ON COMPLETE GRADIENT SHRINKING RICCI SOLITONS

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ABSTRACT. In this paper we derive a precise estimate on the growth of potential functions of complete noncompact shrinking solitons. Based on this, we prove that a complete noncompact gradient shrinking Ricci soliton has at most Euclidean volume growth. The latter result can be viewed as an analog of the well-known theorem of Bishop that a complete noncompact Riemannian manifold with nonnegative Ricci curvature has at most Euclidean volume growth.

1. The results

A complete Riemannian metric g_{ij} on a smooth manifold M^n is called a *gradient* shrinking Ricci soliton if there exists a smooth function f on M^n such that the Ricci tensor R_{ij} of the metric g_{ij} is given by

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}$$

for some positive constant $\rho > 0$. The function f is called a *potential function* of the shrinking soliton. Note that by scaling g_{ij} one can normalize $\rho = \frac{1}{2}$ so that

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij}. \tag{1.1}$$

Gradient shrinking Ricci solitons play an important role in Hamilton's Ricci flow as they correspond to self-similar solutions, and often arise as Type I singularity models. In this paper, we investigate the asymptotic behavior of potential functions and volume growth rates of complete noncompact gradient shrinking solitons. Our main results are:

Theorem 1.1. Let (M^n, g_{ij}, f) be a complete noncompact gradient shrinking Ricci soliton satisfying (1.1). Then, the potential function f satisfies the estimates

$$\frac{1}{4}(r(x) - c_1)^2 \le f(x) \le \frac{1}{4}(r(x) + c_2)^2.$$

Here $r(x) = d(x_0, x)$ is the distance function from some fixed point $x_0 \in M$, c_1 and c_2 are positive constants depending only on n and the geometry of g_{ij} on the unit ball $B_{x_0}(1)$.

Remark 1.1. In view of the Gaussian shrinker, namely the flat Euclidean space (\mathbb{R}^n, g_0) with the potential function $|x|^2/4$, the leading term $\frac{1}{4}r^2(x)$ for the lower and upper bounds on f in Theorem 1.1 is optimal. We also point out that it has been known, by the work of Ni-Wallach [12] and Cao-Chen-Zhu [3], that any 3-dimensional complete noncompact non-flat shrinking gradient soliton is necessarily the round cylinder $\mathbb{S}^2 \times \mathbb{R}$ or one of its \mathbb{Z}_2 quotients.

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Remark 1.2. When the Ricci curvature of (M^n, g_{ij}, f) is assumed to be bounded, Theorem 1.1 was shown by Perelman [13]. Also, under the assumption of $Rc \geq 0$, a lower estimate of the form

$$f(x) \ge \frac{1}{8}r^2(x) - c_1'$$

was shown by Ni [11]. Moreover, the upper bound in Theorem 1.1 was essentially observed in [3], while a rough quadratic lower bound, as pointed out by Carrillo-Ni [5], could follow from the argument of Fang-Man-Zhang in [7].

Theorem 1.2. Let (M^n, g_{ij}, f) be a complete noncompact gradient shrinking Ricci soliton. Then, there exists some positive constant $C_1 > 0$ such that

$$Vol(B_{x_0}(r)) \le C_1 r^n$$

for r > 0 sufficiently large.

Remark 1.3. In an earlier version of the paper, we had an extra assumption

$$R(x) \le \alpha r^2(x) + A(r(x) + 1),$$
 (1.2)

with $0 \le \alpha < \frac{1}{4}$ and A > 0, on the scalar curvature R. However, as observed by Ovidiu Munteanu, assumption (1.2) actually is not needed in our proof because there holds estimate (3.4) on the average of the scalar curvature in general. Note that, as stated in Lemma 2.3, $R(x) \le \frac{1}{4}(r(x) + c)^2$ holds for any complete noncompact gradient shrinking soliton. It remains interesting to find out whether R is bounded from above by a constant.

