definition 21 (Moments): The second moment of a lattice Λ is defined as the second moment per dimension of a uniform distribution over the fundamental Voronoi region \mathcal{V} :

$$\sigma_\Lambda^2 = \frac{1}{n \text{Vol}(\mathcal{V})} \int_{\mathcal{V}} \|\mathbf{x}\|^2 d\mathbf{x}. \quad (27)$$

The normalized second moment of a lattice is given by:

$$G(\Lambda) = \frac{\sigma_\Lambda^2}{(\text{Vol}(\mathcal{V}))^{2/n}} \quad (28)$$

The following three definitions are the basis for proving AWGN channel coding theorems using nested lattice codes.

Definition 22 (Covering Goodness): A sequence of lattices $\Lambda^{(n)} \subset \mathbb{R}^n$ is covering good if:

$$\lim_{n \rightarrow \infty} \frac{r_{\text{cov}}^{(n)}}{r_{\text{EFFEC}}^{(n)}} = 1. \quad (29)$$

Such lattices were shown to exist by Rogers [52].

Definition 23 (Quantization Goodness): A sequence of lattices $\Lambda^{(n)} \subset \mathbb{R}^n$ is good for mean-squared error (MSE) quantization if:

$$\lim_{n \rightarrow \infty} G(\Lambda^{(n)}) = \frac{1}{2\pi e}. \quad (30)$$

Zamir, Feder, and Poltyrev showed that sequences of such lattices exist in [53].

Definition 24 (AWGN Goodness): Let \mathbf{z} be a length- n random vector with distribution $\mathcal{N}(0, \sigma_Z^2 \mathbf{I}^{n \times n})$. The volume-to-noise ratio of a lattice is given by:

$$\mu(\Lambda, P_e) = \frac{(\text{Vol}(\mathcal{V}))^{2/n}}{\sigma_Z^2} \quad (31)$$

where σ_Z^2 is chosen such that $\Pr\{\mathbf{z} \notin \mathcal{V}\} = P_e$. A sequence of lattices $\Lambda^{(n)}$ is AWGN good if

$$\lim_{n \rightarrow \infty} \mu(\Lambda^{(n)}, P_e) = 2\pi e \quad \forall P_e \in (0, 1) \quad (32)$$

and for fixed volume-to-noise ratio greater than $2\pi e$, the probability of error decays exponentially in n . In [54], Poltyrev demonstrated the existence of such lattices.

B. Lattice Constructions

Our nested lattice codes are a slight variant of those used by Erez and Zamir to approach the capacity of a point-to-point AWGN channel [13]. As in their considerations, we will have a coarse lattice that is covering, quantization, and AWGN good and a fine lattice that is AWGN good. We generalize this construction to include multiple nested fine lattices all of which are AWGN good. This will allow each transmitter to operate at a different rate.

Lemma 1 (Erez-Litsyn-Zamir): There exists a sequence of lattices $\Lambda^{(n)}$ that is simultaneously covering, quantization, and AWGN good.

This is a corollary of their main result which develops lattices that are good in all the above senses as well as for packing [55, Theorem 5]. Note that these lattices are built using Construction A which is described below.

We will use a coarse lattice Λ of dimension n from Lemma 1 scaled such that its second moment is equal to $\frac{\text{SNR}}{2}$. Let $\mathbf{L} \in \mathbb{R}^{n \times n}$ denote the generator matrix of this lattice. Our fine lattices are defined using the following procedure (the first three steps of which are often referred to as Construction A [55], [56]):

- 1) Draw a matrix $\mathbf{G}_L \in \mathbb{F}_p^{n \times k_L}$ with every element chosen i.i.d. according to the uniform distribution over $\{0, 1, 2, \dots, p-1\}$. Recall that p is prime.
- 2) Define the codebook \mathcal{C}_L as follows:

$$\mathcal{C}_L = \left\{ \mathbf{c} = \mathbf{G}_L \mathbf{w} : \mathbf{w} \in \mathbb{F}_p^{k_L} \right\}. \quad (33)$$

All operations in this step are over \mathbb{F}_p .

- 3) Form the lattice $\tilde{\Lambda}_L$ by projecting the codebook into the reals by $g(\cdot)$, scaling down by a factor of p , and placing a copy at every integer vector. This tiles the codebook over \mathbb{R}^n :

$$\tilde{\Lambda}_L = p^{-1}g(\mathcal{C}_L) + \mathbb{Z}^n \quad (34)$$

- 4) Rotate $\tilde{\Lambda}_L$ by the generator matrix of the coarse nested lattice to get the fine lattice for transmitter L :

$$\Lambda_L = \mathbf{L} \tilde{\Lambda}_L \quad (35)$$

- 5) Repeat steps 1) - 4) for each transmitter $\ell = 1, 2, \dots, L-1$ by replacing \mathbf{G}_L with \mathbf{G}_ℓ which is defined to be the first k_ℓ columns of \mathbf{G}_L .

Clearly, any pair of fine lattices $\Lambda_{\ell_1}, \Lambda_{\ell_2}, 1 \leq \ell_1 < \ell_2 < L$ are nested since all elements of \mathcal{C}_{ℓ_1} can be found from \mathcal{C}_{ℓ_2} by multiplying by all $\mathbf{w} \in \mathbb{F}_p^{n \times k_{\ell_2}}$ with zeros in the last $\ell_2 - \ell_1$ elements. Also note that $\Lambda = \mathbf{L}\mathbb{Z}^n$ is nested within each fine lattice by construction. Finally, we get the desired set of nested lattices $\Lambda \subseteq \Lambda_1 \subseteq \dots \subseteq \Lambda_L$.

By the union bound, we get that:

$$\begin{aligned} \Pr \left(\bigcup_{\ell=1}^L \{ \text{rank}(\mathbf{G}_\ell) < k_\ell \} \right) &\leq \sum_{\ell=1}^L \sum_{\substack{\mathbf{w} \in \mathbb{F}_p^{k_\ell} \\ \mathbf{w} \neq \mathbf{0}}} \Pr \{ \mathbf{G}_\ell \mathbf{w} = \mathbf{0} \} \\ &\leq p^{-n} \sum_{\ell=1}^L (p^{k_\ell} - 1) \end{aligned} \quad (36)$$

Thus, so long as p or k_1, \dots, k_L grow appropriately with n , all matrices $\mathbf{G}_1, \dots, \mathbf{G}_L$ are full rank with probability that goes to 1. Note that if \mathbf{G}_ℓ has full rank, then the number of fine lattice points in the fundamental Voronoi region \mathcal{V} of the coarse lattice is given by $|\Lambda_\ell \cap \mathcal{V}| = p^{k_\ell}$ so that the rate of Λ_ℓ is

$$R_\ell = \frac{1}{n} \log |\Lambda_\ell \cap \mathcal{V}| = \frac{k_\ell}{n} \log_2 p = \frac{1}{2} r_\ell \quad (37)$$

as desired. Furthermore, each message vector $\mathbf{w} \in \mathbb{F}_p^{k_\ell}$ can be put into one-to-one correspondence with a point in $\Lambda_\ell \cap \mathcal{V}$ as shown in Lemma 5. In Appendix II, we show that the fine lattices are AWGN good so long as $\frac{n}{p} \rightarrow 0$ as n grows. It is clear that one can choose p, k_1, \dots, k_L so that the fine lattices have all the desired properties. One possibility is to let p grow like $n \log n$ and set $k_\ell = \lfloor n R_\ell (\log p)^{-1} \rfloor$.

Lemma 2: Any lattice Λ that results from Construction A has a full-rank generator matrix \mathbf{L} .

Proof: Note that $\mathbb{Z}^n \subset \Lambda$ so that Λ contains all of the unit vectors by default. Thus, \mathbf{L} spans \mathbb{R}^n and is full rank. \blacksquare

Remark 8: We require that the fine lattices are generated from full-rank submatrices of the same finite field codebook so that it is possible to compute linear equations over messages with different rates. The full rank condition on the coarse lattice allows us to move between lattice equations and equations of finite field messages.

In [55], [57], some useful properties of nested lattices derived from Construction A are established. These apply to our construction as well and we repeat them below.

Lemma 3: Choose any fine lattice Λ_ℓ from the construction above and let $\Lambda_\ell(i)$ denote the i^{th} point in $\Lambda_\ell \cap \mathcal{V}$ for $i = 0, 1, 2, \dots, p^{k_\ell} - 1$. We have that:

- $\Lambda_\ell(i)$ is uniformly distributed over $p^{-1}\Lambda \cap \mathcal{V}$.
 - For any $i_1 \neq i_2$, $[\Lambda_\ell(i_1) - \Lambda_\ell(i_2)] \bmod \Lambda$ is uniformly distributed over $\{p^{-1}\Lambda\} \cap \mathcal{V}$.
- Thus, each fine lattice can be interpreted as a diluted version of a scaled down coarse lattice $p^{-1}\Lambda$.

IV. COMPUTE-AND-FORWARD

We now develop our main result which provides achievable rates for computing linear functions of messages across AWGN networks. As we will see, these rates are often far higher than those available either by first recovering all messages then computing the function or by standard i.i.d. random coding arguments.

First, we will state the achievable computation rate and show how to optimally choose the free parameter in the rate expression. Next, we will provide some illustrative examples. In Section IV-A, we give a detailed proof of the main result.

Let $(x)^+ \triangleq \max(x, 0)$.

Theorem 1: For channel vectors $\mathbf{h}_1, \dots, \mathbf{h}_M \in \mathbb{C}^L$, the relays can recover any set of linear equations with coefficient vectors $\mathbf{a}_1, \dots, \mathbf{a}_M \in \{\mathbb{Z} + j\mathbb{Z}\}^L$ so long as the message rates are less than the computation rate

$$r_\ell < \min_{m: \mathbf{a}_m \neq \mathbf{0}} (R(\mathbf{h}_m, \mathbf{a}_m))^+ \quad (38)$$

$$R(\mathbf{h}_m, \mathbf{a}_m) = \log \left(\frac{\text{SNR}}{|\alpha_m|^2 + \text{SNR} \|\alpha_m \mathbf{h}_m - \mathbf{a}_m\|^2} \right) \quad (39)$$

for some choice of $\alpha_m \in \mathbb{C}$.

Theorem 2: For a given $\mathbf{h}_m \in \mathbb{C}^L$, $\mathbf{a}_m \in \{\mathbb{Z} + j\mathbb{Z}\}^L$, the computation rate given in Theorem 1 is uniquely maximized by choosing α_m to be the MMSE coefficient

$$\alpha_{\text{MMSE}} = \frac{\text{SNR} \mathbf{h}_m^* \mathbf{a}_m}{1 + \text{SNR} \|\mathbf{h}_m\|^2} \quad (40)$$

which results in a computation rate

$$R(\mathbf{h}_m, \mathbf{a}_m) = \log \left(\left(\|\mathbf{a}_m\|^2 - \frac{\text{SNR} |\mathbf{h}_m^* \mathbf{a}_m|^2}{1 + \text{SNR} \|\mathbf{h}_m\|^2} \right)^{-1} \right) \quad (41)$$

Proof: Note that the denominator of (39) is quadratic in α_m . Thus, it can be uniquely minimized by setting the first derivative to zero.

$$\begin{aligned} f(\alpha_m) &= \alpha_m^* \alpha_m + \text{SNR} (\alpha_m \mathbf{h}_m - \mathbf{a}_m)^* (\alpha_m \mathbf{h}_m - \mathbf{a}_m) \\ \frac{df}{d\alpha_m} &= 2\alpha_m + \text{SNR} (2\alpha_m \mathbf{h}_m^* \mathbf{h}_m - 2\mathbf{h}_m^* \mathbf{a}_m) = 0 \end{aligned} \quad (42)$$

$$\alpha_m (2 + 2\text{SNR} \|\mathbf{h}_m\|^2) = 2\text{SNR} \mathbf{h}_m^* \mathbf{a}_m \quad (43)$$

We solve this to get α_{MMSE} and plug back into $f(\alpha_m)$.

$$\begin{aligned} f(\alpha_{\text{MMSE}}) &= \frac{\text{SNR}^2 |\mathbf{h}_m^* \mathbf{a}_m|^2}{(1 + \text{SNR} \|\mathbf{h}_m\|^2)^2} + \frac{\text{SNR}^3 \|\mathbf{h}_m\|^2 |\mathbf{h}_m^* \mathbf{a}_m|^2}{(1 + \text{SNR} \|\mathbf{h}_m\|^2)^2} \dots \\ &\dots - 2 \frac{\text{SNR}^2 |\mathbf{h}_m^* \mathbf{a}_m|^2}{1 + \text{SNR} \|\mathbf{h}_m\|^2} + \text{SNR} \|\mathbf{a}_m\|^2 \end{aligned} \quad (44)$$

$$= - \frac{\text{SNR}^2 |\mathbf{h}_m^* \mathbf{a}_m|^2}{1 + \text{SNR} \|\mathbf{h}_m\|^2} + \text{SNR} \|\mathbf{a}_m\|^2 \quad (45)$$

Substituting this into $\log\left(\frac{\text{SNR}}{f(\alpha_{\text{MMSE}})}\right)$ yields the desired computation rate. ■

The above two theorems can be interpreted in the following sense. Each transmitter uses a fixed (lattice) codebook with a specified rate. By setting $\alpha_m = 1$ and decoding to the closest lattice point (in Euclidean distance), the relays could recover any linear equation so long as the message rates are less than

$$\log\left(\frac{\text{SNR}}{1 + \text{SNR}\|\mathbf{h}_m - \mathbf{a}_m\|^2}\right) \tag{46}$$

However, if each relay scales its channel output by the appropriate α prior to decoding, then higher rates are possible. This is because the channel output $\alpha_m \mathbf{y}_m = \sum \alpha_m h_{m\ell} \mathbf{x}_\ell + \alpha_m \mathbf{z}_m$ can be equivalently written as a channel output $\tilde{\mathbf{y}}_m = \sum \tilde{h}_{m\ell} \mathbf{x}_\ell + \tilde{\mathbf{z}}_m$ where $\tilde{h}_{m\ell} = \alpha_m h_{m\ell}$ and $\tilde{\mathbf{z}}_m$ is i.i.d. circularly symmetric Gaussian noise with variance $|\alpha_m|^2$. Since there is a rate penalty both for noise and for non-integer channel coefficients, then α should be used to optimally balance between the two as in Theorem 2. This is quite similar to the role of the MMSE scaling coefficient used by Erez and Zamir to achieve the capacity of the point-to-point AWGN channel in [13].

