

# Explicit a priori bounds on transfer operator eigenvalues

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ABSTRACT. We provide explicit bounds on the eigenvalues of transfer operators defined in terms of holomorphic data.

Linear operators of the form  $\mathcal{L}f = \sum_{i \in \mathcal{I}} w_i \cdot f \circ T_i$ , so-called *transfer operators* (see e.g. [Bal, Rue1, Rue2]), arise in a number of problems in dynamical systems. If the  $T_i$  are inverse branches of an expanding map  $T$ , and the weight functions  $w_i$  are positive, the spectrum of  $\mathcal{L}$  has well-known interpretations in terms of the exponential mixing rate of an invariant Gibbs measure (see [Bal]). Applications also arise when the  $w_i$  are real-valued (e.g. [CCR, JMS, Pol]) or complex-valued (e.g. [Dol, PS]).

In this article we suppose that  $T_i$  and  $w_i$  are analytic functions of  $d$  variables, for each  $i$  in some countable<sup>1</sup> index set  $\mathcal{I}$ . Under suitable hypotheses on  $T_i$  and  $w_i$  the transfer operator  $\mathcal{L}$  defines a compact operator on Hardy space  $H^2(B)$ , and we can give completely explicit bounds on its eigenvalue sequence<sup>2</sup>  $\{\lambda_n(\mathcal{L})\}_{n=1}^\infty$ :

**THEOREM 1.** *Suppose there is a complex Euclidean ball  $B \subset \mathbb{C}^d$  such that each  $w_i : B \rightarrow \mathbb{C}$  is holomorphic with  $\sum_{i \in \mathcal{I}} \sup_{z \in B} |w_i(z)| < \infty$ , and each  $T_i : B \rightarrow B$  is holomorphic with  $\cup_{i \in \mathcal{I}} T_i(B)$  contained in the ball concentric with  $B$  whose radius is  $r < 1$  times that of  $B$ .*

*Then  $\mathcal{L} : H^2(B) \rightarrow H^2(B)$  is compact and*

$$|\lambda_n(\mathcal{L})| < \frac{W \sqrt{d}}{r^d (1-r^2)^{d/2}} n^{(d-1)/(2d)} r^{\frac{d}{d+1} (d!)^{1/d} n^{1/d}} \quad \text{for all } n \geq 1, \quad (1)$$

where  $W := \sup_{z \in B} \sum_{i \in \mathcal{I}} |w_i(z)|$ .

If  $d = 1$  then

$$|\lambda_n(\mathcal{L})| \leq \frac{W}{\sqrt{1-r^2}} r^{(n-1)/2} \quad \text{for all } n \geq 1. \quad (2)$$

**REMARK 2.**

- (i) An estimate of the form  $|\lambda_n(\mathcal{L})| \leq C\theta^{n^{1/d}}$  for some (undefined) constants  $C > 0, \theta \in (0, 1)$  is asserted, either implicitly or explicitly, in the work of several authors (e.g. [FR, Fri, GLZ]); the novelty here is that careful derivation of this bound renders explicit the constants  $C, \theta$ .
- (ii) Using different techniques, the bound  $|\lambda_n(\mathcal{L})| \leq C\theta^{n^{1/d}}$  can also be established in the case where  $B$  is an arbitrary open subset of  $\mathbb{C}^d$  (see [BJ]), though here our expressions for  $C, \theta$  are more complicated.

**EXAMPLE 3.** If  $\mathcal{L}f(z) = \sum_{n=1}^\infty \left(\frac{1}{n+z}\right)^2 f\left(\frac{1}{n+z}\right)$  (the Perron-Frobenius operator for the Gauss map  $x \mapsto 1/x \pmod{1}$ ), cf. [May],  $B \subset \mathbb{C}$  may be chosen as the open disc of radius  $3/2$

<sup>1</sup>Subsequent results are new even when  $\mathcal{I}$  is finite, but it is convenient to also allow countably infinite  $\mathcal{I}$ .