Remark 1.4. Feldman-Ilmanen-Knopf [8] constructed a complete noncompact gradient Kähler shrinker on the tautological line bundle $\mathcal{O}(-1)$ of the complex projective space $\mathbb{C}P^{n-1}$ $(n \geq 2)$ which has Euclidean volume growth, quadratic curvature decay, and with Ricci curvature changing signs. This example shows that the volume growth rate in Theorem 1.2 is optimal. Note that Carrillo-Ni [5] showed that any non-flat gradient shrinking soliton with nonnegative Ricci curvature $Rc \geq 0$ must have zero asymptotic volume ratio, i.e., $\lim_{r\to\infty} \operatorname{Vol}(B_{x_0}(r))/r^n = 0$.

Combining Theorem 1.1 and Theorem 1.2, we also have the following consequence, which was obtained previously in [10] and [15] respectively.

Corollary 1.1. Let (M^n, g_{ij}, f) be a complete noncompact gradient shrinking Ricci soliton. Then we have

$$\int_{M} |u| e^{-f} dV < +\infty$$

for any function u on M with $|u(x)| \leq Ae^{\alpha r^2(x)}$, $0 \leq \alpha < \frac{1}{4}$ and A > 0. In particular, the weighted volume of M is finite,

$$\int_{M} e^{-f} dV < +\infty.$$

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2. Asymptotic behavior of the potential function

In this section, we investigate the asymptotic behavior of the potential function of an arbitrary complete noncompact gradient shrinking Ricci solitons and prove Theorem 1.1.

First of all, we need a few useful facts about complete gradient shrinking solitons. The first basic result is due to Hamilton (cf. Theorem 20.1 in [9]).

Lemma 2.1. Let (M^n, g_{ij}, f) be a complete gradient shrinking soliton satisfying (1.1). Then we have

$$\nabla_i R = 2R_{ij} \nabla_j f,$$

and

$$R + |\nabla f|^2 - f = C_0$$

for some constant C_0 . Here R denotes the scalar curvature of g_{ij} .

As a consequence, by adding the constant C_0 to f, we can assume

$$R + |\nabla f|^2 - f = 0. {(2.1)}$$

From now on we will make this normalization on f throughout the paper.

We will also need the following useful result, which is a special case of a more general result on complete ancient solutions due to B.-L. Chen [6] (cf. Proposition 5.5 in [1]).

Lemma 2.2. Let (M^n, g_{ij}, f) be a complete shrinking Ricci soliton. Then g_{ij} has nonnegative scalar curvature $R \ge 0$.

As an immediate consequence of (2.1) and Lemma 2.2, one gets the following result, which was essentially observed by Cao-Chen-Zhu [3] (cf. p.78-79 in [3]).

Lemma 2.3. Let (M^n, g_{ij}, f) be a complete shrinking Ricci soliton satisfying (1.1) and (2.1). Then,

$$f(x) \le \frac{1}{4}(r(x) + 2\sqrt{f(x_0)})^2,$$
 (2.2)

$$|\nabla f|(x) \le \frac{1}{2}r(x) + \sqrt{f(x_0)},$$
 (2.3)

and

$$R(x) \le \frac{1}{4}(r(x) + 2\sqrt{f(x_0)})^2.$$
 (2.4)

Here $r(x) = d(x_0, x)$ is the distance function from some fixed point $x_0 \in M$.

Proof. By Lemma 2.2 and (2.1),

$$0 \le |\nabla f|^2 \le f$$
, or $|\nabla \sqrt{f}| \le \frac{1}{2}$ (2.5)

whenever f > 0. Thus \sqrt{f} is an Lipschitz function and

$$|\sqrt{f(x)} - \sqrt{f(x_0)}| \le \frac{1}{2}r(x).$$

Hence

$$\sqrt{f(x)} \le \frac{1}{2}r(x) + \sqrt{f(x_0)},$$

or

$$f(x) \le \frac{1}{4}(r(x) + 2\sqrt{f(x_0)})^2.$$

This proves (2.2), from which (2.3) and (2.4) follow immediately.

Now (2.2) provides the upper estimate on f in Theorem 1.1. However, proving the lower estimate turns out to be more subtle.

Proposition 2.1. Let (M^n, g_{ij}, f) be a complete noncompact gradient shrinking Ricci soliton satisfying the normalization conditions (1.1) and (2.1). Then, f satisfies the estimate

$$f(x) \ge \frac{1}{4}(r(x) - c_1)^2$$

where c_1 is a positive constant depending only on n and the geometry of g_{ij} on the unit ball $B_{x_0}(1)$.