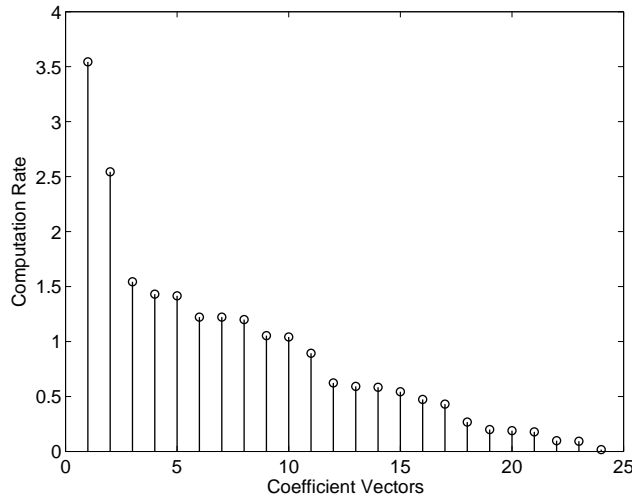


Fig. 3. Computation rate at $\text{SNR} = 5\text{dB}$ for different coefficient vectors $\mathbf{a} \in \{\mathbb{Z} + j\mathbb{Z}\}^2$ for channel $\mathbf{h} = [-1.1744 + j2.1496 \ 1.2512 - j1.6335]^T$. The coefficient vector with the highest rate is $\mathbf{a} = [1 \ -1]^T$ with $R = 3.5436$. Using standard multiple-access codes, this equation can only be computed at rate $R_{\text{MAC}} = 2.5301$.

We will often be interested in finding the coefficient vectors with the highest computation rates. This does not require a search over all integer vectors as most vectors trivially give zero rate.

Lemma 4: For a given channel vector $\mathbf{h} \in \mathbb{C}^L$, the computation rate from Theorem 2 is zero if the coefficient vector $\mathbf{a} \in \{\mathbb{Z} + j\mathbb{Z}\}^L$ satisfies:

$$\|\mathbf{a}\|^2 \geq 1 + \|\mathbf{h}\|^2 \text{SNR}. \tag{47}$$

Proof: Note that $|\mathbf{h}_m^* \mathbf{a}_m|^2 \leq \|\mathbf{h}_m\|^2 \|\mathbf{a}_m\|^2$ by the Cauchy-Schwarz inequality. Using this, we can upper bound the achievable rate from Theorem 2:

$$\log\left(\left(\|\mathbf{a}_m\|^2 - \frac{\text{SNR} |\mathbf{h}_m^* \mathbf{a}_m|^2}{1 + \text{SNR}\|\mathbf{h}_m\|^2}\right)^{-1}\right) \tag{48}$$

$$= \log\left(\frac{1 + \text{SNR}\|\mathbf{h}_m\|^2}{\|\mathbf{a}_m\|^2 + \text{SNR}\|\mathbf{h}_m\|^2 \|\mathbf{a}_m\|^2 - \text{SNR} |\mathbf{h}_m^* \mathbf{a}_m|^2}\right) \tag{49}$$

$$\leq \log\left(\frac{1 + \text{SNR}\|\mathbf{h}_m\|^2}{\|\mathbf{a}_m\|^2}\right).$$

The result follows immediately. ■

In Figure 3, we have plotted the computation rates for coefficient vectors with non-zero entries for an example channel of length 2. The vectors are sorted by descending rate and we only include one of the four possible rotations of each vector, \mathbf{a} , $-\mathbf{a}$, $j\mathbf{a}$, and $-j\mathbf{a}$, as these yield identical rates.

Remark 9: Note that each relay is free to decode more than one equation, so long as all the appropriate computation rates are satisfied. In some cases, it may be beneficial to recover a desired equation by first decoding equations of subsets of messages and then combining them.

Example 1: Let the channel matrix take values on the complex integers, $\mathbf{H} \in \{\mathbb{Z} + j\mathbb{Z}\}^{M \times L}$, and assume that each relay wants a linear equation with a coefficient vector that corresponds exactly to the channel coefficients, $\mathbf{a}_m = \mathbf{h}_m$. Using Theorem 2 the achievable computation rate is:

$$R = \log \left(\left(\|\mathbf{h}_m\|^2 - \frac{\text{SNR}\|\mathbf{h}_m\|^4}{1 + \text{SNR}\|\mathbf{h}_m\|^2} \right)^{-1} \right) \quad (50)$$

$$= \log \left(\frac{1 + \text{SNR}\|\mathbf{h}_m\|^2}{\|\mathbf{h}_m\|^2 + \text{SNR}\|\mathbf{h}_m\|^4 - \text{SNR}\|\mathbf{h}_m\|^4} \right) \quad (51)$$

$$= \log \left(\frac{1}{\|\mathbf{h}_m\|^2} + \text{SNR} \right) \quad (52)$$

Remark 10: One interesting special case of Example 1 is computing the sum of codewords $\mathbf{w}_1 \oplus \mathbf{w}_2$ over a two-user Gaussian multiple-access channel $\mathbf{y} = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{z}$. To date, the best known achievable rate for this scenario is $\log(\frac{1}{2} + \text{SNR})$. Several papers (including our own) have studied this special case and it is an open problem as to whether the best known outer bound $\log(1 + \text{SNR})$ is achievable [28]–[30]. Clearly, one can do better in the low SNR regime using standard multiple-access codes to recover all the messages then compute the sum to get $\frac{1}{2} \log(1 + 2\text{SNR})$.

Example 2: Assume there are L transmitters and L relays. Receiver m wants to recover the message from transmitter m . This corresponds to setting the desired coefficient vector to be a unit vector $\mathbf{a}_m = [0 \cdots 0 \ 1 \ 0 \cdots 0]^T$ where the m^{th} element is 1 and the rest are 0. Substituting this choice of \mathbf{a}_m into Theorem 2, we get that the messages can be decoded if their rates satisfy:

$$r_m < \log \left(\left(1 - \frac{\text{SNR}|h_{mm}|^2}{1 + \text{SNR}\|\mathbf{h}_m\|^2} \right)^{-1} \right) \quad (53)$$

$$= \log \left(\left(\frac{1 + \text{SNR} \sum_{\ell \neq m} |h_{m\ell}|^2}{1 + \text{SNR}\|\mathbf{h}_m\|^2} \right)^{-1} \right) \quad (54)$$

$$= \log \left(1 + \frac{\text{SNR}|h_{mm}|^2}{1 + \text{SNR}\|\mathbf{h}_m\|^2} \right) \quad (55)$$

This is exactly the rate achievable with standard multiple-access techniques if the relays ignore all other messages as noise. In Section VI, we will use successive cancellation of lattice equations to show that if a relay wants all of the messages, any point in the Gaussian multiple-access rate region is achievable with compute-and-forward.

The following example shows that it is useful to allow for a different rate at each transmitter.

Example 3: We have $L = 4$ transmitters and $M = 2$ relays. The channel vectors are $\mathbf{h}_1 = [4 \ -4 \ 1 \ -1]^T$ and $\mathbf{h}_2 = [1 \ 1 \ 2 \ 2]^T$. The desired coefficient vectors are $\mathbf{a}_1 = \mathbf{h}_1$ and $\mathbf{a}_2 = [0 \ 0 \ 1 \ 1]$. These equations can be reliably recovered so long as the message rates satisfy:

$$r_\ell < \begin{cases} \log \left(\frac{1}{34} + \text{SNR} \right) & \ell = 1, 2 \\ \log \left(\frac{1}{2} + \frac{4\text{SNR}}{1+2\text{SNR}} \right) & \ell = 3, 4 \end{cases} \quad (56)$$

A. Proof of Theorem 1

We now provide a detailed description of our encoding and decoding scheme. The following four steps are a basic outline:

- 1) Each transmitter maps its message from the finite field onto an element of a nested lattice code.
- 2) Lattice codewords are dithered and transmitted over the channel.

- 3) Receivers reliably decode a linear equation of the lattice codewords.
- 4) These lattice equations are mapped back to the finite field to get the desired linear combination of messages.

We will first show how to go from messages to lattice codewords and from lattice equations back to linear combinations of messages. Then, we prove that linear equations of lattice codewords can be recovered at the desired computation rates. Finally, we will combine all of these steps to get our compute-and-forward scheme.

The following lemma provides a mapping from messages over the finite field to points in a nested lattice code.

Lemma 5: The function $\phi_\ell : \mathbb{F}_p^{k_\ell} \rightarrow \Lambda_\ell \cap \mathcal{V}$ is one-to-one where:

$$\phi_\ell(\mathbf{w}) = [\mathbf{L}p^{-1}g(\mathbf{G}_\ell\mathbf{w})] \bmod \Lambda. \quad (57)$$

Proof: Since \mathbf{G}_ℓ is assumed to be full rank, it takes \mathbf{w} to a unique point in \mathcal{C}_ℓ . Thus, $p^{-1}g(\mathbf{G}_\ell\mathbf{w})$ maps \mathbf{w} to a unique point in $[0, 1)^n$. Lemma 2 shows that \mathbf{L} is full rank so we just need show that the mod Λ operation is a bijection between $\mathbf{L}[0, 1)^n$ and \mathcal{V} . Assume, for the sake of a contradiction, $\exists x, y \in \mathbf{L}[0, 1)^n, x \neq y$ such that $[x] \bmod \Lambda = [y] \bmod \Lambda$. This implies that:

$$\begin{aligned} x - Q_\Lambda(x) &= y - Q_\Lambda(y) \\ [\mathbf{L}^{-1}(x - Q_\Lambda(x))] \bmod \mathbb{Z}^n &= [\mathbf{L}^{-1}(y - Q_\Lambda(y))] \bmod \mathbb{Z}^n \\ [\mathbf{L}^{-1}x] \bmod \mathbb{Z}^n &= [\mathbf{L}^{-1}y] \bmod \mathbb{Z}^n \\ x &= y \end{aligned}$$

where the third step follows since for any $\lambda \in \Lambda$, $\mathbf{L}^{-1}\lambda \in \mathbb{Z}^n$. A contradiction has been reached which, combined with the fact that $|\mathbb{F}_p^{k_\ell}| = |\Lambda_\ell \cap \mathcal{V}| = p^{k_\ell}$, shows that ϕ_ℓ is a one-to-one map. \blacksquare

Assume the messages are encoded onto nested lattice codes using ϕ_ℓ from Lemma 5 and that the relays successfully recover a linear combination of lattice points. The following lemma shows how to convert these lattice equations back to the desired messages equations.

Lemma 6: For messages $\mathbf{w}_\ell \in \mathbb{F}_p^{k_\ell}$ zero-padded to length k_m , let $\mathbf{u} = \sum q_\ell \mathbf{w}_\ell$ for some coefficients $q_\ell \in \mathbb{F}_p$. Also, set $\mathbf{t}_\ell = \phi_\ell(\mathbf{w}_\ell)$ and $\mathbf{v} = [\sum a_\ell \mathbf{t}_\ell] \bmod \Lambda$ for some $a_\ell \in \mathbb{Z}$ such that $q_\ell = g^{-1}([a_\ell] \bmod p)$. We have that $\mathbf{u} = \phi_m^{-1}(\mathbf{v})$ where

$$\phi_m^{-1}(\mathbf{v}) = (\mathbf{G}_m^T \mathbf{G}_m)^{-1} \mathbf{G}_m^T g^{-1}(p[\mathbf{L}^{-1}\mathbf{v}] \bmod \mathbb{Z}^n). \quad (58)$$

Proof: Recall that since \mathbf{L} is the generator matrix of Λ , $\mathbf{L}^{-1}\Lambda = \mathbb{Z}^n$. Also note that since \mathbf{w}_ℓ is zero-padded to length k_m , then multiplying by \mathbf{G}_m has the same effect as multiplying the original message by \mathbf{G}_ℓ . We have that:

$$[\mathbf{L}^{-1}\mathbf{v}] \bmod \mathbb{Z}^n \quad (59)$$

$$= \left[\mathbf{L}^{-1} \sum_{\ell=1}^L a_\ell \mathbf{t}_\ell + \mathbf{L}^{-1} Q_\Lambda \left(\sum_{\ell=1}^L a_\ell \mathbf{t}_\ell \right) \right] \bmod \mathbb{Z}^n \quad (60)$$

$$= \left[\mathbf{L}^{-1} \sum_{\ell=1}^L a_\ell \mathbf{t}_\ell \right] \bmod \mathbb{Z}^n \quad (61)$$

$$= \left[\sum_{\ell=1}^L a_\ell (p^{-1}g(\mathbf{G}_m \mathbf{w}_\ell) - \mathbf{L}^{-1}Q_\Lambda(\mathbf{L}p^{-1}g(\mathbf{G}_m \mathbf{w}_\ell))) \right] \bmod \mathbb{Z}^n \quad (62)$$

$$= \left[\sum_{\ell=1}^L a_\ell p^{-1}g(\mathbf{G}_m \mathbf{w}_\ell) \right] \bmod \mathbb{Z}^n \quad (63)$$

Multiplying by p we get:

$$p[\mathbf{L}^{-1}\mathbf{v}] \bmod \mathbb{Z}^n = \left[\sum_{\ell=1}^L a_\ell g(\mathbf{G}_m \mathbf{w}_\ell) \right] \bmod p\mathbb{Z}^n \quad (64)$$

$$= \left[g \left(\bigoplus_{\ell=1}^L q_\ell \mathbf{G}_m \mathbf{w}_\ell \right) \right] \bmod p\mathbb{Z}^n \quad (65)$$

$$= g \left(\bigoplus_{\ell=1}^L q_\ell \mathbf{G}_m \mathbf{w}_\ell \right) \quad (66)$$

Applying g^{-1} to move back to the finite field we get:

$$g^{-1} (p[\mathbf{L}^{-1}\mathbf{v}] \bmod \mathbb{Z}^n) = \mathbf{G}_m \bigoplus_{\ell=1}^L q_\ell \mathbf{w}_\ell \quad (67)$$

Finally, note that $(\mathbf{G}_m^T \mathbf{G}_m)^{-1} \mathbf{G}_m^T$ is the left-inverse of \mathbf{G}_m and we get that $\phi_m^{-1}(\mathbf{v}) = \mathbf{u}$. \blacksquare

The proofs for our lattice coding scheme will make use of random dither vectors which are available at the transmitters and relays as common randomness. Since our scheme works with respect to expectation over these vectors, then it is clear that (at least) one set of good fixed dither vectors exists. See Appendix III for more details. The following lemma from [13] captures a key property of dithered nested lattice codes.

