Dynamical laws of superenergy in General Relativity

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Abstract. Bel and Bel-Robinson tensors were introduced nearly fifty years ago in an attempt to generalize to gravitation the energy-momentum tensor of electromagnetism. This generalization was successful from the mathematical point of view because these tensors share mathematical properties which are remarkably similar to those of the energy-momentum tensor of electromagnetism. However, the physical role of these tensors in General Relativity has remained obscure and no interpretation has achieved wide acceptance. In principle, they cannot represent *energy* and the term *superenergy* has been coined for the hypothetical physical magnitude lying behind them. In this work we try to shed light on the true physical meaning of *superenergy* by following the same procedure which enables us to give an interpretation of the electromagnetic energy. This procedure consists in performing an orthogonal splitting of Bel and Bel-Robinson tensors and analysing the different parts resulting from the splitting. In the electromagnetic case such splitting gives rise to the electromagnetic *energy density*, the Poynting vector and the electromagnetic stress tensor, each of them having a precise physical interpretation which is deduced from the *dynamical laws* of electromagnetism (Poynting theorem). The full orthogonal splitting of Bel and Bel-Robinson tensors is more complex but, as expected, similarities with electromagnetism are present. Also the covariant divergence of Bel tensor is analogous to the covariant divergence of the electromagnetic energy-momentum tensor and the orthogonal splitting of the former is found. The ensuing equations are to the superenergy what the Poynting theorem is to electromagnetism. Some consequences of these *dynamical laws of superenergy* are explored, among them the possibility of defining *superenergy radiative states* for the gravitational field.

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1. Introduction

General Relativity is, in some aspects, a peculiar theory. In it the spacetime itself is part of the degrees of freedom and this fact brings to General Relativity some complications not present in other theories where the fields are set in a fixed spacetime background. One of these complications is the impossibility of defining a local invariant concept of *gravitational energy density*. The main argument to sustain this assertion relies on the equivalence principle which states that the dynamical effects of gravity can be always transformed away at a point. The consequence of this is that any geometric object representing gravitational "energy-momentum" can always be set to zero in an suitable coordinate system or frame and this property cannot be fulfilled by a tensor. Only a *pseudo-tensor* can accomplish this duty but gravitational energy-momentum pseudo-tensors are not unequivocally defined because, by the very nature of a pseudo-tensor, they are always tied to a given frame or coordinate system. The use of a pseudo-tensor makes very difficult to address problems such as the calculation of the gravitational energy radiated by a source.

Different approaches to the "gravitational energy problem" in General Relativity have been provided along the years and no general formalism has emerged (although formalisms tailored for particular important cases do exist). One of these approaches seeks to enhance the formal similarities between electromagnetism and gravitation in order to find a replacement for the missing "gravitational energy-momentum tensor". The idea is to take the electromagnetic energy-momentum tensor and translate it into a gravitational counterpart by somehow replacing the Faraday tensor by the Riemann tensor in the expression giving the energy-momentum tensor for electromagnetism. This translation is by no means straightforward due to the different nature of Riemann and Faraday tensors but it can be certainly accomplished. The result of this translation is a four index tensor quadratic in the Riemann tensor which was first found by Bel [3]. Bel tensor has mathematical properties which are remarkably similar to the electromagnetic energy-momentum tensor (see theorem 3 for a summary). An important particular case arises if we replace in Bel tensor definition Riemann by Weyl tensor which results in Bel-Robinson tensor [2].

From the above considerations it is clear that Bel tensor will represent a magnitude which is different from energy. Such new magnitude was called "superenergy" by Bel and its status in General Relativity has been subject to much debate and no widely accepted conclusions have been reached. A simple dimensional analysis shows that in geometrized units the physical dimension of superenergy is L^{-4} where L represents length. Another important property is the tensorial character of superenergy. This means that if we work with *gravitational superenergy* instead of *gravitational energy* we can avoid all the technical complications arising when one works with pseudotensors. The main goal of this paper is to show what the consequences are of considering superenergy as a measurable physical magnitude. In order to carry out our program we need to find the orthogonal splitting with respect to an observer of Bel tensor (so we will be able to explain the observer what will be obtained when measuring superenergy) and we need to find the variation of the different parts of the orthogonal splitting found along the observer's path. The outcome of this last part is a set of equations which we call *the dynamical laws of superenergy* and they are one of the main results of this paper.

We may examine at this point what the above procedure yields in the case of electromagnetism. In this case we are working with a magnitude with dimensions of energy instead of dimensions of superenergy but this is now of no relevance. The different parts resulting from the orthogonal splitting of the electromagnetic energy-momentum tensor are the electromagnetic energy density, the Poynting vector and the electromagnetic stress-tensor. The utility of each of these parts is explained in basic electrodynamics textbooks. The *dynamical laws of electromagnetic energy* are contained in the Poynting theorem and it is through this theorem that the electromagnetic energy density and the Poynting vector gain their full physical understanding as measurable quantities. The Poynting theorem is nothing less than the orthogonal splitting of the covariant divergence of electromagnetic energy-momentum tensor. The parts of this splitting are the variation of the electromagnetic energy density and the Poynting vector along the observer's path. The Poynting theorem enables us to draw conclusions so important as the characterization of radiative electromagnetic fields or the expression for the total force acting on an electromagnetic system.

In General Relativity we may consider the expression for the covariant divergence of Bel tensor as the gravitational counterpart for the covariant divergence of the energy momentum tensor of electromagnetism. Therefore if we perform the orthogonal splitting of the former we will obtain a set of equations which can be regarded as the counterpart of the Poynting theorem. We call these set of equations the *dynamical laws of superenergy*. As we may expect, the dynamical laws of superenergy are far more complex than electromagnetism's Poynting theorem but still we can follow the same pattern as in electromagnetism to draw some conclusions. For example, we can decide when a gravitational system is radiating superenergy (intrinsic superenergy radiative state). This was already attempted by Bel in the late fifties but since the full set of dynamical laws of superenergy was not available, Bel's result does not apply to general enough cases.

The paper is organized as follows: in section 2 we review the notation and the essential concepts about orthogonal splittings. In section 3 we find the orthogonal splitting of the electromagnetic energy-momentum dynamical laws for an arbitrary observer in a general spacetime (theorem 2). This is the complete version of the classical Poynting theorem and some of its consequences are discussed. In section 4 we present the Bel and Bel-Robinson tensors and their essential mathematical properties are summarized in theorem 3. Section 5 contains the orthogonal splitting of Bel-Robinson tensor and we study the basic mathematical properties of the different parts of the orthogonal splitting. Since these parts are expressed in terms of the electric and magnetic parts of Weyl tensor, we can obtain particular *canonical forms* valid for some Petrov types (subsection 5.1). Section 6 is devoted to the orthogonal splitting of Bel tensor and section 7 contains the main result of this paper which is theorem 4. This theorem spells out the different parts of the orthogonal decomposition of the covariant divergence of Bel tensor (see equation (39)) which as explained above are the dynamical laws of superenergy. In section 8 we study the radiation of superenergy from a general point of view. To that end the definition of *intrinsic superenergy radiative state* is put forward (definition 3). Finally in section 9 we suggest a possible relationship between the concepts of *energy* and *superenergy*.

The main results of this paper rely on heavy tensor calculations which can only be carried out with the aid of a computer algebra system. All the calculations of this paper have been undertaken with the computer program *xAct* [33]. *xAct* is a suite of MATHEMATICA packages devised to make calculations in General Relativity and Differential Geometry. Among the many features of the *xAct* system we stress its ability to canonicalize tensor expressions by means of powerful algorithms based in permutation group theory (package *xperm*), the excellent implementation of tensor calculus (package *xTensor*) and the possibility of working with frames and tensor components (package *xCoba*). In appendix A we provide further details about how *xAct* has been used in this paper. Currently, no other computer algebra system, either free or commercial, has the capabilities to perform the calculations needed in this paper.

2. The orthogonal splitting

We start by introducing the basic notation and conventions which will be adopted in this paper. We shall work in a four dimensional smooth Lorentzian manifold V which we will call *spacetime*. The abstract index notation is followed throughout to denote tensors on V with Latin lowercase letters reserved for the abstract indexes. We use bold typeface for component indexes. Round (square) brackets enclosing indexes denote index symmetrization (antisymmetrization). Unless otherwise stated all tensors are assumed smooth and defined globally on V. The metric tensor is g_{ab} and our signature convention is $(-,+,+,+)$. This metric is used to raise and lower indexes in the usual way. Associated to the metric is the *volume element* which we denote by η_{abcd}. The Levi-Civita connection compatible with g_{ab} is the only affine connection ∇_a satisfying $\nabla_a \mathrm{g}_{bc} = 0$ and our convention for the curvature tensor of this connection is fixed by the Ricci identity

$$\nabla_a \nabla_b X^c - \nabla_b \nabla_a X^c = X^d R_{bad}{}^c.$$

The Ricci tensor and the scalar curvature are $R_{bd} \equiv R_{bad}{}^a$ and $R \equiv R^a{}_a$ respectively. From these, the Einstein tensor is defined by the familiar formula $G_{ab} \equiv R_{ab} - R\mathrm{g}_{ab}/2$. The Lie derivative with respect to any vector field X^a is the differential operator $\pounds_X$. Geometrized units with $8\pi G = c = 1$ are used unless otherwise stated. End of a proof is marked with $\Box$.

Specially important for us are unit timelike vector fields. For any such vector field, the family of its integral curves defines a *timelike congruence* or *observer set*. This unit timelike vector enables us to perform an *orthogonal splitting* (also called *3+1 decomposition*) of any tensor on V. The orthogonal splitting lies in the basis of many studies and formalisms in General Relativity and has been extensively studied in the literature but since it will be used in this work many times we review next its essentials (good accounts can be found in [18, 32]). Let n^a be any vector field with $n_a n^a = -1$ and define the *spatial metric* h_{ab} by

$$h_{ab} \equiv g_{ab} + n_a n_b, \;\; h_{ab} h^b{}_c = h_{ac}, \;\; h^a{}_a = 3. \tag{1}$$

The tensor h_{ab} has the properties of an orthogonal projector. We shall call a covariant tensor $T_{a_1 \ldots a_m}$ *spatial* with respect to h_{ab} if it is invariant under $h^a{}_b$ i.e. if

$$h^{a_1}{}_{b_1} \cdots h^{a_m}{}_{b_m} T_{a_1 \cdots a_m} = T_{b_1 \cdots b_m},$$

with the obvious generalization for any mixed tensor. This property implies that the inner contraction of n^a with $T_{a_1 \ldots a_m}$ (taken on any index) vanishes. We introduce next the orthogonal projection operator defined by

$$P_h(L_{a_1 \ldots a_m}) \equiv h^{s_1}{}_{a_1} \cdots h^{s_m}{}_{a_r} L_{s_1 \ldots s_m}, \tag{2}$$

where $L_{a_1 \ldots a_m}$ is an arbitrary tensor. Clearly $P_h(L_{a_1 \ldots a_m})$ is a spatial tensor. Another definition which we need is the *generalized inner contraction* of the tensor $L_{a_1 \ldots a_m}$ with the unit normal which is given by

$$n^J(L_{a_1 \ldots a_m}) \equiv n^{s_1} \cdots n^{s_J} L_{\ldots s_1 \ldots s_2 \ldots s_J \ldots}.$$

Here J is an ordered subset of the set of abstract indexes $\{a_1 \ldots a_m\}$ and the dummies $\{s_1 \ldots s_J\}$ are placed in those slots of L indicated by J. Therefore $n^J(L_{a_1 \ldots a_m})$ has $m - \#J$ free indexes given by the complementary list of J with respect to $\{a_1 \ldots a_m\}$.