<sup>2</sup>Precisely,  $\{\lambda_n(\mathcal{L})\}_{n=1}^\infty$  denotes the sequence of all eigenvalues of  $\mathcal{L}$  counting algebraic multiplicities and ordered by decreasing modulus, with the usual convention (see e.g. [Pie, 3.2.20]) that distinct eigenvalues with the same modulus can be written in any order.

centred at the point 1. In this case  $W = \sup_{z \in B} \sum_{n=1}^{\infty} |n+z|^{-2} = \sum_{n=1}^{\infty} (n-1/2)^{-2} = \pi^2/2$  and  $r = 2/3$ , so (2) yields

$$|\lambda_n(\mathcal{L})| \leq \frac{3\pi^2}{2\sqrt{5}} (2/3)^{(n-1)/2} \quad \text{for all } n \geq 1.$$

NOTATION 4. For an open ball  $D \subset \mathbb{C}^d$ , let  $H^\infty(D)$  denote the Banach space consisting of all bounded holomorphic  $\mathbb{C}$ -valued functions on  $D$ , with norm  $\|f\|_{H^\infty(D)} := \sup_{z \in D} |f(z)|$ .

*Hardy space*  $H^2(D)$  (see [Kra, Ch. 8.3]) is the  $L^2(\partial D, \sigma)$ -closure of the set of those  $f \in H^\infty(D)$  which extend continuously to the boundary  $\partial D$ , where  $\sigma$  denotes  $(2d-1)$ -dimensional Lebesgue measure on  $\partial D$ , normalised so that  $\sigma(\partial D) = 1$ . In particular,  $H^2(D)$  is a Hilbert subspace of  $L^2(\partial D, \sigma)$  with each element  $f \in H^2(D)$  having a natural holomorphic extension to  $D$  (see [Kra, Ch. 1.5]).

In the sequel, no generality is lost by taking  $B$  in the statement of Theorem 1 to be the unit ball  $B_1$ , and the smaller concentric ball to be  $B_r$ , the ball of radius  $r$  centred at 0.

If  $L : X_1 \rightarrow X_2$  is a continuous operator between Banach spaces then for  $k \geq 1$ , its  $k$ -th approximation number  $a_k(L)$  is defined as

$$a_k(L) = \inf\{\|L - K\| \mid K : X_1 \rightarrow X_2 \text{ linear and continuous with } \text{rank}(K) < k\}.$$

The proof of Theorem 1 hinges on the following two lemmas.

LEMMA 5. *If  $J : H^2(B_1) \hookrightarrow H^\infty(B_r)$  denotes the canonical embedding, then  $J$  and  $\mathcal{L}$  are compact and for all  $n \geq 1$*

$$|\lambda_n(\mathcal{L})| \leq W \prod_{k=1}^n a_k(J)^{1/n}. \quad (3)$$

PROOF. If  $f \in H^2(B_1)$  and  $z \in B_r$  then  $|f(z)| \leq (2/(1-r))^{d/2}$  by [Rud, Thm. 7.2.5], so  $\{f \mid \|f\|_{H^2(B_1)} \leq 1\}$  is a normal family in  $H^\infty(B_r)$ , hence relatively compact in  $H^\infty(B_r)$  by Montel's Theorem (see [Nar, Ch. 1, Prop. 6]), thus  $J$  is compact.

