Optical Solitons in an Anisotropic Medium with Arbitrary Dipole Moments

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ABSTRACT

We find the Lax pair for a system of reduced Maxwell–Bloch equations that describes the propagation of two-component extremely short electromagnetic pulses through the medium containing two-level quantum particles with arbitrary dipole moments.

Keywords: extremely short pulse, optical anisotropy, self-induced transparency, soliton

1. INTRODUCTION

Generation of extremely short pulses[1–5] (ESP) with a duration of a few periods of light oscillations has offered a strong incentive for theoretical studies of their interaction with matter (see reviews[6,7] and references therein). For obvious reasons, the slowly varying envelope approximation commonly exploited in the nonlinear optics of quasi-monochromatic (or ultrashort) pulses cannot be applied to investigate the propagation of ESP.

The slowly varying envelope approximation was not used in the case of the ultrashort pulses in Ref. 8, where an alternative approach to the theory of self-induced transparency[9,10] (SIT) was developed. This approach was based on the assumption of low density of the quantum particles, which was consistent with conditions of the SIT experiments. Since the backscattered wave is weak in that case, the order of derivatives in the wave equation for the pulse field can be reduced by using the unidirectional propagation (UP) approximation.[11] The resulting so-called reduced Maxwell-Bloch (RMB) equations[8] , as well as the SIT equations[10] , are integrable by the inverse scattering transformation (IST) method.[12–14] This method is widely recognized as one of the most powerful tools in studying the nonlinear phenomena. In particular, the pulse dynamics in the integrable models of nonlinear optics is described by the soliton solutions.

In the last years, the theoretical investigation of coherent nonlinear effects in anisotropic media attracts great attention.[15–26] This is caused by the significant development of the nanotechnologies and the methods of producing the low-dimensional quantum structures. Unlike the case of isotropic media, the parity of the stationary states of the quantum particles of the anisotropic medium is not well defined. That is why the diagonal elements of the matrix of the dipole moment operator and their difference, which is called the permanent dipole moment (PDM) of the transition, may not vanish. An optical pulse propagating in such a medium not only induces transitions between these states, but also shifts the transition frequency via dynamic Stark effect. Owing to this, the ultrashort pulses can propagate in the medium in the regimes that differ from the SIT one.[17,20]

The dynamics of one-component ESP in the anisotropic media was studied in papers[15,16,18,19,21,23]. It was revealed[16] that the scalar RMB equations with PDM are integrable in the frameworks of the IST. The numerical investigation of the pulse formation governed by these equations displayed an existence of solitary stable bipolar signal with nonzero time area.[19] Its solitonic nature was established in Refs. 21 and 23.

The propagation through anisotropic media of the two-component ESP was investigated in[22,24–26]. The case, where one of the pulse components causes the quantum transitions, while another one shifts the energy levels, was considered in Ref. 24. Corresponding two-component RMB equations differ by notations from the system describing the transverse-longitudinal acoustic pulse propagation in the low-temperature paramagnetic crystal

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and are integrable.[27] An application of the spectral overlap approximation to these equations gives one more integrable model.[25,26] If both components of the ESP excite the quantum transitions only (i.e., PDM of the transition is equal to zero), the proper system of the two-component RMB equations is also integrable.[22]

We see that different particular cases of the two-component RMB equations are integrable by means of the IST method. The main aims of the present study are to join these cases and to find more general conditions, under which the integrability of the RMB equations takes place.

2. TWO-COMPONENT SYSTEM OF THE MAXWELL–BLOCH EQUATIONS

Let us consider the medium containing two-level quantum particles. Assume for simplicity that the medium is isotropic, i.e. an anisotropy is induced by the quantum particles. Let the plane ESP propagate through the medium in the positive direction of y axes of the Cartesian coordinate system. Then the Maxwell equations yield the following system for projections E_x and E_z of the electric field:

$$\frac{\partial^2 E_x}{\partial y^2} - \frac{n^2}{c^2}\frac{\partial^2 E_x}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 P_x}{\partial t^2}, \tag{1}$$

$$\frac{\partial^2 E_z}{\partial y^2} - \frac{n^2}{c^2}\frac{\partial^2 E_z}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 P_z}{\partial t^2}, \tag{2}$$

where P_x and P_z are the components of polarization connected with the two-level particles; n is the refractive index of the medium; c is the speed of light in free space.

To describe the evolution of quantum particles, we exploit the von Neumann equation for density matrix $\hat{\rho}$:

$$i\hbar\frac{\partial\hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}]. \tag{3}$$

Here Hamiltonian H of the two-level particle is written as follows

$$\hat{H} = \mathrm{diag}(0, \hbar\omega_0) - \hat{d}_x E_x - \hat{d}_z E_z, \tag{4}$$

where ω_0 is the resonant frequency of quantum transition; $\hat{d}_x$ and $\hat{d}_z$ are the matrices of the projection of the dipole moment operator on x and z axes, respectively; $\hbar$ is the Plank's constant.