Using the orthogonal projection operator and the generalized inner contraction we find that any tensor $L_{a_1\dots a_m}$ can be written in the following way

$$L_{a_1\dots a_m}=\sum_{J\in\mathscr{P}(\{a_1\dots a_m\})}(-1)^{\#J}n_J P_h(n^J(L_{a_1\dots a_m})),\tag{3}$$

where $\mathscr{P}(\{a_1\dots a_m\})$ is the power set of $\{a_1\dots a_m\}$ and

$$n_J\equiv n_{a_q}\cdots n_{a_p},\ J=\{a_q,\dots,a_p\}\in\mathscr{P}(\{a_1\dots a_m\}).$$

The right hand side of (3) is called the *orthogonal splitting* of $L_{a_1\dots a_m}$ with respect to the unit normal n^a (we will just speak of orthogonal splitting of a tensor if the unit normal is understood). The orthogonal splitting given by (3) is unique and the set of spatial tensors $\{P_h(n^J(L_{a_1\dots a_m}))\}$ contains all the information about $L_{a_1\dots a_m}$. A trivial example of orthogonal splitting is that of the metric tensor itself which is obtained from the first expression in (1). Another important example of orthogonal splitting which is easily deduced from (3) is

$$\eta_{abcd}=-n_a\varepsilon_{bcd}+n_b\varepsilon_{acd}-n_c\varepsilon_{abd}+n_d\varepsilon_{abc},$$

where ε_{abc} is the *spatial volume element* and is defined by

$$\varepsilon_{abc}\equiv n^d\eta_{dabc}.$$

2.1. Kinematical quantities

As we explained above, the set of integral curves of n^a represents a family of observers. In physical applications it is important to introduce quantities describing the *relative motion* of each curve of the family and this is the role of the kinematical quantities. To define them we find the orthogonal splitting of $\nabla_a n_b$ which is

$$\nabla_a n_b=-A_b n_a+\frac{1}{3}\theta h_{ab}+\sigma_{ab}+\omega_{ab}.\tag{4}$$

The tensor A_b is the acceleration, the scalar θ is the expansion and σ_{ab}, ω_{ab} are the shear and the rotation respectively. From previous equation it is easy to obtain expressions for the kinematical quantities in terms of n^a

$$A^b=n^a\nabla_a n^b,\ \theta=\nabla_a n^a,\ \omega_{ab}=h_{[a}{}^{d}h_{b]}{}^{c}\nabla_d n_c,\ \sigma_{ab}=h_{(a}{}^{d}h_{b)}{}^{c}\nabla_d n_c-\frac{\theta}{3}.\tag{5}$$

Straightforward properties of the kinematical quantities are

$$\sigma_{(ab)}=\sigma_{ab},\ \omega_{[ab]}=\omega_{ab},\ \sigma^a{}_a=0,\tag{6}$$

Sometimes the rotation is replaced by the *vorticity* which is defined as follows

$$\omega_a\equiv\frac{1}{2}\varepsilon_{abc}\omega^{bc},\ \omega_{ab}=\varepsilon_{abc}\omega^c.\tag{7}$$

Each of the kinematical quantities has a precise interpretation which deals with the *relative motion* of the observers of the congruence (see e.g. [18, 21] for a more detailed description of these concepts).

2.2. Cattaneo operator

Another, very important object, which is needed when working with orthogonal splittings is the *Cattaneo operator* also known as *spatial connection* [16]. If $L_{a_1 \dots a_m}$ is a covariant tensor then we define the linear operator

$$D_a L_{a_1 \dots a_m} \equiv P_h(\nabla_a L_{a_1 \dots a_m}), \tag{8}$$

with obvious definitions for contravariant and mixed tensors. The Cattaneo operator is not a linear connection on the spacetime manifold V because it does not satisfy the Leibnitz rule unless both factors of the product upon which D_a acts are spatial. From its definition, it is clear that $D_a T_{b_1 \dots b_m}$ is a spatial tensor. Important additional properties of the Cattaneo operator are

$$D_a h_{bc} = 0,\ D_a D_b \varphi - D_b D_a \varphi = 2n^c \omega_{ab} \nabla_c \varphi = 2\omega_{ab} \pounds_n \varphi,\ \varphi \in C^1(V). \tag{9}$$

The Cattaneo operator enables us to write in a compact form the orthogonal splitting of any expression involving derivatives. Specially important in this work is to find the orthogonal splitting of the covariant derivative of a spatial tensor. To illustrate how this works, let us study a particular simple example. Consider $\nabla_a L_b$, where L_b is an arbitrary spatial covector ($n^a L_a = 0$). In this case formula (3) yields

$$\nabla_a L_b = D_a L_b - n_a P_h(n^p \nabla_p L_b) - n_b P_h(n^p \nabla_a L_p) + n_a n_b n^p n^q \nabla_p L_q. \tag{10}$$

Next we use in this equation the relations

$$n^c \nabla_c L_a = \pounds_n L_a - A_c \nabla_a n^c,\ n^p \nabla_a L_p = -L_p \nabla_a n^p, \tag{11}$$

and replace in (10) the covariant derivatives of the unit normal by the expression given in (4). After some manipulations equation (10) becomes

$$\begin{aligned}\nabla_a L_b = -A^c L_c n_b n_a + \left(\frac{1}{3} L_b \theta - \pounds_n L_b + L^c(\sigma_{bc} + \omega_{bc})\right) n_a + D_a L_b + \\ +n_b \left(\frac{1}{3} L_a \theta + L^c\left(\sigma_{ac} + \omega_{ac}\right)\right),\end{aligned} \tag{12}$$

which has the form of (3) and hence is the complete orthogonal splitting of $\nabla_a L_b$. Note that $\pounds_n L_a$ is a spatial covector if L_a is spatial, due to the property $\pounds_n n^a = 0$. The procedure followed to obtain (12) is easily generalized for the covariant derivative of any spatial tensor (see appendix A for more examples). This kind of calculation is extensively used in section 7.

3. Electromagnetism as a working example

As a preparation to the study which we are going to undertake for the gravitational field, we analyse first the case of electromagnetism. The electromagnetic field is described by an antisymmetric rank-2 tensor F_{ab} (the Faraday or electromagnetic tensor) which satisfies the *Maxwell equations*

$$\nabla_{[a} F_{bc]} = 0,\ \nabla_a F^a_{\ b} = j_b, \tag{13}$$

where j^b is *the charge current* (we follow in the sequel the Heaviside-Lorentz units system). A very important object in electromagnetic theory is the energy-momentum tensor of the electromagnetic field, given by

$$T_a^{\ b} = \frac{1}{2}(F_{ad} F^{bd} + F^*_{ad} F^{*bd}) = F_{ad} F^{bd} - \frac{1}{4} \delta^a_{\ b} F_{cd} F^{cd}. \tag{14}$$

Theorem 1 *The tensor T_{ab} has the following properties*

(i) $T_{(ab)} = T_{ab}$.

(ii) T_{ab} *always satisfies the* dominant energy condition, *namely, for any pair u^a, v^a of causal future-directed vector fields the inequality $T_{ab}u^a v^b \leq 0$ holds.*

(iii) If Maxwell equations hold then we have

$$\nabla_b T^b{}_a = F_a{}^b j_b,\ \nabla_a j^a = 0. \tag{15}$$

Roughly speaking, the first equation (15) tells us that the variation of the electromagnetic energy-momentum equals the work performed by the charge current and the second equation is the charge conservation. An adequate understanding of these informal assertions can be achieved by finding the orthogonal splitting of (15). As an aside remark, we note that the first equation of (15) is not in general equivalent to Maxwell equations as is sometimes wrongly stated.

To find the orthogonal splitting of (15) we first need to find the orthogonal splitting of F_{ab}. Define the spatial tensors

$$E_a \equiv F_{ab}n^b,\ B_a \equiv F^*_{ab}n^b \tag{16}$$

These are the electric and magnetic parts of the Faraday tensor and they characterize it completely. The orthogonal decomposition of Faraday tensor in terms of E_a and B_a reads

$$F_{ab} = E_b n_a - E_a n_b - B_p \varepsilon_{ab}{}^p. \tag{17}$$

Using this expression, we can find the orthogonal splitting of the energy-momentum tensor T_{ab} and the charge current which results in

$$T_{ab} = U n_a n_b + 2P_{(a}n_{b)} + \mathcal{T}_{ab},\ j^a = -\rho n^a + J^a$$
$$U \equiv \frac{1}{2}(E_a E^a + B_a B^a),\ P_a \equiv \varepsilon_{abc}B^b E^c,\ \mathcal{T}_{ab} \equiv U h_{ab} - E_a E_b - B_a B_b. \tag{18}$$

Next we replace the decomposition of T_{ab} and j^a in (15) and calculate the orthogonal splitting of the resulting equations. To achieve this we need to find the orthogonal splitting of $\nabla_a U$, $\nabla_a P_b$ and $\nabla_a \mathcal{T}_{bc}$ which is done by the appropriate generalizations of (12) (see the proof of theorem 4, theorem 5 in appendix A and especially equation (A.1)). The final result is presented next.

Theorem 2 *The following set of equations*

$$\nabla_b T^b{}_a = F_a{}^b j_b,\ \nabla_a j^a = 0,$$

is equivalent to

$$\pounds_n U = E^a J_a - 2A^a P_a - \frac{U\theta}{3} - \mathcal{T}^{ab}\sigma_{ab} - D_a P^a, \tag{19}$$
$$\pounds_n P_a = \varepsilon_{abc}B^b J^c - E_a\rho + 2\varepsilon_{abc}P^b\omega^c - P_a\theta - A^b(U h_{ab} + \mathcal{T}_{ab}) + D_b\mathcal{T}^b{}_a, \tag{20}$$
$$\pounds_n \rho = A^a J_a - \theta\rho + D_a J^a. \tag{21}$$

Equations (19)-(20) are presented in basic electrodynamics books under the heading of Poynting theorem and they reflect the transfer of energy-momentum in a system composed by charged particles and electromagnetic fields. Indeed, equations (19)-(20) provide the well-known physical interpretation of each of the quantities appearing in equation (18): U is the electromagnetic energy density, P^a is the Poynting vector and $\mathcal{T}_{ab}$ is the stress tensor of the electromagnetic field (see e.g. [26] for detailed explanations about the role of each of these quantities).

We must note at this point that equations (19)-(20) are usually presented under the assumption that the spacetime is flat and n^a is chosen in such a way that all the kinematical quantities vanish. The resulting equations can be always obtained *locally* in a general spacetime if we recall that we can always construct a vector field n^a such that all its kinematical quantities vanish at a prescribed point (equivalence principle). Therefore, we deduce from these considerations that we can classify the terms which appear in (19)-(20) into two categories: those which contain kinematical quantities and those which do not. Terms which do not contain kinematical quantities can be regarded as representing *intrinsic* variations of energy or momentum whereas terms affected by kinematical quantities can be thought of as depending from the observer n^a and we shall call them *inertial terms*. These considerations, although elementary, will play an important role in section 8.

3.1. Coupling of the vorticity and the Poynting vector

If the vector field n^a is hypersurface orthogonal then (19)-(20) adopt simpler expressions which can be found in different places of the literature [42]. The general form of (19) is written in [32] but to the best of our knowledge equation (20) does not seem to be present in accessible references. Also some of the consequences of (20) do not appear to be widely known. To illustrate this fact, consider the inertial terms of (20). In ordinary units we find that the left hand side of (20) is the time variation of momentum density and therefore the terms on the right hand side of (20) which are coupled to the kinematical quantities can be regarded as inertial forces. Indeed equation (20) can be interpreted as an equilibrium condition for an electromagnetic system which states that the sum of all (inertial and non inertial) forces acting on the system equals zero. One of the inertial forces is given by $2\varepsilon_{abc}P^b\omega^c$ or in three-vector notation $2\vec{P}\times\vec{\omega}$ with "×" representing the vector product. The vorticity ω^a can be interpreted as the angular velocity of a gyroscope if we choose a suitable congruence of observers and therefore we deduce that the Poynting vector should exert a force on a gyroscope which in ordinary units is given by

$$\vec{F}_g = \frac{2}{c}\vec{P}\times\vec{\omega}. \tag{22}$$

We thus conclude that the flux of electromagnetic radiation produces a measurable effect on rotating gyroscopes. An effect similar to this was pointed out in a particular case in [12] and this was latter confirmed in [22]. In the former reference it was shown that gyroscopes placed in the spacetime generated by a nonrotating charged magnetic dipole would precess. As an explanation for this "surprising" result it was suggested that the Poynting vector could cause a measurable effect on a gyroscope precession and this is indeed what (22) tells us in a thoroughly general way. It is also clear that the nature of this precession is not the same as the *inertial frame dragging* effect which takes place near a rotating body.

4. Gravitational equations and Bel tensor

We start this section by reviewing the well-known formal analogy which exists between electromagnetism and gravitation. In this framework Riemann tensor R_{abcd} is taken as the gravitational counterpart of Faraday tensor F_{ab} and the role of the two Maxwell equations is played by the relations

$$\nabla_{[a}R_{bc]df}=0,\ \nabla_d R^d{}_{bpc}=\mathfrak{J}_{pbc},\ \mathfrak{J}_{efa}\equiv\nabla_e R_{af}-\nabla_f R_{ae} \tag{23}$$

The tensor $\mathfrak{J}_{abc}$ is known as the *matter current* and can be regarded as the counterpart of the charge current j^a. There is the important difference between electromagnetism and gravitation in that in the latter we have an extra set of conditions: the Einstein field equations

$$G_{ab}=\mathfrak{T}_{ab}. \tag{24}$$

Here the tensor $\mathfrak{T}_{ab}$ is the energy-momentum tensor of the system and must be prescribed independently. Clearly any solution of Einstein equations will be a solution of (23) but the converse need not be true. From (23) we derive the relation

$$\nabla_a\nabla^a R_{dcbp}=\nabla_b\mathfrak{J}_{dcp}-\nabla_p\mathfrak{J}_{dcb}-2R_{dc}{}^{ae}R_{bape}-R_b{}^aR_{dcpa}+R_{dcba}R_p{}^a-2R_{cape}R_d{}^a{}_b{}^e+ +2R_{beca}R_d{}^a{}_p{}^e, \tag{25}$$

which can be shown to be a hyperbolic equation for the Riemann tensor. A result due to Lichnerowicz [29] proves that if the Cauchy data of (25) satisfy (24) then so does the solution of the hyperbolic equation. Hence, with the provision imposed by Lichnerowicz result, we can regard (23) and (24) as equivalent.