Lemma 7 (Erez-Zamir): Let \mathbf{t} be a random vector taking values on \mathbb{R}^n . If \mathbf{d} is independent of \mathbf{t} and uniformly distributed over \mathcal{V} , then $[\mathbf{t} - \mathbf{d}] \bmod \Lambda$ is also independent of \mathbf{t} and uniformly distributed over \mathcal{V} .

Theorem 3: For any $\epsilon > 0$ and n large enough, there exist nested lattice codes $\Lambda \subseteq \Lambda_1 \subseteq \dots \subseteq \Lambda_L$ with rates R_1, \dots, R_L , such that for all channel vectors $\mathbf{h}_1, \dots, \mathbf{h}_M \in \mathbb{C}^L$ and coefficient vectors $\mathbf{a}_1, \dots, \mathbf{a}_M \in \{\mathbb{Z} + j\mathbb{Z}\}^L$, each relay can decode lattice equations $\mathbf{v}_m^R, \mathbf{v}_m^I$ where

$$\mathbf{v}_m^R = \left[\sum_{\ell=1}^L \operatorname{Re}(a_{m\ell}) \mathbf{t}_\ell^R - \operatorname{Im}(a_{m\ell}) \mathbf{t}_\ell^I \right] \bmod \Lambda \quad (68)$$

$$\mathbf{v}_m^I = \left[\sum_{\ell=1}^L \operatorname{Im}(a_{m\ell}) \mathbf{t}_\ell^R + \operatorname{Re}(a_{m\ell}) \mathbf{t}_\ell^I \right] \bmod \Lambda \quad (69)$$

of transmitted lattice points $\mathbf{t}_\ell^R, \mathbf{t}_\ell^I \in \Lambda_\ell \cap \mathcal{V}$ with total probability of error ϵ so long as:

$$R_\ell < \min_{m: a_{m\ell} \neq 0} (R(\mathbf{h}_m, \mathbf{a}_m))^+ \quad (70)$$

$$R(\mathbf{h}_m, \mathbf{a}_m) = \frac{1}{2} \log \left(\frac{\text{SNR}}{|\alpha_m|^2 + \text{SNR} \|\alpha_m \mathbf{h}_m - \mathbf{a}_m\|^2} \right) \quad (71)$$

for some choice of $\alpha_m \in \mathbb{C}$.

Proof: Each encoder is given two dither vectors, \mathbf{d}_ℓ^R and \mathbf{d}_ℓ^I , which are independently drawn according to a uniform distribution over \mathcal{V} . All dither vectors are made available to each relay. Encoder ℓ generates a channel input:

$$\mathbf{x}_\ell = [\mathbf{t}_\ell^R - \mathbf{d}_\ell^R] \bmod \Lambda + j[\mathbf{t}_\ell^I - \mathbf{d}_\ell^I] \bmod \Lambda. \quad (72)$$

By Lemma 7, the real and imaginary parts of \mathbf{x}_ℓ are independent and uniform over \mathcal{V} so $\frac{1}{n} E[\|\mathbf{x}_\ell\|^2] = \text{SNR}$, with expectation taken over the dithers. In Appendix III, we argue that there exist fixed dithers that meet the power constraint set forth in (3).

The channel output at relay m is:

$$\mathbf{y}_m = \sum_{\ell=1}^L h_{m\ell} \mathbf{x}_\ell + \mathbf{z}_m. \quad (73)$$

Let $\ell(m) = \max \{\ell : a_{m\ell} \neq 0\}$ and let Q_m denote the lattice quantizer for $\Lambda_{\ell(m)}$. Note that $\ell(m)$ is the highest rate message in the equation and thus the rate of the equation that relay m wants to decode. Each relay computes:

$$\mathbf{s}_m^R = \text{Re}(\alpha_m \mathbf{y}_m) + \sum_{\ell=1}^L \text{Re}(a_{m\ell}) \mathbf{d}_\ell^R - \text{Im}(a_{m\ell}) \mathbf{d}_\ell^I \quad (74)$$

$$\mathbf{s}_m^I = \text{Im}(\alpha_m \mathbf{y}_m) + \sum_{\ell=1}^L \text{Im}(a_{m\ell}) \mathbf{d}_\ell^R + \text{Re}(a_{m\ell}) \mathbf{d}_\ell^I. \quad (75)$$

To get the estimates of the lattice equations, these vectors are quantized onto $\Lambda_{\ell(m)}$ modulo the coarse lattice Λ :

$$\hat{\mathbf{v}}_m^R = [Q_m(\mathbf{s}_m^R)] \bmod \Lambda \quad (76)$$

$$\hat{\mathbf{v}}_m^I = [Q_m(\mathbf{s}_m^I)] \bmod \Lambda. \quad (77)$$

Note that by (20) we have:

$$[Q_m(\mathbf{s}_m^R)] \bmod \Lambda = [Q_m([\mathbf{s}_m^R] \bmod \Lambda)] \bmod \Lambda. \quad (78)$$

We now show that $[\mathbf{s}_m^R] \bmod \Lambda$ is equivalent to \mathbf{v}_m^R plus some noise terms.

$$\begin{aligned} & [\mathbf{s}_m^R] \bmod \Lambda \\ &= \left[\sum_{\ell=1}^L \text{Re}(\alpha_m h_{m\ell}) \text{Re}(\mathbf{x}_\ell) - \text{Im}(\alpha_m h_{m\ell}) \text{Im}(\mathbf{x}_\ell) + \text{Re}(a_{m\ell}) \mathbf{d}_\ell^R - \text{Im}(a_{m\ell}) \mathbf{d}_\ell^I + \text{Re}(\alpha_m \mathbf{z}_m) \right] \bmod \Lambda \\ &= \left[\sum_{\ell=1}^L \text{Re}(a_{m\ell}) (\text{Re}(\mathbf{x}_\ell) + \mathbf{d}_\ell^R) - \text{Im}(a_{m\ell}) (\text{Im}(\mathbf{x}_\ell) + \mathbf{d}_\ell^I) + \text{Re}((\alpha_m h_{m\ell} - a_{m\ell}) \mathbf{x}_\ell + \alpha_m \mathbf{z}_m) \right] \bmod \Lambda \\ &= \left[\sum_{\ell=1}^L \text{Re}(a_{m\ell}) ([\mathbf{t}_\ell^R - \mathbf{d}_\ell^R] \bmod \Lambda + \mathbf{d}_\ell^R) - \text{Im}(a_{m\ell}) ([\mathbf{t}_\ell^I - \mathbf{d}_\ell^I] \bmod \Lambda + \mathbf{d}_\ell^I) \dots \right. \\ & \quad \left. \dots + \text{Re}((\alpha_m h_{m\ell} - a_{m\ell}) \mathbf{x}_\ell + \alpha_m \mathbf{z}_m) \right] \bmod \Lambda \end{aligned} \quad (79)$$

$$= \left[\sum_{\ell=1}^L \text{Re}(a_{m\ell}) \mathbf{t}_\ell^R - \text{Im}(a_{m\ell}) \mathbf{t}_\ell^I + \text{Re}((\alpha_m h_{m\ell} - a_{m\ell}) \mathbf{x}_\ell + \alpha_m \mathbf{z}_m) \right] \bmod \Lambda \quad (80)$$

$$= \left[\mathbf{v}_m^R + \text{Re}(\alpha_m \mathbf{z}_m) + \sum_{\ell=1}^L \text{Re}(\alpha_m h_{m\ell} - a_{m\ell}) [\mathbf{t}_\ell^R - \mathbf{d}_\ell^R] \bmod \Lambda - \text{Im}(\alpha_m h_{m\ell} - a_{m\ell}) [\mathbf{t}_\ell^I - \mathbf{d}_\ell^I] \bmod \Lambda \right] \bmod \Lambda$$

Using similar manipulations, it can be shown that $[\mathbf{s}_m^I] \bmod \Lambda$ is equivalent to \mathbf{v}_m^I plus some noise terms as well. Using Lemma 7, we can show that the channels from \mathbf{v}_m^R to $\hat{\mathbf{v}}_m^R$ and from \mathbf{v}_m^I to $\hat{\mathbf{v}}_m^I$ are equivalent in distribution to:

$$\hat{\mathbf{v}}_m^R = [Q_m(\mathbf{v}_m^R + \mathbf{z}_{eq,m}^R)] \bmod \Lambda \quad (81)$$

$$\hat{\mathbf{v}}_m^I = [Q_m(\mathbf{v}_m^I + \mathbf{z}_{eq,m}^I)] \bmod \Lambda \quad (82)$$

$$\mathbf{z}_{eq,m}^R = \text{Re}(\alpha_m \mathbf{z}_m) + \sum_{\ell=1}^L \theta_{m\ell}^R \tilde{\mathbf{d}}_\ell^R - \theta_{m\ell}^I \tilde{\mathbf{d}}_\ell^I \quad (83)$$

$$\mathbf{z}_{eq,m}^I = \text{Im}(\alpha_m \mathbf{z}_m) + \sum_{\ell=1}^L \theta_{m\ell}^I \tilde{\mathbf{d}}_\ell^R + \theta_{m\ell}^R \tilde{\mathbf{d}}_\ell^I \quad (84)$$

$$\theta_{m\ell}^R = \text{Re}(\alpha_m h_{m\ell} - a_{m\ell}) \quad (85)$$

$$\theta_{m\ell}^I = \text{Im}(\alpha_m h_{m\ell} - a_{m\ell}) \quad (86)$$

where each $\tilde{\mathbf{d}}_\ell^R$ and $\tilde{\mathbf{d}}_\ell^I$ is drawn independently and uniformly from \mathcal{V} .

Using Lemma 8 from Appendix I, we have that the densities of both $\mathbf{z}_{eq,m}^R$ and $\mathbf{z}_{eq,m}^I$ are upper bounded (times a constant) by the density of an i.i.d. zero-mean Gaussian vector \mathbf{z}_m^* whose variance σ_m^2 approaches

$$N_{eq,m} = \frac{|\alpha_m|^2}{2} + \frac{\text{SNR}}{2} \left((\theta_{m\ell}^R)^2 + (\theta_{m\ell}^I)^2 \right) \quad (87)$$

$$= \frac{|\alpha_m|^2}{2} + \frac{\text{SNR}}{2} \|\alpha_m \mathbf{h}_m - \mathbf{a}_m\|^2 \quad (88)$$

as $n \rightarrow \infty$. We also show in Appendix II that $\Lambda_1, \Lambda_2, \dots, \Lambda_L$ are AWGN good. From Definition 24, this means that $\Pr(\mathbf{z}_m^* \notin \mathcal{V}_{\ell(m)})$ goes to zero exponentially in n so long as the volume-to-noise ratio satisfies $\mu(\Lambda_{\ell(m)}, P_e) > 2\pi e$. Since decoding is only in error if $\mathbf{z}_{eq,m}^R$ or $\mathbf{z}_{eq,m}^I$ leave the fundamental Voronoi region of the fine lattice $\Lambda_{\ell(m)}$ and the probability of this event is upper bounded by $\Pr(\mathbf{z}_m^* \notin \mathcal{V}_{\ell(m)})$ times a constant, we get that $\Pr(\hat{\mathbf{v}}_m^R \neq \mathbf{v}_m^R)$ and $\Pr(\hat{\mathbf{v}}_m^I \neq \mathbf{v}_m^I)$ go to zero exponentially in n .

To ensure that the probability of error goes to zero for all desired equations, we get that the volume of $\mathcal{V}_{\ell(m)}$ must satisfy

$$2\pi e < \mu(\Lambda_{\ell(m)}, P_e) = \frac{(\text{Vol}(\mathcal{V}_{\ell(m)}))^{2/n}}{\sigma_m^2} \quad (89)$$

for all relays with $a_{m\ell} \neq 0$. If we set the volume of \mathcal{V}_ℓ as follows, the constraints are always met:

$$\text{Vol}(\mathcal{V}_\ell) > \left(2\pi e \max_{m:a_{m\ell} \neq 0} \sigma_m^2 \right)^{n/2} \quad (90)$$

Recall that the rate of a nested lattice code is

$$R_\ell = \frac{1}{n} \log \left(\frac{\text{Vol}(\mathcal{V})}{\text{Vol}(\mathcal{V}_\ell)} \right). \quad (91)$$

Using (28), we can solve for the volume of the fundamental Voronoi region of the coarse lattice:

$$\text{Vol}(\mathcal{V}) = \left(\frac{\text{SNR}/2}{G(\Lambda)} \right)^{n/2} \quad (92)$$

It follows that we can achieve any rate less than:

$$R_\ell < \min_{m:a_{m\ell} \neq 0} \frac{1}{2} \log \left(\frac{\text{SNR}/2}{G(\Lambda)2\pi e\sigma_m^2} \right) \quad (93)$$

while driving the probability of error to zero. Choose $\delta > 0$. Since Λ is good for quantization, for n large enough, we have that $G(\Lambda)2\pi e < (1 + \delta)$. We also know that σ_m^2 converges to $N_{eq,m}$ so for n large enough we have $\sigma_m^2 < (1 + \delta)N_{eq,m}$. Finally, we get that the rate of each nested lattice code is at least:

$$\min_{m:a_{m\ell} \neq 0} \frac{1}{2} \log \left(\frac{\text{SNR}}{|\alpha_m|^2 + \text{SNR}\|\alpha_m \mathbf{h}_m - \mathbf{a}_m\|^2} \right) - \log(1 + \delta)$$

Thus, by choosing δ small enough, we can approach the computation rates as closely as desired. ■

Remark 11: Note that (71) differs from (39) by a factor of $\frac{1}{2}$. This is because (71) is the rate of the nested lattice code which operates in the reals and (39) is the rate of the message which has a real and an imaginary component.

