ON THE EXISTENCE OF EXPONENTIALLY DECREASING SOLUTIONS OF THE NONLINEAR LANDAU DAMPING PROBLEM

Hyung Ju Hwang[1] and Juan J. L. Velázquez[2]

Abstract.- In this paper we prove the existence of a large class of periodic solutions of the Vlasov-Poisson in one space dimension that decay exponentially as $t \to \infty$. The exponential decay is well known for the linearized version of the Landau damping problem. The results in this paper provide the first example of solutions of the whole nonlinear Vlasov-Poisson system that exhibit such rate of decay.

Keywords.- Landau damping, Vlasov-Poisson system, exponential decay, analiticity properties of the solutions.

1 Introduction.

Landau damping is a remarkable property of collisionless plasmas. This effect was discovered in [8] and it consists in the exponential damping of charge oscillations in the plasma due to the combined effect of the electrical fields generated by the charges and the dispersion in the particle velocities.

In more mathematical terms, this effect is usually studied using the Vlasov-Poisson system for negatively charged particles moving in a constant background of positive charges that makes the whole system electrically neutral. A linearized Vlasov-Poisson problem is then derived assuming that the inhomogeneities of the charge distribution are small. For rather general initial distributions of particles and initial velocities the corresponding charge disturbances of this linear system decay exponentially fast.

The mathematical theory of the Vlasov-Poisson equation, including global existence results for general initial data in space dimension $N \leq 3$ has been established in several papers (cf. [5], [9], [10]).

Landau damping has been also extensively studied in the mathematical and physical literature (cf. [3], [4], [6], [7], [11], [12]). All the results in these papers hold only for the linearized version of the problem. The derivation of the exponential decay for the linearized Vlasov-Poisson system, in the cases where such a decay takes place, relies heavily on the analytic properties of the initial data (cf. [8], [7], [11]). However, it has been proved in [12] that exponential

[1]Department of Mathematics, Pohang University of Science and Technology, Pohang 790-784, Republic of Korea.

[2]ICMAT (CSIC-UAM-UC3M-UCM). Facultad de Matemáticas, Universidad Complutense, Madrid 28040, Spain.

decay cannot be expected, even for the linearized problem, for nonanalytic initial data.

There are also physical arguments that explain why the energy of the charge waves tends to be transferred to the distribution of velocities of the particles (cf. [3], [7]), yielding exponential decay of the charge disturbances. In these arguments the linearity of the problem plays an essential role, in particular it is possible to decompose an arbitrary disturbance in monochromatic waves using Fourier transform. However, it is not obvious how to generalize the arguments to the whole nonlinear problem, even for small perturbations of the linear case. Nevertheless, the numerical simulations in [15] indicate that exponential decay of the charge disturbances might be expected for the whole nonlinear Vlasov-Poisson model and a large class of analytic initial data.

Assuming that there is a background of charge n_0 to make the system electrically neutral, the mathematical problem that is usually considered in the study of Landau damping is the following one:

$$f_t + vf_x + Ef_v = 0 \ \ , \ \ -\infty < v < \infty \ \ , \ \ 0 < x < L \ \ , \ \ t > 0 \tag{1.1}$$

$$f(0,v,t) = f(L,v,t) \tag{1.2}$$

$$\phi_{xx}(x,t) = \rho(x,t) = \int_{-\infty}^{\infty} f(x,v,t)\,dv - n_0 \ \ , \ \ 0 < x < L \ \ , \ \ t > 0 \tag{1.3}$$

$$\phi(0,t) = \phi(L,t) \ \ , \ \ \phi_x(0,t) = \phi_x(L,t) \ \ , \ \ t > 0 \tag{1.4}$$

$$E = \phi_x \ \ , \ \ 0 < x < L \ \ , \ \ t > 0 \tag{1.5}$$

$$f(x,v,t) = f_0(x,v) = f_e(v) + g_0(x,v)\,, \quad t = 0 \tag{1.6}$$

where $f_e(v)$ satisfies:

$$\int_{-\infty}^{\infty} f_e(v)\,dv = n_0 \tag{1.7}$$

The physical basis on assuming the existence of the background of charge n_0 is that the system contains two types of particles with very different masses. The lighter ones that can move easily are described by means of the distribution $f(x,v,t)$. On the other hand, there are some heavier "ions" that cannot move so easily and are replaced by the charge background n_0.

Without loss of generality we can assume, using suitable units;

$$n_0 = 1 \ \ , \ \ L = 2\pi \tag{1.8}$$

Notice that (1.7) implies that f_e is a stationary solution of (1.1)-(1.5). Moreover, (1.6) and (1.7) imply:

$$\int_0^{2\pi} \int_{-\infty}^{\infty} g_0\,dx\,dv = 0 \tag{1.9}$$

In this paper we will obtain a large class of periodic solutions of the nonlinear Vlasov-Poisson system (1.1)-(1.6) in a bounded domain for which the corresponding charge density ρ and electric field E decay exponentially as $t \to \infty$.

The result is a perturbative one, in the sense that the derived solutions will be close to solutions of the linearized equation. However, even under these restrictions, these results are, to our knowledge, the first ones showing rigorously exponential decay of the electric field for the nonlinear Vlasov-Poisson equation.

The main difficulty of the problem under consideration, even with the above mentioned smallness conditions, is that a naive linearization near an equilibrium distribution $f_e(v)$, say a maxwellian distribution, does not allow to show that the remaining nonlinear terms are small. This problem will be explained in detail in Section 2. The rationale behind this problem is that, although the electrical field and the charge density converge to zero, the distribution function $f(x,v,t)$ for the particles does not converge necessarily to the equilibrium distribution $f_e(v)$. On physical grounds this might not be expected in the absence of collisions. Actually, the dissipation of the energy contained in the field $E(x,t)$ must result in the gain of kinetic energy of the particles of the system, as it was noticed in [3] (cf. also [7] for a clear explanation of this). However, in the absence of dissipative mechanisms there is no reason to expect for the long time distribution of particle velocities to approximate to the equilibrium distribution $f_e(v)$ or even to be spatially homogeneous. In fact, if the field $E(x,t)$ vanishes fast enough as $t \to \infty$, the only restriction that we can expect for the long time asymptotics of $f(x,v,t)$ is to behave like a "free streaming" function $f_\infty(x - vt, v)$.

The previous discussion suggests not to linearize around the equilibrium distribution, but near a free streaming function $f_\infty(x - vt, v)$. It turns out that under suitable analyticity assumptions on f_∞ and suitable smallness conditions ensuring that f_∞ is close to a stable equilibrium $f_e(v)$, it is possible to obtain solutions of the Vlasov-Poisson system defined in $t \in (0, \infty)$, such that $f(x,v,t) \sim f_\infty(x - vt, v)$ as $t \to \infty$, and $E(x,t) \to 0$ as $t \to \infty$.

It is interesting to notice that the linearization near the "free streaming" distribution $f_\infty(x - vt, v)$ has some peculiar features. Indeed, the charge density $\rho_\infty(x,t) \equiv \int f_\infty(x - vt, v)\, dv - 1$ is not identically zero for any $t \geq 0$, as it could be expected from the fact that $f \to f_\infty$ and $E \to 0$ as $t \to \infty$. It turns out that $\rho_\infty(x,t)$ approaches to zero exponentially fast as $t \to \infty$ under suitable analyticity assumptions on f_∞ that will be made precise later. The decay to zero of the density for the free streaming problem, due to the dispersion of the velocities, is a well known fact (cf. [2]). The argument in this paper indicates that this convergence to zero plays a crucial role in the Landau damping problem. However, the averaging property is not the only ground explaining Landau damping, since some additional stability conditions must be satisfied by the distribution f_e. The conditions on f_e that are required to obtain Landau damping have been explained by means of intuitive physical arguments (cf. [3], [7]). However, the role played by the averaging of the asymptotic distribution of particle velocities has not been considered, to our knowledge, in the physical study of this problem.

The main result of this paper is the following:

Theorem 1 *There exist initial data* $f_0(x,v) = f_e(v) + g_0(x,v)$ *with* g_0 *satisfying (1.9) such that the corresponding solution of the system (1.1)-(1.6) satisfies:*

$$|E(x,t)| + |\rho(x,t)| \leq Ce^{-\gamma t} \quad , \quad 0 \leq t < \infty$$

for some $\gamma > 0,\ C > 0.$

More precise conditions on $f_e,\ g_0$ will be given in Section 6, and Theorem 17.

The plan of the paper is the following. We recall in Section 2 some general properties of the classical linearized Vlasov-Poisson system. In particular, the difficulties that arise in trying to handle nonlinear terms with this linearization procedure will be explained. Section 3 describes formally the new linearization procedure. Section 4 formulates in detail a set of conditions for $f_e,\ g_\infty$ that will play a crucial role in the rest of the paper. Section 5 analyses in detail mathematical properties of the linearized operator. Section 6 contains the proof of exponential decay for the nonlinear Vlasov-Poisson problem, in particular the proof of Theorem 1.

2 The classical Landau linearized problem.

In this Section we describe the linearized problem that was originally considered by Vlasov (cf. [14]) and Landau (cf. [8]) and that has been studied also in the mathematical and physical literature with different levels of rigor [3], [4], [7], [11], [12]). The goal is to explain shortly the kind of difficulties that arise in trying to derive exponential decay for the field E associated to the nonlinear Vlasov-Poisson problem taking as a starting point the linearized problem. More precisely, we will explain why a naive linearization argument does not allow to derive exponential decay for the field. A linearization procedure that allows to obtain such exponential decay estimates for the nonlinear problem will be introduced in Section 3. The procedure that we will use to solve the linearized problem is equivalent to the one used by Landau, although it is different, but closer to the approach that we will use in the forthcoming sections to study the nonlinear problem.

We write:

$$f(x,v,t) = f_e(v) + g(x,v,t)$$

We assume that g is a small perturbation of the steady solution $f_e(v)$. Neglecting formally quadratic terms in g we obtain the following problem that is the one usually considered in the study of the linearized Landau damping problem:

$$g_t + vg_x + Ef_{e,v} = 0 \ , \ 0 < x < L \ , \ t > 0 \ , \ v \in \mathbb{R} \tag{2.1}$$

$$\phi_{xx} = \int g dv \ , \ E = \phi_x \ , \ 0 < x < L \ , \ t > 0 \tag{2.2}$$

$$g(x,v,0) = g_0(x,v) \ , \ 0 < x < L \ , \ v \in \mathbb{R} \tag{2.3}$$

$$\phi(0,t) = \phi(L,t) \ , \ \phi_x(0,t) = \phi_x(L,t) \ , \ t > 0 \tag{2.4}$$

$$g(0,v,t) = g(L,v,t) \ , \ v \in \mathbb{R} \ , \ t > 0 \tag{2.5}$$

Then, using Duhamel's principle to solve (2.1), (2.3) we obtain:

$$g(x,v,t) = g_0(x - vt, v) - \int_0^t E(x - v(t-s), s) f_{e,v}(v) ds$$

where we assume that g_0, E are extended periodically to $x \in \mathbb{R}$ with period L.

Plugging this identity into (2.2) we obtain the following integro-differential equation for the electric field:

$$E_x(x,t) = \int_{-\infty}^{\infty} g(x,v,t) dv = \int_{-\infty}^{\infty} g_0(x - vt, v) dv -$$
$$- \int_0^t \left[\int_{-\infty}^{\infty} E(x - v(t-s), s) f_{e,v}(v) dv \right] ds$$

It is easier to study this equation using Fourier variables:

$$E(x,t) = \sum_{n=-\infty}^{\infty} b_n(t) e^{inx} \tag{2.6}$$
$$g_0(x,v) = \sum_{n=-\infty}^{\infty} g_n(v) e^{inx}$$

where the periodicity of ϕ requires $b_0(t) = 0$.