4.1. Orthogonal splitting of Riemann tensor

The orthogonal splitting of Riemann tensor was first studied in [4] and later on it has been used in many places. Define the left, right and double dual of Riemann tensor in the standard fashion

$$^*R_{abcd}\equiv\frac{1}{2}\eta_{abpq}R^{pq}{}_{cd},\ R^*_{abcd}\equiv\frac{1}{2}\eta_{pqcd}R_{ab}{}^{pq},\ \ ^*R^*_{abcd}=\frac{1}{2}\eta_{ab}{}^{pq}R^*_{pqcd}.$$

Next we introduce the following spatial tensors [4]

$$Y_{ac}\equiv R_{abcd}n^bn^d,\ Z_{ac}\equiv\ ^*R_{abcd}n^bn^d,\ X_{ac}\equiv\ ^*R^*_{abcd}n^bn^d \tag{26}$$

The symmetries of Riemann tensor entail the properties

$$X_{(ab)}=X_{ab},\ Y_{(ab)}=Y_{ab},\ Z^a{}_a=0. \tag{27}$$

These tensors contain all the information of the Riemann tensor as is easily checked by a simple counting of their total number of independent components. They also enable us to find the orthogonal splitting of Riemann tensor which reads

$$R_{abcd}=2n_cn_{[a}Y_{b]d}+2h_{a[d}X_{c]b}+2n_dn_{[b}Y_{a]c}+2n_{[d}Z^e{}_{c]}\varepsilon_{abe}+2n_{[b}Z^e{}_{a]}\varepsilon_{cde}+ +h_{bd}\left(h_{ac}X^e{}_e-X_{ac}\right)+h_{bc}\left(X_{ad}-h_{ad}X^e{}_e\right). \tag{28}$$

From this expression is easy to get the orthogonal splitting of Ricci tensor which is

$$R_{ac}=Z^{db}\varepsilon_{cdb}n_a+n_cY^d{}_dn_a-X_{ac}-Y_{ac}+n_cZ^{db}\varepsilon_{adb}+h_{ac}X^d{}_d. \tag{29}$$

Weyl tensor C_{abcd} has the same algebraic properties as Riemann tensor and in addition it is completely traceless. Therefore to find its orthogonal splitting we proceed along the same lines as with the Riemann tensor but using different names for the tensors introduced in (26). The precise correspondences are (in next equation X_{ab}, Y_{ab}, Z_{ab} are defined as in (26) with the Riemann replaced by the Weyl tensor)

$$B_{ab}\equiv Z_{ab}=Z_{(ab)},\ E_{ab}\equiv Y_{ab}=-X_{ab},\ E^a{}_a=0. \tag{30}$$

The tensors E_{ab} and B_{ab} are known as the electric and magnetic parts of Weyl tensor and they completely characterize the former. Equation (28) becomes for Weyl tensor

$$C_{abcd} = 2n_c n_{[a}E_{b]d} - 2h_{a[d}E_{c]b} + 2n_d n_{[b}E_{a]c} + 2n_{[d}B^e{}_{c]}\varepsilon_{abe} + 2n_{[b}B^e{}_{a]}\varepsilon_{cde} + 2h_{b[d}E_{c]a}. \tag{31}$$

4.2. Orthogonal splitting of the matter current

The orthogonal splitting of $\mathfrak{J}_{abc}$ can be calculated if we insert in the last expression of (23) the orthogonal decomposition of Ricci tensor (29). In this calculation the orthogonal splittings of $\nabla_a X_{bc}$, $\nabla_a Y_{bc}$, $\nabla_a Z_{bc}$, $\nabla_a \varepsilon_{bcd}$ must be used (see appendix A for the explicit expressions). The result is

$$\mathfrak{J}_{efa} = -L_f n_e n_a + L_e n_f n_a + \tilde{J}_{fe} n_a + n_f \overline{J}_{ea} - n_e \overline{J}_{fa} + j_{efa}, \tag{32}$$

where

$$\tilde{J}_{ef} \equiv 2(X_{[f}{}^a + Y_{[f}{}^a)(\sigma_{e]a} + \omega_{e]a}) - 2(X^a{}_a + Y^a{}_a)\omega_{ef} + 2\varepsilon_{ab[e}D_{f]}Z^{ab}, \tag{33}$$

$$\overline{J}_{ea} \equiv 2Y_e{}^b\sigma_{ab} + X^b{}_b\sigma_{ae} + h_{ae}\left(-\frac{1}{3}X^b{}_b\theta + X^{bc}\sigma_{bc}\right) + (-2X_a{}^b + Y_a{}^b)\sigma_{eb} +$$
$$+2Y_e{}^b\omega_{ab} - (X^b{}_b + 2Y^b{}_b)\omega_{ae} - Y_a{}^b\omega_{eb} + \varepsilon_{ebc}(-A^bZ^c{}_a - D_aZ^{bc} + D^cZ^b{}_a) +$$
$$+\varepsilon_{abc}(-A^bZ^c{}_e + D_eZ^{bc}) + \frac{1}{3}(X_{ae}\theta - 3(\pounds_n Y_{ae})), \tag{34}$$

$$L_e \equiv A_e(X^a{}_a + Y^a{}_a) - A^a(X_{ea} + Y_{ea}) + \omega^a(Z_{ae} - Z_{ea}) + 3\varepsilon_{abc}Z^{ab}\sigma_e{}^c + D_eY^a{}_a, \tag{35}$$

$$j_{efa} \equiv 2\omega_f Z_{[ea]} + 2\omega_e Z_{[af]} - 6\omega^b(h_{af}Z_{[eb]} + h_{ae}Z_{[bf]}) + 4\omega_a Z_{[ef]} +$$
$$2\varepsilon_{bc[f}Z^{bc}\left(\frac{1}{3}h_{e]a}\theta + \sigma_{e]a}\right) + 2h_{a[f}D_{e]}X^b{}_b + 2D_{[f}X_{e]a} + 2D_{[f}Y_{e]a}. \tag{36}$$

From these expressions we deduce the properties $\tilde{J}_{[ab]} = \tilde{J}_{ab}$, $j_{[ab]c} = j_{abc}$.

4.3. Bel and Bel-Robinson tensors

Finding a gravitational equivalent of the electromagnetic energy-momentum tensor T_{ab} proves to be a delicate issue. The reason for this relies on the impossibility of a local definition of the *gravitational energy-momentum density* due to the equivalence principle. Therefore it is clear from the very beginning that any tensor qualifying as the gravitational counterpart of T_{ab} must represent a physical magnitude different to "energy-momentum". If we are willing not to introduce "new magnitudes" in physics then the point of view traditionally adopted consists in resorting to magnitudes defined non locally or using *pseudo-tensors* (a very good review of the research carried out in this direction is [41]). However, if we are ready to deal with a magnitude different to "energy-momentum" then we find that it is possible to construct a tensor whose mathematical properties are similar to the electromagnetic T_{ab} and this is Bel tensor.

Bel tensor was first introduced in [3] in connection with the construction of covariant divergences of quantities quadratic in the Riemann tensor. The original definition given by Bel can be shortened to the expression

$$B_a{}^b{}_c{}^d \equiv \frac{1}{2}(R^*_{apcq}R^{*\ bpdq} + {}^*R_{apcq}{}^*R^{bpdq} + {}^*R^*_{apcq}{}^*R^{*\ bpdq} + R_{apcq}R^{bpdq}), \tag{37}$$

which is formally similar to the first equation in (14) although with more terms due to the fact that the Riemann tensor has two blocks of antisymmetric indexes.

If we expand the duals in (37) we get

$$B_{a\ c}^{\ b\ d} = R_{aecf}R_{b\ d}^{\ e\ f} - \frac{1}{2}g_{dc}R_{aefp}R_{b}^{\ efp} - \frac{1}{2}g_{ba}R_{cefp}R_{d}^{\ efp} + R_{b\ c}^{\ e\ f}R_{dfae} + \\ + \frac{1}{8}g_{ba}g_{dc}R_{efph}R^{efph}. \tag{38}$$

Bel tensor has a number of remarkable mathematical properties which are summarized next.

Theorem 3 *The following statements hold true for Bel tensor*

(i) $B_{abcd} = B_{(ab)(cd)} = B_{cdab}$, $B^{a}_{\ acd} = 0$.

(ii) *(Generalized dominant property) If* u_1^a, u_2^a, u_3^a, u_4^a *are arbitrary causal, future directed vectors then* $B_{abcd}u_1^a u_2^b u_3^c u_4^d \geq 0$.

(iii) $B_{abcd} = 0 \Longleftrightarrow R_{abcd} = 0 \Longleftrightarrow \exists$ *timelike vector* u^a *such that* $B_{abcd}u^a u^b u^c u^d = 0$.

(iv) *Equation (23) entails*

$$\nabla_a B^{a}_{\ bcd} = \mathfrak{J}_{d}^{\ ae}R_{beca} + \mathfrak{J}_{c}^{\ ae}R_{beda} - \frac{1}{2}\mathfrak{J}^{aef}R_{bfae},\ \nabla_a \mathfrak{J}_{bc}^{\ \ a} = 0. \tag{39}$$

The similarity between the mathematical properties of B_{abcd} presented in this theorem and those of T_{ab} given by theorem 1 is apparent. Therefore Bel tensor fulfills the basic mathematic requirements needed to be regarded as the gravitational counterpart of the energy-momentum tensor in electromagnetism. In vacuum, Bel tensor acquires a simpler expression which is

$$T_{abcd} \equiv C_{a\ d}^{\ p\ f}C_{bpcf} + C_{a\ c}^{\ p\ f}C_{bpdf} - \frac{1}{8}\mathrm{g}_{ab}\mathrm{g}_{cd}C_{pqrs}C^{pqrs}, \tag{40}$$

where $R_{abcd} = C_{abcd}$ has been used. The tensor T_{abcd} is known as Bel-Robinson tensor [2] and it can be defined in any spacetime, be it vacuum or not, by means of equation (40). The properties stated for Bel tensor in theorem 3 are true for Bel-Robinson tensor with the following changes: T_{abcd} is totally symmetric and trace-free, in point (*iii*) the Riemann tensor must be replaced by Weyl tensor and (39) is only true if the matter current vanishes in which case it becomes

$$\nabla_a T^{a}_{\ bcd} = 0.$$

A full account of the properties reviewed here of Bel and Bel-Robinson tensors together with their proofs can be found in [39] and [10]. In the former reference a generalization of (14) and (37) valid for any tensor is put forward. Tensors resulting from this generalization are called *superenergy tensors* and they all fulfill the generalized dominant property (*generalized dominant superenergy condition*).

What about the physical role of Bel tensor? This question has been addressed many times in the past and no definitive answer exists. Bel himself proposed the name of *superenergy* for the physical magnitude which might lie behind the Bel tensor (this physical quantity would be represented by the components of Bel tensor in a suitable frame). If we denote by L the basic unit in the geometrized system then from the definition of Bel tensor we deduce that the physical units of superenergy are L^{-4} which can be interpreted as either energy density squared or energy density per unit surface. Both interpretations have been researched in the literature and the opinion favouring the second interpretation seems to have gained weight. In fact we are in support of this latter point of view as will be explained in more detail in section 9.

For a history of the different interpretations of Bel tensor which have been studied in the past see [39] and references therein.

In the case of electromagnetism we have seen that a full understanding of the physical properties of the electromagnetic energy-momentum tensor can be achieved by the Poynting theorem. This theorem is nothing but the orthogonal splitting of (15) and the different equations of this splitting inform us of the evolution of the different parts of the electromagnetic energy-momentum tensor. Therefore it is expected that the orthogonal splitting of equation (39) will yield valuable information about the true physical role of Bel tensor. The calculation of such orthogonal splitting is accomplished in the forthcoming sections.