Remark 12: The proof of Theorem 3 only requires channel state information at the relays. In particular, each relay only needs to know the channel coefficients from the transmitters to itself \mathbf{h}_m .

We now put all of these ingredients together to prove Theorem 1. See Figures 4 and 5 for block diagrams of the encoding and decoding process.

Encoder ℓ maps its finite field message vectors, $\mathbf{w}_\ell^R, \mathbf{w}_\ell^I \in \mathbb{F}_p^{k_\ell}$, to lattice points, $\mathbf{t}_\ell^R, \mathbf{t}_\ell^I \in \Lambda_\ell \cap \mathcal{V}$, using $\phi_\ell(\cdot)$ from Lemma 5.

$$\mathbf{t}_\ell^R = \phi_\ell(\mathbf{w}_\ell^R) \quad (94)$$

$$\mathbf{t}_\ell^I = \phi_\ell(\mathbf{w}_\ell^I) \quad (95)$$

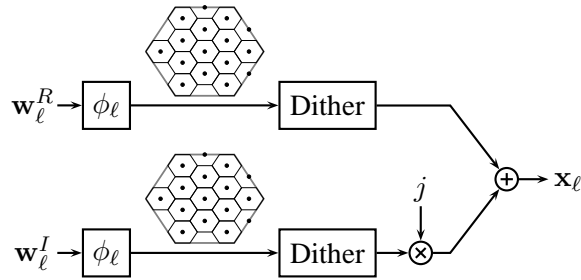


Fig. 4. Block diagram of the compute-and-forward encoder for transmitter ℓ , \mathcal{E}_ℓ . Messages from a finite field are mapped onto a nested lattice code, dithered, and transmitted across the channel.

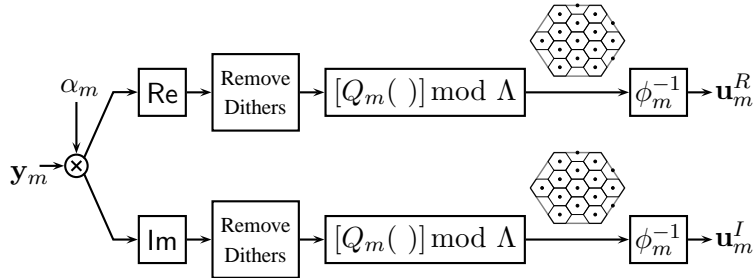


Fig. 5. Block diagram of the compute-and-forward decoder for relay m , \mathcal{D}_m . The channel observation is scaled and decomposed into its real and imaginary components. The decoder then removes the dithers, quantizes onto the appropriate fine lattice, and takes the modulus over the coarse lattice. This results in an equation of lattice codewords which is then mapped into an equation of messages over the finite field.

Using Theorem 3, these lattice points are transmitted across the channel. Since the message rates are twice the lattice code rates, $r_\ell = 2R_\ell$, the relays can recover lattice equations

$$\mathbf{v}_m^R = \left[\sum_{\ell=1}^L \text{Re}(a_{m\ell}) \mathbf{t}_\ell^R - \text{Im}(a_{m\ell}) \mathbf{t}_\ell^I \right] \bmod \Lambda \quad (96)$$

$$\mathbf{v}_m^I = \left[\sum_{\ell=1}^L \text{Im}(a_{m\ell}) \mathbf{t}_\ell^R + \text{Re}(a_{m\ell}) \mathbf{t}_\ell^I \right] \bmod \Lambda \quad (97)$$

for any $\mathbf{a}_1, \dots, \mathbf{a}_M \in \{\mathbb{Z} + j\mathbb{Z}\}^L$ so long as

$$r_\ell < \min_{m: a_{m\ell} \neq 0} \left(\log \left(\frac{\text{SNR}}{|\alpha_m|^2 + \text{SNR} \|\alpha_m \mathbf{h}_m - \mathbf{a}_m\|^2} \right) \right)^+$$

for some $\alpha_1, \dots, \alpha_M \in \mathbb{C}$. Finally, using ϕ_m^{-1} from Lemma 6, each relay can decode the desired linear combination of messages:

$$\mathbf{u}_m^R = \phi_m^{-1}(\mathbf{v}_m^R) = \bigoplus_{\ell=1}^L q_{m\ell}^R \mathbf{w}_\ell^R \oplus (-q_{m\ell}^I) \mathbf{w}_\ell^I \quad (98)$$

$$\mathbf{u}_m^I = \phi_m^{-1}(\mathbf{v}_m^I) = \bigoplus_{\ell=1}^L q_{m\ell}^I \mathbf{w}_\ell^R \oplus q_{m\ell}^R \mathbf{w}_\ell^I \quad (99)$$

B. Real-Valued Channels

We now restate our main results for real-valued channels.

Corollary 1: For real-valued channel vectors $\mathbf{h}_1, \dots, \mathbf{h}_M \in \mathbb{R}^L$, the relays can recover any set of linear equations with coefficient vectors $\mathbf{a}_1, \dots, \mathbf{a}_M \in \mathbb{Z}^L$ so long as the message rates are less than the computation rate

$$r_\ell < \min_{m: a_{m\ell} \neq 0} (R(\mathbf{h}_m, \mathbf{a}_m))^+ \quad (100)$$

$$R(\mathbf{h}_m, \mathbf{a}_m) = \frac{1}{2} \log \left(\frac{\text{SNR}}{\alpha_m^2 + \text{SNR} \|\alpha_m \mathbf{h}_m - \mathbf{a}_m\|^2} \right) \quad (101)$$

for some choice of $\alpha_m \in \mathbb{R}$.

The coding scheme is quite similar to that for complex-valued channels: encoder ℓ maps its finite field message vector, $\mathbf{w}_\ell \in \mathbb{F}_p^{k_\ell}$, to a lattice point, $\mathbf{t}_\ell \in \Lambda_\ell \cap \mathcal{V}$, using $\phi_\ell(\cdot)$ from Lemma 5.

$$\mathbf{t}_\ell = \phi_\ell(\mathbf{w}_\ell) \quad (102)$$

It follows from the proof of Theorem 3 that if these lattice points are transmitted across the channel then each relay can recover a lattice equation

$$\mathbf{v}_m = \left[\sum_{\ell=1}^L a_{m\ell} \mathbf{t}_\ell \right] \bmod \Lambda \quad (103)$$

for any $\mathbf{a}_1, \dots, \mathbf{a}_M \in \mathbb{Z}^L$ so long as the message rates satisfy

$$r_\ell < \min_{m: a_{m\ell} \neq 0} \frac{1}{2} \left(\log \left(\frac{\text{SNR}}{\alpha_m^2 + \text{SNR} \|\alpha_m \mathbf{h}_m - \mathbf{a}_m\|^2} \right) \right)^+$$

for some $\alpha_1, \dots, \alpha_M \in \mathbb{R}$. Finally, using ϕ_m^{-1} from Lemma 6, each relay can decode the desired linear combination of messages:

$$\mathbf{u}_m = \phi_m^{-1}(\mathbf{v}_m) = \bigoplus_{\ell=1}^L q_{m\ell} \mathbf{w}_\ell \quad (104)$$

Corollary 2: For a given $\mathbf{h}_m \in \mathbb{R}^L$, $\mathbf{a}_m \in \mathbb{Z}^L$, the computation rate given in Corollary 1 is uniquely maximized by choosing α_m to be the MMSE coefficient

$$\alpha_{\text{MMSE}} = \frac{\text{SNR} \mathbf{h}_m^T \mathbf{a}_m}{1 + \text{SNR} \|\mathbf{h}_m\|^2} \quad (105)$$

which results in a computation rate

$$R(\mathbf{h}_m, \mathbf{a}_m) = \frac{1}{2} \log \left(\left(\|\mathbf{a}_m\|^2 - \frac{\text{SNR} (\mathbf{h}_m^T \mathbf{a}_m)^2}{1 + \text{SNR} \|\mathbf{h}_m\|^2} \right)^{-1} \right)$$

C. Multi-Stage Networks

The framework developed in this section can easily be applied to AWGN networks with more than one layer of relays. Once the first layer has recovered its equations, it can just treat them as a set of messages for the second layer. The second layer simply decodes equations with coefficients that are close to the channel coefficients. This process repeats until the equations reach a destination. Since these layered equations are all linear, they can be expressed as linear equations over the original messages.

V. RECOVERING MESSAGES

The primary goal of compute-and-forward is to enable higher achievable rates across an AWGN network. Relays decode linear equations of transmitted messages and pass them towards the destination nodes which, upon receiving enough equations, attempt to solve for their desired messages. In this section, we give sufficient conditions for recovering messages from a given set of equations.

It will be useful to represent the equations in matrix form. Let $\mathbf{Q}^R = \{q_{m\ell}^R\}$ and $\mathbf{Q}^I = \{q_{m\ell}^I\}$. Now let \mathbf{Q} denote the coefficient matrix where

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}^R & -\mathbf{Q}^I \\ \mathbf{Q}^I & \mathbf{Q}^R \end{bmatrix}. \quad (106)$$

It is clear that we can write the linear combinations of messages \mathbf{u}_m^R and \mathbf{u}_m^I as

$$\begin{bmatrix} \mathbf{u}_1^R \\ \vdots \\ \mathbf{u}_M^R \\ \mathbf{u}_1^I \\ \vdots \\ \mathbf{u}_M^I \end{bmatrix} = \mathbf{Q} \begin{bmatrix} \mathbf{w}_1^R \\ \vdots \\ \mathbf{w}_M^R \\ \mathbf{w}_1^I \\ \vdots \\ \mathbf{w}_M^I \end{bmatrix}. \quad (107)$$

Using this representation, we can easily obtain the following two theorems for recovering messages.

Theorem 4: Given L linear combinations of messages with coefficient matrix $\mathbf{Q} \in \mathbb{F}_p^{2L \times 2L}$, a destination can recover all messages if \mathbf{Q} is full rank over \mathbb{F}_p .

The proof follows by noting that if \mathbf{Q} is full rank, the destination can simply apply the inverse matrix to the vector of equations in (107) to recover the original messages.

Recall that δ_ℓ is the unit vector with 1 in the ℓ^{th} entry and 0 elsewhere.

Theorem 5: Given K linear combinations of messages with coefficient matrix $\mathbf{Q} \in \mathbb{F}_p^{2K \times 2L}$, a destination can recover the message from encoder ℓ if there exists a $2 \times 2K$ matrix Φ such that

$$\Phi \mathbf{Q} = \begin{bmatrix} \delta_\ell^T \\ \delta_{\ell+M}^T \end{bmatrix} \quad (108)$$

Again, it is clear from (107) that applying Φ to the vector of equations allows the destination to recover \mathbf{w}_ℓ^R and \mathbf{w}_ℓ^I since they occupy positions ℓ and $\ell + M$ in the vector of messages.

Remark 13: These conditions can also be easily stated in terms of the coefficient vectors $\mathbf{a}_1, \dots, \mathbf{a}_M \in \{\mathbb{Z} + j\mathbb{Z}\}^L$. First set $\mathbf{A} = [\mathbf{a}_1 \ \dots \ \mathbf{a}_M]^T$ and let $\tilde{\mathbf{A}}$ denote its real-valued decomposition:

$$\tilde{\mathbf{A}} = \begin{bmatrix} \text{Re}(\mathbf{A}) & -\text{Im}(\mathbf{A}) \\ \text{Im}(\mathbf{A}) & \text{Re}(\mathbf{A}) \end{bmatrix}. \quad (109)$$

Now one can replace \mathbf{Q} with $\tilde{\mathbf{A}}$ in Theorems 4 and 5 so long as all operations are taken modulo p .

It may be more convenient to evaluate the rank of the coefficients directly on the complex field. This is possible, given some assumptions on the equation coefficients.

Theorem 6: The destination is given L linear equations with coefficient vectors $\mathbf{a}_1, \dots, \mathbf{a}_L \in \mathbb{F}_p^{2L \times 2L}$. Assume that magnitude of each equation coefficient is upper bounded by a constant, $|a_{m\ell}| < a_{\text{MAX}}$. Then, there exists an n_0 such that for all blocklengths $n \geq n_0$, the destination can recover all messages if $\mathbf{A} = [\mathbf{a}_1 \ \dots \ \mathbf{a}_L]^T$ is full rank over the complex field.

Proof: \mathbf{A} is full rank over the complex field if and only if its real-valued decomposition $\tilde{\mathbf{A}}$ from (109) is full rank over the reals. Recall that a matrix is full rank only if its determinant is non-zero. We will now show that for sufficiently large p , if the determinant of $\tilde{\mathbf{A}}$ is non-zero over the reals it is non-zero modulo p . The determinant over \mathbb{R} can be written as:

$$\det(\tilde{\mathbf{A}}) = \sum_{\sigma \in \mathcal{S}} \text{sgn}(\sigma) \prod_{m=1}^{2L} \tilde{a}_{m\sigma(m)} \quad (110)$$

where \mathcal{S} is the set of all permutations of $\{1, 2, \dots, 2L\}$, $\text{sgn}(\sigma)$ is the signature of the permutation which is equal to 1 if it is an even permutation and -1 if it is an odd permutation, and $\tilde{a}_{m\ell}$ are the entries of $\tilde{\mathbf{A}}$. Using the bound on $a_{m\ell}$ and the fact that $|\mathcal{S}| = (2L)!$, the determinant is lower and upper bounded as follows:

$$-(2L)!(a_{\text{MAX}})^{2L} \leq \det(\tilde{\mathbf{A}}) \leq (2L)!(a_{\text{MAX}})^{2L} \quad (111)$$

The determinant under modulo p arithmetic can be written as:

$$\left[\sum_{\sigma \in \mathcal{S}} \text{sgn}(\sigma) \prod_{m=1}^{2L} \tilde{a}_{m\sigma(m)} \right] \bmod p \quad (112)$$

Since the underlying field size $p \rightarrow \infty$ as $n \rightarrow \infty$, for large enough blocklength n , we can use the bounds on $\det(\tilde{\mathbf{A}})$ to show that the determinant modulo p does not wrap around zero. This immediately implies that it is zero if and only the determinant is zero over the reals. ■

Remark 14: Theorem 6 can also be stated in terms of bounds on the channel coefficients. For instance, if $|h_{m\ell}| < h_{\text{MAX}}$, then we can use the bound in Lemma 4, to show that $|a_{m\ell}|$ is bounded as well. More generally, the result holds if the channel coefficients are drawn from a distribution such that $Pr(\cup_{m\ell}\{|h_{m\ell}| > h_{\text{MAX}}\}) \rightarrow 0$ as $h_{\text{MAX}} \rightarrow \infty$. In this case, we choose h_{MAX} such that this probability is very small and can be absorbed into the total probability of error for our scheme. The result follows by taking an appropriate increasing sequence of h_{MAX} .