Then the linearized problem becomes:

$$inb_n(t) = \int_{-\infty}^{\infty} dv g_n(v) e^{-invt} - \int_0^t b_n(s) \left[\int_{-\infty}^{\infty} e^{-inv(t-s)} f_{e,v}(v) dv \right] ds$$

We write:

$$K(\xi) = \int_{-\infty}^{\infty} e^{-i\xi v} f_{e,v}(v) dv \tag{2.7}$$

Then:

$$inb_n(t) + \int_0^t b_n(s) K(n(t-s)) ds = G_n(t) \equiv \int_{-\infty}^{\infty} dv g_n(v) e^{-invt} \tag{2.8}$$

In order to solve this equation we introduce a family of fundamental solutions given by:

$$inB_n(t;t_0)+\int_0^t B_n(s;t_0)\,K(n(t-s))\,ds=\delta(t-t_0) \tag{2.9}$$

where $t_0 \geq 0$. It then follows that:

$$b_n(t)=\int_0^\infty B_n(t;s)\,G_n(s)\,ds \tag{2.10}$$

This family of convolution integral equations (2.9) can be solved using Laplace transforms. Given a function $\varphi(t)$ in $\{t>0\}$ we define:

$$\tilde{\varphi}(z)=\int_0^\infty \varphi(t)\,e^{-zt}dt$$

We recall that the Laplace transform of the convolution:

$$(\varphi*\psi)(t)=\int_0^t \varphi(t-s)\,\psi(s)\,ds$$

is given by:

$$\widetilde{(\varphi*\psi)}(t)=\tilde{\varphi}(z)\,\tilde{\psi}(z)$$

On the other hand the Laplace transforms of the sequence of functions $K_n(t)=K(nt)$ are given by:

$$\widetilde{K_n}(t)=\int_0^\infty K(nt)\,e^{-zt}dt=\frac{1}{n}\tilde{K}\left(\frac{z}{n}\right)$$

Taking the Laplace transform of (2.9) it then follows that:

$$in\widetilde{B_n}(z;t_0)+\frac{1}{n}\tilde{K}\left(\frac{z}{n}\right)\widetilde{B_n}(z;t_0)=e^{-zt_0}$$

Then:

$$\widetilde{B_n}(z;t_0)=\frac{e^{-zt_0}}{in+\frac{1}{n}\tilde{K}\left(\frac{z}{n}\right)} \tag{2.11}$$

Notice that the convolution structure of (2.9) implies that $B_n(t;s)=B_n(t-s;0)$. This can be seen also using (2.11). Moreover, classical properties of analytic functions imply that the absence of zeros of $Q_n(z)\equiv in+\frac{1}{n}\tilde{K}\left(\frac{z}{n}\right)$ in the half-plane $\{\mathrm{Re}(z)>-\gamma\}$ for some $\gamma>0$ implies the exponential decay of $B_n(t;0)$ as $t\to\infty$. The function $Q_n(z)$ is, up to linear changes of variables, the so-called Landau function, has been studied extensively in several papers. Notice that an equivalent way of writing it, using (2.7), is the following one:

$$\tilde{K}(z)=\int_0^\infty d\xi e^{-\xi z}\int_{-\infty}^\infty e^{-i\xi v}f_{e,v}(v)\,dv=\int_{-\infty}^\infty \frac{f_{e,v}(v)\,dv}{z+iv}=i\int_{-\infty}^\infty \frac{f_e(v)\,dv}{(z+iv)^2}$$

$$Q_n(z)=i\left(n+\frac{1}{n}\int_{-\infty}^\infty \frac{f_e(v)\,dv}{\left(\frac{z}{n}+iv\right)^2}\right)$$

The following results can be derived from the results in the paper [11] (cf. also [7] for related, although more formal results).

Theorem 2 *Suppose that the function $f_e(v)$ is analytic in the strip $|\mathrm{Im}(v)| < A$, $g_0(x,v)$ is analytic in $|\mathrm{Im}(v)| < A$, $|\mathrm{Im}(x)| < A$ and satisfy*

$$|f_e(v)| \leq \frac{B}{1+|v|^{\alpha}} \quad , \quad \alpha > 1 \quad , \quad |\mathrm{Im}(v)| < A$$

$$|g_0(x,v)| \leq \frac{B}{1+|v|^{\alpha}} \quad , \quad \alpha > 1 \quad , \quad |\mathrm{Im}(v)| < A, \; |\mathrm{Im}(x)| < A$$

for some $A, B > 0$. Assume also that the function $Q_n(z)$ does not have zeros in the half-plane $\mathrm{Im}(z) \geq 0$ for some value of n. Then the n-th Fourier coefficient of $E(x,t)$ defined in (2.6) decreases exponentially as $t \to \infty$.

Theorem 3 *Suppose that f_e satisfy the assumptions in Theorem 2. Assume that the function $Q_n(z)$ has a zero in the half-plane $\mathrm{Im}(z) > 0$ for some $n \in \mathbb{Z}$. Then, there exists $g_0(x,v)$ satisfying the assumptions in Theorem 2 such that:*

$$\|E(\cdot,t)\|_{L^p(0,2\pi)} \geq Ce^{\gamma t} \quad , \quad 1 \leq p \leq \infty$$

for some $\gamma > 0$, $C > 0$.

Remark 4 *Notice that the assumption on the zeros of $Q_n(z)$ in Theorem 2 is not satisfied for all the nonnegative initial distributions $f_e(v)$. There are several examples of such functions yielding instabilities in [7], [11]. For instance, an example of function satisfying the analyticity assumptions on Theorem 2 and where the corresponding function $Q_n(z)$ has zeros in the half-plane $\mathrm{Im}(z) > 0$ for $n \neq 0$ is (cf. [11]):*

$$f_e(v) = \frac{4a^{5/2}}{3\pi^{\frac{1}{2}}} v^4 e^{-av^2}$$

There exist distributions $f_e(v)$ for which the corresponding functions $Q_n(z)$ yield stability. For instance, the maxwellian distribution $f_e(v) = \frac{e^{-v^2}}{\sqrt{\pi}}$ has been extensively studied in the physical literature and it has been shown to satisfy the assumption in Theorem 2 (cf. [7], [11]). On the other hand there is a proof in [7] of the fact that all the distributions of $f_e(v)$ with only one maximum yield stability for the corresponding linearized problem.

An example, not so relevant physically, but where the condition on the roots of $Q_n(z)$ in Theorem 2 can be checked easily is:

$$f_e(v) = \frac{1}{1+v^2}$$

In this case, the function $Q_n(z)$ can be computed using residues:

$$Q_n(z) = i\left[n + \frac{1}{n}\frac{\pi}{\left(\frac{z}{n} + sgn\left(\mathrm{Re}\left(\frac{z}{n}\right)\right)\right)^2}\right]$$

the zeroes of $Q_n(z)$ are:

$$z = -|n| \pm \frac{\sqrt{\pi}}{n} i$$

The following result can be obtained using the inversion formula for the Laplace transform, as well as the methods in [11].

Theorem 5 *Suppose that f_e satisfies the assumptions in Theorem 2 and the function $Q_n(z)$ does not have zeroes in $\mathrm{Im}(z) \geq 0$. Then the function $B_n(t;t_0)$ defined by means of (2.9) satisfies:*

$$|B_n(t;t_0)| \leq \frac{C}{n} e^{-\gamma(t-t_0)} \quad , \quad t \geq t_0 \quad , \; B_n(t;t_0) = 0 \quad , \quad t < t_0$$

This estimate for the fundamental solution $B_n(t;t_0)$ implies an exponential decay for E and for initial data g_0 satisfying the assumptions in Theorem 2. Indeed, these assumptions on g_0 imply:

$$|G_n(t)| \leq \left| \int dv g_n(v) e^{-invt} \right| \leq C_n \varepsilon e^{-\frac{A}{2}|n|t}$$

where the constants C_n decrease faster than any power law if the function g_0 is assumed to be C^∞ in x.

Then, using (2.10):

$$|b_n(t)| \leq C_n \varepsilon e^{-bt}$$

for some $b > 0$, whence the exponential decay of E follows.

Let us now indicate the difficulty that arises in trying to use the results for the linearized problem in order to derive decay of the solutions for the whole nonlinear problem. Suppose that we keep the neglected quadratic terms in the linearized equation (2.1). Then:

$$g_t + vg_x + Ef_{e,v} + Eg_v = 0 \quad , \quad 0 < x < L \quad , \quad t > 0 \; , \; v \in \mathbb{R} \tag{2.12}$$

Using Duhamel's formula it then follows that:

$$g(x,v,t) = g_0(x - vt, v) - \int_0^t E(x - v(t-s), s) f_{e,v}(v) ds$$
$$- \int_0^t E(x - v(t-s), s) g(v) ds$$

Plugging this identity into (2.2) as before it follows that:

$$E_x(x,t) = \int g(x,v,t)\,dv = \int g_0(x-vt,v)\,dv -$$
$$-\int_0^t \left[\int E(x-v(t-s),s)\, f_{e,v}(v)\,dv\right] ds -$$
$$-\int_0^t \left[\int E(x-v(t-s),s)\, g(x,v,s)\,dv\right] ds$$

Notice that, due to the presence of the last term containing the nonlinear part of the integration $\int_0^t [...]\,ds$ it might not be possible to ensure that this term converges to zero exponentially fast. Actually, this does not happen in general. Therefore, it might not be possible to treat this term as a small quadratic perturbation of the above considered problem. This difficulty is not resolved by using more sophisticated linearization procedures, for instance approximating the characteristics of the whole nonlinear problem by the free streaming characteristics plus a corrective term and expanding by using the Taylor series in powers of the corrective term. The reason behind the failure of these approximation procedures is that the function $g(x,v,t)$ does not converge to zero, although the electric field E does. As a consequence, all these linearization procedures contain terms that may not be expected to be small as $t \to \infty$ and some method for handling them must be found.

3 Linearization near infinity: Formal computation.

In this Section we describe at a formal level a more convenient way of linearizing the Vlasov-Poisson system in order to derive estimates for the nonlinear terms. The key idea is to linearize around the expected asymptotics of the solutions as $t \to \infty$. However, the function $f(x,v,t)$ is expected to be oscillatory as $t \to \infty$, since in the absence of the field its dynamics would be given by the transport equation, and then we might expect:

$$f(x,v,t) \sim F(x-vt,v) \quad \text{as} \quad t \to \infty$$

for some function F. In order to obtain a function converging to a limit as $t \to \infty$ it is convenient to introduce a new set of variables:

$$z = x - vt \tag{3.1}$$
$$\tilde{f}(x-vt,v,t) = f(x,v,t) \tag{3.2}$$

Then:

$$f_t = \tilde{f}_t - v\tilde{f}_z \quad , \quad f_x = \tilde{f}_z \quad , \quad f_v = \tilde{f}_v - t\tilde{f}_z$$

Then (1.1) becomes:

$$\tilde{f}_t(z,v,t) - tE(z+vt,t)\,\tilde{f}_z(z,v,t) + E(z+vt,t)\,\tilde{f}_v(z,v,t) = 0$$

On the other hand we can rewrite (1.3) as:

$$E_x(x,t) = \int_{-\infty}^{\infty} dv f(x,v,t) - 1 = \int_{-\infty}^{\infty} \tilde{f}(x-wt,w,t)\,dw - 1$$

where we have used that $n_0 = 1$. Also:

$$E_z(z,t) = \int_{-\infty}^{\infty} \tilde{f}(z-wt,w,t)\,dw - 1$$

From now on, we will drop the tilde from $\tilde{f}$ in order to simplify the notation. Therefore we need to study the problem:

$$f_t(z,v,t) - tE(z+vt,t)\,f_z(z,v,t) + E(z+vt,t)\,f_v(z,v,t) = 0 \tag{3.3}$$

$$E_z(z,t) = \int_{-\infty}^{\infty} f(z-wt,w,t)\,dw - 1 \tag{3.4}$$

More precisely, we will construct solutions $f(z,v,t)$ satisfying:

$$f(z,v,t) \to f_e(v) + g_\infty(z,v) \equiv f_\infty(z,v) \quad \text{as} \quad t \to \infty \tag{3.5}$$

where the functions $f_e(v)$, $g_\infty(x,v)$ will be assumed to satisfy suitable analyticity assumptions that will be made precise later, and the function $g_\infty(x,v)$ is assumed to be of order of the small parameter ε and:

$$\int_0^{2\pi} g_\infty(z,v)\,dz = 0 \tag{3.6}$$

The characteristic equations associated to (3.3) satisfy:

$$Z(t,t;z,v) = z \quad , \quad V(t,t;z,v) = v \tag{3.7}$$

and:

$$\frac{\partial Z}{\partial s}(s,t;z,v) = -sE(Z+Vs,s) \quad , \quad \frac{\partial V}{\partial s}(s,t;z,v) = E(Z+Vs,s) \tag{3.8}$$