Proof. Consider any minimizing normal geodesic $\gamma(s)$, $0 \le s \le s_0$ for some arbitrary large $s_0 > 0$, starting from $x_0 = \gamma(0)$. Denote by $X(s) = \dot{\gamma}(s)$ the unit tangent vector along γ . Then, by the second variation of arc length, we have

$$\int_0^{s_0} \phi^2 Rc(X, X) ds \le (n - 1) \int_0^{s_0} |\dot{\phi}(s)|^2 ds \tag{2.6}$$

for every nonnegative function $\phi(s)$ defined on the interval $[0, s_0]$. Now, following Hamilton [9], we choose $\phi(s)$ by

$$\phi(s) = \begin{cases} s, & s \in [0, 1], \\ 1, & s \in [1, s_0 - 1], \\ s_0 - s, & s \in [s_0 - 1, s_0]. \end{cases}$$

Then

$$\begin{split} \int_0^{s_0} Rc(X,X)ds &= \int_0^{s_0} \phi^2 Rc(X,X)ds + \int_0^{s_0} (1-\phi^2)Rc(X,X)ds \\ &\leq (n-1)\int_0^{s_0} |\dot{\phi}(s)|^2 ds + \int_0^{s_0} (1-\phi^2)Rc(X,X)ds \\ &\leq 2(n-1) + \max_{B_{x_0}(1)} |Rc| + \max_{B_{\gamma(s_0)}(1)} |Rc|. \end{split}$$

On the other hand, by (1.1), we have

$$\nabla_X \dot{f} = \nabla_X \nabla_X f = \frac{1}{2} - Rc(X, X). \tag{2.7}$$

Integrating (2.7) along γ from 0 to s_0 , we get

$$\dot{f}(\gamma(s_0)) - \dot{f}(\gamma(0)) = \frac{1}{2}s_0 - \int_0^{s_0} Rc(X, X)ds$$

$$\geq \frac{s_0}{2} - 2(n-1) - \max_{B_{x_0}(1)} |Rc| - \max_{B_{\gamma(s_0)}(1)} |Rc|.$$

In case g_{ij} has bounded Ricci curvature $|Rc| \leq C$ for some constant C > 0, then it would follow that

$$\dot{f}(\gamma(s_0)) \ge \frac{1}{2}s_0 - \dot{f}(\gamma(0)) - 2(n-1) - 2C = \frac{1}{2}(s_0 - c),$$
 (2.8)

and that

$$f(\gamma(s_0)) \ge \frac{1}{4}(s_0 - c)^2 - f(x_0) - \frac{c^2}{4},$$

proving what we wanted.²

However, since we do not assume any curvature bound in Theorem 1.1, we have to modify the above argument.

First of all, by integrating (2.7) along γ from s = 1 to $s = s_0 - 1$ instead and using (2.6) as before, we have

$$\dot{f}(\gamma(s_0 - 1)) - \dot{f}(\gamma(1)) = \int_1^{s_0 - 1} \nabla_X \dot{f}(\gamma(s)) ds
= \frac{1}{2} (s_0 - 2) - \int_1^{s_0 - 1} Rc(X, X) ds
= \frac{1}{2} (s_0 - 2) - \int_1^{s_0 - 1} \phi^2(s) Rc(X, X) ds
\ge \frac{s_0}{2} - 2n + 1 - \max_{B_{x_0}(1)} |Rc| + \int_{s_0 - 1}^{s_0} \phi^2 Rc(X, X) ds.$$

Next, using (2.7) and integration by parts one more time as in [7], we obtain

$$\int_{s_0-1}^{s_0} \phi^2 Rc(X,X) ds = \frac{1}{2} \int_{s_0-1}^{s_0} \phi^2(s) ds - \int_{s_0-1}^{s_0} \phi^2(s) \nabla_X \dot{f}(\gamma(s)) ds$$
$$= \frac{1}{6} + \dot{f}(\gamma(s_0-1)) - 2 \int_{s_0-1}^{s_0} \phi(s) \dot{f}(\gamma(s)) ds.$$