Next observe that if  $f \in H^\infty(B_1)$  then  $f \in H^2(B_1)$  by [Rud, Thm. 5.6.8] and the canonical embedding  $\hat{J} : H^\infty(B_1) \hookrightarrow H^2(B_1)$  is continuous of norm 1, because  $\sigma(\partial B_1) = 1$ . We claim that  $\hat{\mathcal{L}}f := \sum_{i \in \mathcal{I}} w_i \cdot f \circ T_i$  defines a continuous operator  $\hat{\mathcal{L}} : H^\infty(B_r) \rightarrow H^\infty(B_1)$ . To see this, fix  $f \in H^\infty(B_r)$  and note that  $w_i \cdot f \circ T_i \in H^\infty(B_1)$  with  $\|w_i \cdot f \circ T_i\|_{H^\infty(B_1)} \leq \|w_i\|_{H^\infty(B_1)} \|f\|_{H^\infty(B_r)}$  for every  $i \in \mathcal{I}$ . But since  $\|\hat{\mathcal{L}}f\|_{H^\infty(B_1)} \leq \sum_{i \in \mathcal{I}} \|w_i\|_{H^\infty(B_1)} \|f\|_{H^\infty(B_r)}$  and  $\sum_{i \in \mathcal{I}} \|w_i\|_{H^\infty(B_1)} < \infty$  by hypothesis, we conclude that  $\hat{\mathcal{L}}f \in H^\infty(B_1)$  and that  $\hat{\mathcal{L}}$  is continuous. Now  $|f(T_i(z))| \leq \|f\|_{H^\infty(B_r)}$  for every  $z \in B_1$ ,  $i \in \mathcal{I}$ , so  $\|\hat{\mathcal{L}}f\|_{H^\infty(B_1)} = \sup_{z \in B_1} |(\hat{\mathcal{L}}f)(z)| \leq \sup_{z \in B_1} \sum_{i \in \mathcal{I}} |w_i(z)| |f(T_i(z))| \leq W \|f\|_{H^\infty(B_r)}$ , and hence  $\|\hat{\mathcal{L}}\| \leq W$ . Now clearly  $\mathcal{L} = \hat{J}\hat{\mathcal{L}}J$ , so  $\mathcal{L}$  is compact, and

$$a_k(\mathcal{L}) \leq \|\hat{J}\hat{\mathcal{L}}\| a_k(J) \leq W a_k(J) \quad \text{for all } k \geq 1, \quad (4)$$

since in general  $a_k(L_1 L_2) \leq \|L_1\| a_k(L_2)$  whenever  $L_1$  and  $L_2$  are bounded operators between Banach spaces (see [Pie, 2.2]). Moreover, since  $\mathcal{L}$  is a compact operator on Hilbert space, Weyl's inequality (see [Pie, 3.5.1], [Wey]) asserts that  $\prod_{k=1}^n |\lambda_k(\mathcal{L})| \leq \prod_{k=1}^n a_k(\mathcal{L})$  for all  $n \geq 1$ . Together with (4) this yields (3), because  $|\lambda_n(\mathcal{L})| \leq \prod_{k=1}^n |\lambda_k(\mathcal{L})|^{1/n}$ .  $\square$

LEMMA 6. *If  $h_d(k) := \binom{k+d}{d}$  then for all  $n \geq 1$ ,*

$$a_n(J)^2 \leq \sum_{l=k}^{\infty} h_{d-1}(l) r^{2l} \quad \text{where } k \geq 0 \text{ is such that } h_d(k-1) < n \leq h_d(k). \quad (5)$$

PROOF.  $H^2(B_1)$  has reproducing kernel  $K(z, \zeta) = (1 - (z, \zeta)_{\mathbb{C}^d})^{-d}$  (see [Kra, Thm. 1.5.5]<sup>3</sup>), where  $(\cdot, \cdot)_{\mathbb{C}^d}$  denotes the Euclidean inner product, and  $K(z, \zeta) = \sum_{n=1}^{\infty} p_n(z) \overline{p_n(\zeta)}$  whenever  $\{p_n\}_{n=1}^{\infty}$  is an orthonormal basis for  $H^2(B_1)$ , the series converging pointwise for every  $(z, \zeta) \in B_1 \times B_1$  (see [Hal, p. 19]).