We define the functions:

$$Z(\infty,t;z,v) = Z_\infty(t;z,v) \quad , \quad V(\infty,t;z,v) = V_\infty(t;z,v) \tag{3.9}$$

Then, the solution of (3.3) can be written as:

$$f(z,v,t) = f_\infty(Z_\infty(t;z,v), V_\infty(t;z,v))$$

Using (3.5) it follows that:

$$f(z,v,t) = f_e(V_\infty(t;z,v)) + g_\infty(Z_\infty(t;z,v), V_\infty(t;z,v))$$

Therefore:

$$E_z(z,t) = \int_{-\infty}^{\infty} [f(z-wt,w,t) - f_e(w)]\, dw$$

whence:

$$E_z(z,t) = \int_{-\infty}^{\infty} [f_e(V_\infty(t;z-wt,w)) - f_e(w)]\, dw + \int_{-\infty}^{\infty} g_\infty(Z_\infty(t;z-wt,w), V_\infty(t;z-wt,w))\, dw \tag{3.10}$$

In order to linearize (3.10) we argue as follows. We are trying to construct solutions where E decreases exponentially as $t \to \infty$. Moreover, the field can be expected to be of order ε. In particular the functions Z, V can be expected to be nearly constant. We can then approximate the solution of the equations (3.8) to the leading order as:

$$Z_\infty(t;z,v) - z = -\int_t^{\infty} sE(z+vs,s)\, ds \tag{3.11}$$

$$V_\infty(t;z,v) - v = \int_t^{\infty} E(z+vs,s)\, ds \tag{3.12}$$

Linearizing the functions f_e, g_∞ using Taylor's theorem, and taking into account (3.11), (3.12) we obtain:

$$\begin{aligned} E_z(z,t) = &\int_t^{\infty} ds \int_{-\infty}^{\infty} f_{e,v}(w)\, E(z-wt+ws,s)\, dw + \int_{-\infty}^{\infty} g_\infty(z-wt,w)\, dw \\ &- \int_t^{\infty} sds \int_{-\infty}^{\infty} \frac{\partial g_\infty}{\partial z}(z-wt,w)\, E(z-wt+ws,s)\, dw + \\ &+ \int_t^{\infty} ds \int_{-\infty}^{\infty} \frac{\partial g_\infty}{\partial w}(z-wt,w)\, E(z-wt+ws,s)\, dw \end{aligned} \tag{3.13}$$

Equation (3.13) is the linearized problem as $t \to \infty$. Our goal is to show that this problem provides a good approximation for the solutions of the original problem (1.1)-(1.6) as $t \to \infty$.

It is relevant to notice that (3.13) contains terms that can be expected to be of order ε^2, namely the last two ones. On the other hand, the first two terms on the right hand side of (3.13) can be expected to be of order ε. The reason for keeping the last two terms is that due to the presence of the term s in the third term of (3.13) it is not "a priori" clear if this term can be neglected for times $t \approx \frac{1}{\varepsilon}$ or larger. The last term in (3.13) has been kept to ensure that for the resulting problem $\int_0^{2\pi} E_z(z,t)\,dz = 0$. Indeed, assuming that this identity is satisfied and taking into account that $\int_0^{2\pi} E(z,t)\,dz = 0$ as well as (3.6) it follows, after integrating (3.13) in $z \in [0, 2\pi]$:

$$
\begin{aligned}
0 = & -\int_0^{2\pi} dz \int_t^{\infty} sds \int_{-\infty}^{\infty} \frac{\partial g_\infty}{\partial z}(z-wt,w)\, E(z-wt+ws,s)\, dw+ \\
& + \int_0^{2\pi} dz \int_t^{\infty} ds \int_{-\infty}^{\infty} \frac{\partial g_\infty}{\partial w}(z-wt,w)\, E(z-wt+ws,s)\, dw
\end{aligned} \tag{3.14}
$$

In order to check this inequality we notice that:

$$
\frac{Dg_\infty}{Dw}(z-wt,w) = -t\frac{\partial g_\infty}{\partial z}(z-wt,w) + \frac{\partial g_\infty}{\partial w}(z-wt,w) \tag{3.15}
$$

Therefore, the right hand side of (3.14) can be written as:

$$
\int_t^{\infty} ds \int_0^{2\pi} dz \int_{-\infty}^{\infty} dw E(z-wt+ws,s)\left[(t-s)\frac{\partial g_\infty}{\partial z}(z-wt,w) + \frac{Dg_\infty}{Dw}(z-wt,w)\right] \tag{3.16}
$$

and, integrating by parts we can transform (3.16) into:

$$
\begin{aligned}
& -\int_t^{\infty} ds \int_0^{2\pi} dz \int_{-\infty}^{\infty} dw g_\infty(z-wt,w)\, \cdot \\
& \cdot \left[(t-s) E_z(z-wt+ws,s) - (t-s) E_z(z-wt+ws,s)\right] = 0
\end{aligned}
$$

whence (3.14) holds.

To conclude this section, we remark that there is a way of writing (3.13) where it becomes apparent that the last two terms are really small perturbations. Indeed, using (3.15) we can rewrite (3.13) as:

$$
\begin{aligned}
E_z(z,t) = & \int_t^{\infty} ds \int_{-\infty}^{\infty} f_{e,v}(w)\, E(z-wt+ws,s)\, dw + \int_{-\infty}^{\infty} g_\infty(z-wt,w)\, dw \\
& + \int_t^{\infty} \frac{s}{t} ds \int_{-\infty}^{\infty} \left[\frac{Dg_\infty}{Dw}(z-wt,w)\right] E(z-wt+ws,s)\, dw+ \\
& + \int_t^{\infty} ds \int_{-\infty}^{\infty} \left(1-\frac{s}{t}\right) \frac{\partial g_\infty}{\partial w}(z-wt,w)\, E(z-wt+ws,s)\, dw
\end{aligned}
$$

and, after integrating by parts in the third equation:

$$
\begin{aligned}
E_z(z,t) = & \int_t^{\infty} ds \int_{-\infty}^{\infty} f_{e,v}(w)\, E(z-wt+ws,s)\, dw + \int_{-\infty}^{\infty} g_\infty(z-wt,w)\, dw \\
& + \int_t^{\infty} (t-s)\frac{s}{t} ds \int_{-\infty}^{\infty} g_\infty(z-wt,w)\, E_z(z-wt+ws,s)\, dw+ \\
& + \int_t^{\infty} ds \int_{-\infty}^{\infty} \left(1-\frac{s}{t}\right) \frac{\partial g_\infty}{\partial w}(z-wt,w)\, E(z-wt+ws,s)\, dw
\end{aligned} \tag{3.17}
$$

Suppose now that we study (3.17) in a space of functions satisfying $|E(z,t)| + |E_z(z,t)| \leq Me^{-\gamma t}$ for some suitable $\gamma > 0,\ M > 0$. Then, the third and fourth terms on the right side of (3.17) can be estimated as:

$$C\varepsilon M e^{-\gamma t}$$

It then follows that, for a suitable choice of γ, the last two terms in (3.17) can be expected to be small perturbative terms. This fact will be made rigorous in Section 6.

Therefore, we expect to be able to approximate (3.17) as:

$$E_z(z,t) = \int_t^{\infty} ds \int_{-\infty}^{\infty} f_{e,v}(w)\, E(z - wt + ws, s)\, dw + \int_{-\infty}^{\infty} g_{\infty}(z - wt, w)\, dw \tag{3.18}$$

Equation (3.18) is basically equivalent to the linearized Landau damping problem studied in Section 2, except for the fact that the linearization has been made at $t = \infty$. We will study first some properties of the linearized problem (3.18) using Fourier analysis. We will then study the whole nonlinear problem (1.1)-(1.6) using a suitable functional framework.

4 Analyticity assumptions for f_e and g_∞ and some consequences.

We make precise now the assumptions made on the functions $f_e,\ g_\infty$.

Assumptions (A):

- The function $f_e(v)$ is analytic in the strip $|\mathrm{Im}(v)| \leq A$ and it satisfies in that set
$$|f_e(v)| \leq \frac{B}{1 + |v|^{\alpha}} \quad , \quad \alpha > 1 \quad , \quad \alpha \neq 2 \tag{4.1}$$
- The function $g_\infty(x,v)$ is analytic in the sets $|\mathrm{Im}(x)| \leq A,\ |\mathrm{Im}(v)| \leq A$ and it satisfies in this set:
$$|g_\infty(x,v)| \leq \frac{\varepsilon}{1 + |v|^{\alpha}} \quad , \quad \alpha > 1 \quad , \quad \alpha \neq 2 \tag{4.2}$$
- The function $g_\infty(x,v)$ is periodic in the x variable with period 2π and it satisfies:
$$\int_0^{2\pi} g_\infty(x,v)\, dx = 0 \tag{4.3}$$

The requirement $\alpha \neq 2$ seems at a first glance a bit artificial. This assumption has been made in order to avoid the onset of logarithmic terms that would introduce nonessential technical difficulties. Notice that we can always assume that $\alpha \neq 2$ reducing the value of α a bit if needed.

There is a consequence of (4.2) that will be used repeatedly in the following.

Lemma 6 *Suppose that (4.2), (4.3) are satisfied. Then the function*

$$H(z,t)=\int_{-\infty}^{\infty} g_{\infty}(z-wt,w)\,dw \tag{4.4}$$

satisfies:

$$|H(z,t)|\leq C\varepsilon e^{-\gamma t}\quad,\quad z\in[0,2\pi] \tag{4.5}$$

where $\gamma>0$ can be chosen arbitrarily close to A, $\gamma<A$ and $C>0$ depends only on α,γ,A.

Proof. Due to the analyticity properties of the function g_{∞} we can rewrite $H(z,t)$ for $|\mathrm{Im}(z)|<A+At$ using contour deformation as:

$$H(z,t)=\int_{\Gamma(z,t)} g_{\infty}(z-wt,w)\,dw \tag{4.6}$$

where $\Gamma(z,t)=\{w\in\mathbb{C}:\mathrm{Im}(w)=a(z,t)\}$ with $|\mathrm{Im}(z)-ta(z,t)|<A$, $|a(z,t)|<A$. It is readily seen that for any z satisfying $|\mathrm{Im}(z)|<A+At$ it is possible to choose such $a(z,t)$, although in a nonunique way. For instance, given z in $|\mathrm{Im}(z)-rt|=\beta<A$ with $|r|<A$, $|\beta|<A$ we can choose $a(z,t)=r$. It then follows from the representation formula (4.6) that $H(z,t)$ is analytic in $|\mathrm{Im}(z)|<A+At$. Moreover, using (4.2) it follows that

$$|H(z,t)|\leq C\varepsilon \tag{4.7}$$

in $|\mathrm{Im}(z)|<A+At$.