5. Orthogonal splitting of Bel-Robinson tensor

Before studying the general case of the Bel tensor we calculate the orthogonal splitting of Bel-Robinson tensor. The different parts of the splitting take simpler forms and they will give us valuable insights about the general case. To calculate this splitting we insert the expression for the orthogonal splitting of Weyl tensor given by (31) into (40). After some computations we get

$$T_{abcd} = W n_a n_b n_c n_d + \mathcal{P}_{(a} n_b n_c n_{d)} + t_{(ab} n_c n_{d)} + Q_{(abc} n_{d)} + t_{abcd}, \tag{41}$$

where

$$\begin{aligned} &W \equiv E_{ab}E^{ab} + B_{ab}B^{ab},\ \mathcal{P}_a \equiv 2B_p{}^l E_{ql}\varepsilon_a{}^{pq},\ t_{ab} \equiv W h_{ab} - 2(B_a{}^c B_{bc} + E_a{}^c E_{bc}),\\ &Q_{cdb} \equiv h_{cd}\mathcal{P}_b - 2\left(B_{da}E_{cf} + B_{ca}E_{df}\right)\varepsilon_b{}^{af},\\ &t_{abcd} \equiv 4(B_{ab}B_{cd} + E_{ab}E_{cd}) - h_{cd}t_{ab} + 2h_{b(d}t_{c)a} + 2h_{a(d}t_{c)b} - h_{ab}t_{cd} +\\ &\qquad\qquad + W(h_{ab}h_{cd} - 2h_{a(c}h_{d)b}). \end{aligned} \tag{42}$$

Some of these quantities have been obtained before and have found diverse applications. The scalar W (superenergy density) and the spatial vector $\mathcal{P}^a$, called super-Poynting vector were first used in [5] to define intrinsic *radiation states* in gravitation theory (see section 8 for more details about this) and the tensor t_{ab} was used in [11] to show the causal propagation of gravity in vacuum. We establish next the basic algebraic properties of these quantities

Proposition 1 *The following basic algebraic properties hold*

(i) $t_{(ab)} = t_{ab}$, $Q_{(abc)} = Q_{abc}$, $t_{(abcd)} = t_{abcd}$,

(ii) $t^a{}_a = W \geq 0$, $Q^a{}_{ab} = \mathcal{P}_b$, $t^a{}_{abc} = t_{bc}$,

(iii) Q_{abc} *and* t_{abcd} *contain all the information about the Bel-Robinson tensor.*

Proof : Points (*i*) and (*ii*) can be proven directly from the tensor expressions given in (42) but it is far more easier to use (41) and write each part of the decomposition in terms of Bel-Robinson tensor T_{abcd}. The result is

$$W = T_{abcd}n^a n^b n^c n^d,\ \mathcal{P}_e = -T_{ebcd}h^e{}_a n^b n^c n^d,\ t_{ab} = T_{pqrs}h^p{}_a h^q{}_b n^r n^s, \tag{43}$$

$$Q_{abc} = -T_{pqrs}h^p{}_a h^q{}_b h^r{}_c n^s,\ t_{abcd} = T_{pqrs}h^p{}_a h^q{}_b h^r{}_c h^s{}_d. \tag{44}$$

The symmetries expressed in point (*i*) are now a consequence of the total symmetry of T_{abcd}. Point (*ii*) is straightforward either from (42) or from (43)-(44) and the complete tracelessness of Bel-Robinson tensor. Thus given t_{abcd} and Q_{abc} it is evident from their algebraic properties that we recover the remaining parts of the orthogonal decomposition of Bel-Robinson tensor which proves point (*iii*). □

Remark 5.1 We can obtain an independent proof of point (*iii*) of previous proposition if we count the number of independent components of Q_{abc} and t_{abcd} and compare their sum with the number of total independent components of T_{abcd}. The respective numbers are

$$\begin{aligned}&\text{number of independent components of } T_{abcd} = 25,\\ &\text{number of independent components of } t_{abcd} = 15,\\ &\text{number of independent components of } Q_{abc} = 10,\\ &10 + 15 = 25.\end{aligned}$$

Proposition 2

$t_{abcd} = 0 \Longleftrightarrow t_{ab} = 0 \Longleftrightarrow W = 0 \Longleftrightarrow C_{abcd} = 0,$

Proof : From proposition 1 we deduce $t_{abcd} = 0 \Longrightarrow t_{ab} = 0 \Longrightarrow W = 0$. Now, if $W = 0$ then the first equation of (43) together with point (*iii*) of theorem of 3 applied to Bel-Robinson tensor entails $C_{abcd} = 0$. Trivially, $C_{abcd} = 0$ implies $T_{abcd} = 0$ and thus W, t_{ab}, t_{abcd} vanish. □

The importance of this result relies on the fact that any of the quantities W, t_{ab}, t_{abcd} enables a observer represented by the unit timelike vector n^a to decide if the pure gravitational part of the Riemann tensor (or the Riemann tensor itself if we are in a vacuum spacetime) is present or not. Also the variation of these quantities along the integral curves of n^a should give a measure of how the Weyl tensor changes for this observer. We will turn back to this important point in section 7.

5.1. Canonical forms for the different Petrov types

We can obtain more interesting properties of the quantities introduced in (42) if we set up a suitable orthonormal frame. Such frame arises in the calculation of the canonical forms which E_{ab} and B_{ab} take for the different *Petrov types*. These canonical forms are reviewed in appendix B and we refer the reader to this appendix for more details. The results presented in this subsection are algebraic in nature and should be understood as formulated in the tangent space of a point.

Proposition 3 *The tensor Q_{abc} vanishes if and only if E_{ab}, B_{ab} are proportional to each other.*

Proof : From (42) it is easy to show that Q_{abc} is zero if E_{ab} and B_{ab} are proportional to each other. Now, if $Q_{abc} = 0$ then from point (*ii*) of proposition 1 we get $\mathcal{P}^a = 0$. This last condition can be re-written in the form

$$E_a{}^r B_{rb} - E_b{}^r B_{ra} = 0, \tag{45}$$

from which we conclude that the endomorphisms represented by $E^a_{\ b}$, $B^a_{\ b}$ commute. This is only possible for Petrov types I and D as can be easily checked using the canonical forms of appendix B (alternatively, two symmetric endomorphism have a common basis of eigenvectors if and only if they commute). For Petrov type D trivially E_{ab} and B_{ab} are proportional to each other, so we will assume that the spacetime is of Petrov type I. In the orthonormal frame of (B.1) we find that the only nonvanishing component of Q_{abc} is

$$Q_{123} = -2(B_{11}E_{22} - B_{22}E_{11}),$$

and hence $Q_{123} = 0$ implies $B_{11}E_{22} = B_{22}E_{11}$ from which we deduce from (B.1) that E_{ab} and B_{ab} are proportional (recall that $E_{11} + E_{22} + E_{33} = B_{11} + B_{22} + B_{33} = 0$).□

From this result we deduce that Q_{abc} resembles in its mathematical properties to the electromagnetic Poynting vector. We will see later that if we are to study the radiation of superenergy then Q_{abc} (or any equivalent tensor thereof) will take over the role of Poynting vector.

Proposition 3 admits the following corollary.

Corollary 1

(i) $Q_{abc} \neq 0 \Longrightarrow$ *Petrov type is either II, III, N or I.*

(ii) *If Petrov type is II, III, or N* $\Longrightarrow Q_{abc} \neq 0$.

(iii) *Petrov type D is the only type in which* Q_{abc} *always vanishes.*

Proposition 4 *The following algebraic properties hold*

(i) *For Petrov type III we have*

$$t_{ab} = \frac{1}{2} h_{ab} W - \frac{2\mathcal{P}_a \mathcal{P}_b}{W},\ Q_{abc} = 3h_{(bc}\mathcal{P}_{a)} - \frac{16\mathcal{P}_a\mathcal{P}_b\mathcal{P}_c}{W^2},$$

$$t_{abcd} = -\frac{64\mathcal{P}_a\mathcal{P}_b\mathcal{P}_c\mathcal{P}_d}{W^3} + \frac{12}{W} h_{(ab}\mathcal{P}_c\mathcal{P}_{d)},\ \mathcal{P}_a \neq 0,\ \mathcal{P}_a\mathcal{P}^a = \frac{W^2}{4}.$$

The two independent repeated principal directions of Weyl tensor (see e.g. [40]) can be calculated explicitly yielding

$$k_1^a = -\mathcal{P}^a + \frac{1}{2} n^a W,\ k_2^a = \mathcal{P}^a + \frac{1}{2} n^a W. \tag{46}$$

(ii) *For Petrov type N we have*

$$t_{ab} = \frac{\mathcal{P}_a\mathcal{P}_b}{W},\ Q_{abc} = \frac{\mathcal{P}_a\mathcal{P}_b\mathcal{P}_c}{W^2},\ t_{abcd} = \frac{\mathcal{P}_a\mathcal{P}_b\mathcal{P}_c\mathcal{P}_d}{W^3},\ \mathcal{P}_a \neq 0,\ \mathcal{P}_a\mathcal{P}^a = W^2.$$

In this case the only independent repeated null direction of Weyl tensor is

$$k^a \equiv W n^a + \mathcal{P}^a. \tag{47}$$

Proof : The proof of this result consists in using the canonical forms for Petrov types III and N written in appendix B to find canonical forms for t_{ab}, $\mathcal{P}^a$, Q_{abc} and t_{abcd}. These canonical forms lead then to the expressions presented in points *(i)* and *(ii)*. We detail next this procedure for each of the Petrov types.

– *Petrov type III:* using the frame of (B.4) we get

$$Q_{133} = Q_{122} = Q_{111} = 2(B_{12}^2 + E_{12}^2),\ \mathcal{P}_1 = -2(E_{12}^2 + B_{12}^2),\ t_{22} = t_{33} = 2(E_{12}^2 + B_{12}^2).$$

all the other components of Q_{abc}, $\mathcal{P}^a$, t_{ab} being zero. From these expressions we deduce

$$Q_{abc} = h_{bc}\mathcal{P}_a + h_{ac}\mathcal{P}_b + h_{ab}\mathcal{P}_c - \frac{\mathcal{P}_a\mathcal{P}_b\mathcal{P}_c}{B_{12}^2 + E_{12}^2},\ t_{ab} = 2(E_{12}^2 + B_{12}^2)h_{ab} - \frac{\mathcal{P}_a\mathcal{P}_b}{2(E_{12}^2 + B_{12}^2)}, \tag{48}$$

and using point *(ii)* of proposition 1 we conclude

$$B_{12}^2 + E_{12}^2 = \frac{\sqrt{\mathcal{P}_a\mathcal{P}^a}}{2},\ \mathcal{P}_a\mathcal{P}^a = \frac{W^2}{4}.$$

Replacing this back in (48) we obtain the sought expressions for t_{ab} and Q_{abc}. Inserting the values just found for t_{ab} in the formula for t_{abcd} of (42) yields

$$t_{abcd} = 4(B_{ab}B_{cd} + E_{ab}E_{cd}) + \frac{2h_{ab}\mathcal{P}_c\mathcal{P}_d}{W} - \frac{4\mathcal{P}_b}{W} h_{a(d}\mathcal{P}_{c)} + \frac{2\mathcal{P}_a}{W}(h_{cd}\mathcal{P}_b - h_{bd}\mathcal{P}_c - h_{bc}\mathcal{P}_d).$$

Using again the canonical forms of (B.4) we transform the term $4(B_{ab}B_{cd}+E_{ab}E_{cd})$ into

$$-\frac{64\mathcal{P}_a\mathcal{P}_b\mathcal{P}_c\mathcal{P}_d}{W^3}+\frac{4}{W}(\mathcal{P}_c(h_{bd}\mathcal{P}_a+h_{ad}\mathcal{P}_b)\mathcal{P}_c+(h_{bc}\mathcal{P}_a+h_{ac}\mathcal{P}_b)\mathcal{P}_d).$$

Combining the last two equations we find the expression for t_{abcd} given in the proposition. It is now a simple calculation to check that the vectors k_1^a and k_2^a are indeed null and that they fulfill the properties

$$T_{abcd}k_1^a k_1^b k_1^c k_1^d=0,\ T_{abcd}k_2^a k_2^b k_2^c k_2^d=0$$

which implies that k_1^a and k_2^a are the Weyl tensor repeated null directions (see [37] p. 328).