Define  $J_n : H^2(B_1) \rightarrow H^\infty(B_r)$  by  $J_n f = \sum_{k=1}^{n-1} (f, p_k) p_k$ . If  $z \in B_r$  then

$$\begin{aligned} |Jf(z) - J_n f(z)|^2 &= |f(z) - J_n f(z)|^2 = \left| \sum_{k=n}^{\infty} (f, p_k) p_k(z) \right|^2 \\ &\leq \sum_{k=n}^{\infty} |(f, p_k)|^2 \sum_{k=n}^{\infty} |p_k(z)|^2 \leq \|f\|_{H^2(B_1)}^2 \left( K(z, z) - \sum_{k=1}^{n-1} |p_k(z)|^2 \right), \end{aligned}$$

so

$$a_n(J)^2 \leq \sup_{z \in B_r} \left( K(z, z) - \sum_{k=1}^{n-1} |p_k(z)|^2 \right). \quad (6)$$

If  $n = 1$  then  $k = 0$ , in which case (5) follows from (6) since  $\sum_{l=0}^{\infty} h_{d-1}(l) r^{2l} = (1 - r^2)^{-d}$ . Now define the orthonormal basis  $\{p_{\underline{n}} \mid \underline{n} \in \mathbb{N}_0^d\}$  by (cf. [Rud, Prop. 1.4.8, 1.4.9])

$$p_{\underline{n}}(z) = K_{\underline{n}} z^{\underline{n}} \quad (\underline{n} \in \mathbb{N}_0^d),$$

where  $K_{\underline{n}} = \sqrt{\frac{(|\underline{n}|+d-1)!}{(d-1)! \underline{n}!}}$ ,  $\underline{n} = (n_1, \dots, n_d)$ ,  $z^{\underline{n}} = z_1^{n_1} \dots z_d^{n_d}$ ,  $\underline{n}! = n_1! \dots n_d!$ ,  $|\underline{n}| = n_1 + \dots + n_d$ .

If  $n \geq 2$  then there are  $\binom{k+d-1}{d}$  multinomials of degree less than or equal to  $k-1$ , so

$$a_n(J)^2 \leq \sup_{z \in B_r} \left( K(z, z) - \sum_{|\underline{n}| \leq k-1} |p_{\underline{n}}(z)|^2 \right) = \sup_{z \in B_r} \sum_{l=k}^{\infty} \sum_{|\underline{n}|=l} |p_{\underline{n}}(z)|^2 \leq \sum_{l=k}^{\infty} \frac{(l+d-1)!}{(d-1)! l!} r^{2l}$$

for all  $n > \binom{k+d-1}{d}$ , because  $\sum_{|\underline{n}|=l} \frac{1}{\underline{n}!} |z^{\underline{n}}|^2 \leq \frac{1}{l!} r^{2l}$  for  $z \in B_r$  by the multinomial theorem.  $\square$

PROOF OF THEOREM 1. By Lemma 5 it suffices to bound the geometric means  $(\prod_{k=1}^n a_k)^{1/n}$ , where  $a_k := a_k(J)$ . From Lemma 6 it follows that

$$a_n^2 \leq \tilde{\alpha}_n \frac{r^{2\tilde{\beta}_n}}{(1-r^2)^d} \quad \text{for all } n \geq 1, \quad (7)$$

where

$$\begin{aligned} \tilde{\alpha}_n &:= h_{d-1}(k) & \text{for } h_d(k-1) < n \leq h_d(k), \\ \tilde{\beta}_n &:= k \end{aligned}$$

because

$$\sum_{l=k}^{\infty} h_{d-1}(l) r^{2l} = h_{d-1}(k) r^{2k} \sum_{l=0}^{\infty} \frac{h_{d-1}(l+k)}{h_{d-1}(k)} r^{2l} \leq h_{d-1}(k) r^{2k} \sum_{l=0}^{\infty} h_{d-1}(l) r^{2l} = h_{d-1}(k) \frac{r^{2k}}{(1-r^2)^d}.$$