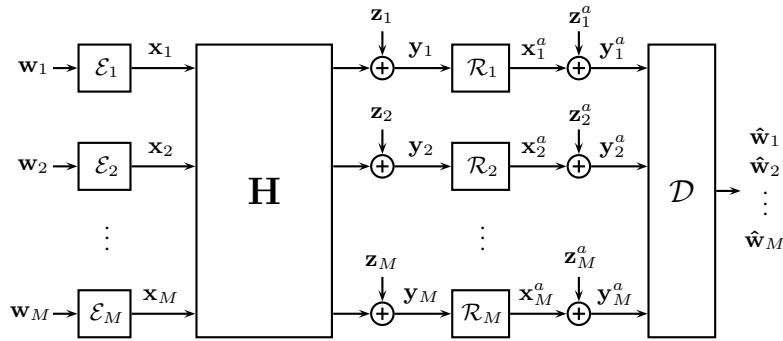


Fig. 6. A linear relay network where compute-and-forward is beneficial.

Example 4: Consider the AWGN network in Figure 6. Encoder $\mathcal{E}_1, \dots, \mathcal{E}_M$ send messages through a channel \mathbf{H} to relays $\mathcal{R}_1, \dots, \mathcal{R}_M$. Each relay has a point-to-point AWGN channel to the final decoder \mathcal{D} which wants to recover all of the messages at the highest possible symmetric rate. Each channel input has power at most SNR and all noise terms are i.i.d. circularly symmetric Gaussian with variance 1. Let \mathbf{H} be an $M \times M$ Hadamard matrix. (We assume that M is chosen such that a Hadamard matrix of that size exists.) Recall that a Hadamard matrix has ± 1 entries such that $\mathbf{H}\mathbf{H}^T = M\mathbf{I}$.

Using Theorems 2 and 6 and setting the coefficient vectors equal to the channel vectors, $\mathbf{a}_m = \mathbf{h}_m$, compute-and-forward can achieve

$$R_{\text{COMP}} = \log \left(\frac{1}{M} + \text{SNR} \right) \quad (113)$$

bits per channel use per user since \mathbf{H} is full rank. It can be shown that decode-and-forward, amplify-and-forward, and compress-and-forward (with i.i.d. Gaussian codebooks) can achieve

$$R_{\text{DF}} = \frac{1}{M} \log(1 + M\text{SNR}) \quad (114)$$

$$R_{\text{AF}} = R_{\text{CF}} = \log \left(1 + \text{SNR} \left(\frac{\text{SNR}}{M\text{SNR} + 1} \right) \right) \quad (115)$$

bits per channel use per user. Compute-and-forward is the dominant strategy except at very low SNR and it approaches the upper bound $R_{\text{UPPER}} = \log(1 + \text{SNR})$ as $\text{SNR} \rightarrow \infty$. As M increases the rates of decode-and-forward, amplify-and-forward, and compress-and-forward go to 0.

Example 5: There are 3 destinations which each want to decode the message associated with their index. Assume that the underlying field size is $p = 3$. The first destination receives equations with coefficient vectors $\mathbf{a}_1 = [4 \ 2 \ 2]^T$ and $\mathbf{a}_2 = [5j \ -1 \ -1]^T$, the second receives $\mathbf{b}_1 = [1 \ 2 \ 1]^T$ and $\mathbf{b}_2 = [-1 \ 0 \ -1]^T$, and the third receives

$\mathbf{c} = [0 \ 0 \ 1 + j]^T$. The finite field matrix representation of these equations is:

$$\mathbf{Q}_a^R = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} \quad \mathbf{Q}_a^I = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \quad (116)$$

$$\mathbf{Q}_b^R = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix} \quad \mathbf{Q}_b^I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (117)$$

$$\mathbf{Q}_c^R = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad \mathbf{Q}_c^I = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad (118)$$

Using Theorem 5 with the following matrices, each destination can recover its desired message:

$$\Phi_a = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 \end{bmatrix} \quad \Phi_b = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} \quad (119)$$

$$\Phi_c = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} \quad (120)$$

VI. SUCCESSIVE CANCELLATION

Once a relay has recovered an equation of messages, it can subtract it from its channel observation. This results in a residual channel output from which it can extract a different equation, potentially with a higher rate than possible over the original channel. One key difference from standard applications of successive cancellation is that the relay cannot completely cancel out all channel inputs associated with the decoded equation. This is because in the first step, it only decodes an integer combination of the messages, which is often not the same as the linear combination taken by the channel.

We demonstrate an achievable region for decoding two different equations using successive cancellation at each relay. This can be easily generalized to more than two equations.

Theorem 7: For channel vectors $\mathbf{h}_1, \dots, \mathbf{h}_M \in \mathbb{C}^L$, the relays can first decode any set of linear equations with coefficient vectors $\mathbf{a}_1, \dots, \mathbf{a}_M \in \{\mathbb{Z} + j\mathbb{Z}\}^L$ and then any set with coefficient vectors $\mathbf{b}_1, \dots, \mathbf{b}_M \in \{\mathbb{Z} + j\mathbb{Z}\}^L$ so long as the message rates are less than the computation rates:

$$r_\ell < \min \left(\min_{m: a_{m\ell} \neq 0} R_1(\mathbf{h}_m, \mathbf{a}_m), \min_{m: b_{m\ell} \neq 0} R_2(\mathbf{h}_m, \mathbf{a}_m, \mathbf{b}_m) \right)$$

$$R_1(\mathbf{h}_m, \mathbf{a}_m) = \log \left(\frac{\text{SNR}}{|\alpha_m|^2 + \text{SNR} \|\alpha_m \mathbf{h}_m - \mathbf{a}_m\|^2} \right)$$

$$R_2(\mathbf{h}_m, \mathbf{a}_m, \mathbf{b}_m) = \begin{cases} R_{B1}, & \mathbf{a}_m = \delta_i \text{ for some } i, \\ R_{B2}, & \text{otherwise.} \end{cases}$$

$$R_{B1} = \log \left(\frac{\text{SNR}}{|\beta_m|^2 + \text{SNR} \sum_{\ell \neq i} |\beta_m h_{m\ell} - b_{m\ell}|^2} \right)$$

$$R_{B2} = \log \left(\frac{\text{SNR}}{|\beta_m|^2 + \text{SNR} \|\beta_m \mathbf{h}_m - \tau_m \mathbf{a}_m - \mathbf{b}_m\|^2} \right)$$

for some choice of $\alpha_m, \beta_m \in \mathbb{C}$ and $\tau_m \in \mathbb{Z} + j\mathbb{Z}$.

Proof: All messages are mapped on to lattice points, dithered, and transmitted across the channel as in the proof of Theorem 1. The first set of equations can be reliably decoded using the procedure from Theorem 1 as well. Now, we condition on the event that each relay has successfully recovered the equation with coefficient vectors \mathbf{a}_m .

Consider the case where the first coefficient vector at relay m is a unit vector $\mathbf{a}_m = \delta_i$. This means that relay m can successfully decode the message $(\mathbf{w}_i^R, \mathbf{w}_i^I)$ from encoder i . It can then replicate the encoding process to get \mathbf{x}_i . Now, the relay computes

$$\mathbf{y}_m - h_{mi} \mathbf{x}_i = \sum_{\ell \neq i} h_{m\ell} \mathbf{x}_\ell + \mathbf{z}_m \quad (121)$$

and uses this as a channel output for Theorem 1 to get the equation with coefficient vector $\tilde{\mathbf{b}}_m$ which is equal to \mathbf{b}_m except that it has 0 in the i^{th} position. It then adds \mathbf{w}_i^R and \mathbf{w}_i^I to the recovered equation to get \mathbf{b}_m .

If \mathbf{a}_m is not a unit vector, the decoder has access to the lattice equations:

$$\mathbf{v}_m^R = \left[\sum_{\ell=1}^L \text{Re}(a_{m\ell}) \mathbf{t}_\ell^R - \text{Im}(a_{m\ell}) \mathbf{t}_\ell^I \right] \text{ mod } \Lambda \quad (122)$$

$$\mathbf{v}_m^I = \left[\sum_{\ell=1}^L \text{Im}(a_{m\ell}) \mathbf{t}_\ell^R + \text{Re}(a_{m\ell}) \mathbf{t}_\ell^I \right] \text{ mod } \Lambda \quad (123)$$

From which it computes

$$\begin{aligned} \tilde{\mathbf{v}}_m^R &= \left[\mathbf{v}_m^R - \left(\sum_{\ell=1}^L \text{Re}(a_{m\ell}) \mathbf{d}_\ell^R - \text{Im}(a_{m\ell}) \mathbf{d}_\ell^I \right) \right] \text{ mod } \Lambda \\ &= \left[\sum_{\ell=1}^L \text{Re}(a_{m\ell}) \mathbf{x}_\ell \right] \text{ mod } \Lambda \end{aligned} \quad (124)$$

$$\begin{aligned} \tilde{\mathbf{v}}_m^I &= \left[\mathbf{v}_m^I - \left(\sum_{\ell=1}^L \text{Im}(a_{m\ell}) \mathbf{d}_\ell^R + \text{Re}(a_{m\ell}) \mathbf{d}_\ell^I \right) \right] \text{ mod } \Lambda \\ &= \left[\sum_{\ell=1}^L \text{Im}(a_{m\ell}) \mathbf{x}_\ell \right] \text{ mod } \Lambda \end{aligned} \quad (125)$$

$$\begin{aligned} \tilde{\mathbf{y}}_m^R &= [\text{Re}(\beta_m \mathbf{y}_m) - \text{Re}(\tau_m) \tilde{\mathbf{v}}_m^R + \text{Im}(\tau_m) \tilde{\mathbf{v}}_m^I] \text{ mod } \Lambda \\ &= \left[\text{Re} \left(\sum_{\ell=1}^L (\beta_m h_{m\ell} - \tau_m a_{m\ell}) \mathbf{x}_\ell + \mathbf{z}_m \right) \right] \text{ mod } \Lambda \\ \tilde{\mathbf{y}}_m^I &= [\text{Im}(\beta_m \mathbf{y}_m) - \text{Im}(\tau_m) \tilde{\mathbf{v}}_m^R - \text{Re}(\tau_m) \tilde{\mathbf{v}}_m^I] \text{ mod } \Lambda \\ &= \left[\text{Im} \left(\sum_{\ell=1}^L (\beta_m h_{m\ell} - \tau_m a_{m\ell}) \mathbf{x}_\ell + \mathbf{z}_m \right) \right] \text{ mod } \Lambda \end{aligned}$$

Now we can follow the steps in the proof of Theorem 3. In (74), replace $\text{Re}(\alpha_m \mathbf{y}_m)$ with $\tilde{\mathbf{y}}_m^R$ and in (75), replace $\text{Im}(\alpha_m \mathbf{y}_m)$ with $\tilde{\mathbf{y}}_m^I$. In all steps of the proof, we substitute $a_{m\ell}$ with $b_{m\ell}$, $\alpha_m h_{m\ell}$ with $\beta_m h_{m\ell} - \tau_m a_{m\ell}$, and, if has not already been replaced, α_m with β_m . ■

Remark 15: Given, \mathbf{a}_m , \mathbf{b}_m , and τ_m , we can solve for the optimal β_m following the steps of the proof of Theorem 2.

Example 6: There are $L = 4$ transmitters and $M = 1$ relay and the channel vector is $\mathbf{h}_1 = [10 \ 10 \ 8j \ 8j]^T$. The relay wants to first decode the equation with coefficient vector $\mathbf{a}_1 = [1 \ 1 \ j \ j]^T$ and then with coefficient vector $\mathbf{b}_1 = [1 \ 1 \ -j \ -j]^T$. Using Theorem 7, this is possible if the message rates satisfy:

$$r_\ell < \min \left(\log \left(\frac{1}{4} + \frac{81\text{SNR}}{1 + 4\text{SNR}} \right), \log \left(\frac{1}{328} + \text{SNR} \right) \right)^+$$

by using $\tau_1 = 9$ so that $\mathbf{h}_1 - \tau_1 \mathbf{a}_1 = \mathbf{b}_1$. Note that if we applied Theorem 1 to decode \mathbf{b}_1 , we would not be able to get a positive rate.

Remark 16: As noted in Remark 9, it may be more efficient to recover an equation piecewise by recovering equations of subsets of messages and taking an appropriate linear combination of these equations. Theorem 7 is strictly better for this process than Theorem 1.