Therefore,

$$2\int_{s_0-1}^{s_0} \phi(s)\dot{f}(\gamma(s))ds \ge \frac{s_0}{2} - 2n + \frac{7}{6} - \max_{B_{x_0}(1)} |Rc| + \dot{f}(\gamma(1)). \tag{2.9}$$

Furthermore, by (2.5) we have

$$|\dot{f}(\gamma(s))| \le \sqrt{f(\gamma(s))},$$

and

$$|\sqrt{f(\gamma(s))} - \sqrt{f(\gamma(s_0))}| \le \frac{1}{2}(s_0 - s) \le \frac{1}{2},$$

whenever $s_0 - 1 \le s \le s_0$. Thus,

$$\max_{s_0 - 1 \le s \le s_0} |\dot{f}(\gamma(s))| \le \sqrt{f(\gamma(s_0))} + \frac{1}{2}.$$
 (2.10)

Combining (2.9) and (2.10), and noting $2 \int_{s_0-1}^{s_0} \phi(s) ds = 1$, we conclude that

$$\sqrt{f(\gamma(s_0))} \ge \frac{1}{2}(s_0 - c_1)$$

for some constant c_1 depending only on n and the geometry of g_{ij} on the unit ball $B_{x_0}(1)$. This completes the proof of Proposition 2.1 and Theorem 1.1.

²Indeed, the above argument was essentially sketched by Perelman (see, p.3 of [13]), and a detailed argument was presented in [4] (p.385-386).

3. Volume growth of complete gradient shrinking solitons

In this section, we examine the volume growth of geodesic balls of complete noncompact gradient shrinking Ricci solitons.

Let us define

$$\rho(x) = 2\sqrt{f(x)}.$$

Then, by Theorem 1.1, we have

$$r(x) - c \le \rho(x) \le r(x) + c \tag{3.1}$$

with $c = \max\{c_1, c_2\} > 0$. Also, we have

$$\nabla \rho = \frac{\nabla f}{\sqrt{f}}$$
 and $|\nabla \rho| = \frac{|\nabla f|}{\sqrt{f}} \le 1.$ (3.2)

Denote by

$$D(r) = \{x \in M : \rho(x) < r\}$$
 and $V(r) = \int_{D(r)} dV$.

Then, by the Co-Area formula (cf. [14]), we have,

$$V(r) = \int_0^r ds \int_{\partial D(s)} \frac{1}{|\nabla \rho|} dA.$$

Hence,

$$V'(r) = \int_{\partial D(r)} \frac{1}{|\nabla \rho|} dA = \frac{r}{2} \int_{\partial D(r)} \frac{1}{|\nabla f|} dA. \tag{3.3}$$

Here we have used (3.2) in deriving the last identity in (3.3).

Lemma 3.1.

$$nV(r) - rV'(r) = 2 \int_{D(r)} RdV - 2 \int_{\partial D(r)} \frac{R}{|\nabla f|} dV.$$

Proof. Taking the trace in (1.1), we have

$$R + \Delta f = \frac{n}{2}.$$

Thus,

$$\begin{split} nV(r) - 2\int_{D(r)} RdV &= 2\int_{D(r)} \Delta f dV \\ &= 2\int_{\partial D(r)} \nabla f \cdot \frac{\nabla \rho}{|\nabla \rho|} \\ &= 2\int_{\partial D(r)} |\nabla f| dV \\ &= 2\int_{\partial D(r)} \frac{f - R}{|\nabla f|} dV \\ &= rV'(r) - 2\int_{\partial D(r)} \frac{R}{|\nabla f|} dV. \end{split}$$

Remark 3.1. As pointed out to us by Ovidiu Munteanu, we have also actually shown that

$$\int_{D(r)} RdV \le \frac{n}{2}V(r). \tag{3.4}$$

Namely, the average scalar curvature over D(r) is bounded by n/2.

Now we are ready to prove **Theorem 1.2**.