Combining (7) with Lemma 5 gives, for all  $n \geq 1$ ,

$$|\lambda_n(\mathcal{L})| \leq W \alpha_n \frac{r^{\beta_n}}{(1-r^2)^{d/2}}, \quad (8)$$

where

$$\alpha_n := \prod_{l=1}^n \tilde{\alpha}_l^{1/(2n)}, \quad \beta_n := \frac{1}{n} \sum_{l=1}^n \tilde{\beta}_l.$$

<sup>3</sup>Note that the extra factor  $(d-1)/(2\pi^d)$  appearing in [Kra, Thm. 1.5.5] is due to a different normalisation of the measure  $\sigma$  on  $\partial B_1$ .

To obtain (1) and (2) from (8) we require an upper bound on  $\alpha_n$  and a lower bound on  $\beta_n$ . We start with the bounds for  $\alpha_n$ . Observe that

$$\tilde{\alpha}_1 = h_{d-1}(0) = 1, \text{ and } \tilde{\alpha}_l \leq d(l-1)^{1-1/d} \text{ for } l \geq 2. \quad (9)$$

To see this note that

$$\frac{h_{d-1}(k)}{h_d(k-1)^{1-1/d}} = \frac{(d!)^{1-1/d}}{(d-1)!} \left( \frac{\prod_{l=1}^{d-1} (k+l)^d}{\prod_{l=0}^{d-1} (k+l)^{d-1}} \right)^{1/d} = \frac{(d!)^{1-1/d}}{(d-1)!} \prod_{l=1}^{d-1} \left( 1 + \frac{l}{k} \right)^{1/d}$$

is decreasing in  $k$ , so if  $h_d(k-1) < n \leq h_d(k)$  then  $\frac{\tilde{\alpha}_l}{(l-1)^{1-1/d}} \leq \frac{h_{d-1}(k)}{h_d(k-1)^{1-1/d}} \leq \frac{h_{d-1}(1)}{h_d(0)^{1-1/d}} = d$ .

The estimate (9) now yields the upper bound

$$\alpha_n = \prod_{i=1}^n \tilde{\alpha}_i^{1/(2n)} \leq \sqrt{d}((n-1)!)^{(d-1)/(2dn)} \leq \sqrt{d} \left( 2 \left( \frac{n}{e} \right)^n \right)^{(d-1)/(2dn)} \leq \sqrt{d} n^{(d-1)/(2d)}, \quad (10)$$

where, for  $n > 1$ , we have used the estimate  $(n-1)! \leq 2 \left( \frac{n}{e} \right)^n$  (i.e.  $\log(n-1)! \leq \int_{x=2}^n \log x \, dx \leq n \log n - n + \log 2$ ).

We now turn to the bounds for  $\beta_n$ . If  $h_d(k-1) < l \leq h_d(k)$ , so that  $\tilde{\beta}_l = k$ , then  $l \leq h_d(k) \leq (d!)^{-1} (k+d)^d$ , which implies  $\tilde{\beta}_l = k \geq (d!)^{1/d} l^{1/d} - d$ . Therefore

$$\beta_n = \frac{1}{n} \sum_{l=1}^n \tilde{\beta}_l \geq -d + (d!)^{1/d} \frac{1}{n} \sum_{l=1}^n l^{1/d} > -d + (d!)^{1/d} \frac{d}{d+1} n^{1/d}, \quad (11)$$

where we have used  $\sum_{l=1}^n l^{1/d} > \int_{x=0}^n x^{1/d} \, dx = \frac{d}{d+1} n^{1+1/d}$ .

Assertion (1) now follows from (8), (10), and (11). Finally, if  $d = 1$  then  $\beta_n = \frac{1}{n} \sum_{l=1}^n \tilde{\beta}_l = \frac{1}{n} \sum_{l=1}^n (l-1) = (n-1)/2$ , and (10) becomes  $\alpha_n \leq 1$ , so substituting into (8) yields (2).  $\square$

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