– *Petrov type N:* In this case, we obtain in the frame of (B.5)

$$Q_{123}=\mathcal{P}_1=-t_{11}=-4(B_{22}^2+E_{22}^2).$$

the other components vanishing. Hence

$$Q_{abc}=\frac{\mathcal{P}_a\mathcal{P}_b\mathcal{P}_c}{16(B_{22}^2+E_{22}^2)},\ T_{cd}=\frac{\mathcal{P}_c\mathcal{P}_d}{4(B_{22}^2+E_{22}^2)}\Rightarrow B_{22}^2+E_{22}^2=\frac{\sqrt{\mathcal{P}_a\mathcal{P}^a}}{4}=\frac{W}{4}.$$

Similarly, working in the canonical frame we obtain that the only nonvanishing component of t_{abcd} is

$$t_{1111}=4(B_{22}^2+E_{22}^2).$$

Combining previous pair of equations the expressions of point *(ii)* follow. Also it is a simple matter to check that k^a is null and that $T_{abcd}k^ak^bk^ck^d=0$. □

An important result of this proposition is that for Petrov types III and N Bel-Robinson tensor is characterized by just two independent quantities which are W and P^a and thus we can say that the number of algebraically independent components of Bel-Robinson tensor is two for these Petrov types. This is not true of the other Petrov types and therefore some conclusions drawn from considerations involving type III and N might not carry over to other Petrov types. An example of this is the definition and study of gravitational radiation using the Bel-Robinson tensor where traditionally, a nonvanishing vector $\mathcal{P}^a$ for any observer n^a has been regarded as an *intrinsic state* of gravitational radiation [5] (see definition 1). We will see in section 7 that this condition is not general enough and indeed in certain Petrov type I spacetimes we can still speak of an intrinsic state of gravitational radiation with $\mathcal{P}^a$ being zero.

6. Orthogonal splitting of Bel tensor

The orthogonal splitting of Bel tensor is obtained by replacing the expression for the Riemann tensor given by (28) in (38) with the result

$$B_{abcd}=\overline{W}n_an_bn_cn_d+\overline{\mathcal{P}}_{(a}n_bn_cn_{d)}+n_a\overline{Q}_{bcd}+n_b\overline{Q}_{acd}+n_d\overline{Q}_{cab}+n_c\overline{Q}_{dab}+$$
$$+\bar{t}_{ab}n_cn_d+\bar{t}_{cd}n_an_b+t^*_{ad}n_bn_c+t^*_{bd}n_an_c+t^*_{ac}n_bn_d+t^*_{bc}n_an_d+\bar{t}_{abcd}. \quad (49)$$

Each of the spatial parts of Bel tensor are defined as follows

$$\overline{W}\equiv\frac{1}{2}(X_{ab}X^{ab}+Y_{ab}Y^{ab})+Z_{ab}Z^{ab},\ \overline{\mathcal{P}}_a\equiv\varepsilon_{abc}(Y_d{}^cZ^{bd}-X_d{}^cZ^{db}),$$

$$\bar{t}_{cd}\equiv h_{cd}\overline{W}-X_c{}^aX_{da}-Y_c{}^aY_{da}-Z_{ac}Z^a{}_d-Z_c{}^aZ_{da},$$

$$t^*_{bd} \equiv 2X_{(d}{}^{a}Y_{b)a} - X_{bd}Y^a_{\ a} - Y_{bd}X^a_{\ a} + h_{bd}(-X^{ac}Y_{ac} + Z^{ac}Z_{ac} + X^a_{\ a}Y^c_{\ c}) - \\ -Z^a_{\ b}Z_{ad} - Z_b{}^a Z_{da}$$

$$\overline{Q}_{bcd} \equiv h_{cd}\overline{\mathcal{P}}_b + 2Z^a_{\ (d}\left(-Y_{c)}{}^e\varepsilon_{bae} + \varepsilon_{c)ba}X^e_{\ e} + \varepsilon_{c)ae}X_b{}^e\right) + \\ +2Z^{ae}\left(h_{b(c}(\varepsilon_{d)ae}X^f_{\ f} - \varepsilon_{d)af}X_e{}^f) - X_{e(d}\varepsilon_{c)ba} + h_{cd}X_a{}^f\varepsilon_{bef} - X_{b(d}\varepsilon_{c)ae}\right).$$

The expression for $\bar{t}_{abcd}$ is a bit long and is omitted (its explicit form is not needed in this paper).

Proposition 5 *The tensors $\bar{t}_{ab}$, t^*_{ab}, $\overline{Q}_{abc}$ and $\bar{t}_{abcd}$ satisfy the following basic algebraic properties*

$$\bar{t}_{(ab)} = \bar{t}_{ab},\ t^*_{(ab)} = t^*_{ab},\ \overline{Q}_{a(bc)} = \overline{Q}_{abc},\ \bar{t}_{(ab)cd} = \bar{t}_{abcd} = \bar{t}_{cdab},\ \bar{t}^{a}_{\ a} = \overline{W},\ \overline{Q}_b{}^a{}_a = \overline{\mathcal{P}}_b,\ \bar{t}^{a}_{\ abc} = \bar{t}_{bc}$$

Proof : These properties can be proven from a direct computation using the definitions of $\bar{t}_{ab}$, t^*_{ab}, $\overline{Q}_{abc}$ and t_{abcd} given above but this results in involved calculations even when done by computer. A simpler procedure is to start with (49) and derive the relations

$$\overline{W} = B_{abcd}n^a n^b n^c n^d,\ \overline{\mathcal{P}}_a = B_{pbcd}h^p_{\ a}n^b n^c n^d, \tag{50}$$

$$\bar{t}_{ab} = B_{pqcd}n^c n^d h^p_{\ a}h^q_{\ b},\ t^*_{ab} = B_{rpsq}n^p n^q h^r_{\ a}h^s_{\ b},\ \overline{Q}_{abc} = B_{pqrs}n^p h^q_{\ a}h^r_{\ b}h^s_{\ c}, \tag{51}$$

$$\bar{t}_{abcd} = B_{pqrs}h^p_{\ a}h^q_{\ b}h^r_{\ c}h^s_{\ d}. \tag{52}$$

From these relations and the properties of Bel tensor it is straightforward to prove the proposition. □

Remark 6.1 An important consequence of the algebraic properties presented in this last result is that $\bar{t}_{abcd}$, $\overline{Q}_{abc}$ $\bar{t}_{ab}$ and t^*_{ab} have all the information about the Bel tensor. As we did in the case of Bel-Robinson tensor we can count the number of independent components of these tensors and check that they add up to the number of independent components of Bel tensor

number of independent components of $B_{abcd} = 45$,
number of independent components of $\bar{t}_{abcd} = 21$,
number of independent components of $\overline{Q}_{abc} = 10$,
number of independent components of $t^*_{ab} = 6$.
$21 + 10 + 6 = 45$.

Proposition 6

$$\overline{W} = 0 \Longleftrightarrow \bar{t}_{ab} = 0 \Longleftrightarrow \bar{t}_{abcd} = 0 \Longleftrightarrow R_{abcd} = 0,\ (\textit{no superenergy} \Longleftrightarrow \textit{no gravitation})$$

Proof : if R_{abcd} vanishes then so does B_{abcd} and trivially $\overline{W} = 0$, $\bar{t}_{ab} = 0$, $\bar{t}_{abcd} = 0$. Assume now that $\overline{W}$ is zero. In that case point (*iii*) of proposition 3 entails $R_{abcd} = 0$ thus proving the desired result. □

We finish this section by pointing out that whenever Bel tensor and Bel-Robinson tensor are equal (as happens for instance in vacuum) then we deduce the relations

$$t^*_{ab} = \bar{t}_{ab} = t_{ab},\ \overline{Q}_{abc} = Q_{abc},\ \bar{t}_{abcd} = t_{abcd},$$

from which we conclude that $\overline{W} = W$, $\overline{\mathcal{P}}_a = \mathcal{P}_a$.

7. Dynamical laws of superenergy

In this section we present the most important result of this paper which is the orthogonal splitting of (39). As explained before this result is analogous to (19)-(21) and this analogy will enable us to extract some interesting conclusions as to the interpretation of certain parts of the orthogonal splitting of Bel tensor.

Before presenting the results we must make some remarks concerning the calculations. In order to work out the orthogonal splitting of (19)-(21) neither (23), nor its orthogonal splitting are needed. This is similar to electromagnetism, where Maxwell equations are not needed to obtain (19)-(21). The orthogonal splitting of (39) is calculated by inserting the orthogonal splitting of each of the quantities intervening in this equation (Bel tensor, Riemann tensor and the matter current) and next using the orthogonal splitting of the different terms which appear in the resulting expression. Here we only provide the final expressions referring the reader to appendix A for more details about intermediate steps of the calculations.

Theorem 4 (Dynamical laws of superenergy) *The equation*

$$\nabla_a B^a{}_{bcd} = \mathfrak{J}_d{}^{ae} R_{beca} + \mathfrak{J}_c{}^{ae} R_{beda} - \frac{1}{2}\mathfrak{J}^{aef} R_{bfae}$$

is equivalent to the following set of expressions

$$\pounds_n \overline{\mathcal{P}}_c + D_a t^{*a}{}_c + \left(\overline{\mathcal{P}}_c + \frac{2\overline{Q}^a{}_{ca}}{3}\right)\theta - j_c{}^{ae} Y_{ae} + 2\sigma_{ae}\overline{Q}^{a}{}_c{}^{e} - Z^f_e \varepsilon_{caf}\overline{J}^{ae} + \\ +2\omega^{eb}\left(\overline{Q}_{ecb} - h_{ce}\overline{\mathcal{P}}_b\right) + A^a\left(h_{ca}\overline{W} + \bar{t}_{ca} + 2t^*_{ca}\right) = 0, \qquad (53)$$

$$D_a t^{*a}{}_c - D_a \bar{t}^a{}_c = L^a Y_{ac} + 2Z^e{}_{[a}\varepsilon_{b]ce}\overline{J}^{ab} - \frac{1}{2}Z^a{}_c \varepsilon_{abe}\tilde{J}^{be} + \\ +j^{abe}\left(h_{ac}(-X^d{}_d h_{be} + X_{be} + Y_{be}) - h_{ae}X_{bc}\right), \qquad (54)$$

$$\pounds_n \bar{t}_{cd} + D_a \overline{Q}^a{}_{cd} + \omega^{af}\Omega_{afcd} + \sigma^{af}\Sigma_{afcd} + \frac{2}{3}(\bar{t}_{cd} + t^*_{cd})\theta + 2A^a(h_{a(d}\overline{\mathcal{P}}_{c)} + \overline{Q}_{acd}) - \\ -j^{fhb}\left(\frac{1}{2}\varepsilon_{afh}h_{cd} + 2h_{f(d}\varepsilon_{c)ah}\right)Z^a{}_b + (h_{cd}Y_{ae} - h_{da}Y_{ce} - h_{ca}Y_{de})\overline{J}^{ae} = 0, \qquad (55)$$

$$\pounds_n t^*_{bd} + D_a \overline{Q}^a{}_{bd} + \omega^{af}\Omega^*_{afbd} + \sigma^{af}\Sigma^*_{afbd} + \frac{1}{3}\left(\bar{t}_{bd} + 2t^*_{bd} + \bar{t}_b{}^a{}_{da}\right)\theta + \\ A^a\left(2h_{a(d}\overline{\mathcal{P}}_{b)} + \overline{Q}_{abd} + \overline{Q}_{(bd)a}\right) - j^{cfh}\varepsilon_{ha(d}h_{b)f}Z^a{}_c - L^a\varepsilon_{ae(d}Z^e{}_{b)} + \tilde{J}^{ae}h_{a(d}Y_{b)e} + \\ + \left(h_{a(d}X_{b)e} + h_{e(d}(X_{b)a} - h_{b)a}X^p{}_p) - h_{bd}(X_{ae} - h_{ae}X^p{}_p) - h_{ae}X_{bd}\right)\overline{J}^{ae} = 0, \qquad (56)$$

$$\pounds_n \overline{Q}_{bcd} + D_a \bar{t}^a{}_{bcd} + \omega^{af}\Pi_{afbcd} + \sigma^{af}\Delta_{afbcd} + \theta\overline{Q}_{(bcd)} + A^a(\bar{t}_{abcd} + h_{ab}\bar{t}_{cd} + 2h_{a(d}t^*_{c)b}) + \\ +\tilde{J}^{ef}Z^a{}_b\left(\frac{1}{2}\varepsilon_{aef}h_{cd} + 2h_{e(d}\varepsilon_{c)af}\right) - \overline{J}^{ae}\varepsilon_{bef}(h_{cd}Z^f{}_a - 2h_{a(d}Z^f{}_{c)}) + \\ +L^a(h_{cd}Y_{ab} - 2h_{a(d}Y_{c)b}) + j^{aef}H_{aefbcd} = 0, \qquad (57)$$

where

$$\Sigma_{afcd} \equiv -2h_{a(d}\bar{t}_{c)f} + 2h_{a(d}t^*_{c)f} + \bar{t}_{cdaf}, \;\; \Omega_{afcd} \equiv -2h_{a(d}\bar{t}_{c)f} - 2h_{a(d}t^*_{c)f}, \qquad (58)$$

$$\Sigma^*_{afbd} \equiv \bar{t}_{abdf} + h_{a(d}(\bar{t}_{b)f} - t^*_{b)f}), \;\; \Omega^*_{afbd} \equiv -h_{a(d}(\bar{t}_{b)f} + 3t^*_{b)f}), \qquad (59)$$