The periodicity of g_{∞} in z implies that $H(z,t)$ is periodic in z for any $t\geq 0$ and (4.3) implies

$$\int_0^{2\pi} H(z,t)\,dz=0$$

We can then write $H(z,t)$ using the following Fourier series:

$$H(z,t)=\sum_{n\neq 0}\frac{e^{inz}}{2\pi}\int_0^{2\pi} e^{-in\xi}H(\xi,t)\,d\xi \tag{4.8}$$

Using the analyticity properties of $H(\cdot,t)$ we can rewrite the Fourier coefficients in (4.8) as:

$$\int_0^{2\pi} e^{-in\xi}H(\xi,t)\,d\xi=\int_{[0,2\pi]+iL} e^{-in\xi}H(\xi,t)\,d\xi$$

where $L\in(-A-At,A+At)$. Choosing $sign(L)=-sign(n)$ and choosing $L=\gamma+\gamma t$ with $\gamma<A$ arbitrarily close to A it then follows from (4.7) that:

$$\left|\int_0^{2\pi} e^{-in\xi}H(\xi,t)\,d\xi\right|\leq 2\pi C\varepsilon e^{-\gamma|n|(t+1)}$$

and plugging this estimate into (4.8) we obtain (4.5). ■

5 Construction of the fundamental solution of the linearized problem at $t = \infty$.

Equation (3.18) as well as the form of the nonlinear terms that have been neglected in the derivation of it suggest to study the following problem:

$$E_z(z,t) = \int_t^\infty ds \int_{-\infty}^\infty f_{e,v}(w) E(z - wt + ws, s)\, dw + h(z,t) \tag{5.1}$$

where $h(z,t)$ is a bounded function decreasing sufficiently fast as $t \to \infty$ and satisfying

$$\int_0^{2\pi} h(z,t)\, dz = 0 \tag{5.2}$$

In order to study this problem we will construct a fundamental solution associated with it. More precisely we will derive, using Fourier analysis, an explicit formula for a solution of:

$$G_z(z,t) = \int_t^\infty ds \int_{-\infty}^\infty f_{e,v}(w) G(z - wt + ws, s)\, dw + \left[\delta(z) - \frac{1}{2\pi}\right] \delta(t), \tag{5.3}$$

$$t \in \mathbb{R},\ z \in \mathbb{R}$$

satisfying:

$$G(z,t) = 0 \ ,\ \ t > 0\ ,\ \ z \in \mathbb{R} \tag{5.4}$$

$$G(z,t) = G(z + 2\pi, t) \tag{5.5}$$

Using (5.2) as well as the invariance of the homogeneous part of (5.3) under spatial and time translations we obtain that a solution $E(z,t)$ of (5.1) can be written as:

$$\begin{aligned} E(z,t) &= \int_{-\infty}^\infty ds \int_0^{2\pi} d\xi G(z - \xi, t - s)\, h(\xi, s) \\ &= \int_t^\infty ds \int_0^{2\pi} d\xi G(z - \xi, t - s)\, h(\xi, s) \end{aligned} \tag{5.6}$$

since, due to (5.2):

$$h(z,t) = \int_{-\infty}^\infty ds \int_0^{2\pi} d\xi \left[\delta(z - \xi) - \frac{1}{2\pi}\right] \delta(t - s)\, h(\xi, s)$$

There is a function that plays a crucial role in the whole theory of Landau damping and that appears in slightly different forms in different papers devoted to this subject. This function, that is usually referred as the Landau function takes the following form in our setting:

$$\Phi(\eta; n) = \int_{\mathbb{R}} \frac{f_{e,v}(w)}{w - \eta} dw - n^2 \tag{5.7}$$

This function is defined in $\eta \in \mathbb{C}\backslash\mathbb{R}$, $n = \pm1, \pm2, ...$ If f_e satisfies (4.1), the function Φ can be extended analytically to the domain $\{\mathrm{Im}(\eta) > -A\}$ for any $n = \pm1, \pm2, ...$ It is worth mentioning that the function $\Phi(\eta; n)$ is discontinuous for $\eta \in \mathbb{R}$. Indeed, due to the Plemej-Sokolski formula (cf. [1]):

$$\Phi(\eta_0 + i0) - \Phi(\eta_0 - i0) = 2\pi i f_{e,v}(\eta_0)$$

Therefore the analytic continuation of the function Φ to the domain $\{\mathrm{Im}(\eta) > -A\}$ is not given by the integral formula (5.7). This is a well known property of the Landau function (cf. [7]).

The main result of this Section is summarized in the following Theorem:

Theorem 7 *Suppose that f_e satisfies (4.1). Let $0 < \delta < A$. Define two functions $\psi_\pm(\eta)$ by means of:*

$$\psi_\pm(\eta) = \int_{C_{\pm\delta}} \frac{f_{e,v}(w)}{\eta \pm w} dw, \tag{5.8}$$

where $C_{\pm\delta} = \mathbb{R} \pm i\delta$.

We define a function $Q(z,t)$ by means of:

$$Q_z(z,t) = \frac{1}{(2\pi)^2} \sum_{n=-\infty\ ,\ n\neq0}^{\infty} \frac{e^{inz}}{n} \int_{-\infty}^{\infty} \frac{e^{i|n|\eta t}\psi_{sign(n)}(\eta)}{\left(1 - \frac{\psi_{sign(n)}(\eta)}{|n|n}\right)} d\eta \tag{5.9}$$

$$Q(z,t) = \int_0^z Q_z(\xi,t)\, d\xi + \frac{1}{2\pi}\int_0^{2\pi} \xi Q_z(\xi,t)\, d\xi \tag{5.10}$$

as well as a tempered distribution G by means of:

$$G_z(z,t) = Q_z(z,t) + \delta(t)\left[\sum_{\ell=-\infty}^{\infty} \delta(z + 2\pi\ell) - \frac{1}{2\pi}\right] \tag{5.11}$$

$$G(z,t) = \int_0^z G_z(\xi,t)\, d\xi + \frac{1}{2\pi}\int_0^{2\pi} \xi G_z(\xi,t)\, d\xi \tag{5.12}$$

Suppose that the Landau function defined in (5.7) does not have zeroes in the half-plane $\{\mathrm{Im}(\eta) \geq 0\}$. Then, the series defining the function Q_z in (5.9) converges for any $z \in \mathbb{R}$, $t \in \mathbb{R}$. The distributions G, Q are supported in the set $t \leq 0$ and they are periodic in z with period 2π. We have $Q \in C^\infty(\mathbb{R}\times(-\infty,0))$. Moreover, the following estimates hold:

$$|Q(z,t)| + |Q_z(z,t)| \leq \frac{C}{\left(1 + \left|\frac{z}{t}\right|^\alpha\right)}\ \ ,\ \ 0 \leq t \leq 1\ \ ,\ \ -\pi \leq z \leq \pi \tag{5.13}$$

$$|Q(z,t)| + |Q_z(z,t)| \leq Ce^{-a|t|}\ \ ,\ \ t \geq 1\ \ ,\ \ z \in \mathbb{R} \tag{5.14}$$

where α is as in (4.1) and the constants C, a depend on B, α.

The distribution G solves (5.3)-(5.5).

In order to prove Theorem 7 we begin by deriving the analyticity properties of the functions $\psi_{\pm}$ as well as some basic estimates for them.

Lemma 8 *Suppose that f_e satisfies (4.1) and let δ be as in Theorem 7. Then the functions $\psi_{\pm}(\eta)$ defined in (5.8) are analytic in $\{\operatorname{Im}(\eta) > -\delta\}$ and they satisfy the following estimates:*

$$\left|\psi_{\pm}(\eta)\right| \leq \frac{C}{1+|\eta|^{\beta}} \ \ , \ \ \operatorname{Im}(\eta) > -\frac{\delta}{2} \ \ , \tag{5.15}$$

$$\left|D_{\eta}^{k}\psi_{\pm}(\eta)\right| \leq \frac{C_k}{1+|\eta|^{\beta}} \ \ , \ \ \operatorname{Im}(\eta) > -\frac{\delta}{2} \ , \ \ k=1,2,... \tag{5.16}$$

where $1 < \beta = \min\{\alpha, 2\} - \varepsilon_0$ with $\varepsilon_0 > 0$ that might be chosen arbitrarily small, C depends only on α, B, δ, ε_0 and C_k depends on α, B, δ, ε_0, k.

Proof. The analyticity of $\psi_{\pm}(\eta)$ in $\{\operatorname{Im}(\eta) > -\delta\}$ is just a consequence of the analyticity of the functions $\frac{1}{\eta \pm w}$ for each $w \in C_{\pm\delta}$ in the half-plane $\operatorname{Im}(\eta) > -\delta$ as well as the assumption (4.1).

To derive (5.15) we use the inequality:

$$\left|\frac{1}{\eta \pm w} - \frac{1}{\eta}\right| \leq \frac{|w|}{|\eta|\,|\eta \pm w|}$$

Then:

$$\left|\psi_{\pm}(\eta) - \frac{1}{\eta}\int_{C_{\pm\delta}} f_{e,v}(w)\,dw\right| \leq \frac{B}{|\eta|}\int_{C_{\pm\delta}} \frac{|w|}{|\eta \pm w|}\frac{1}{1+|w|^{\alpha}}\,|dw| \tag{5.17}$$

Using the fact that $\int_{C_{\pm\delta}} f_{e,v}(w)\,dw = 0$ and splitting the integral on the right side of (5.17) in the regions where $|w| \leq \frac{|\eta|}{2}$ and $|w| > \frac{|\eta|}{2}$ respectively, we obtain:

$$\begin{aligned}\left|\psi_{\pm}(\eta)\right| \leq \frac{B}{|\eta|^{2}}\int_{C_{\pm\delta}\cap\left\{|w|\leq\frac{|\eta|}{2}\right\}} \frac{|w|}{1+|w|^{\alpha}}\,|dw| + \\ + \frac{B}{|\eta|}\int_{C_{\pm\delta}\cap\left\{|w|>\frac{|\eta|}{2}\right\}} \frac{1}{|\eta \pm w|\,|w|^{\alpha-1}}\,|dw|\end{aligned} \tag{5.18}$$

We can estimate the first integral on the right side of (5.18) by a constant if $\alpha > 2$ and as $C|\eta|^{-2+\alpha}$ if $\alpha < 2$. (Notice that $\alpha \neq 2$, and therefore the logarithmic case does not occur). On the other hand we can estimate the second integral on the right hand side of (5.18) introducing the rescaling $w = |\eta|\,\zeta$. Then:

$$\left|\psi_{\pm}(\eta)\right| \leq \frac{C}{|\eta|^{\beta}} + \frac{B}{|\eta|^{\alpha}}\int_{\frac{C_{\pm\delta}}{|\eta|}\cap\left\{|\zeta|>\frac{1}{2}\right\}} \frac{1}{\left|\frac{\eta}{|\eta|} \pm \zeta\right|\,|\zeta|^{\alpha-1}}\,|d\zeta|$$

The integral in this formula is convergent, since $\alpha > 1$. The main contribution to this integral for large $|\eta|$ is due to the region where $\left|\frac{\eta}{|\eta|} \pm \zeta\right|$ is small. The closest distance between ζ and $\mp\frac{\eta}{|\eta|}$ is of order $\frac{1}{|\eta|}$. Therefore:

$$\int_{\frac{C_{\pm\delta}}{|\eta|}\cap\left\{|\zeta|>\frac{1}{2}\right\}} \frac{1}{\left|\frac{\eta}{|\eta|} \pm \zeta\right| |\zeta|^{\alpha-1}} |d\zeta| \leq C_\delta \left[|\log(|\eta|)| + 1\right]$$

whence, choosing $\varepsilon_0 > 0$ arbitrarily small, (5.15) follows.

Estimate (5.16) can be proved combining (5.15) as well as the fact that the functions $\psi_\pm(\eta)$ are analytic in $\{\mathrm{Im}(\eta) > -\delta\}$ and the classical Cauchy's inequalities for analytic functions (cf. [1]). ■

The functions $\psi_\pm(\eta)$ are closely related to the Landau function Φ (cf. (5.7)). We reformulate some properties of the functions $\psi_\pm(\eta)$ in terms of properties of the function Φ that have been studied often in the literature (cf. for instance [3]).

Lemma 9 *Suppose that* f_e *satisfies (4.1). Suppose that the Landau function* $\Phi(\eta; n)$ *defined in (5.7) does not have zeroes in the region* $\{\mathrm{Im}(\eta) \geq 0\}$ *for any* $n = \pm 1, \pm 2, \ldots.$ *Then, there exist* $\nu_0 > 0$ *and* $\theta > 0$ *such that*

$$\left|1 - \frac{\psi_{sign(n)}(\eta)}{|n|\, n}\right| \geq \theta \tag{5.19}$$

for $\mathrm{Im}(\eta) > -\nu_0$.