Assume there is only one relay and that it wants to recover all transmitted messages. This is the standard Gaussian multiple-access problem whose capacity region is well-known to be the set of all rate tuples (r_1, \dots, r_L) satisfying:

$$\sum_{\ell \in S} r_\ell < \log \left(1 + \text{SNR} \sum_{\ell \in S} |h_{1\ell}|^2 \right) \quad (126)$$

for all subsets $S \subseteq \{1, 2, \dots, L\}$ [58, Theorem 14.3.5]. We now show that compute-and-forward includes the multiple-access capacity region as a special case. First, we consider the corner point of the capacity region associated with decoding the messages in ascending order. From Example 2, this is possible if:

$$r_1 < \log \left(1 + \frac{|h_{11}|^2 \text{SNR}}{1 + \text{SNR} \sum_{i=2}^L |h_{1i}|^2} \right). \quad (127)$$

Using Theorem 7, the relay removes \mathbf{x}_1 from the channel observation to get $\sum_{\ell=2}^L h_{1\ell} \mathbf{x}_\ell + \mathbf{z}_1$. It then repeats the above procedure for each message in ascending order to get:

$$r_\ell < \log \left(1 + \frac{|h_{1\ell}|^2 \text{SNR}}{1 + \text{SNR} \sum_{i=\ell+1}^L |h_{1i}|^2} \right) \quad (128)$$

This is clearly a corner point of the multiple-access capacity region. By changing the decoding order, any corner point is achievable. Note that any point on the boundary of the capacity region is achievable by time-sharing between two corner points.

VII. SUPERPOSITION

In the previous section, we considered the scenario where each relay decodes several equations, but the transmitters each use a single codebook (as in Theorem 1). However, when decoding multiple equations, it is sometimes useful to superimpose multiple codebooks. We investigate this possibility in this section.

We will assume that there are two levels A and B and that each relay wants to recover an equation from both levels. (If it is not interested in a level, it can just set its desired coefficients to zero.)

Each encoder has two messages $\mathbf{w}_{\ell A} = (\mathbf{w}_{\ell A}^R, \mathbf{w}_{\ell A}^I)$ and $\mathbf{w}_{\ell B} = (\mathbf{w}_{\ell B}^R, \mathbf{w}_{\ell B}^I)$ with rates $r_{\ell A}$ and $r_{\ell B}$ respectively. Each relay wants to decode equations $\mathbf{u}_{m A}^R, \mathbf{u}_{m A}^I$ and $\mathbf{u}_{m B}^R, \mathbf{u}_{m B}^I$ with coefficient vectors \mathbf{a}_m and \mathbf{b}_m respectively. In the theorem below, we give achievable rates for this scenario by combining superposition and successive cancellation. The basic idea is to superimpose two lattice codes at each receiver (in both the real and imaginary dimensions) scaled by $\gamma_{\ell A}$ and $\gamma_{\ell B}$ to ensure that the power constraint is met.

Theorem 8: Choose $\gamma_{\ell A}, \gamma_{\ell B}$ such that $|\gamma_{\ell A}|^2 + |\gamma_{\ell B}|^2 = 1$. For channel vectors $\mathbf{h}_1, \dots, \mathbf{h}_M \in \mathbb{C}^L$, the relays can first decode any set of linear equations over $\mathbf{w}_{\ell A}$ with coefficient vectors $\mathbf{a}_1, \dots, \mathbf{a}_M \in \{\mathbb{Z} + j\mathbb{Z}\}^L$ and then any set of linear equations over $\mathbf{w}_{\ell B}$ with coefficient vectors $\mathbf{b}_1, \dots, \mathbf{b}_M \in \{\mathbb{Z} + j\mathbb{Z}\}^L$ so long as the message rates are less than the computation rates:

$$\begin{aligned} \mathbf{h}_{m A} &= [\gamma_{1A} h_{m1} \ \cdots \ \gamma_{LA} h_{mL}]^T \\ \mathbf{h}_{m B} &= [\gamma_{1B} h_{m1} \ \cdots \ \gamma_{LB} h_{mL}]^T \\ r_{\ell A} &< \min_{m: \mathbf{a}_m \neq \mathbf{0}} (R_1(\mathbf{h}_{m A}, \mathbf{h}_{m B}, \mathbf{a}_m))^+ \\ R_1(\mathbf{h}_{m A}, \mathbf{h}_{m B}, \mathbf{a}_m) &= \log \left(\frac{\text{SNR}}{|\alpha_m|^2 (1 + \text{SNR} \|\mathbf{h}_{m B}\|^2) + \text{SNR} \|\alpha_m \mathbf{h}_{m A} - \mathbf{a}_m\|^2} \right) \\ r_{\ell B} &< \min_{m: \mathbf{b}_m \neq \mathbf{0}} (R_2(\mathbf{h}_{m A}, \mathbf{h}_{m B}, \mathbf{a}_m, \mathbf{b}_m))^+ \\ R_2(\mathbf{h}_{m A}, \mathbf{h}_{m B}, \mathbf{a}_m, \mathbf{b}_m) &= \begin{cases} R_{B1}, & \mathbf{a}_m = \delta_i \text{ for some } i, \\ R_{B2}, & \text{otherwise.} \end{cases} \\ R_{B1} &= \log \left(\frac{\text{SNR}}{|\beta_m|^2 (1 + \text{SNR} \sum_{\ell \neq i} |\gamma_{\ell A} h_{m\ell}|^2) + \text{SNR} \|\beta_m \mathbf{h}_{m B} - \mathbf{b}_m\|^2} \right) \\ R_{B2} &= \log \left(\frac{\text{SNR}}{|\beta_m|^2 + \text{SNR} \|\beta_m \mathbf{h}_{m A} - \tau_m \mathbf{a}_m\|^2 + \text{SNR} \|\beta_m \mathbf{h}_{m B} - \mathbf{b}_m\|^2} \right) \end{aligned}$$

for some choice of $\alpha_m, \beta_m \in \mathbb{C}$ and $\tau_m \in \mathbb{Z} + j\mathbb{Z}$.

Proof: Choose a set of nested lattices $\Lambda, \Lambda_{1A}, \dots, \Lambda_{LA}, \Lambda_{1B}, \dots, \Lambda_{LB}$ with appropriate rates. Each encoder maps its messages onto lattices and dithers them with $\mathbf{d}_{\ell A}^R, \mathbf{d}_{\ell A}^I, \mathbf{d}_{\ell B}^R, \mathbf{d}_{\ell B}^I$ generated independently and uniformly

from \mathcal{V} :

$$\mathbf{t}_{\ell A}^R = \phi_{\ell A}(\mathbf{w}_{\ell A}^R) \quad \mathbf{t}_{\ell A}^I = \phi_{\ell A}(\mathbf{w}_{\ell A}^I) \quad (129)$$

$$\mathbf{t}_{\ell B}^R = \phi_{\ell B}(\mathbf{w}_{\ell B}^R) \quad \mathbf{t}_{\ell B}^I = \phi_{\ell B}(\mathbf{w}_{\ell B}^I) \quad (130)$$

$$\mathbf{x}_{\ell A} = [\mathbf{t}_{\ell A}^R - \mathbf{d}_{\ell A}^R] \bmod \Lambda + j[\mathbf{t}_{\ell A}^I - \mathbf{d}_{\ell A}^I] \bmod \Lambda \quad (131)$$

$$\mathbf{x}_{\ell B} = [\mathbf{t}_{\ell B}^R - \mathbf{d}_{\ell B}^R] \bmod \Lambda + j[\mathbf{t}_{\ell B}^I - \mathbf{d}_{\ell B}^I] \bmod \Lambda \quad (132)$$

It then combines $\mathbf{x}_{\ell A}$ and $\mathbf{x}_{\ell B}$ according to $\gamma_{\ell A}$ and $\gamma_{\ell B}$ which guarantees the power constraint is met:

$$\mathbf{x}_{\ell} = \gamma_{\ell A}\mathbf{x}_{\ell A} + \gamma_{\ell B}\mathbf{x}_{\ell B} \quad (133)$$

$$\frac{1}{n}E[\|\mathbf{x}_{\ell}\|^2] = (|\gamma_{\ell A}|^2 + |\gamma_{\ell B}|^2)\text{SNR} = \text{SNR} \quad (134)$$

At each receiver, we can just treat the channel output as if it came from $2L$ transmitters labelled $1A, \dots, LA, 1B, \dots, LB$.

We can write the channel to receiver m and the desired coefficient vectors as:

$$\tilde{\mathbf{h}}_m = \begin{bmatrix} \mathbf{h}_{mA} \\ \mathbf{h}_{mB} \end{bmatrix} \quad \tilde{\mathbf{a}}_m = \begin{bmatrix} \mathbf{a}_m \\ \mathbf{0} \end{bmatrix} \quad \tilde{\mathbf{b}}_m = \begin{bmatrix} \mathbf{0} \\ \mathbf{b}_m \end{bmatrix}. \quad (135)$$

We can now directly apply Theorem 7 with $\tilde{\mathbf{h}}_m$, $\tilde{\mathbf{a}}_m$, and $\tilde{\mathbf{b}}_m$ to get the desired result. ■

Remark 17: In order to focus on concepts and keep notation manageable, we have chosen to present our result as above in Theorem 8. There are several immediate extensions, including:

- More than two levels.
- Allowing a different decoding order at each relay.
- Equations spanning different levels.

Example 7: There are $L = 3$ transmitters and $M = 1$ relay and the channel vector is $\mathbf{h}_1 = [1 \ 1 \ \sqrt{2}]^T$. Levels A and B that use scaling coefficients $\gamma_{1A} = \gamma_{2A} = 0$, $\gamma_{1B} = \gamma_{2B} = 1$, and $\gamma_{3A} = \gamma_{3B} = 1/\sqrt{2}$. The relay wants to first decode the equation with coefficient vector $\mathbf{a}_1 = [0 \ 0 \ 1]^T$ from level A and then the equation with coefficient vector $\mathbf{b}_1 = [1 \ 1 \ 1]^T$ from level B . Using Theorem 8, this is possible if the message rates satisfy:

$$r_{3A} < \log \left(1 + \frac{\text{SNR}}{1 + 3\text{SNR}} \right) \quad (136)$$

$$r_{\ell B} < \left(\log \left(\frac{1}{3} + \text{SNR} \right) \right)^+ \quad \ell = 1, 2, 3. \quad (137)$$

Remark 18: It can be shown that nested lattice codes can approach the capacity region of the standard Gaussian broadcast problem. See [14] for more details.

Remark 19: For an application of this superposition scheme to a backhaul-limited cellular uplink network, see [41].

VIII. SLOW FADING AND DISTRIBUTED MIMO

So far, we have considered fixed channel coefficients. Now, we demonstrate that our scheme can be applied to the slow fading scenario, even if the transmitters do not know the channel realization. Under a slow fading model, the channel matrix \mathbf{H} is chosen according to some probability distribution and then remains fixed for all time. As a result, we must accept some probability that the rate used by the transmitters is above the maximum rate permitted for those channel coefficients. For an achievable strategy with rate $R_{\text{SCHEME}}(\mathbf{H})$ for fixed \mathbf{H} , this *outage probability* is given by:

$$\rho_{\text{OUT}}(R) = P_{\mathbf{H}}(R_{\text{SCHEME}}(\mathbf{H}) < R) \quad (138)$$

We can also characterize the performance of a given strategy by its *outage rate*:

$$R_{\text{OUT}}(\rho) = \sup\{R : \rho_{\text{OUT}}(R) \leq \rho\}. \quad (139)$$

We will now compare the outage performance of compute-and-forward to the performance of classical relaying strategies over a simple network. First, it is useful to note that usually, the exact choice of the coefficient vector at a relay is not important, so long as the resulting equations can be solved for the desired messages. Thus, to

maximize the performance of our overall scheme, we should have each relay decode the equation that is available at the *highest rate*.

Consider the two user distributed MIMO network in Figure 7. There are two sources, two relays, and one destination. The relays see the transmitters through \mathbf{H} whose entries are i.i.d. Rayleigh. Each relay is given a bit pipe with rate C bits per channel use to the destination. The destination would like to recover both message w_1 and w_2 at highest possible symmetric outage rate. Recall that for a symmetric rate point to be achievable, both transmitters must be able to communicate their messages with at least that rate.

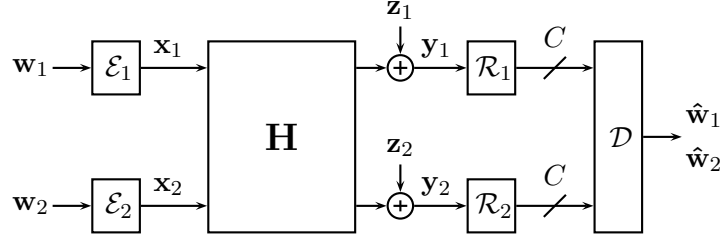


Fig. 7. Two transmitters communicate to a distributed MIMO receiver with two antennas. Each antenna has a rate C bit pipe to the receiver.

The basic compute-and-forward strategy has each relay decode the equation with the highest rate and pass that to the destination. If the equations received by the destination are full rank, decoding is successful. Unfortunately, at low SNR, the probability that the equations are not full rank is quite high as shown in Figure 9. One simple solution is to force each relay to choose an equation with $a_{mm} \neq 0$. This results in equations that are far more likely to be solvable at the expense of slightly lower computation rates. The achievable rates for these two strategies are given below and are plotted in Figure 8 for $C = 2$ and outage probability $\rho = 1/4$.

$$R_{\text{MAX},m} = \max_{\mathbf{a}_m} \log \left(\left(\|\mathbf{a}_m\|^2 - \frac{\text{SNR} |\mathbf{h}_m^* \mathbf{a}_m|^2}{1 + \text{SNR} \|\mathbf{h}_m\|^2} \right)^{-1} \right)$$

$$R_{\text{NZ},m} = \max_{\substack{\mathbf{a}_m \\ a_{mm} \neq 0}} \log \left(\left(\|\mathbf{a}_m\|^2 - \frac{\text{SNR} |\mathbf{h}_m^* \mathbf{a}_m|^2}{1 + \text{SNR} \|\mathbf{h}_m\|^2} \right)^{-1} \right)$$

$$R_{\text{COMP}}(\mathbf{H}) = \begin{cases} \min_m R_{\text{MAX},m} & \text{if } \mathbf{A} \text{ is full rank,} \\ 0 & \text{otherwise.} \end{cases} \quad (140)$$

$$R_{\text{COMP,NZ}}(\mathbf{H}) = \begin{cases} \min_m R_{\text{NZ},m} & \text{if } \mathbf{A} \text{ is full rank,} \\ 0 & \text{otherwise.} \end{cases} \quad (141)$$

For decode-and-forward, we require that each relay is responsible for a single message. It attempts to recover this message either by treating the other message as noise or decoding both messages. The rate for this strategy is evaluated below and plotted in Figure 8. For more details on decode-and-forward for multiple relays (as well as compress-and-forward and cut-set upper bounds), see [3].