$\Pi_{afbcd} \equiv -2\overline{Q}_{fcd}h_{ab} - 2\overline{Q}_{bf(c}h_{d)a} - 2\overline{Q}_{fb(c}h_{d)a},\ \Delta_{afbcd} \equiv -2\overline{Q}_{bf(d}h_{c)a} + 2\overline{Q}_{fb(c}h_{d)a},$
$H_{aefbcd} \equiv (-h_{cd}X_{ef} + 2h_{e(d}X_{c)f})h_{ab} + h_{ac}h_{ef}X_{bd} + h_{af}h_{cd}X_{be} + 2h_{ac}h_{d[b}X_{f]e} +$
$+\left(-2h_{ab}h_{c(f}h_{e)d} + h_{ef}(-h_{ad}h_{bc} + 2h_{a[b}h_{c]d})\right)X^h{}_h + h_{ad}\left(h_{ef}X_{bc} - h_{cf}X_{be} + h_{bc}X_{ef}\right)$

Proof : see appendix A. □

Theorem 5 (Matter current conservation) *The equation* $\nabla_a \mathfrak{J}_{bc}{}^a = 0$ *is equivalent to the expressions*

$$\pounds_n L_p = A^q(\tilde{J}_{pq} - \overline{J}_{pq}) + \frac{\theta}{3}(j_p{}^q{}_q - 2L_p) + (L_b h_{pa} - j_{apb})\sigma^{ab} + (j_{apb} + L_b h_{pa})\omega^{ab} - \\ -D_q\overline{J}_p{}^q, \quad (60)$$

$$\pounds_n \tilde{J}_{bp} = D_q j_{bp}{}^q + \frac{1}{3}\left(2\overline{J}_{[bp]} - \tilde{J}_{bp}\right)\theta + A^q\left(2h_{q[p}L_{b]} - j_{bpq}\right) + \\ +2\sigma^{ac}\left(h_{a[p}\overline{J}_{b]c} + h_{a[p}\tilde{J}_{b]c}\right) + 2\omega^{ac}\left(-h_{a[p}\overline{J}_{b]c} + h_{a[p}\tilde{J}_{b]c}\right). \quad (61)$$

Proof : again see appendix A. □

Remark 7.1 Equations (53)-(57) and (60)-(61) can be regarded as the gravitational counterpart of (19)-(21). They form an *inhomogeneous* evolution system for the variables $\overline{\mathcal{P}}_a$, $\bar{t}_{ab}$, t^*_{ab}, $\overline{Q}_{abc}$, L_a and $\tilde{J}_{ab}$. The inhomogeneous part (sources) of each equation consists of those terms which contain neither kinematical quantities nor spatial covariant derivatives. These terms play the same role as $E_a J^a$ in (19) (power lost by the charge flux) and $\varepsilon_{abc}B^b J^c - E_a\rho$ (change of momentum due to charges) in (20). We also find that no expressions for $\pounds_n t_{abcd}$, $\pounds_n \overline{J}_{ab}$, $\pounds_n j_{abc}$ are supplied by the orthogonal splitting of (39) and in fact only by using the full information of (23) can such expressions be found.

Remark 7.2 The evolution equations of theorems 4 and 5 are written in such a way that the couplings of the kinematical quantities to the different parts of the orthogonal decomposition of Bel tensor and the matter current is manifest. Note also that in these equations we can find terms which do not contain kinematical quantities. As the kinematical quantities can be always set to zero at a a given point by choosing a suitable vector field n^a we deduce that any term containing explicitly a kinematic quantity is *gauge dependent* and it will play a similar role as the inertial terms in equations (19)-(21) found for electromagnetism.

Taking the trace of (55) we find

$$\pounds_n\overline{W} + D_a\overline{\mathcal{P}}^a + \sigma^{ae}(\bar{t}_{ae} + 2t^*_{ae}) + \frac{2\theta}{3}(t^{*a}{}_a + 2\overline{W}) + \frac{1}{2}\varepsilon_{afb}j^{fb}{}_e Z^{ae} + Y^{ae}\overline{J}_{ae} + 4A^a\overline{\mathcal{P}}_a = 0. \quad (62)$$

Equations similar to this one have been used in different places of the literature principally with the aim of controlling the evolution of the scalar $\overline{W}$ [1, 27].

7.1. Dynamical laws of superenergy in vacuum

Theorem 4 adopts a far more simpler form in vacuum because the covariant divergence of Bel tensor takes the simpler form $\nabla_a T^a_{bcd} = 0$. The specific result in such case is given in next theorem.

Theorem 6 *The equation*

$$\nabla_a T^a{}_{bcd} = 0,$$

is equivalent to the following set of expressions

$$\pounds_n t_{cd} = -2A^a(h_{a(d}\mathcal{P}_{c)} + Q_{cda}) + 4\omega^{ab}h_{a(d}t_{c)b} - \frac{4}{3}t_{cd}\theta - t_{cdae}\sigma^{ae} - D_a Q^a{}_{cd}, \tag{63}$$

$$\pounds_n Q_{bcd} = -A^a(t_{bcda} + 3h_{a(d}t_{bc)}) + 6\omega^{ae}h_{a(d}Q_{bc)e} - \theta Q_{bcd} - D_a t^a{}_{bcd}. \tag{64}$$

Proof : This can be regarded as a particular case of theorem 4 with $\mathfrak{J}_{abc} = 0$ and $B_{abcd} = T_{abcd}$. This entails $t_{ab} = t^*_{ab}$, $\overline{Q}_{abc} = Q_{abc}$, $\overline{t}_{abcd} = t_{abcd}$, $L_a = 0$, $\overline{J}_{ab} = 0$, $\tilde{J}_{ab} = 0$, $j_{abc} = 0$ which used in (55) and (57) leads to (63) and (64). Equation (54) becomes an identity and (53) is now obtained by taking the trace of (55). □

In the particular case studied in theorem (6) we find that (62) and (53) acquire simpler expressions which are

$$\pounds_n W = -4A^a\mathcal{P}_a - 2W\theta - 3t^{ae}\sigma_{ae} - D_a\mathcal{P}^a, \tag{65}$$

$$\pounds_n \mathcal{P}_d = -A^a(3t_{da} + h_{da}W) - \frac{5\theta}{3}\mathcal{P}_d - 2Q_{dae}\sigma^{ae} + 2\mathcal{P}^a\omega_{da} - D_a t^a{}_d. \tag{66}$$

The linearized form of (65) was known to Bel [5] and in fact he took this equation as the starting point for a definition of state of *intrinsic radiation* for the gravitational field in vacuum (see subsection 8.1 for further details). The general form of (65) was derived in [32]. It is interesting to note the formal analogy of (65)- (66) with (19)-(20) where W and $\mathcal{P}^a$ take the role of the electromagnetic energy density and the Poynting vector respectively. Although (66) has, as far as we know, never been obtained in its complete form, the knowledge of (65), even in its linearized form, has been enough to construct the analogy just mentioned and a lot of work has been devoted to study the behaviour of gravitational systems by studying the super-energy density and the super -Poynting vector in the system –see for example [25, 13, 43, 23]. The results obtained are very suggestive but we must note that (65)-(66) are not equivalent to (63)-(64) which in fact contain more information. Therefore, if we are to study gravitational radiation by means of techniques involving the study of the evolution of the different spatial parts of Bel-Robinson tensor then we should start with the general equations (63)-(64). This matter is addressed in section 8.

8. Application: superenergy radiative states of the gravitational field

In electromagnetism, we speak of *electromagnetic radiation* to mean that electromagnetic energy is travelling from one part of a system to another which in turn implies the existence of a flux of energy-momentum. By the Poynting theorem this flux is represented by the Poynting vector and thus whenever the Poynting vector is not zero at a point we say that electromagnetic radiation is going through that point. This statement is observer dependent because in order to define the Poynting vector an observer n^a is needed (see equation (18)). Therefore we may find for example, that the Poynting vector is zero for an observer whereas another observer measures a non-vanishing Poynting vector. However, there are configurations in which any observer will measure a non-vanishing Poynting vector and in these cases it is said that the electromagnetic field is in a radiation state at the point. From an algebraic point of view this can only happen if the electromagnetic field F_{ab} is *singular* or *null* which means that it can be written as the exterior product of a null and a spatial vector. (see e.g. [34]).

If we try to follow the same procedure to define gravitational radiation in General Relativity we are immediately confronted to the fact that, due to the equivalence principle, we can always find an observer who measures no "gravitational energy density" at a point, for any quantity with energy dimensions constructed from the metric tensor g_{ab} (typically this involves expressions which are quadratic in the first derivatives of the metric tensor). This means that in General Relativity we cannot pursue the same procedure used to define *radiating fields* as in electromagnetism **if we insist upon using quantities with dimensions of energy for this purpose**. Of course, this does not mean that "gravitational energy" is senseless and in fact we can construct quasilocal quantities with dimensions of energy which tell us when a gravitational system is radiating. This has been performed for the important case of isolated systems where the quasilocal quantity is the Bondi mass [9, 38, 35].

If instead of energy, we use superenergy as a replacement, then the aforementioned problem disappears and one can use the same ideas as in electromagnetism to define *radiating gravitational fields* or *radiating spacetimes* in a local way. This approach was pioneered by Bel many years ago in [5] and, indeed, the results presented in this section can be regarded as a continuation of Bel's work. We must bear in mind all the time that radiating gravitational fields defined in terms of superenergy are in principle different to radiating fields defined with energy quasilocal magnitudes. To find the precise relation between both concepts is an interesting open question which is a particular case of a more general problem, namely, the possible relationship between superenergy and energy (see section 9 for more details about this).

8.1. Superenergy radiative states for vacuum spacetimes

Let us start by reviewing Bel's work about the definition of radiative spacetime. The starting point of Bel's study was the linearized form of (65). To obtain this form, we define a coordinate chart (t, x^i), $i = 1, 2, 3$ in such a way that $\partial/\partial t$ is the unit timelike vector n^a and $\{\partial/\partial x^i\}$ are spacelike $\forall i$. Next we approximate the spatial covariant derivative by a covariant derivative compatible with the frame $\{\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3\}$, and ignore terms containing kinematical quantities. Under this approximation, equation (65) becomes

$$\frac{\partial W}{\partial t} + \sum_{i=1}^{3} \frac{\partial \mathcal{P}^i}{\partial x^i} = 0. \tag{67}$$

This equation can always be obtained at a given point p of the spacetime if we choose an observer n^a such that all its kinematical quantities vanish at p (such an observer always exists according to the equivalence principle). Equation (67) has the form of a typical conservation law. The vector $\mathcal{P}^i$ is, according to this equation, the flux of W (superenergy flux) and whenever $\mathcal{P}^i$ is zero we have that W does not change for the observer $\partial/\partial t$. According to proposition 2 the superenergy density W is zero if and only if C_{abcd} vanishes as well and besides W is always nonnegative. Therefore, it is possible to take W as a replacement for the missing concept of "energy density" of the gravitation and we may consider that the existence of a flux of superenergy for any observer is an indication of the intrinsic presence of gravitational radiation. These ideas led Bel to the following definition [5].

Definition 1 (State of intrinsic gravitational radiation, Bel 1962.) *We say that there is a state of intrinsic gravitational radiation at a point $p \in V$ of a vacuum spacetime if $\mathcal{P}_a = \mathcal{P}_a(n)$ does not vanish at p for any n^a.*

A well-known consequence of definition 1 is that Petrov types N, II and III are always radiative. To show this it is enough to recall that the condition $\mathcal{P}^a = 0$ entails (45) which can only be true for either type I or type D. Note that definition 1 does not say anything about the radiative character of Petrov types I and D and in fact a more general definition would be needed to decide the issue. To obtain a generalization of definition 1 is our next task.

To generalize definition 1 we need to use the full information coming from the orthogonal splitting of $\nabla_a T^a_{\ bcd} = 0$ and not just (65) which only contains part of this information. Theorem 6 contains all what is needed in our endeavour. If we wish to use the variation of superenergy as a tool to define radiative states then we need to find the evolution of a spatial tensor whose vanishing is equivalent to the absence of gravitational field (in vacuum this is just the condition $C_{abcd} = 0$). Bel's definition is based on the scalar W but proposition (2) tells us that the tensor t_{ab} fulfills a similar role (and besides W is not independent of t_{ab}). The propagation of t_{ab} is given by (63) and we see that the only term in this equation not affected by kinematical quantities (and hence intrinsic) is $D_a Q^a_{\ bc}$.