Proof. Using (5.8) we have:

$$1 - \frac{\psi_{sign(n)}(\eta)}{|n|\, n} = 1 - \frac{1}{n^2}\int_{C_{\pm\delta}} \frac{f_{e,v}(w)}{w + sign(n)\,\eta} dw \equiv -\frac{1}{n^2}\Phi(-sign(n)\,\eta; n) \tag{5.20}$$

Notice that (5.7) as well as the fact that $f_e(w)$ takes real values for $w \in \mathbb{R}$ implies:

$$\Phi(\bar{\eta}; n) = \overline{(\Phi(\eta; n))} \tag{5.21}$$

Therefore, if $\Phi(\eta; n)$ does not have zeroes for $\{\mathrm{Im}(\eta) \geq 0\}$, it does not have zeroes for $\{\mathrm{Im}(\eta) \leq 0\}$ either. Moreover, due to (5.21) we can restrict our attention to the case $sign(n) = -1$. Due to (4.1) we can extend analytically the function $\Phi(\eta; n)$ to the domain $\{\mathrm{Im}(\eta) > -A\}$ by means of the formula:

$$\Phi(\eta; n) = \int_{\mathbb{R}-Ai} \frac{f_{e,v}(w)}{w - \eta} dw - n^2$$

It then follows, due to (4.1) that for $|\eta| > \rho$ with ρ independent of n:

$$\left|\frac{1}{n^2}\Phi(-sign(n)\,\eta; n)\right| \geq \frac{1}{2}$$

On the other hand, since $\Phi(\eta;n)$ does not have zeroes for $\eta \in \{\operatorname{Im}(\eta) \geq 0\}$, $|\eta| \leq \rho$ it follows by continuity that the analytic extension of $\frac{1}{n^2}\Phi(\eta;n)$ to $\{\operatorname{Im}(\eta) > -A\}$ does not have zeroes in the region $-\nu_0 \leq \operatorname{Im}(\eta) \leq 0$, $|\eta| \leq \rho$ for some $\nu_0 > 0$ sufficiently small. Notice that ν_0 can be chosen uniformly in n. Using (5.20) the result follows. ■

Remark 10 *The absence of zeros of the Landau function $\Phi(\eta;n)$ in the half plane $\{\operatorname{Im}(\eta) \geq 0\}$ for any $n = \pm 1, \pm 2, ...$ is precisely the condition required for the stability of the solutions of the linearized Landau problem studied in Section 2. Therefore, the functions $f_e(v)$ yielding stability for the problem considered there yield also stability for the linearized problem considered in this Section. In particular, the examples of stability and instability in Remark 4 are also valid for the linearized problem considered in this Section.*

We now derive some estimates on the Fourier coefficients in the series (5.9) that will ensure the convergence of the series.

From now on we will write by shortness $\pm$ instead of $sign(n)$.

Lemma 11 *Suppose that f_e satisfies (4.1) and let δ be as in Theorem 7. Suppose that the Landau function $\Phi(\eta;n)$ defined in (5.7) does not have zeroes in the region $\{\operatorname{Im}(\eta) \geq 0\}$ for any $n = \pm 1, \pm 2, ...$ Then:*

$$\left| \int_{-\infty}^{\infty} \frac{e^{i|n|\eta t}\psi_{\pm}(\eta)}{\left(1 - \frac{\psi_{\pm}(\eta)}{|n|n}\right)} d\eta \right| \leq \frac{C_\gamma}{|n|^\gamma} \frac{1}{|t|^\gamma} \ \ , \ t \in \mathbb{R} \ , \ \ \gamma > 0 \ , \ n \neq 0 \tag{5.22}$$

where C_γ depends on α, B, δ, γ. Moreover, we have:

$$\int_{-\infty}^{\infty} \frac{e^{i|n|\eta t}\psi_{\pm}(\eta)}{\left(1 - \frac{\psi_{\pm}(\eta)}{|n|n}\right)} d\eta = 0 \ \ for \ \ t > 0 \tag{5.23}$$

and

$$\left| \int_{-\infty}^{\infty} \frac{\psi_{\pm}(\eta)}{\left(1 - \frac{\psi_{\pm}(\eta)}{|n|n}\right)} d\eta \right| \leq \frac{C}{n^2} \ \ , \ n \neq 0 \tag{5.24}$$

where C depends on α, B, δ, γ.

Proof. Notice that, for any $\ell = 1, 2, ...$ we have:

$$e^{i|n|\eta t} = \frac{1}{(i|n|t)^\ell} \frac{\partial^\ell}{\partial \eta^\ell}\left(e^{i|n|\eta t}\right)$$

Then, integrating by parts we obtain:

$$\int_{-\infty}^{\infty} \frac{e^{i|n|\eta t}\psi_{\pm}(\eta)}{\left(1 - \frac{\psi_{\pm}(\eta)}{|n|n}\right)} d\eta = \frac{(-1)^\ell}{(i|n|t)^\ell} \int_{-\infty}^{\infty} e^{i|n|\eta t} \frac{\partial^\ell}{\partial \eta^\ell} \left(\frac{\psi_{\pm}(\eta)}{\left(1 - \frac{\psi_{\pm}(\eta)}{|n|n}\right)} \right) d\eta \tag{5.25}$$

Using (5.19) in Lemma 9, it then follows from (5.16) that:

$$\left|\frac{\partial^{\ell}}{\partial\eta^{\ell}}\left(\frac{\psi_{\pm}(\eta)}{\left(1-\frac{\psi_{\pm}(\eta)}{|n|n}\right)}\right)\right|\leq\frac{C_{\ell}}{1+|\eta|^{\beta}} \tag{5.26}$$

with β as in Lemma 8. Combining (5.25), (5.19), (5.26) we obtain

$$\left|\int_{-\infty}^{\infty}\frac{e^{i|n|\eta t}\psi_{\pm}(\eta)}{\left(1-\frac{\psi_{\pm}(\eta)}{|n|n}\right)}d\eta\right|\leq\frac{C_{\ell}}{(|n|\,t)^{\ell}}\quad,\quad t\in\mathbb{R}\ ,\ n\neq0\ ,\ \ell=0,1,2,... \tag{5.27}$$

Notice that for $\ell=0$ (5.27) follows immediately from (5.15). Estimate (5.22) follows for $\gamma\in(\ell-1,\ell)$ by interpolation.

The identity (5.23) can be obtained using the fact that in the half plane $\operatorname{Im}(\eta)>0$ where the function $\frac{\psi_{\pm}(\eta)}{\left(1-\frac{\psi_{\pm}(\eta)}{|n|n}\right)}$ is analytic and it is bounded as $\frac{C_{\ell}}{1+|\eta|^{\beta}}$ we have also the estimate $\left|e^{i|n|\eta t}\right|=e^{-|n|\operatorname{Im}(\eta)t}\leq1$. Then (5.23) follows by deforming the contour of integration $(-\infty,\infty)$ to $(-\infty,\infty)+iR$ with $R>0$ and taking the limit $R\to\infty$.

Finally, we prove (5.24) using the identity:

$$\int_{-\infty}^{\infty}\frac{\psi_{\pm}(\eta)}{\left(1-\frac{\psi_{\pm}(\eta)}{|n|n}\right)}d\eta=\int_{-\infty}^{\infty}\psi_{\pm}(\eta)\,d\eta+\frac{1}{|n|\,n}\int_{-\infty}^{\infty}\frac{\left(\psi_{\pm}(\eta)\right)^{2}}{\left(1-\frac{\psi_{\pm}(\eta)}{|n|n}\right)}d\eta \tag{5.28}$$

The first term on the right hand side of (5.28) can be computed using (5.8):

$$\int_{-\infty}^{\infty}\psi_{\pm}(\eta)\,d\eta=\lim_{R\to\infty}\int_{-R}^{R}\int_{C_{\pm\delta}}\frac{f_{e,v}(w)}{\eta\pm w}dwd\eta=\lim_{R\to\infty}\int_{C_{\pm\delta}}dwf_{e,v}(w)\int_{-R}^{R}\frac{d\eta}{\eta\pm w}$$

Since $\int_{-R}^{R}\frac{d\eta}{\eta\pm w}=\log\left(\frac{R\pm w}{-R\pm w}\right)$ is bounded for large R and $w\in C_{\pm\delta}$ we can apply Lebesgue dominated convergence Theorem to obtain:

$$\int_{-\infty}^{\infty}\psi_{\pm}(\eta)\,d\eta=\int_{C_{\pm\delta}}dwf_{e,v}(w)\lim_{R\to\infty}\left[\log\left(\frac{R\pm w}{-R\pm w}\right)\right]=-\pi i\int_{C_{\pm\delta}}dwf_{e,v}(w)=0 \tag{5.29}$$

Using (5.15), (5.19) and (5.29) in (5.28) we obtain (5.24). ■

We can now prove the convergence of the series defining $Q_z(z,t)$ in (5.9).

Lemma 12 *Suppose that the assumptions of Theorem 7 are satisfied. The series on the right hand side of (5.9) is convergent for any* $t\in\mathbb{R}$*. The function* Q_z *defined by means of (5.9) is identically zero for* $t>0$, $Q_z(\cdot,t)\in C^{\infty}(\mathbb{R})$ *for any* $t<0$ *and* $Q_z(\cdot,t)\in C^{1}(\mathbb{R})$ *for* $t=0$*.* $Q_z(\cdot,t)$ *defined by means of (5.9) is periodic with period* 2π *for any* $t\in\mathbb{R}$ *and it satisfies* $\int_0^{2\pi}Q_z(z,t)\,dz=0$.

Proof. This Lemma is just a consequence of (5.22)-(5.24). ■

Notice that (5.22)-(5.24) are not strong enough to derive uniform estimates for the function $Q_z(z,t)$ or its derivatives if t is close to zero. This is made in the following Lemma, where the self-similar structure of $Q_z(z,t)$ is derived.

Lemma 13 *Suppose that the assumptions of Theorem 7 are satisfied. Then*

$$Q_z(z,t) = f_e\left(\frac{z}{t}\right) + R(z,t) \quad , \quad z \in \mathbb{R} \, , \ |z| \leq \pi \, , \ \ -1 \leq t < 0$$

where:

$$|R(z,t)| + |R_z(z,t)| \leq C \ \ , \ \ z \in \mathbb{R} \, , \ \ t \in \mathbb{R}$$

Proof. We can write

$$\int_{-\infty}^{\infty} \frac{e^{i|n|\eta t}\psi_{\pm}(\eta)}{\left(1 - \frac{\psi_{\pm}(\eta)}{|n|n}\right)} d\eta = \int_{-\infty}^{\infty} e^{i|n|\eta t}\psi_{\pm}(\eta)\, d\eta + \Omega_n(t) \tag{5.30}$$

where:

$$\Omega_n(t) = \frac{1}{|n|\, n} \int_{-\infty}^{\infty} \frac{e^{i|n|\eta t}\left(\psi_{\pm}(\eta)\right)^2}{\left(1 - \frac{\psi_{\pm}(\eta)}{|n|n}\right)} d\eta \tag{5.31}$$

Using (5.15), (5.19), (5.24) we have:

$$|\Omega_n(t)| \leq \frac{C}{n^2} \tag{5.32}$$

The term $\int_{-\infty}^{\infty} e^{i|n|\eta t}\psi_{\pm}(\eta)\, d\eta$ can be computed explicitly. To this end we derive a more convenient formula for this integral. Using (5.8) we obtain:

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{i|n|\eta t}\psi_{\pm}(\eta)\, d\eta \\
&= \lim_{R\to\infty} \int_{-R}^{R} e^{i|n|\eta t}\psi_{\pm}(\eta)\, d\eta \\
&= \lim_{R\to\infty} \int_{-R}^{R} e^{i|n|\eta t} \int_{C_{\pm\delta}} \frac{f_{e,v}(w)}{\eta \pm w} dw d\eta \\
&= \lim_{R\to\infty} \int_{C_{\pm\delta}} dw f_{e,v}(w) \int_{-R}^{R} \frac{e^{i|n|\eta t}}{\eta \pm w} d\eta \\
&= \int_{C_{\pm\delta}} dw f_{e,v}(w) \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i|n|\eta t}}{\eta \pm w} d\eta
\end{aligned}$$

We just need to compute the integral for $t < 0$. This computation can be made by using residues. More precisely, we can replace the contour of integration $[-R,R]$ by $[-R,R] \cup \left\{z : z = Re^{i\theta},\ \theta \in [-\pi, 0]\right\}$. The contribution to the integral of the half-circle disappears as $R \to \infty$, Indeed, the integrand can be estimated as $\frac{C}{R}$ if $|\mathrm{Im}(\eta)| \leq A$ is of order one, and as $\frac{Ce^{-A|nt|}}{R}$ for $\mathrm{Im}(\eta) \leq -A$. Using the fact that the length of the half circle is bounded by CR, it follows that the contribution of the integral to the half-circle disappears taking the limits $R \to \infty$, $A \to \infty$. Thus:

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i|n|\eta t}}{\eta \pm w} d\eta = -2\pi i e^{\mp i|n|wt} = -2\pi i e^{-inwt}$$

whence, using also the analyticity properties of f_e:

$$\int_{-\infty}^{\infty} e^{i|n|\eta t} \psi_{\pm}(\eta)\, d\eta = -2\pi i \int_{-\infty}^{\infty} dw f_{e,v}(w)\, e^{-inwt} \tag{5.33}$$