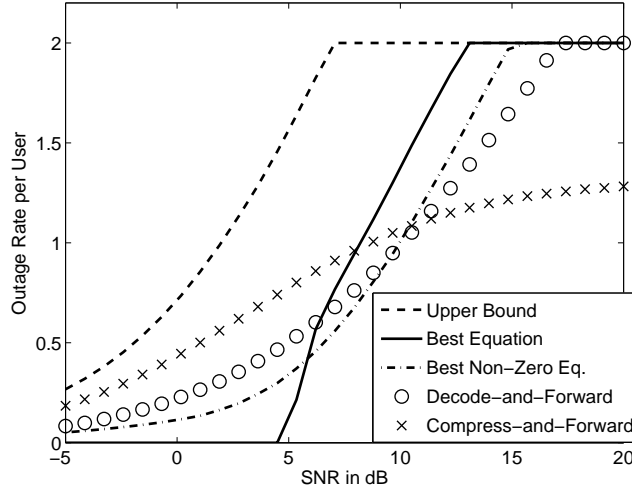


Fig. 8. Symmetric outage rates for the 2-user distributed MIMO multiple-access channel with i.i.d. Rayleigh fading only known at the receivers. Here, we set $C = 2$ and outage probability $\rho = 1/4$.

$$R_{\text{ignore},1} = \log \left(1 + \frac{|h_{11}|^2 \text{SNR}}{1 + |h_{12}|^2 \text{SNR}} \right) \quad (142)$$

$$R_{\text{ignore},2} = \log \left(1 + \frac{|h_{22}|^2 \text{SNR}}{1 + |h_{21}|^2 \text{SNR}} \right) \quad (143)$$

$$R_{\text{decode},m} = \min \left(\log (1 + |h_{m1}|^2 \text{SNR}), \right. \\ \left. \log (1 + |h_{m2}|^2 \text{SNR}), \right. \\ \left. \frac{1}{2} \log (1 + \|\mathbf{h}_m\|^2 \text{SNR}) \right) \quad (144)$$

$$R_{ii} = \min(R_{\text{ignore},1}, R_{\text{ignore},2}) \quad (145)$$

$$R_{id} = \min(R_{\text{ignore},1}, R_{\text{decode},2}) \quad (146)$$

$$R_{di} = \min(R_{\text{decode},1}, R_{\text{ignore},2}) \quad (147)$$

$$R_{dd} = \min(R_{\text{decode},1}, R_{\text{decode},2}) \quad (148)$$

$$R_{DF}(\mathbf{H}) = \min(\max(R_{ii}, R_{id}, R_{di}, R_{dd}), C) \quad (149)$$

For our upper bound, we use a cut-set bound that either groups the relays with the sources or with the destination. This yields the following bound on the symmetric rate:

$$R_{\text{MIMO}}(\mathbf{H}) = \min \left(\log (1 + (|h_{11}|^2 + |h_{21}|^2) \text{SNR}), \right. \\ \left. \log (1 + (|h_{12}|^2 + |h_{22}|^2) \text{SNR}), \right. \\ \left. \frac{1}{2} \log \det (\mathbf{I} + \mathbf{H}\mathbf{H}^* \text{SNR}) \right) \quad (150)$$

$$R_{\text{UPPER}}(\mathbf{H}) = \min(R_{\text{MIMO}}(\mathbf{H}), C) \quad (151)$$

Finally, we consider the performance of compress-and-forward with i.i.d. Gaussian codebooks. The variance of the channel observation at relay m is $1 + \|\mathbf{h}_m\|^2 \text{SNR}$ and we have to compress this using C bits. At the destination,

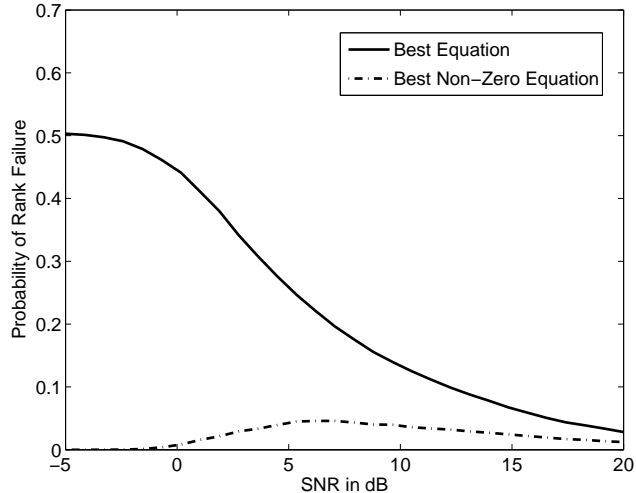


Fig. 9. Probability of rank failure for the 2-user distributed MIMO multiple-access channel by having each relay decode the best equation and the best non-zero equation.

one can equivalently write this as a MIMO channel with channel matrix \mathbf{H}_{CF} :

$$\text{SNR}_{\text{CF},m} = \frac{\text{SNR}(2^C - 1)}{2^C + \text{SNR}\|\mathbf{h}_m\|^2} \quad (152)$$

$$\mathbf{H}_{\text{CF}} = \begin{bmatrix} \sqrt{\text{SNR}_{\text{CF},1}} & 0 \\ 0 & \sqrt{\text{SNR}_{\text{CF},2}} \end{bmatrix} \mathbf{H} \quad (153)$$

$$R_{\text{CF}}(\mathbf{H}) = \min(R_{\text{MIMO}}(\mathbf{H}_{\text{CF}}), C). \quad (154)$$

From Figure 8, we can see that compute-and-forward (with the best equation) outperforms all other strategies starting at approximately 8dB. It also saturates the bit pipes to the destination at an SNR 5dB less than required for decode-and-forward. However, the gains are not as dramatic as observed in Example 4. For non-integer coefficients, we can only decode an integer combination and the remainder acts like additional noise. Despite this penalty, compute-and-forward is the best strategy in the moderate SNR regime. Compress-and-forward is a good strategy at low SNR but since it cannot saturate the bit pipe rate C since it introduces quantization noise. Decode-and-forward suffers at high SNR since it treats interference as noise.

Note that the encoding strategy for compute-and-forward does not depend on the choice of equation coefficients at the relay. Therefore, one can obtain the maximum of the best equation rate and the best non-zero equation rate with the same strategy simply by disallowing certain coefficients at the relays past an appropriate SNR.

Remark 20: Since the channel from the transmitters to the relays is essentially a 2-user interference channel, it may be useful to have each transmitter send out a public and a private message as in the Han-Kobayashi scheme [59]. Such a scheme might improve the performance of both the decode-and-forward strategy and the compute-and-forward strategy (by employing superposition as in Section VII).

IX. WIRELESS NETWORK CODING

In our compute-and-forward strategy, messages and equations thereof take values on a finite field, making it ideally suited for linear network coding. Rather than choosing the network coding coefficients according to a uniform distribution [24], we choose the coefficients to closely match the channel realization. This allows the relay to operate at higher computation rates and, as a result, increases the end-to-end performance of the overall network code.

In previous work, we have shown how to build network codes for finite field and Gaussian multiple-access networks [19], [27], [28]. Using the coding scheme developed in this paper, these results can be extended to include both broadcast constraints and fading. We demonstrate this through an example.

For our example, we will use a slight modification of the well-studied two-way relay network [25], [26], [29], [30], [60]. In the usual considerations, two users want to exchange messages through a relay that sees a noisy combination of their signals. The relay can communicate to the users through a broadcast channel and the goal is to maximize the symmetric rate. It is well-known that for the Gaussian two-way relay channel case without fading, nested lattice codes rapidly approach the upper bound as the SNR increases [29], [30]. Here, we consider a two-way relay channel with i.i.d. Rayleigh fading on all links (see Figure 10). We assume that the channel coefficients are only known at the receiving end and evaluate the outage performance as in Section VIII. The two users have power constraint SNR and the relay has power constraint SNR_{BC} .

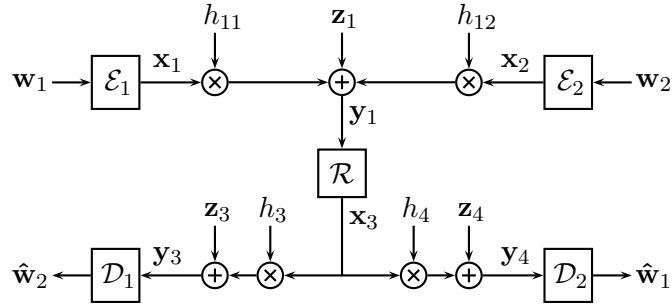


Fig. 10. Two-way relay channel with fading. Two users exchange messages with the help of an intermediate relay.

The multicast rate from the relay to two users for a given channel realization is just the minimum of the two point-to-point rates:

$$R_{\text{BC}} = \log \left(1 + (\min(|h_3|^2, |h_4|^2)) \text{SNR}_{\text{BC}} \right). \quad (155)$$

For compute-and-forward, we can guarantee that each user can solve for its desired message by decoding a message with non-zero coefficients at the relay. We also let the relay perform successive cancellation to recover the messages individually. This allows it to access the corner points of the multiple-access region. For each channel realization, the relay selects the strategy that results in the highest symmetric rate.

$$R_{\text{NZ}} = \max_{a_{11}, a_{12} \neq 0} \log \left(\left(\|\mathbf{a}_1\|^2 - \frac{\text{SNR} |\mathbf{h}_1^* \mathbf{a}_1|^2}{1 + \text{SNR} \|\mathbf{h}_1\|^2} \right)^{-1} \right)$$

$$R_{\text{corner},1} = \min \left(\log \left(1 + \frac{|h_{11}|^2 \text{SNR}}{1 + |h_{12}|^2 \text{SNR}} \right), \log (1 + |h_{12}|^2 \text{SNR}) \right)$$

$$R_{\text{corner},2} = \min \left(\log \left(1 + \frac{|h_{12}|^2 \text{SNR}}{1 + |h_{11}|^2 \text{SNR}} \right), \log (1 + |h_{11}|^2 \text{SNR}) \right)$$

$$R_{\text{COMP}}(\mathbf{H}) = \min \left(R_{\text{BC}}, \max \left(R_{\text{NZ}}, R_{\text{corner},1}, R_{\text{corner},2} \right) \right)$$

The relay is required to recover both message for decode-and-forward and the rate is simply the maximum

symmetric rate in the multiple-access capacity region.

$$R_{\text{decode}} = \min \left(\begin{aligned} &\log (1 + |h_{11}|^2 \text{SNR}), \\ &\log (1 + |h_{12}|^2 \text{SNR}), \\ &\frac{1}{2} \log (1 + \|\mathbf{h}_1\|^2 \text{SNR}) \end{aligned} \right) \quad (156)$$

$$R_{\text{DF}}(\mathbf{H}) = \min \left(R_{\text{BC}}, R_{\text{decode}} \right) \quad (157)$$

The amplify-and-forward scheme has the relay scale its observation to meet the power constraint and retransmit.

$$R_{\text{AF}}(\mathbf{H}) = \min \left(\begin{aligned} &\log \left(1 + \frac{|h_{12}h_3|^2 \text{SNR} \text{SNR}_{\text{BC}}}{1 + \|\mathbf{h}_1\|^2 \text{SNR} + |h_3|^2 \text{SNR}_{\text{BC}}} \right), \\ &\log \left(1 + \frac{|h_{11}h_4|^2 \text{SNR} \text{SNR}_{\text{BC}}}{1 + \|\mathbf{h}_1\|^2 \text{SNR} + |h_4|^2 \text{SNR}_{\text{BC}}} \right) \end{aligned} \right)$$

Finally, since we are only interested in the symmetric rate, the upper bound is just the minimum over all channel strengths.

$$R_{\text{UPPER}}(\mathbf{H}) = \min \left(\begin{aligned} &R_{\text{BC}}, \log (1 + |h_{11}|^2 \text{SNR}), \\ &\log (1 + |h_{12}|^2 \text{SNR}) \end{aligned} \right) \quad (158)$$

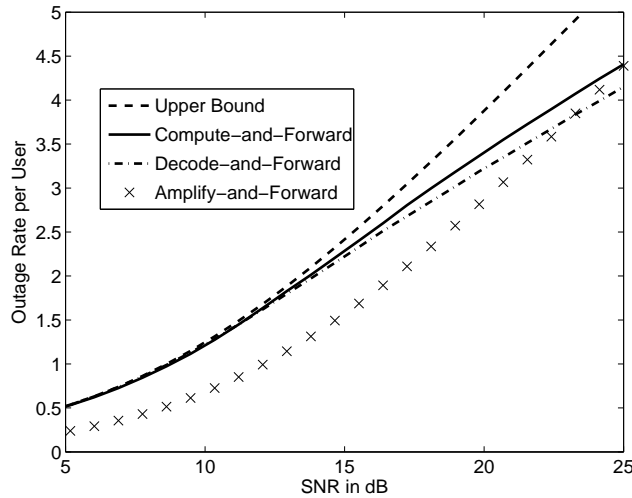


Fig. 11. Outage rate per user for the two way relay channel with i.i.d. Rayleigh fading only known at the receivers. Here, we set $\text{SNR}_{\text{BC}} = 2\text{SNR}$ and outage probability $\rho = 1/3$.

In Figure 11, we plot the performance of these schemes for $\text{SNR}_{\text{BC}} = 2\text{SNR}$ and outage probability $\rho = 1/3$. Note that here there is no possibility of rank failure for compute-and-forward and it is always at least as good as decode-and-forward. However, unlike the no fading case, compute-and-forward does not approach the upper bound as the SNR increases. In fact, past a certain SNR (in this case 25dB), compress-and-forward is the best strategy. This is primarily due to the noise penalty for decoding an integer combination from non-integer channel coefficients. For the no fading case, the channels coefficients are assumed to be unity and this penalty does not appear.

The basic insight is that, for certain network topologies, the non-integer penalty can become significant. This penalty can be managed in several ways, including:

- Adjusting the channel at the transmitters if channel state information is available.