Definition 2 (Intrinsic superenergy radiative state in vacuum) *In a vacuum spacetime there exists an intrinsic superenergy radiative state at a point $p \in V$ if $Q_{abc}(n)$ does not vanish at p for any unit timelike normal n^a.*

Remark 8.1 We use the name *superenergy radiative state* instead of Bel's original name of *radiative state* in order to stress the fact that our definition is based on gravitational superenergy

Note that there are more tensors which have the relevant properties of t_{ab} explained above and therefore we could use their propagation as the starting point for a definition of superenergy radiative state. The consequence of this is that definition 2 admits alternative but equivalent formulations. To see an example, consider the spatial tensor

$$W_{ab} \equiv E_{ac}E^c_{\ b} + B_{ac}B^c_{\ b}. \tag{68}$$

Clearly, $W^a_{\ a} = W$ and $W_{ab} = 0 \iff C_{abcd} = 0$. Moreover, for any spatial vector x^a $W_{ab}x^a x^b$ is non-negative (proposition 7), a property which will be important in section 9. We find that in terms of W_{ab} equation (63) takes the equivalent form

$$\begin{aligned}\pounds_n W_{cd} = {}& A^a(-h_{cd}\mathcal{P}_a + \frac{1}{2}h_{ad}\mathcal{P}_c + \frac{1}{2}h_{ac}\mathcal{P}_d + 2S_{cda}) - 4\varepsilon_{ab(d}W_{c)}^{\ \ b}\omega^a + \frac{4}{3}\theta W_{cd} + \\ &+ \left(\frac{t_{cdab}}{2} + h_{ca}h_{db}W + 3h_{cd}W_{ab}\right)\sigma^{ab} + D_a S_{cd}^{\ \ a},\end{aligned} \tag{69}$$

where

$$S_{cda} \equiv \frac{1}{2}B_{b(d}E_{c)e}\varepsilon_a^{\ eb}. \tag{70}$$

In view of (69) we deduce that definition 2 can be formulated replacing Q_{abc} by S_{abc}. In fact it is not difficult to check that both S_{abc} and Q_{abc} have the same information and thus are completely equivalent. We may expect that any reasonable definition of superenergy radiative state should be formulated in terms of a spatial tensor being equivalent to Q_{abc}. Any such tensor can be regarded as the gravitational equivalent of electromagnetism's Poynting vector. The tensor S_{abc} seems to be the simplest choice and we will adopt it as the basic geometric object measuring "superenergy flux".

Another interesting aspect about (63)-(64) or (69), already pointed out in remark 7.2 is the fact that they are written in such a way that the couplings of the kinematical quantities to the different spatial parts of the decomposition of Bel-Robinson tensor are apparent. In our present context these couplings could be interpreted as the effect on the superenergy radiation due to the acceleration, the expansion, the shear and the rotation (see definition 4). At this point it is instructive to compare equation (63) (or its equivalent (69)) with its electromagnetic counterpart which is (19). In the electromagnetic case we realize that the vorticity has no effect whatsoever on the radiation of electromagnetic energy whereas it certainly influences the radiation of superenergy because ω^a (or equivalently ω_{ab}) appears explicitly in (63).

8.2. Superenergy radiative states for general spacetimes

Using the ideas explained in previous section we can formulate a definition of intrinsic superenergy radiative state similar to definition 2 but valid for a general spacetime. In this case we need to study the evolution of a spatial quantity which is zero if and only if the Riemann tensor vanishes. As stated in proposition 6 the tensor $\bar{t}_{ab}$ has the required properties and hence the terms appearing in the evolution equation of $\bar{t}_{ab}$ should enable us to define the concept of intrinsic radiative state. The sought evolution equation is (55) and hence the inspection of such equation leads us to the following

Definition 3 (Intrinsic superenergy radiative state in a general spacetime)

There exists an intrinsic superenergy radiative state at a point $p \in V$ if for any unit timelike vector n^a we have that $\overline{Q}_{abc}(n)$ does not vanish at p.

Similar considerations as in the case of definition 2 apply here.

Besides the definition of intrinsic superenergy radiative state just introduced, we can obtain other interesting results from the study of equation (55). These results deal with the possibility of defining superenergy radiative states associated to the rotation, the shear, the expansion and the matter current.

Definition 4 *Let n^a be a unit timelike vector and p a point. We say that*

- *the rotation of n^a is in a superenergy radiative state at p if the tensor $\Omega_{[af]cd}$ (see equation (58)) does not vanish at p.*
- *The shear of n^a is in a superenergy radiative state at p if the tensor $\Sigma_{(ab)cd}$ (see equation (58)) does not vanish at p.*
- *The expansion of n^a is in a superenergy radiative state at p if the tensor $2(t_{ab} + t^*_{ab})/3$ does not vanish at p.*
- *The acceleration of n^a is in a superenergy radiative state at p if the tensor $2(h_{a(d}\mathcal{P}_{c)} + \overline{Q}_{acd})$ does not vanish at p.*

Using equation (55) we can also decide when the matter is in a intrinsic superenergy radiative state.

Definition 5 *We say that the matter is in a intrinsic superenergy radiative state at a point p if for any observer n^a the tensor*

$$S^{matter}_{cd}(n^a) \equiv -j^{fhb}\left(\frac{1}{2}\varepsilon_{afh}h_{cd} + 2h_{f(d}\varepsilon_{c)ah}\right)Z^a_{\ b} + (h_{cd}Y_{ae} - h_{da}Y_{ce} - h_{ca}Y_{de})\,\overline{J}^{ae},$$

does not vanish at p.

9. Relationship between superenergy and energy

The possible meaning of superenergy has been largely researched and no conclusive results have been obtained. One of the aspects which has received wider attention is the possibility that some sort of relation exists between superenergy and energy. Here we indicate a possible way to answer this question which takes advantage of the orthogonal splittings calculated in sections 5 and 6. We explain first the general idea and later show how it is used in a particular example.

Consider the tensors t_{ab}, $\bar{t}_{ab}$ which appear in the orthogonal decomposition of Bel and Bel-Robinson tensor respectively and construct from them the spatial tensors

$$W_{ab} = \frac{1}{2}(Wh_{ab} - t_{ab}),\ \overline{W}_{ab} \equiv \frac{1}{2}(h_{ab}\overline{W} - \bar{t}_{ab}) \tag{71}$$

Proposition 7 *For any spatial vector x^a, we have $W_{ab}x^ax^b \geq 0$, $\overline{W}_{ab}x^ax^b \geq 0$.*

Proof : W_{ab} was defined in equation (68) and $\overline{W}_{ab}$ is found to be

$$\overline{W}_{ab} = \frac{1}{2}(X_a^{\ d}X_{bd} + Y_a^{\ d}Y_{bd} + Z_{da}Z^d_{\ b} + Z_a^{\ d}Z_{bd}).$$

The claimed properties are now a direct consequence of previous equation and (68).□

Proposition 8 *If the Weyl tensor is of Petrov type D or III then $W_{ab}x^ax^b > 0$ for any spatial vector x^a.*

Proof : We need to show that W_{ab} is strictly positive definite when regarded as a spatial tensor. In the case of Petrov type D it is enough to calculate W_{ab} in the canonical frame of appendix B and obtain

$$W_{ab} = \mathrm{diag}(E^2_{11} + B^2_{11}, 4(E^2_{11} + B^2_{11}), 4(E^2_{11} + B^2_{11})),\ E_{11} \neq 0,\ B_{11} \neq 0.$$

To prove the result in the case of type III we use the expressions of t_{ab} found in proposition 4 which yield

$$W_{ab} = \frac{1}{4}Wh_{ab} + \frac{\mathcal{P}_a\mathcal{P}_b}{W}$$

from which is obvious that W_{ab} must be strictly positive definite (recall that $W > 0$). □

Suppose now that the vector n^a is hypersurface orthogonal and denote by $\mathcal{M} \subset V$ one of the hypersurfaces n^a is orthogonal to. Let $\mathcal{S} \subset \mathcal{M}$ be a surface and define a measure on $\mathcal{S}$ in the following way (the tensor $\mathfrak{U}_{ab}$ will be used to denote either W_{ab} or $\overline{W}_{ab}$)

$$\mu(\mathcal{S}') = \int_{\mathcal{S}'\subset\mathcal{S}} d\mu = \int_{S'} \sqrt{\det((i^*\mathfrak{U})_{\mathbf{ab}})}d\chi,$$

where $i : \mathcal{S} \to \mathcal{M}$ is an embedding, $i^*\mathfrak{U}_{ab}$ is the pull-back of $\mathfrak{U}_{ab}$ and $d\chi$ is the canonical measure induced by the local coordinate system or frame set on $\mathcal{S}$ in which the scalar $\det((i^*\mathfrak{U})_{\mathbf{ab}})$ is calculated. Elementary considerations imply that μ neither depends on the embedding i, nor on the coordinates or frame mentioned before.

The tensor $\mathfrak{U}_{ab}$ represents superenergy and hence the scalar $\mu(\mathcal{S}')$ has dimensions of energy density. We may expect that $\mu(\mathcal{S}')$ is somehow measuring the average *energy density* of an open set relative to $\mathcal{M}$ which contains $\mathcal{S}'$. Next definition makes this idea more precise.

Definition 6 *Let $\mathcal{W} \subset \mathcal{M}$ be an open set relative to $\mathcal{M}$ with compact closure and consider $\mathcal{S} = \partial\mathcal{W}$. The average energy density in the set $\mathcal{W}$ is defined by the quantity $\mu(\mathcal{S})$. The total energy E of the open set $\mathcal{W}$ is defined by*

$$E(\mathcal{W}) = \mu(\mathcal{S})vol(\mathcal{W}),$$

where $vol(\mathcal{W})$ is the measure of $\mathcal{W}$ induced by the spatial metric h_{ab} (3-volume).

Remark 9.1 The quantity $E(\mathcal{W})$ is a quasilocal quantity which approaches zero if $\mathcal{W}$ shrinks to a point. The concept of quasilocal energy has been considered many times in the past starting with the work of Hawking in [20] and many modifications have been pursued afterwards [14, 30, 31, 36]. A complete history of the different proposals put forward can be found in [41]. Unlike definition 6, some of the definitions of quasilocal energy only consider special geometries (or topologies) for $\mathcal{W}$, typically spheres. In this last case it is possible to relate the limit of the quotient between many of these quasilocal energies and certain power of the sphere radius to the scalar W [24, 7, 6, 15].

To investigate further the meaning of $E(\mathcal{W})$ we calculate this quantity in a particular but relevant case.

Example 9.1 (Schwarzschild interior solution) The Schwarzschild interior solution is a static spherically symmetric solution of the Einstein field equations for a perfect fluid with constant density ρ_0. The line element is

$$ds^2 = -\left(\frac{3}{2}\sqrt{1-\frac{R^2\rho_0}{3}} - \frac{1}{2}\sqrt{1-\frac{\rho_0 r^2}{3}}\right)^2 dt^2 + \frac{1}{1-\frac{\rho_0 r^2}{3}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (72)$$

where R is a constant (star radius) and $-\infty < t < \infty$, $0 < r \leq R < \sqrt{3/\rho_0}$, $0 < \theta < \pi$, $0 < \varphi < 2\pi$. This spacetime can be matched to Schwarzschild vacuum through the hypersurface $r = R$. Let us take as the unit timelike normal n^a the vector field

$$\frac{2\sqrt{3}}{-3\sqrt{3-R^2\rho_0} + \sqrt{3-r^2\rho_0}}\frac{\partial}{\partial t},$$

which is parallel to the static Killing vector $\partial/\partial t$. The aim of this example is the calculation of $E(\mathcal{W})$ when we take as $\mathcal{W}$ any of the hypersurfaces orthogonal to $\partial/\partial t$ which are given by the condition $t = const$. Since $\partial/\partial t$ is a Killing vector, this quantity does not depend on the chosen hypersurface. After a calculation we find that

$$\bar{t}_{ab}dx^a dx^b =$$

$$= \frac{\rho_0^2\left(2\rho_0 r^2 + 9R^2\rho_0 + 6\sqrt{(3-R^2\rho_0)(3-r^2\rho_0)} - 33\right)}{6\left(r^2\rho_0 - 3\right)\left(\sqrt{3-r^2\rho_0} - 3\sqrt{3-R^2\rho_0}\right)^2}(dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi)),$$

$$h_{ab}dx^a dx^b = \frac{3}{3-\rho_0 r^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