The expressions for the polarization components read as

$$P_x = N\,\mathrm{Tr}\,(\hat{\rho}\hat{d}_x), \tag{5}$$

$$P_z = N\,\mathrm{Tr}\,(\hat{\rho}\hat{d}_z), \tag{6}$$

where N is the concentration of the quantum particles. We suppose that the matrices of the dipole moment are defined as given

$$\hat{d}_x = \begin{pmatrix} D_x & d_1 \\ d_1 & 0 \end{pmatrix}, \tag{7}$$

$$\hat{d}_z = \begin{pmatrix} D_z & \delta + i d_2 \\ \delta - i d_2 & 0 \end{pmatrix}, \tag{8}$$

where d_1, d_2, δ, D_x and D_z are real parameters. This representation for the dipole moment matrices is general, but we have reduced it to a simpler form. Quantities D_x and D_z are nothing but the PDM projections.

Using equations (3)–(8), we find

$$\frac{\partial W}{\partial t} = 2\frac{d_2}{\hbar}\,E_z U - 2\left(\frac{d_1}{\hbar}\,E_x + \frac{\delta}{\hbar}\,E_z\right)V, \tag{9}$$

$$\frac{\partial U}{\partial t} = -\left(\omega_0 + \frac{D_x}{\hbar}\,E_x + \frac{D_z}{\hbar}\,E_z\right)V - 2\frac{d_2}{\hbar}\,E_z W, \tag{10}$$

$$\frac{\partial V}{\partial t} = \left(\omega_0 + \frac{D_x}{\hbar}\,E_x + \frac{D_z}{\hbar}\,E_z\right)U + 2\left(\frac{d_1}{\hbar}\,E_x + \frac{\delta}{\hbar}\,E_z\right)W, \tag{11}$$

where

$$W = \frac{\rho_{22} - \rho_{11}}{2}, \quad U = \frac{\rho_{12} + \rho_{21}}{2}, \quad V = \frac{\rho_{12} - \rho_{21}}{2i}$$

are the components of the Bloch vector; ρ_{jk} $(j, k = 1, 2)$ are the elements of the density matrix.

Let the concentration of the quantum particles be small: $N\|\hat{d}_{x,\,z}\|^2/\hbar\omega_0 \ll 1$, where $\|C\|$ is the norm of the matrix C. Then we are able to reduce the order of derivatives in the wave equations for the electric field components. An application of the UP approximation[11] to equations (1), (2) and exclusion of the time derivatives of the elements of the density matrix with the help of (9)–(11) give

$$\frac{\partial E_x}{\partial y} + \frac{n}{c}\frac{\partial E_x}{\partial t} = \frac{4\pi N}{nc\hbar}\Big[SE_z + \hbar\omega_0 d_1 V\Big], \tag{12}$$

$$\frac{\partial E_z}{\partial y} + \frac{n}{c}\frac{\partial E_z}{\partial t} = -\frac{4\pi N}{nc\hbar}\Big[SE_x + \hbar\omega_0(d_2 U - \delta V)\Big], \tag{13}$$

where $S = 2d_1 d_2 W + d_2 D_x U + (d_1 D_z - \delta D_x)V$.

The two-component system of RMB equations (9)–(13) describes the propagation of vector ESP in the medium containing two-level quantum particles with arbitrary dipole moments. It is seen that both electric field components, as well as any superposition of them, fulfill two different functions in the general case: they excite the quantum transitions and shift the energy levels due to PDM. Obviously, the system obtained coincides with the RMB equations for isotropic medium[8] if we put PDM equal to zero ($D_x = D_z = 0$) and $E_z = d_2 = 0$ (or $E_x = d_1 = 0$). Other integrable cases of equations (9)–(13) were studied in[16,22,24].