Using (5.30), (5.33) we can rewrite (5.9) as:

$$Q_z(z,t) = \omega_0(z,t) + \omega_1(z,t) \quad , \quad t < 0 \tag{5.34}$$

$$\omega_0(z,t) = -\frac{i}{(2\pi)} \sum_{n=-\infty\ ,\ n\neq 0}^{\infty} \frac{e^{inz}}{n} \int_{-\infty}^{\infty} dw f_{e,v}(w)\, e^{-inwt}$$

$$\omega_1(z,t) = \frac{1}{(2\pi)^2} \sum_{n=-\infty\ ,\ n\neq 0}^{\infty} \frac{\Omega_n(t)}{n} e^{inz}$$

Notice that due to (5.32) the function $\omega_1(z,t)$ satisfies:

$$|\omega_1(z,t)| + |\omega_{1,z}(z,t)| \leq C \quad , \quad z \in \mathbb{R}\ , \quad t \in \mathbb{R} \tag{5.35}$$

On the other hand the function $\omega_0(z,t)$ can be explicitly computed. Indeed, let us define the distribution:

$$T(z,t) \equiv -\frac{i}{(2\pi)} \sum_{n=-\infty\ ,\ n\neq 0}^{\infty} \frac{e^{in(z-wt)}}{n}$$

We have, in the sense of distributions:

$$\frac{\partial T}{\partial z}(z,t;w) = \frac{1}{2\pi} \sum_{n=-\infty\ ,\ n\neq 0}^{\infty} e^{inz} e^{-inwt} = \sum_{\ell=-\infty}^{\infty} \delta(z - wt + 2\pi\ell) - \frac{1}{2\pi}$$

whence, using the formula:

$$T(z,t;w) = \int_0^z T_z(\xi,t;w)\, d\xi + \frac{1}{2\pi} \int_0^{2\pi} \xi T_z(\xi,t;w)\, d\xi$$

we obtain:

$$T(z,t;w) = \sum_{\ell=-\infty}^{\infty} \left[\chi(z - wt + 2\pi\ell) - \chi(-wt + 2\pi\ell)\right] - \left(\frac{z}{2\pi} + \frac{1}{2}\right) +$$
$$+ \frac{1}{2\pi} \sum_{\ell=-\infty}^{\infty} (wt - 2\pi\ell) \left[\chi(2\pi(\ell+1) - wt) - \chi(2\pi\ell - wt)\right]$$

where $\chi(\cdot)$ denotes the characteristic function whose support is $(0,\infty)$.

Writing

$$\omega_0(z,t) = \int_{-\infty}^{\infty} dw f_{e,v}(w) T(z,t;w)$$

it then follows that:

$$\omega_0(z,t) = \sum_{\ell=-\infty}^{\infty} \int_{\frac{2\pi\ell}{t}}^{\frac{z+2\pi\ell}{t}} f_{e,v}(w)\,dw + \frac{1}{2\pi} \sum_{\ell=-\infty}^{\infty} \int_{\frac{2\pi\ell}{t}}^{\frac{2\pi(\ell+1)}{t}} f_{e,v}(w)(wt-2\pi\ell)\,dw$$

whence, after some integration:

$$\omega_0(z,t) = \sum_{\ell=-\infty}^{\infty} \left[f_e\left(\frac{z+2\pi\ell}{t}\right) - f_e\left(\frac{2\pi\ell}{t}\right)\right] +$$
$$+ \frac{z}{2\pi} \sum_{\ell=-\infty}^{\infty} f_e\left(\frac{2\pi(\ell+1)}{t}\right) - \frac{t}{2\pi} \sum_{\ell=-\infty}^{\infty} \int_{\frac{2\pi\ell}{t}}^{\frac{2\pi(\ell+1)}{t}} f_e(w)\,dw$$

Using (5.34) we obtain:

$$Q_z(z,t) = f_e\left(\frac{z}{t}\right) + R(z,t)$$

$$R(z,t) = -f_e(0) + \sum_{\ell=-\infty,\ell\neq 0}^{\infty} \left[f_e\left(\frac{z+2\pi\ell}{t}\right) - f_e\left(\frac{2\pi\ell}{t}\right)\right] +$$
$$+ \frac{z}{2\pi} \sum_{\ell=-\infty}^{\infty} f_e\left(\frac{2\pi(\ell+1)}{t}\right) - \frac{t}{2\pi} \sum_{\ell=-\infty}^{\infty} \int_{\frac{2\pi\ell}{t}}^{\frac{2\pi(\ell+1)}{t}} f_e(w)\,dw + \omega_1(z,t)$$

Using (4.1) and (5.35) it follows that:

$$|R(z,t)| + |R_z(z,t)| \leq C \quad , \quad z \in \mathbb{R}\ ,\ |z| \leq \pi\ ,\ -1 \leq t < 0$$

Thus the result follows. ■

We can now derive estimates for $Q_z(z,t)$ for t large.

Lemma 14 *Suppose that the assumptions of Theorem 7 hold. Then:*

$$|Q(z,t)| + |Q_z(z,t)| \leq Ce^{at}$$

for $t < -1$, with $a > 0$ and $C > 0$.

Proof. Combining (5.15) and (5.19) we can write using contour deformation:

$$\int_{-\infty}^{\infty} \frac{e^{i|n|\eta t}\psi_{\pm}(\eta)}{\left(1 - \frac{\psi_{\pm}(\eta)}{|n|n}\right)} d\eta = \int_{\mathbb{R}-i\gamma} \frac{e^{i|n|\eta t}\psi_{\pm}(\eta)}{\left(1 - \frac{\psi_{\pm}(\eta)}{|n|n}\right)} d\eta$$

thus we have

$$\left|\int_{-\infty}^{\infty}\frac{e^{i|n|\eta t}\psi_{\pm}(\eta)}{\left(1-\frac{\psi_{\pm}(\eta)}{|n|n}\right)}d\eta\right|\leq Ce^{\gamma|n|t}\quad,\quad t<0$$

for some $\gamma>0$. It then follows from (5.9) that $Q_z(z,t)$ is periodic in z with period 2π, as well as analytic and bounded in $|\mathrm{Im}(z)|\leq\frac{\gamma t}{2}$, $t<-1$. Arguing as in the proof of Lemma 6 we obtain that $Q_z(z,t)$ satisfies:

$$|Q_z(z,t)|\leq Ce^{at}\quad,\ t\leq-1$$

for some $a>0$. A similar estimate for $Q(z,t)$ then follows from the formula

$$Q(z,t)=\int_0^z Q_z(\xi,t)\,d\xi+\frac{1}{2\pi}\int_0^{2\pi}\xi Q_z(\xi,t)\,d\xi$$

and the result follows. ■

To conclude the proof of Theorem 7 it only remains to show that the distribution G defined in (5.11), (5.12) solves (5.3)-(5.5) in the sense of distributions. To this end we will reformulate (5.3) in term of the Fourier transform of G and we will verify that the solution of the resulting equation is the one given by the Fourier transform of G that can be computed using (5.9)-(5.12).

Lemma 15 *Suppose that the assumptions of Theorem 7 are satisfied. Then the distribution $G(z,t)$ defined by means of (5.11), (5.12) solves (5.3)-(5.5) in the sense of distributions.*

Proof. We expand $G(z,t)$ using Fourier series:

$$G(z,t)=\sum_{n=-\infty}^{n=\infty}g_n(t)\,e^{inz}\tag{5.36}$$

Plugging (5.36) into (5.3) we obtain, after some computations that the functions $g_n(t)$ satisfy:

$$ing_n(t)=\int_t^{\infty}dsg_n(s)\int_{-\infty}^{\infty}f_{e,v}(w)\,e^{-inw(t-s)}dw+\frac{1}{2\pi}\delta(t)\quad,\quad n\neq0\tag{5.37}$$

In order to solve these equations we compute the Fourier transform of the functions $g_n(t)$ that we define by means of:

$$\tilde{g}_n(\theta)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g_n(t)\,e^{-it\theta}dt$$

We define also the functions:

$$\Phi_{\pm}(\theta)=\int_{-\infty}^{\infty}\varphi_{\pm}(t)\,e^{-it\theta}dt\tag{5.38}$$

where:

$$\varphi_{\pm}(t) = \int_{-\infty}^{\infty} f_{e,v}(w)\, e^{\mp iwt} dw \quad , \quad t < 0$$
$$\varphi_{\pm}(t) = 0 \quad , \quad t \geq 0$$

The functions $\varphi_{\pm}$ decrease exponentially as $t \to -\infty$. This can be seen using (4.1) and contour deformation to derive the following representation formula for these functions:

$$\varphi_{\pm}(t) = \int_{C_{\pm\delta}} f_{e,v}(w)\, e^{-iwt} dw \quad , \quad t < 0 \tag{5.39}$$

where $C_{\pm\delta} = \mathbb{R} \pm i\delta$, $0 < \delta < A$.

Applying the Fourier transform to (5.37) and using the fact that a convolution is transformed to a product by this transformation we obtain:

$$in\tilde{g}_n(\theta) = \frac{1}{|n|}\Phi_{\pm}\left(\frac{\theta}{|n|}\right)\tilde{g}_n(\theta) + \frac{1}{2\pi}\frac{1}{\sqrt{2\pi}} \tag{5.40}$$

On the other hand, using (5.39) we can rewrite the functions $\Phi_{\pm}$ in the following form:

$$\Phi_{\pm}(\theta) = \int_{-\infty}^{0} e^{-it\theta}\left[\int_{C_{\pm\delta}} f_{e,v}(w)\, e^{\mp itw} dw\right] dt = \int_{C_{\pm\delta}} f_{e,v}(w)\left[\int_{-\infty}^{0} e^{-it(\theta \pm w)} dt\right] dw$$
$$= i\int_{C_{\pm\delta}} \frac{f_{e,v}(w)}{\theta \pm w} dw$$

Then (5.40) implies:

$$\tilde{g}_n(\theta) = \frac{1}{\sqrt{(2\pi)^3}}\frac{1}{i}\frac{1}{n - \frac{1}{|n|}\left[\int_{C_{\pm\delta}} \frac{f_{e,v}(w)}{\frac{\theta}{|n|} \pm w} dw\right]}$$

or, using the inversion formula for the Fourier transform:

$$g_n(t) = \frac{1}{(2\pi)^2 i}\int_{-\infty}^{\infty} \frac{e^{it\theta} d\theta}{n - \frac{1}{|n|}\left[\int_{C_{\pm\delta}} \frac{f_{e,v}(w)}{\frac{\theta}{n} \pm w} dw\right]} \quad , \quad n \neq 0$$

It is readily seen that this formula is the same that the one for the Fourier coefficients of the function G in (5.12). Notice that the choice (5.12) implies $g_0(t) = 0$. This concludes the proof of the result. ■

End of the Proof of Theorem 7. It is just a consequence of Lemmas 8-15. ■

Using Theorem 7 we can prove the following result that will play a crucial role in the analysis of the nonlinear problem:

Proposition 16 *Suppose that the assumptions of Theorem 7 hold. Let* $h \in C(\mathbb{R} \times \mathbb{R}^+)$ *be a function satisfying (5.2),* $h(z+2\pi, t) = h(z,t)$ *for* $z \in \mathbb{R},\ t \geq 0$, *as well as the estimate:*

$$|h(z,t)| \leq Be^{-\gamma t} \quad , \quad z \in \mathbb{R} \quad , \ t \geq 0$$

with $0 < \gamma < a$.

Then there exists a function $E(z,t) \in C(\mathbb{R} \times \mathbb{R}^+)$ *solving (5.1), satisfying* $\int_0^{2\pi} E(z,t)\, dz = 0,\ E(z+2\pi, t) = E(z,t)$ *and:*

$$|E(z,t)| + |E_z(z,t)| \leq CBe^{-\gamma t}$$

for some $C > 0$ *which depends only on* $A,\ B,\ \alpha$.