The surface $\partial\mathcal{W}$ turns out to be the sphere $r = R$ and hence using the above expression we find

$$i^*\overline{W}_{ab}dx^a dx^b = \frac{5}{72}R^2\rho_0^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

from which we easily conclude that $\mu(\partial\mathcal{W}) = 20\pi R^2\rho_0^2/72$. Also we have the result

$$vol(\mathcal{W}) = 2\pi\left(\frac{3\sqrt{3}\arcsin\left(\frac{R\sqrt{\rho_0}}{\sqrt{3}}\right)}{\rho_0^{3/2}} - \frac{\sqrt{3}R\sqrt{3 - R^2\rho_0}}{\rho_0}\right),$$

and thus

$$E(\mathcal{W}) = \frac{5\pi^2R^2\rho_0^2}{9}\left(\frac{3\sqrt{3}\arcsin\left(\frac{R\sqrt{\rho_0}}{\sqrt{3}}\right)}{\rho_0^{3/2}} - \frac{\sqrt{3}R\sqrt{3 - R^2\rho_0}}{\rho_0}\right).$$

If $\rho_0 \ll 1$ (Newtonian limit) then we find that previous expression can be approximated by

$$E(\mathcal{W}) = \frac{10\pi^2R^5\rho_0^2}{9}. \tag{73}$$

It is interesting to compare this expression with the result we would obtain using Newtonian gravitation theory. In Newtonian gravitation one can define the concept of *gravitational potential energy* of a mass distribution. By definition this is the work required to assemble the mass distribution by bringing its parts together from infinity. If we assume that the Newtonian potential at a point $x \in \mathcal{W}$ is $\Phi(x)$ and we denote by $\rho(x)$ the mass density then the gravitational potential energy of a mass distribution inside the set $\mathcal{W}$ is

$$E_N(\mathcal{W}) = \frac{1}{2}\int_{\mathcal{W}}\rho(x)\Phi(x)d^3x.$$

For a spherical body of constant density ρ_0 and radius R the Newtonian potential in standard spherical coordinates is $\Phi(r) = \rho_0 r^2/6$ and therefore using previous formula we get

$$E_N(\mathcal{W}) = \frac{\rho_0^2R^5}{15}. \tag{74}$$

We see that the Newtonian result is, up to a factor, the value given by equation (73). The discrepancy factor can be fixed by redefining the parameter ρ_0 in (72) and thus we may conclude that in this particular example the Newtonian limit of $E(\mathcal{W})$ agrees with the Newtonian gravitational potential energy.

10. Conclusions and open issues

In this work we have obtained the full orthogonal splitting of Bel tensor and its covariant divergence and we have particularized it for the important case of vacuum spacetimes where Bel tensor becomes Bel-Robinson tensor. This gives rise to the dynamical laws of the superenergy. The concept of superenergy radiative state has been introduced and a possible relationship between energy and superenergy suggested. The work just presented opens new research lines which we believe are worth exploring. Perhaps one of the most interesting issues is a *global* formulation of the dynamical laws of superenergy complementing the local formulation of theorem

4. Such global formulation would enable us to apply our techniques to realistic astrophysical settings such as oscillating stars, rotating bodies or radiating binary systems. Using similar ideas as those of section 9 we can entertain the possibility of constructing other quasilocal magnitudes related for instance to the total force which acts on a gravitating system.

In this paper we have restricted ourselves to gravitational superenergy but one can define tensors representing superenergy from a general field resulting in the *superenergy tensor* of that field [39]. In this framework it is possible to calculate the covariant divergence of a superenergy tensor and obtain an expression similar to the first equation in (39) with the Bel tensor replaced by a suitable superenergy tensor. The orthogonal splitting of such equation would yield the dynamical laws of the superenergy associated to that particular field. An interesting example concerns the electromagnetic field. In this case a possible superenergy tensor is the Chevreton tensor which was first introduced in [17] and recently stimulating results about its symmetries and the covariant divergence of its trace have been obtained [8]. Chevreton tensor, like Bel-Robinson tensor, is a four rank tensor and its covariant divergence couples the Riemann tensor with terms which contain covariant derivatives of the Faraday tensor. This suggests a possible exchange between the gravitational and the electromagnetic superenergies [39, 28, 19]. The orthogonal splitting of the covariant divergence of Chevreton tensor might shed light in the nature of this exchange.

Another important issue is the possible relationship between superenergy and any of the available quasilocal concepts of *gravitational energy* which have been developed during the years. This is a topic which has been largely researched in the past and no clear conclusion has been reached. The orthogonal splitting of Bel tensor brings a new point of view to this problem as we explained in section 9. However, a sound physical foundation of the concept presented in definition 6 is still lacking.

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Appendix A. Technical details about the computations

In this appendix we supply details about the calculations required in this work. In order to do so we need to explain some implementation aspects of the system *xAct*. We will limit ourselves to only those issues which are needed in our calculations referring the interested reader to [33] for a full documentation and tutorials about *xAct*.

Orthogonal splittings play an essential part in our work and the implementation of *xAct* of this matter is completely adapted to our requirements. The basic elements of the orthogonal splitting are defined through the command

```
DefMetric[1, metrich[−a, −b], cd, {"|", "D"}, InducedFrom → {g, n}, PrintAs → h].
```

Here `metrich[−a, −b]` represents the spatial metric h_{ab} which is constructed from the spacetime metric g_{ab} (represented in the system by `g[−a, −b]`) and the unit normal vector n^a (represented by `n[a]`). The operator `cd[−a]` is the Cattaneo operator D_a associated to h_{ab}. The system is able to handle all the properties of Cattaneo operator explained in subsection 2.2 in a natural fashion.

The general expression for the orthogonal splitting of any tensor is equation (3). This result is implemented in *xAct* by means of the command

$$\texttt{InducedDecomposition}[\texttt{expr\_}, \{\texttt{metrich}, \texttt{n}\}],$$

where `expr_` represents any tensor. The output of `InducedDecomposition` is the result of applying formula (3) to `expr_`. The orthogonal projector operator P_h which appears in (3) is also implemented in *xAct* by means of the command `Projectormetrich[expr_]` where again `expr_` represents an arbitrary tensor and `metrich` is the name of the spatial metric. The basic commands just explained enable us to find efficiently orthogonal splittings similar to equation (12) with L_a replaced by any spatial tensor of higher rank.

Proof of theorems 4 and 5

To prove theorems 4 and 5 we need to find the orthogonal decomposition of the equations shown in (39). The first step is to replace B_{abcd}, R_{abcd} and $\mathfrak{J}_{abc}$ by their orthogonal splittings, eqs. (49), (28) and (32) respectively. The covariant derivatives of n^a are decomposed according to (4) and $\nabla_a\varepsilon_{bcd}$ is decomposed by means of the formula

$$\nabla_d\varepsilon_{fhl} = 3A^a n_d n_{[h}\varepsilon_{fl]a} - 3n_{[h}\left(\frac{1}{3}\varepsilon_{fl]d}\theta + \varepsilon_{fl]a}(\sigma_d{}^a + \omega_d{}^a)\right) + 3n_d\varepsilon_{a[hl}(\sigma_{f]}^{\ a} + \omega_{f]}^{\ a}).$$

After doing these replacements we obtain expressions which contain $\nabla_a\overline{W}$, $\nabla_a\overline{\mathcal{P}}_b$, $\nabla_a\bar{t}_{bc}$, $\nabla_a t^*_{bc}$, $\nabla_a\bar{t}_{bcde}$, $\nabla_a L_b$, $\nabla_a\tilde{J}_{bc}$, $\nabla_a\overline{J}_{bc}$, $\nabla_a j_{bcd}$. These are further decomposed by following the procedure explained in subsection 2.2. For example, the orthogonal decomposition of $\nabla_a L_b$ is just equation (12) which also holds if we replace L_b by $\overline{\mathcal{P}}_b$. Other needed orthogonal decompositions are

$$\nabla_c\bar{t}_{ab} = -2A^d n_c n_{(b}\bar{t}_{a)d} + 2n_{(a}\left(\frac{1}{3}\bar{t}_{b)c}\theta + \bar{t}_{b)}{}^d(\sigma_{cd} + \omega_{cd})\right) + D_c\bar{t}_{ab} +$$
$$+ n_c\left(\frac{2}{3}\theta\bar{t}_{ab} + 2\bar{t}_{(b}{}^d(\sigma_{a)d} + \omega_{a)d}) - \pounds_n\bar{t}_{ab}\right), \qquad \text{(A.1)}$$

which is also valid if we replace $\bar{t}_{ab}$ by any symmetric spatial tensor and

$$\nabla_a\tilde{J}_{bc} = 2A^d n_a(\tilde{J}_{d[b}n_{c]}) + 2n_{[c}\left(\frac{1}{3}\tilde{J}_{b]a}\theta + \tilde{J}_{b]}{}^d(\sigma_{ad} + \omega_{ad})\right) + D_a\tilde{J}_{bc} +$$
$$+ n_a\left(\frac{2}{3}\tilde{J}_{bc}\theta + 2\tilde{J}_{[b]}^{\ c}(\sigma_{c]d} + \omega_{c]d}) - \pounds_n\tilde{J}_{bc}\right), \qquad \text{(A.2)}$$

which is true if we replace $\tilde{J}_{bc}$ by any antisymmetric tensor. The expressions for the orthogonal splitting of the remaining covariant derivatives are very long and we omit them. Inserting the orthogonal splittings in (39) and rearranging the equations obtained as polynomials in n^a we obtain expressions whose generic form looks like

$$An^a n^b n^c n^d + B_1^a n^b n^c n^d + \cdots + B_4^d n^a n^b n^c + C_1^{ab} n^c n^d + \cdots + E^{abcd} = 0, \qquad \text{(A.3)}$$
$$Fn^a n^b n^c + G_1^a n^b n^c + \cdots G_3^c n^a n^b + \cdots + I^{abc} = 0, \qquad \text{(A.4)}$$

where all the tensor coefficients of these polynomials are spatial. This implies that the coefficients of the polynomials must vanish and these conditions lead us to the results stated in theorems 4 and 5. Note that the polynomials (A.3)-(A.4) are equivalent to each of the equations presented in (39) and so are the conditions stemming from setting the polynomial coefficients equal to zero.

Appendix B. Canonical forms for the electric and magnetic parts of Weyl tensor in the different Petrov types.

We present next the canonical forms of the electric and magnetic parts of Weyl tensor for the different Petrov types. We follow [5] in our presentation (see also [40] for an equivalent representation of the canonical forms). All the canonical forms are written with respect to certain orthonormal frame $O \equiv \{e_1^a, e_2^a, e_3^a\}$ of spatial vectors (canonical frame).

Petrov type I

In this type E_{ab} and B_{ab} take the following form in the canonical frame O

$$E_{\mathbf{ab}} = \mathrm{diag}(E_{11}, E_{22}, E_{33}),\ B_{\mathbf{ab}} = \mathrm{diag}(B_{11}, B_{22}, B_{33}), \tag{B.1}$$

with the additional conditions

$$E_{11} + E_{22} + E_{33} = 0,\ B_{11} + B_{22} + B_{33} = 0. \tag{B.2}$$

Petrov type D

This type arises if we set $2E_{11} = E_{22} = E_{33}$, $2B_{11} = B_{22} = B_{33}$ in previous case.

Petrov type II

The canonical forms for E_{ab}, B_{ab} in the frame O are

$$E_{\mathbf{ab}} = \begin{pmatrix} E_{11} & 0 & 0 \\ 0 & -\frac{E_{11}}{2} + B_{23} & E_{23} \\ 0 & E_{23} & -\frac{E_{11}}{2} - B_{23} \end{pmatrix},$$
$$B_{\mathbf{ab}} = \begin{pmatrix} B_{11} & 0 & 0 \\ 0 & -\frac{B_{11}}{2} - E_{23} & B_{23} \\ 0 & B_{23} & -\frac{B_{11}}{2} + E_{23} \end{pmatrix}. \tag{B.3}$$

Petrov type III

The canonical forms for E_{ab}, B_{ab} in the frame O are

$$E_{\mathbf{ab}} = \begin{pmatrix} 0 & E_{12} & -B_{12} \\ E_{12} & 0 & 0 \\ -B_{12} & 0 & 0 \end{pmatrix},\ B_{\mathbf{ab}} = \begin{pmatrix} 0 & B_{12} & E_{12} \\ B_{12} & 0 & 0 \\ E_{12} & 0 & 0 \end{pmatrix}. \tag{B.4}$$

Petrov type N

The canonical forms for E_{ab}, B_{ab} in the frame O are

$$E_{\mathbf{ab}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_{22} & -B_{22} \\ 0 & -B_{22} & -E_{22} \end{pmatrix},\ B_{\mathbf{ab}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & B_{22} & E_{22} \\ 0 & E_{22} & -B_{22} \end{pmatrix}. \tag{B.5}$$

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