- Using multiple antennas at the relay to steer the coefficients towards integers [61].
- Selecting transmitters opportunistically based on how close their coefficients are to integers.

However, despite the non-integer penalty, compute-and-forward offers a strict improvement over classical strategies for reliably recovering equations over a finite field. This, combined with the fact that amplify-and-forward requires fairly high values of SNR to overcome noise build-up, makes compute-and-forward a promising strategy for physical-layer network coding for multi-hop networks.

X. UPPER BOUND

In this section, we give a simple upper bound on the computation rate through a genie-aided argument. This bound does not match our achievable strategy in general and it may be possible to construct tighter outer bounds by taking into account the mismatch between the desired function and the function naturally provided by the channel.

Theorem 9: Assume the channel between the transmitters and the relays is $p(y_1, \dots, y_M | x_1, \dots, x_L)$. If the relays, want equations with coefficient vectors $\mathbf{a}_1, \dots, \mathbf{a}_M \in \{\mathbb{Z} + j\mathbb{Z}\}^L$, the message rates are upper bounded as follows:

$$r_\ell \leq \min_{m: a_{m\ell} \neq 0} I(X_\ell; Y_m | X_1, \dots, X_{\ell-1}, X_{\ell+1}, \dots, X_L)$$

For the Gaussian channel model considered in this paper, with channel vectors $\mathbf{h}_1, \dots, \mathbf{h}_M \in \mathbb{C}^L$, this specializes to:

$$r_\ell \leq \min_{m: a_{m\ell} \neq 0} \log(1 + |h_{m\ell}|^2 \text{SNR}) \quad (159)$$

Proof: To each relay m for which $a_{m\ell} \neq 0$, we provide all messages except that from encoder ℓ as genie-aided side-information. Now, we are left with a multicasting problem from encoder ℓ to all relays with $a_{m\ell} \neq 0$. Clearly, the multicast rate is upper bounded by the lowest rate link. For the Gaussian case, it is easy to show that the mutual information expressions are maximized by the Gaussian distribution. ■

XI. CONCLUSIONS

In this paper, we have developed a new coding scheme that enables intermediate relays to reliably recover equations of the original messages by exploiting the interference structure of the wireless channel. As we have seen, this framework can achieve end-to-end rates across an AWGN network that are not accessible with classical relaying strategies. More generally, the techniques in this paper can be used as building blocks for developing new cooperative communication schemes that exploit both the algebraic and statistical properties of wireless networks. Here, we discussed applications to distributed MIMO and wireless network coding and we believe there are many more scenarios where it will be useful. For instance, in [62], we explored an application to physical-layer gossiping in a sensor network.

Compute-and-forward also adds to the growing pile of evidence that structured codes are a powerful tool for tackling problems in multi-user information theory. Recently, many new inner bounds have emerged that take advantage of the algebraic structure of multi-user problems. The behavior observed in these strategies is not well-captured by the usual cut-set outer bounds. Therefore, new outer bounds that account for algebraic as well as statistical structure will be needed to better characterize the capacity regions of multi-user networks [63].

APPENDIX I

UPPER BOUND ON NOISE DENSITIES

In this appendix, we demonstrate that the densities of the noise terms in Theorem 3 are upper bounded by the density of an i.i.d. Gaussian vector. The proof follows that of Lemmas 6 and 11 from [13].

Lemma 8: Let $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \frac{1}{2}\mathbf{I}^{n \times n})$ and let \mathbf{d}_ℓ be independently generated according to a uniform distribution over \mathcal{V} . Also, let σ_B^2 denote the second moment of an n -dimensional ball whose radius is equal to the covering radius r_{cov} of Λ and let \mathbf{z}_ℓ^* be independently generated according to $\mathcal{N}(\mathbf{0}, \sigma_B^2 \mathbf{I}^{n \times n})$. Now, let

$$\mathbf{z}_{eq} = \alpha \mathbf{z} + \sum_{\ell=1}^L \theta_\ell \mathbf{d}_\ell \quad (160)$$

where $\alpha, \theta_\ell \in \mathbb{R}$. There exists an i.i.d. Gaussian vector

$$\mathbf{z}^* = \alpha \mathbf{z} + \sum_{\ell=1}^L \theta_\ell \mathbf{z}_\ell^* \quad (161)$$

with variance σ^2 satisfying

$$\sigma^2 \leq \frac{\alpha^2}{2} + \left(\frac{r_{\text{COV}}}{r_{\text{EFFEC}}} \right)^2 \frac{\text{SNR}}{2} \sum_{\ell=1}^L \theta_\ell^2 \quad (162)$$

such that the density of \mathbf{z}_{eq} is upper bounded as follows:

$$f_{\mathbf{z}_{eq}}(\mathbf{z}) \leq e^{Lc(n)n} f_{\mathbf{z}^*}(\mathbf{z}) \quad (163)$$

$$c(n) = \ln \left(\frac{r_{\text{COV}}}{r_{\text{EFFEC}}} \right) + \frac{1}{2} \ln 2\pi e G_B^{(n)} + \frac{1}{n} \quad (164)$$

where $G_B^{(n)}$ is the normalized second moment of an n -dimensional ball and r_{EFFEC} is the effective radius of Λ .

Proof: First, we will show that the density of \mathbf{z}_{eq} is upper bounded as desired. From Lemma 11 in [13], we have that:

$$f_{\mathbf{d}_\ell}(\mathbf{z}) \leq e^{c(n)n} f_{\mathbf{z}_\ell^*}(\mathbf{z}) \quad (165)$$

Since $\mathbf{z}, \mathbf{d}_1, \dots, \mathbf{d}_L$ are independent, we can write the density of \mathbf{z}_{eq} as an n -dimensional convolution of the densities of its components:

$$f_{\mathbf{z}_{eq}}(\mathbf{z}) = f_{\alpha \mathbf{z}}(\mathbf{z}) * f_{\theta_1 \mathbf{d}_1}(\mathbf{z}) * \dots * f_{\theta_L \mathbf{d}_L}(\mathbf{z}) \quad (166)$$

Similarly, we can write the density of \mathbf{z}^* as:

$$f_{\mathbf{z}^*}(\mathbf{z}) = f_{\alpha \mathbf{z}}(\mathbf{z}) * f_{\theta_1 \mathbf{z}_1^*}(\mathbf{z}) * \dots * f_{\theta_L \mathbf{z}_L^*}(\mathbf{z}) \quad (167)$$

Since probability densities are non-negative, we can use the upper bound in (165) to get:

$$f_{\alpha \mathbf{z}}(\mathbf{z}) * f_{\theta_\ell \mathbf{d}_\ell}(\mathbf{z}) \leq f_{\alpha \mathbf{z}}(\mathbf{z}) * e^{c(n)n} f_{\theta_\ell \mathbf{z}_\ell^*}(\mathbf{z}) \quad (168)$$

Applying this idea L times to $f_{\mathbf{z}_{eq}}(\mathbf{z})$ yields:

$$f_{\mathbf{z}_{eq}}(\mathbf{z}) \leq e^{Lc(n)n} f_{\mathbf{z}^*}(\mathbf{z}) \quad (169)$$

We must now upper bound the variance of \mathbf{z}^* . By Definition 20, $\text{Vol}(\mathcal{B}(r_{\text{EFFEC}})) = \text{Vol}(\mathcal{V})$. Recall that a ball has the smallest second moment for a given volume. Let \mathbf{b} be generated according to the uniform distribution over $\mathcal{B}(r_{\text{COV}})$. It follows that

$$\frac{\text{SNR}}{2} = \frac{1}{n} E [\|\mathbf{d}_\ell\|^2] \quad (170)$$

$$\geq \frac{1}{n} E \left[\left\| \frac{r_{\text{EFFEC}}}{r_{\text{COV}}} \mathbf{b} \right\|^2 \right] = \left(\frac{r_{\text{EFFEC}}}{r_{\text{COV}}} \right)^2 \sigma_B^2 \quad (171)$$

Finally, we get:

$$\sigma^2 = \frac{1}{n} E [\|\alpha \mathbf{z}\|^2] + \frac{1}{n} \sum_{\ell=1}^L E [\|\theta_\ell \mathbf{z}_\ell^*\|^2] \quad (172)$$

$$= \frac{\alpha^2}{2} + \sigma_B^2 \sum_{\ell=1}^L \theta_\ell^2 \quad (173)$$

$$\leq \frac{\alpha^2}{2} + \left(\frac{r_{\text{COV}}}{r_{\text{EFFEC}}} \right)^2 \frac{\text{SNR}}{2} \sum_{\ell=1}^L \theta_\ell^2 \quad (174)$$

Since and $\frac{r_{\text{COV}}}{r_{\text{EFFEC}}} \rightarrow 1$ and $G_B^{(n)} \rightarrow \frac{1}{2\pi e}$ as $n \rightarrow \infty$, $c(n) \rightarrow 0$ as $n \rightarrow \infty$. AWGN good lattices have a positive error exponent for i.i.d. Gaussian noise with variance smaller than the second moment of the lattice so this means that the probability of error can be driven to zero as the blocklength increases. ■

APPENDIX II
FINE LATTICES ARE AWGN GOOD

We now show that the fine lattices can recover from i.i.d. Gaussian noise.

Lemma 9: $\Lambda_1, \Lambda_2, \dots, \Lambda_L$ are AWGN good with probability that goes to 1 as $n \rightarrow \infty$ so long as $\frac{n}{p} \rightarrow 0$.

Proof: Let \mathbf{z}_ℓ denote an i.i.d. Gaussian vector with zero-mean and any variance σ_ℓ^2 such that the volume-to-noise ratio for Λ_ℓ is greater than $2\pi e$. Consider the following channel from $\mathbf{x}_\ell \in \mathcal{V}$ to $\hat{\mathbf{x}}_\ell \in \mathcal{V}$:

$$\hat{\mathbf{x}}_\ell = [Q_{\Lambda_\ell}(\mathbf{x}_\ell + \mathbf{z}_\ell)] \bmod \Lambda \tag{175}$$

and let $P_{e,\ell} = Pr(\hat{\mathbf{x}}_\ell \neq \mathbf{x}_\ell)$. In Appendix B of [13], it is shown that the random coding error exponent for this channel (with \mathbf{x}_ℓ generated uniformly over \mathcal{V}) is equal to the Poltyrev exponent. This means that $P_{e,\ell}$ decreases exponentially with n for volume-to-noise ratio greater than $2\pi e$. Appendix C of [13] shows that the same performance is possible if \mathbf{x}_ℓ is drawn according to a uniform distribution over $\{p^{-1}\Lambda\} \cap \mathcal{V}$ and $\frac{n}{p} \rightarrow 0$.

From Lemma 3, we know that the marginal distribution of each element of $\Lambda_\ell \cap \mathcal{V}$ is uniform over $\{p^{-1}\Lambda\} \cap \mathcal{V}$. Furthermore, all points in the set $\Lambda_\ell \cap \mathcal{V}$ are pairwise independent. This is all that is required to apply the union bound and obtain the same performance as i.i.d. inputs over $\{p^{-1}\Lambda\} \cap \mathcal{V}$ in terms of the error exponent.

Thus, the probability that Λ_ℓ is AWGN good (with the Poltyrev error exponent) goes to 1 as $n \rightarrow \infty$. It follows from a simple union bound that $\Lambda_1, \dots, \Lambda_L$ are simultaneously AWGN good with high probability as $n \rightarrow \infty$. ■

APPENDIX III
FIXED DITHERS

We now show that there exist fixed dithers that are appropriate for our coding scheme. We begin by showing that with high probability over the dither vectors, the power constraint can be met (since in the proof of Theorem 3 this is only done in expectation).

Choose $\delta > 0$. Scale the second moment of the coarse lattice to be $\sigma_\Lambda^2 = \frac{\text{SNR}}{2} - \delta \left(\frac{r_{\text{EFFEC}}}{r_{\text{COV}}} \right)^2$ and consider the real part of the channel input from encoder ℓ , $[\mathbf{t}_\ell^R - \mathbf{d}_\ell^R] \bmod \Lambda$. By Lemma 7, this is uniformly distributed over \mathcal{V} . Let \mathbf{d} be a vector drawn from the uniform distribution over \mathcal{V} and \mathbf{z}^* be an i.i.d. Gaussian vector with mean 0 and variance $\sigma^2 = \left(\frac{r_{\text{COV}}}{r_{\text{EFFEC}}} \right)^2 \sigma_\Lambda^2$. Since \mathbf{z}^* is i.i.d. Gaussian, by the weak law of large numbers

$$P \left(\left| \frac{1}{n} \|\mathbf{z}^*\|^2 - \sigma^2 \right| > \delta \right) \leq \frac{2\sigma^4}{\delta n}. \tag{176}$$

From Lemma 11 in [13], we have that:

$$f_{\mathbf{d}}(\mathbf{z}) \leq e^{c(n)n} f_{\mathbf{z}^*}(\mathbf{z}). \tag{177}$$

with $c(n)$ as in (164). Note that $\sigma^2 = \left(\frac{r_{\text{COV}}}{r_{\text{EFFEC}}} \right)^2 \frac{\text{SNR}}{2} - \delta$ so the probability that \mathbf{d} violates the power constraint can be upper bounded in terms of σ^2 :

$$P \left(\frac{1}{n} \|\mathbf{d}\|^2 > \frac{\text{SNR}}{2} \right) \leq P \left(\left| \frac{1}{n} \|\mathbf{d}\|^2 - \sigma^2 \right| > \delta \right) \tag{178}$$

$$\leq e^{c(n)n} \frac{2\sigma^4}{\delta n} \tag{179}$$

This goes to 0 as n goes to infinity so, by the union bound, all channel inputs obey the power constraint with high probability. Since Λ is covering good and the rates are continuous in the second moment of the coarse lattice, we can choose a decreasing sequence of δ that approaches the rates achieved when $\sigma_\Lambda^2 = \frac{\text{SNR}}{2}$.

We note that the probability of decoding error in Theorem 3 goes to zero as n increases when using random dithers. Taking a union bound over this error event and the event that any dither exceeds a power constraint, we get that for n large enough there is (at least) one good fixed set of dither vectors.

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