Proof. Let (M^n, g_{ij}, f) be a complete noncompact gradient shrinking Ricci soliton. Denote

$$\chi(r) = \int_{D(r)} RdV.$$

By the Co-Area formula, we have

$$\chi(r) = \int_0^r ds \int_{\partial D(s)} \frac{R}{|\nabla \rho|} dA = \frac{1}{2} \int_0^r s ds \int_{\partial D(s)} \frac{R}{|\nabla f|} dA.$$

Hence

$$\chi'(r) = \frac{r}{2} \int_{\partial D(r)} \frac{R}{|\nabla f|} dA.$$

Therefore, Lemma 3.1 can be rewritten as

$$nV(r) - rV'(r) = 2\chi(r) - \frac{4}{r}\chi'(r).$$
 (3.5)

This implies that

$$(r^{-n}V(r))' = 4r^{-n-2}e^{\frac{r^2}{4}}(e^{-\frac{r^2}{4}}\chi(r))'$$
$$= 4r^{-n-2}\chi'(r) - 2r^{-n-1}\chi(r).$$

Integrating the above equation from r_0 to r, we get

$$r^{-n}V(r) - r_0^{-n}V(r_0) = 4r^{-n-2}\chi(r)\Big|_{r_0}^r + 4(n+2)\int_{r_0}^r r^{-n-3}\chi(r)dr$$
$$-2\int_{r_0}^r r^{-n-1}\chi(r)dr$$
$$= 4r^{-n-2}\chi(r) - 4r_0^{-n-2}\chi(r_0)$$
$$+2\int_{r_0}^r r^{-n-3}\chi(r)(2(n+2) - r^2)dr.$$

Since $\chi(r)$ is positive and increasing in r we have, for $r_0 = \sqrt{2(n+2)}$,

$$\int_{r_0}^r r^{-n-3} \chi(r) (2(n+2) - r^2) dr \le \chi(r_0) \int_{r_0}^r r^{-n-3} (2(n+2) - r^2) dr$$
$$= \chi(r_0) \left(-2r^{-n-2} + \frac{1}{n}r^{-n} \right) \Big|_{r_0}^r.$$

Thus,

$$r^{-n}V(r) - r_0^{-n}V(r_0) \le 4r^{-n-2}(\chi(r) - \chi(r_0)) + \frac{2}{n}\chi(r_0)(r^{-n} - r_0^{-n}).$$

It follows that, for $r \ge r_0 = \sqrt{2(n+2)}$,

$$V(r) \le (r_0^{-n}V(r_0))r^n + 4r^{-2}\chi(r). \tag{3.6}$$

On the other hand, by (3.4) we have

$$4r^{-2}\chi(r) \le 2nr^{-2}V(r) \le \frac{1}{2}V(r),\tag{3.7}$$

for r sufficiently large.

Plugging (3.7) into (3.6), we obtain

$$V(r) \le 2r_0^{-n}V(r_0)r^n.$$

Therefore, by (3.1),

$$Vol(B_{x_0}(r)) \le V(r+c) \le V(r_0)r^n$$

for r large enough. This finishes the proof of Theorem 1.2.

We conclude with the following volume lower estimate.

Proposition 3.1. Let (M^n, g_{ij}, f) be a complete noncompact gradient shrinking Ricci soliton. Suppose the average scalar curvature satisfies the upper bound

$$\frac{1}{V(r)} \int_{D(r)} RdV \le \delta,\tag{3.8}$$

for some positive constant $\delta < n/2$ and all sufficiently large r. Then, there exists some positive constant $C_2 > 0$ such that

$$Vol(B_{x_0}(r)) \ge C_2 r^{n-2\delta}$$

for r sufficiently large.

Proof. Combining the assumption (3.8) with Lemma 3.1 and Lemma 2.2, it follows that

$$(n-2\delta)V(r) \le rV'(r). \tag{3.9}$$

Thus,

$$\int_{1}^{r} \frac{V'(s)}{V(s)} ds \ge \int_{1}^{r} \frac{n - 2\delta}{s} ds.$$

Consequently,

$$V(r) \ge V(1)r^{n-2\delta}$$
.

Therefore, in view of (3.1),

$$Vol(B_{x_0}(r)) \ge V(r-c) \ge 2^{-n}V(1)r^{n-2\delta}$$

for r sufficiently large.

Remark 3.2. X.-P. Zhu and the first author (see Theorem 3.1 in [2]) have shown that a complete noncompact gradient shrinking soliton, without any curvature assumption, must have infinite volume. Their proof is, however, more sophisticated, relying on a logarithmic inequality of Carrillo-Ni [5] and the Perelman type noncollapsing argument for complete gradient shrinking solitons.

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