Proof. The desired solution $E(z,t)$ can be obtained by means of the formula (5.6). Notice that we just need the values of $h(z,t)$ in $t \geq 0$ to obtain $E(z,t)$ for $t \geq 0$. Moreover, due to the linearity of the estimate we can assume $B = 1$. It is readily seen that $\int_0^{2\pi} E(z,t)\, dz = 0$. Differentiating (5.6) with respect to z we obtain:

$$E_z(z,t) = \int_{-\infty}^{\infty} ds \int_0^{2\pi} d\xi G_z(z-\xi, t-s)\, h(\xi, s)$$

Using (5.11) it then follows that:

$$E_z(z,t) = h(z,t) + \int_{-\infty}^{\infty} ds \int_0^{2\pi} d\xi Q_z(z-\xi, t-s)\, h(\xi, s)$$

Using (5.13), (5.14) it then follows that:

$$|E_z(z,t)| \leq Ce^{-\gamma t}$$

Finally we use the representation formula:

$$E(z,t) = \int_0^z E_z(\xi, t)\, d\xi + \frac{1}{2\pi} \int_0^{2\pi} \xi E_z(\xi, t)\, d\xi$$

to obtain:

$$|E(z,t)| \leq Ce^{-\gamma t}$$

whence the result follows. ■

6 Nonlinear problem: On the existence of exponentially decaying solutions.

In this Section we will prove the existence of a large class of solutions of the nonlinear problem (1.1)-(1.6) that decrease exponentially fast as $t \to \infty$.

Theorem 17 *Suppose that $f_e,\ g_\infty$ satisfy Assumptions (A). Assume that the Landau function defined in (5.7) does not have zeroes in the half-plane* $\{\operatorname{Im}(\eta) \geq 0\}$. *Then there exist $\varepsilon_0 = \varepsilon_0(\alpha, A, B) > 0,\ L = L(\alpha, A, B)$ such that for any $f_e,\ g_\infty$ satisfying Assumptions (A) with $\varepsilon \leq \varepsilon_0$ there exists a solution $f(z, v, t)$ of (3.3)-(3.4) defined for $z \in \mathbb{R},\ v \in \mathbb{R}$ and $0 \leq t < \infty$ and satisfying (3.5) as well as:*

$$|E(x,t)| \leq C\varepsilon e^{-Lt} \ \ \textit{for} \ \ 0 \leq t < \infty$$

Using the change of variables (3.1), (3.2) we can prove the following result:

Corollary 18 *There exists a function $f_0 \in C^1(\mathbb{R} \times \mathbb{R})$, with $f_0(x + 2\pi, v) = f_0(x, v)$ such that the corresponding solutions of the system (1.1)-(1.6) are defined for $0 \leq t < \infty$ and they satisfy:*

$$|E(x,t)| \leq C\varepsilon e^{-Lt} \ \ \textit{for} \ \ 0 \leq t < \infty$$

Remark 19 *In all the following C will denote a numerical constant depending only on $\alpha,\ B,\ A$ that might change from line to line.*

Proof of Theorem 17. The strategy is to solve (3.10) by means of a fixed point argument in a space of functions satisfying:

$$\|E\| = \sup_{t>0} \left\{ e^{\gamma t} \left[|E(z,t)| + |E_z(z,t)| \right] \right\} < \infty$$

Suppose that $\|E\| < \infty$. Then, integrating the characteristic equations (3.8) it follows that:

$$|V_\infty(t; z, w) - w| \leq C \|E\| e^{-\gamma t} \ \ , \ \ 0 \leq t < \infty$$
$$|Z_\infty(t; z, w) - z| \leq C \|E\| (t+1) e^{-\gamma t} \ \ , \ \ 0 \leq t < \infty$$

Using Taylor's expansion in (3.10) we obtain:

$$\begin{aligned} E_z(z,t) = & \int_{-\infty}^{\infty} f_{e,v}(w) \left(V_\infty(t; z - wt, w) - w \right) dw + \int_{-\infty}^{\infty} g_\infty(z - wt, w) \, dw + \\ & + \int_{-\infty}^{\infty} \frac{\partial g_\infty}{\partial z}(z - wt, w) \left(Z_\infty(t; z - wt, w) - (z - wt) \right) dw + \\ & + \int_{-\infty}^{\infty} \frac{\partial g_\infty}{\partial w}(z - wt, w) \left(V_\infty(t; z - wt, w) - w \right) dw + R(z,t) \end{aligned}$$

where:

$$|R(z,t)| \leq C \|E\|^2 (t+1)^2 e^{-2\gamma t}$$

In order to derive better approximations to the differences $Z_\infty(t; z - wt, w) - (z - wt)$, $V_\infty(t; z - wt, w) - w$ we need to derive estimates for the solutions of the characteristic equations:

$$\frac{\partial Z}{\partial s} = -sE(Z + Vs, s) \tag{6.1}$$

$$\frac{\partial V}{\partial s} = E(Z + Vs, s) \tag{6.2}$$

$$Z(t,t;z,v)=z \quad , \quad V(t,t;z,v)=v \tag{6.3}$$

with $s>t$.

Using Taylor's expansion we can write:

$$\frac{dZ}{ds}=-sE(z+vs,s)$$
$$+O\left(s\|E\|e^{-\gamma s}|Z-z|+s^2\|E\|e^{-\gamma s}|V-v|+\|E\|se^{-\frac{\gamma}{2}s}\exp\left(-\frac{a}{\|E\|}e^{\gamma t}\right)\right)$$
$$\frac{dV}{ds}=E(z+vs,s)$$
$$+O\left(\|E\|e^{-\gamma s}|Z-z|+s\|E\|e^{-\gamma s}|V-v|+\|E\|e^{-\frac{\gamma}{2}s}\exp\left(-\frac{a}{\|E\|}e^{\gamma t}\right)\right)$$

for some $a>0$. The first two terms of the remainder arise in making a Taylor expansion for the range of values where $s|V-v|\leq 1$. The last term is the contribution from the region where $s|V-v|>1$. Since $|V-v|\leq CM\|E\|e^{-\gamma t}$ this requires s huge, and due to the exponential decay of E the estimate follows. Taking into account the estimates for the differences $|Z-z|,\ |V-v|$ that are similar to the estimates obtained for $Z_\infty,\ V_\infty$ it then follows:

$$Z_\infty(t;z,v)-z=-\int_t^\infty sE(z+vs,s)\,ds+O\left((t+1)^2\|E\|^2e^{-2\gamma t}\right) \tag{6.4}$$
$$V_\infty(t;z,v)-v=\int_t^\infty E(z+vs,s)\,ds+O\left((t+1)\|E\|^2e^{-2\gamma t}\right) \tag{6.5}$$

We can then write the following equation for $E(z,t)$:

$$\begin{aligned}E_z(z,t)=&\int_t^\infty ds\int_{-\infty}^\infty f_{e,v}(w)\,E(z+ws-wt,s)\,dw+\int_{-\infty}^\infty g_\infty(z-wt,w)\,dw\\&-\int_t^\infty sds\int_{-\infty}^\infty \frac{\partial g_\infty}{\partial z}(z-wt,w)\,E(z+ws-wt,s)\,dw+\\&+\int_t^\infty ds\int_{-\infty}^\infty \frac{\partial g_\infty}{\partial w}(z-wt,w)\,E(z+ws-wt,s)\,dw+\tilde{R}(z,t)\end{aligned} \tag{6.6}$$

where we remark that the definition of $\tilde{R}$ is:

$$\begin{aligned}\tilde{R}(z,t)=&\int_{-\infty}^\infty\left[f_e(V_\infty(t;z-wt,w))-f_e(w)\right]dw+\\&+\int_{-\infty}^\infty g_\infty(Z_\infty(t;z-wt,w),V_\infty(t;z-wt,w))\,dw-\\&-\int_t^\infty ds\int_{-\infty}^\infty f_{e,v}(w)\,E(z+ws-wt,s)\,dw-\int_{-\infty}^\infty g_\infty(z-wt,w)\,dw+\\&+\int_t^\infty sds\int_{-\infty}^\infty\frac{\partial g_\infty}{\partial z}(z-wt,w)\,E(z+ws-wt,s)\,dw-\\&-\int_t^\infty ds\int_{-\infty}^\infty\frac{\partial g_\infty}{\partial w}(z-wt,w)\,E(z+ws-wt,s)\,dw\end{aligned} \tag{6.7}$$

Using Assumptions (A), expanding the arguments of $f_e,\ g_\infty$ using Taylor's expansion as well as the estimates (6.4), (6.5) we obtain:

$$\left|\tilde{R}(z,t)\right| \leq C\|E\|^2 (t+1)^2 e^{-2\gamma t} \tag{6.8}$$

We now transform the third term on the right side as suggested above, using integrations by parts. It then follows that (see (3.17)):

$$\begin{aligned} E_z(z,t) &= \int_t^\infty ds \int_{-\infty}^\infty f_{e,v}(w)\, E(z-wt+ws,s)\, dw + \int_{-\infty}^\infty g_\infty(z-wt,w)\, dw \\ &+ \int_t^\infty (t-s)\frac{s}{t} ds \int_{-\infty}^\infty g_\infty(z-wt,w)\, E_z(z-wt+ws,s)\, dw + \\ &+ \int_t^\infty ds \int_{-\infty}^\infty \left(1-\frac{s}{t}\right) \frac{\partial g_\infty}{\partial w}(z-wt,w)\, E(z-wt+ws,s)\, dw + \tilde{R}(z,t) \end{aligned}$$

A crucial estimate is the following:

$$\begin{aligned} &\left|\int_t^\infty (t-s)\frac{s}{t} ds \int_{-\infty}^\infty g_\infty(z-wt,w)\, E_z(z-wt+ws,s)\, dw\right| \\ &\leq C\varepsilon \|E\| e^{-\gamma t} \int_t^\infty (s-t)\frac{s}{t} e^{-\gamma(s-t)} ds \leq C\varepsilon \|E\| e^{-\gamma t} \end{aligned} \tag{6.9}$$

where ε comes from g_∞. A similar estimate can be obtained for the fourth term. The problem is then ready for a fixed point argument in the form

$$E_z(z,t) - \int_t^\infty ds \int_{-\infty}^\infty f_{e,v}(w)\, E(z-wt+ws,s)\, dw = \psi(z,t) + L(z,t) + \tilde{R}(z,t) \tag{6.10}$$

where:

$$\psi(z,t) = \int_{-\infty}^\infty g_\infty(z-wt,w)\, dw \tag{6.11}$$

$$\begin{aligned} L(z,t) &= \int_t^\infty (t-s)\frac{s}{t} ds \int_{-\infty}^\infty g_\infty(z-wt,w)\, E_z(z-wt+ws,s)\, dw + \\ &\int_t^\infty ds \int_{-\infty}^\infty \left(1-\frac{s}{t}\right) \frac{\partial g_\infty}{\partial w}(z-wt,w)\, E(z-wt+ws,s)\, dw \end{aligned} \tag{6.12}$$

More precisely, the fixed point scheme is the following one. Given E with $\|E\| < \infty$ we define L as in (6.12) and $\tilde{R}$ as in (6.7). We then define $\mathcal{T}(E)$ as the solution of (6.10) obtained by means of Proposition 16.

In particular, using Lemma 6, (6.8), (6.9) and Proposition 16 we have the following estimate:

$$\|\mathcal{T}(E)\| \leq C_{g_\infty}\varepsilon + C\varepsilon\|E\| + C\|E\|^2 \tag{6.13}$$

where C_{g_∞} is a constant of order one that comes from g_∞ and depends only on $B,\ \alpha$.

A consequence of (6.13) is that the ball $\|E\| \leq 2C_{g_\infty}\varepsilon$ is transformed by means of the operator $\mathcal{T}$ in a set contained in a ball with radius:

$$C_{g_\infty}\varepsilon + 2CC_{g_\infty}\varepsilon^2 + 4C\left(C_{g_\infty}\right)^2\varepsilon^2$$

and this number is strictly less than $2C_{g_\infty}\varepsilon$ if ε is sufficiently small.

Finally we will prove that the operator $\mathcal{T}$ is contractive in the ball $\|E\| \leq 2C_{g_\infty}\varepsilon$ if ε is sufficiently small. To this end we denote as $Z_1(s,t;z,v)$, $V_1(s,t;z,v)$, $Z_2(s,t;z,v)$, $V_2(s,t;z,v)$ the evolution of the characteristics defined by means of (6.1)-(6.3) with electric fields E_1, E_2 respectively with $\|E_1\| \leq 2C_{g_\infty}\varepsilon$, $\|E_2\| \leq 2C_{g_\infty}\varepsilon$. We will use also the notation $Z_{1,\infty}(t;z,v)$, $V_{1,\infty}(t;z,v)$, $Z_{2,\infty}(t;z,v)$, $V_{2,\infty}(t;z,v)$, $L_1(z,t)$, $L_2(z,t)$, $\tilde{R}_1(z,t)$, $\tilde{R}_2(z,t)$ to denote the corresponding functions associated with E_1, E_2.