3. LAX PAIR

An integrability of the nonlinear equations given by means of the IST method implies an opportunity to represent them as the compatibility condition of the overdetermined system of linear equations (Lax pair). Consider the following Lax pair

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= L(\lambda)\psi(\lambda), \\ \frac{\partial \psi}{\partial y} &= A(\lambda)\psi(\lambda), \end{aligned} \tag{14}$$

where λ is so-called spectral parameter; $L(\lambda)$ and $A(\lambda)$ are matrices; $\psi = \psi(y, t, \lambda)$ is a solution of the overdetermined system. Its compatibility condition reads as

$$\frac{\partial L(\lambda)}{\partial y} - \frac{\partial A(\lambda)}{\partial t} + [L(\lambda), A(\lambda)] = 0. \tag{15}$$

Using the overdetermined linear systems for the cases discussed in[16,22], we can offer possible form of the Lax pair for (particular case of) the equations (9)–(13). This form contains coefficients in matrices $L(\lambda)$ and $A(\lambda)$, which are a subject of subsequent definition, and may be valid only under imposing some constraints on the elements of matrices (7) and (8) of the dipole moment projections. Having written down the compatibility condition and having excluded the derivatives with the help of (9)–(13), we obtain the overdetermined system of algebraic equations on the entered coefficients. The number of the coefficients should be great enough to include into a consideration both the cases we start with. For this reason, the system of the algebraic equations is strongly overdetermined. Nevertheless, we have been able to solve this system after straightforward, but tedious calculations. Moreover, it has been done without imposing additional constraints on the dipole moments of the transition. We have found the following expressions for matrices $L(\lambda)$ and $A(\lambda)$ of system (14):

$$L(\lambda) = \begin{pmatrix} \dfrac{i}{2}\left[\lambda^2 - \dfrac{b}{\lambda^2}\right] & \dfrac{1}{2\sqrt{2}\hbar}\left[\lambda E^* + \dfrac{\delta_2}{\delta_1}\dfrac{E}{\lambda}\right] \\ \dfrac{\delta_1}{2\sqrt{2}\hbar}\left[\lambda E + \dfrac{\delta_2^*}{\delta_1}\dfrac{E^*}{\lambda}\right] & -\dfrac{i}{2}\left[\lambda^2 - \dfrac{b}{\lambda^2}\right] \end{pmatrix}, \tag{16}$$

$$A(\lambda)=\frac{2\pi N}{nc}\frac{1}{\lambda^2+\dfrac{b}{\lambda^2}+B}\begin{pmatrix} \dfrac{i}{\hbar}\left[\lambda^2-\dfrac{b}{\lambda^2}\right]S & \dfrac{\sqrt{2}d_1d_2}{\hbar^2\delta_1}\left[\lambda Q^*+\dfrac{\delta_2}{\delta_1}\dfrac{Q}{\lambda}\right] \\ \dfrac{\sqrt{2}d_1d_2}{\hbar^2}\left[\lambda Q+\dfrac{\delta_2^*}{\delta_1}\dfrac{Q^*}{\lambda}\right] & -\dfrac{i}{\hbar}\left[\lambda^2-\dfrac{b}{\lambda^2}\right]S \end{pmatrix}-\frac{n}{c}L(\lambda), \qquad (17)$$

where

$$E=E_x+iE_z+\frac{\delta_3}{\delta_1},\quad Q=\delta_3W+\delta_4U+\delta_5V,$$

$$\delta_1=\frac{2}{\omega_0d_1d_2}\Big[(4d_1^2+D_x^2)d_2^2+(d_1D_z-\delta D_x)^2\Big],$$

$$\delta_2=2d_2^2+2(\delta+id_1)^2+\frac{(D_z+iD_x)^2}{2},$$

$$\delta_3=\frac{2\hbar}{d_1d_2}\Big[d_2^2D_x+(\delta-id_1)(\delta D_x-d_1D_z)\Big],$$

$$\delta_4=\frac{\hbar}{d_1d_2}\Big[\delta D_xD_z-d_1(4d_2^2+D_z^2)+i(d_1D_z-\delta D_x)D_x\Big],$$

$$\delta_5=\frac{\hbar}{d_1}\Big[4d_1\delta+D_xD_z-i(4d_1^2+D_x^2)\Big],$$

$$b=\frac{|\delta_2|^2}{\delta_1^2},\quad B=\frac{1}{\delta_1}\Big[4(d_1^2+d_2^2+\delta^2)+D_x^2+D_z^2\Big].$$

Substituting (16) and (17) into (15), we see that the equality is fulfilled only if equations (9)–(13) take place. Thus, the two-component RMB equations (9)–(13) belong to the class of nonlinear models integrable by means of the IST method at **any** values of parameters d_1, d_2, δ, D_x and D_z.

Equations (15), (16) and (17) with $\delta=D_x=D_z=0$ give the Lax pair presented in Ref. 22. A connection of the Lax pair we found with the pair obtained in Ref. 16 is not so obvious since $d_1d_2=0$ in the last case.

4. CONCLUSION

We have established the integrability in the frameworks of the IST method of the system of two-component RMB equations for anisotropic medium in the most general case. This implies that these equations have multi-soliton solutions, Darboux and Bäcklund transformations, infinite hierarchies of the conservation laws and infinitesimal symmetries and other attributes of the integrable models. The detailed study of them is a subject of further researches.

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