Notice that (6.1)-(6.3) imply:

$$\frac{\partial(Z_1 - Z_2)}{\partial s} = -s\left[E_1(Z_1 + V_1 s, s) - E_2(Z_2 + V_2 s, s)\right]$$
$$\frac{\partial(V_1 - V_2)}{\partial s} = \left[E_1(Z_1 + V_1 s, s) - E_2(Z_2 + V_2 s, s)\right]$$

with:

$$(Z_1 - Z_2)(t,t;z,v) = 0 \qquad (6.14)$$
$$(V_1 - V_2)(t,t;z,v) = 0$$

We can write the equations as:

$$\frac{\partial(Z_1 - Z_2)}{\partial s} = -s\left[a(s)(Z_1 - Z_2) + b(s)(V_1 - V_2) + \left(E_1(Z_2 + V_2 s, s) - E_2(Z_2 + V_2 s, s)\right)\right] \qquad (6.15)$$

$$\frac{\partial(V_1 - V_2)}{\partial s} = \left[a(s)(Z_1 - Z_2) + b(s)(V_1 - V_2) + \left(E_1(Z_2 + V_2 s, s) - E_2(Z_2 + V_2 s, s)\right)\right] \qquad (6.16)$$

where:

$$a(s) = \frac{E_1(Z_1 + V_1 s, s) - E_1(Z_2 + V_1 s, s)}{Z_1 - Z_2}$$
$$b(s) = \frac{E_1(Z_2 + V_1 s, s) - E_1(Z_2 + V_2 s, s)}{V_1 - V_2}$$

Notice that:

$$|a(s)| + |b(s)| \leq \|E_1\| e^{-\gamma s} \qquad (6.17)$$
$$|E_1(Z_2 + V_2 s, s) - E_2(Z_2 + V_2 s, s)| \leq \|E_1 - E_2\| e^{-\gamma s} \qquad (6.18)$$

Then, combining (6.14)-(6.18) we obtain, using a Gronwall type argument:

$$|(Z_1 - Z_2)(s,t;z,v)| \leq C\|E_1 - E_2\|(t+1)e^{-\gamma t}$$
$$|(V_1 - V_2)(s,t;z,v)| \leq C\|E_1 - E_2\| e^{-\gamma t}$$

whence:

$$\begin{aligned}|(Z_{1,\infty}-Z_{2,\infty})(t;z,v)| &\le C\|E_1-E_2\|(t+1)e^{-\gamma t}\ ,\\ |(V_{1,\infty}-V_{2,\infty})(t;z,v)| &\le C\|E_1-E_2\|e^{-\gamma t}\end{aligned} \tag{6.19}$$

Let us denote as $\tilde{E}_1,\ \tilde{E}_2$ respectively the solutions of (6.10) with sources $L_1,\ \tilde{R}_1$ and $L_2,\ \tilde{R}_2$ respectively.

Notice that:

$$\begin{aligned}&\left(\tilde{E}_1-\tilde{E}_2\right)_z(z,t)-\int_t^\infty ds\int_{-\infty}^\infty f_{e,v}(w)\left(\tilde{E}_1-\tilde{E}_2\right)(z-wt+ws,s)\,dw\\ &=(L_1-L_2)(z,t)+\left(\tilde{R}_1-\tilde{R}_2\right)(z,t)\end{aligned} \tag{6.20}$$

Notice also that, arguing as in the estimates of L above (cf. (6.12) and (6.9)):

$$|(L_1-L_2)(z,t)|\le C\varepsilon\|E_1-E_2\|e^{-\gamma t}\ \ ,\ \ z\in\mathbb{R}\ \ ,\ \ t\ge 0 \tag{6.21}$$

We also need to estimate the difference $\left(\tilde{R}_1-\tilde{R}_2\right)$. Therefore we need to estimate:

$$\begin{aligned}\left(\tilde{R}_1-\tilde{R}_2\right)(z,t)=&\int_{-\infty}^\infty\left[f_e(V_{1,\infty}(t;z-wt,w))-f_e(V_{2,\infty}(t;z-wt,w))\right]dw+\\ &+\int_{-\infty}^\infty\left[g_\infty(Z_{1,\infty}(t;z-wt,w),V_{1,\infty}(t;z-wt,w))-\right.\\ &\left.-g_\infty(Z_{2,\infty}(t;z-wt,w),V_{2,\infty}(t;z-wt,w))\right]dw+\\ &-\int_t^\infty ds\int_{-\infty}^\infty f_{e,v}(w)(E_1(z+ws-wt,s)-E_2(z+ws-wt,s))\,dw+\\ &+\int_t^\infty sds\int_{-\infty}^\infty\frac{\partial g_\infty}{\partial z}(z-wt,w)(E_1(z+ws-wt,s)-E_2(z+ws-wt,s))\,dw-\\ &-\int_t^\infty ds\int_{-\infty}^\infty\frac{\partial g_\infty}{\partial w}(z-wt,w)(E_1(z+ws-wt,s)-E_2(z+ws-wt,s))\,dw\end{aligned} \tag{6.22}$$

Taking into account (4.2) and (6.19) we can estimate the second, fourth and fifth terms in (6.22) as $C\varepsilon\|E_1-E_2\|e^{-\gamma t}$. Then:

$$\begin{aligned}\left|\left(\tilde{R}_1-\tilde{R}_2\right)(z,t)\right|\le&\left|\int_{-\infty}^\infty\left[f_e(V_{1,\infty}(t;z-wt,w))-f_e(V_{2,\infty}(t;z-wt,w))\right]dw-\right.\\ &\left.-\int_t^\infty ds\int_{-\infty}^\infty f_{e,v}(w)(E_1(z+ws-wt,s)-E_2(z+ws-wt,s))\,dw\right|+\\ &+C\varepsilon\|E_1-E_2\|e^{-\gamma t}\end{aligned} \tag{6.23}$$

On the other hand (6.2), (6.3) imply:

$$V_{i,\infty}(t; z - wt, w) - w = \int_t^\infty E_i(Z_i(s) + sV_i(s), s)\, ds \ \ , \ \ i = 1, 2 \qquad (6.24)$$

where, for simplicity $Z_i(s) = Z_i(s, t; z - wt, w)$, $V_i(s) = V_i(s, t; z - wt, w)$, $i = 1, 2$.

Using Taylor's Theorem and (6.19) we can estimate the difference $[f_e(V_{1,\infty}(t; z - wt, w)) - f_e(V_{2,\infty}(t; z - wt, w))]$ as:

$$\begin{aligned}&[f_e(V_{1,\infty}(t; z - wt, w)) - f_e(V_{2,\infty}(t; z - wt, w))]\\ &= f_{e,v}(V_{2,\infty}(t; z - wt, w))(V_{1,\infty}(t; z - wt, w) - V_{2,\infty}(t; z - wt, w)) +\\ &+ O\left(\|E_1 - E_2\|^2 e^{-2\gamma t}\right)\end{aligned}$$

Using this estimate as well as (6.19), (6.24) and the fact that $\|E_i\| \leq 2C_{g_\infty}\varepsilon$, $i = 1, 2$ we obtain:

$$\begin{aligned}&[f_e(V_{1,\infty}(t; z - wt, w)) - f_e(V_{2,\infty}(t; z - wt, w))]\\ &= f_{e,v}(w)(V_{1,\infty}(t; z - wt, w) - V_{2,\infty}(t; z - wt, w)) +\\ &+ O\left(\|E_1 - E_2\|^2 e^{-2\gamma t}\right) + O\left(\varepsilon \|E_1 - E_2\| e^{-\gamma t}\right)\end{aligned}$$

Taking into account (6.24) it then follows that:

$$\begin{aligned}&\left|\int_{-\infty}^\infty [f_e(V_{1,\infty}(t; z - wt, w)) - f_e(V_{2,\infty}(t; z - wt, w))]\, dw - \right.\\ &\left. - \int_t^\infty ds \int_{-\infty}^\infty f_{e,v}(w)(E_1(z + ws - wt, s) - E_2(z + ws - wt, s))\, dw\right|\\ &\leq \left|\int_t^\infty ds \int_{-\infty}^\infty f_{e,v}(w)([E_1(Z_1(s) + sV_1(s), s) - E_1(z + ws - wt, s)] - \right.\\ &\left. - [E_2(Z_2(s) + sV_2(s), s) - E_2(z + ws - wt, s)])\right| +\\ &+ C\left(\|E_1 - E_2\|^2 e^{-2\gamma t} + \varepsilon \|E_1 - E_2\| e^{-\gamma t}\right)\end{aligned}$$

Notice that

$$\begin{aligned}&(E_1(Z_1(s) + sV_1(s), s) - E_1(z + ws - wt, s))\\ &- (E_2(Z_2(s) + sV_2(s), s) - E_2(z + ws - wt, s))\\ &= (E_1(Z_1(s) + sV_1(s), s) - E_1(Z_2(s) + sV_2(s), s)) +\\ &+ (E_1(Z_2(s) + sV_2(s), s) - E_1(z + ws - wt, s))\\ &- (E_2(Z_2(s) + sV_2(s), s) - E_2(z + ws - wt, s))\end{aligned}$$

The term $(E_1(Z_1(s) + sV_1(s), s) - E_1(Z_2(s) + sV_2(s), s))$ can be estimated as:

$$\left|\int_{Z_2(s)+sV_2(s)}^{Z_1(s)+sV_1(s)} E_{1,z}(\xi, s)\, d\xi\right| \leq C \|E_1\| e^{-\gamma s}[|Z_1(s) - Z_2(s)| + s|V_1(s) - V_2(s)|]$$

whence, using $\|E_i\| \leq 2C_{g_\infty}\varepsilon$:

$$|E_1(Z_1(s) + sV_1(s), s) - E_1(Z_2(s) + sV_2(s), s)| \leq C\varepsilon e^{-\gamma s} e^{-\gamma t}(1+s)(\|E_1 - E_2\|) \tag{6.25}$$

On the other hand

$$\begin{aligned}
&(E_1(Z_2(s) + sV_2(s), s) - E_1(z + ws - wt, s)) - \\
&- (E_2(Z_2(s) + sV_2(s), s) - E_2(z + ws - wt, s)) \\
&= \int_{z+ws-wt}^{Z_2(s)+sV_2(s)} [E_{1,z}(\xi, s) - E_{2,z}(\xi, s)] \, d\xi
\end{aligned}$$

thus we have

$$\begin{aligned}
&|(E_1(Z_2(s) + sV_2(s), s) - E_1(z + ws - wt, s)) - \\
&- (E_2(Z_2(s) + sV_2(s), s) - E_2(z + ws - wt, s))| \\
&\leq \|E_1 - E_2\| e^{-\gamma s} [|Z_2(s) - (z + ws - wt)| + s|V_2(s) - w|] \\
&\leq Ce^{-\gamma s} e^{-\gamma t}(1+s)\|E_1 - E_2\| \|E_2\|
\end{aligned} \tag{6.26}$$

Then:

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} [f_e(V_{1,\infty}(t; z - wt, w)) - f_e(V_{2,\infty}(t; z - wt, w))] \, dw - \right. \\
&\left. - \int_t^{\infty} ds \int_{-\infty}^{\infty} f_{e,v}(w)(E_1(z + ws - wt, s) - E_2(z + ws - wt, s)) \, dw \right| \\
&\leq C\varepsilon \|E_1 - E_2\| e^{-\gamma t}
\end{aligned}$$

and therefore:

$$\left| \left(\tilde{R}_1 - \tilde{R}_2\right)(z, t) \right| \leq C\varepsilon \|E_1 - E_2\| e^{-\gamma t} \tag{6.27}$$

Combining (6.19), (6.20), (6.27) and Proposition 16 we obtain:

$$\|\mathcal{T}(E_1) - \mathcal{T}(E_2)\| \leq C\varepsilon \|E_1 - E_2\|$$

for $\|E_i\| \leq 2C_{g_\infty}\varepsilon, \ i = 1, 2$, thus the desired contractibility of $\mathcal{T}$ follows. This concludes the proof. ■

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