SOLVABLE RECTANGLE TRIANGLE RANDOM TILINGS

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We show that a rectangle triangle random tiling with a tenfold symmetric phase is solvable by Bethe Ansatz. After the twelvefold square triangle and the eightfold rectangle triangle random tiling, this is the third example of a rectangle triangle tiling which is solvable. A Bethe Ansatz solution provides in principle an accurate estimate of the entropy and phason elastic constants. In the twelvefold and eightfold cases even *exact* analytic expressions have been obtained from the Bethe Ansatz solution.

1 Introduction

A random tiling ensemble with a tenfold symmetric phase is defined by rectangles and isosceles triangles of sides of length 1 and $l = 2\cos(3\pi/10) = \sqrt{2+\tau}/\tau$, where $\tau = (\sqrt{5}+1)/2$ is the golden mean. This random tiling was used by He *et al.*[1] and by Nissen and Beeli [2] to model a decagonal phase in FeNb, by Oxborrow and Mihalkovič [4] to model disorder in decagonal AlPdMn and by Roth and Henley [3] to model the equilibrium structure resulting from a molecular dynamics simulation. The perfect quasiperiodic tiling corresponds to a maximum possible density of a decagonal disc packing.[5] The aim of this work is to calculate the entropy of the random tiling and its phason elastic constants. To perform this calculation we use a transfer matrix that generates the ensemble of tilings.[6] The logarithm of the largest eigenvalue of this matrix will give the free energy in the thermodynamic limit.

2 Bethe Ansatz

We show that the model is solvable in the sense that its transfer matrix can be diagonalized using Bethe Ansatz techniques. First we deform the tiling such that its vertices are a subset of those of the square lattice. Similar to the twelvefold square-triangle tiling [7,8] and the octagonal rectangle-triangle tiling,[9] we have to decorate the deformed tiles with lines to make this deformation map bijective. An example of the deformation and decoration is given in Fig.1. From this example it is seen that the triangles form domain walls between patches of one type of rectangle. There are three types of domain wall; one type is running from bottom-left to top-right (right mover), denoted by the gray lines in Fig.1b, the other two are running form bottom-right to top-left (left

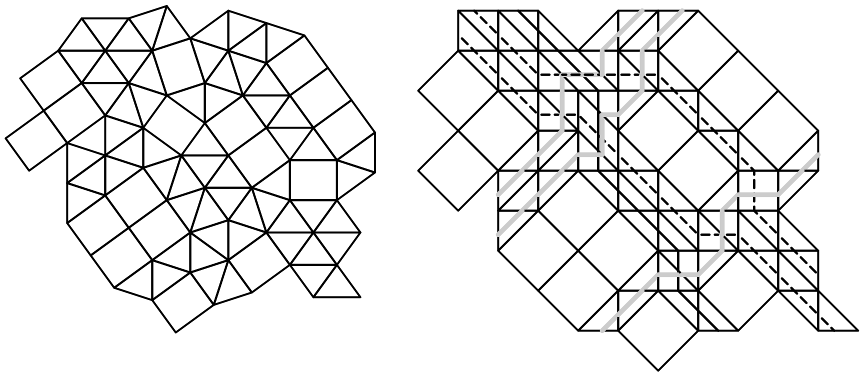

Figure 1: a) Patch of the random tiling. b) Corresponding patch on the square lattice

movers) and are denoted by the solid and dashed lines in Fig.1b. These lines scatter with each other, but they persist through the lattice. This means that the number of each type of domain wall is a conserved quantity under the action of the transfer matrix. We denote the numbers of gray, solid and dashed lines by m, n_1 and n_2 respectively. A state α on a row of the lattice can be specified by the positions $y_1, \ldots, y_m$ of the right movers and by the positions $x_1, \ldots, x_n$ of all the left movers $n = n_1 + n_2$ with the specification that the lines $i_1, \ldots, i_{n_2}$ at positions $x_{i_2}, \ldots, x_{i_{n_2}}$ are dashed lines. Elements $\psi(\alpha)$ of the eigenvector thus can be written explicitly as $\psi(i_1, \ldots, i_{n_2} | x_1, \ldots, x_n; y_1, \ldots, y_m)$. We make the following (plane wave) Ansatz for the form of the eigenvector

$$\psi(i_1, \ldots, i_{n_2} | x_1, \ldots, x_n; y_1, \ldots, y_m) = \sum_{\pi,\rho} \sum_{\mu} A(\Gamma) B(\mu) \prod_{a=1}^{n} z_{\pi_a}^{x_a} \prod_{b=1}^{m} w_{\rho_b}^{y_b} \prod_{c=1}^{n_2} \prod_{r=1}^{i_c} u(\mu_c, \pi_r). \quad (1)$$

The amplitudes A depend on the permutations π and ρ and on the configuration of the various domain walls.[9] Simlarly, the amplitudes B depend on the permutation μ and on the sequence of dashed and solid black lines.

If all the domain walls are separated the action of the transfer matrix is just a shift of each line to the right or to the left. The eigenvalue corresponding to the vector (1) is therefore given by

$$\Lambda = \prod_{i=1}^{n} z_i \prod_{j=1}^{m} w_j^{-1}. \quad (2)$$

At places where different domain walls are close together, the action of the transfer matrix is not given by a mere shift of all domain walls. These more complex processes turn out to be consistent with the form (1) of the eigenvector and impose constraints on the amplitudes A and B and the z's, w's and u's of the eigenvector (1). These equations are the socalled Bethe Ansatz equations and are given by

$$w_j^{-N} = (-)^{m-1} \prod_{k=1}^{m} \left(\frac{w_j}{w_k}\right) \prod_{i=1}^{n} \left(z_i w_j + z_i^{-1} w_j^{-1}\right), \tag{3}$$

$$z_i^N = (-)^{n-1} \prod_{k=1}^{n} \left(\frac{z_k}{z_i}\right) \prod_{j=1}^{m} \left(z_i w_j + z_i^{-1} w_j^{-1}\right) \prod_{l=1}^{n_2} u(l,i), \tag{4}$$

$$(-)^{n_2-1} = \prod_{i=1}^{n} u(l,i), \tag{5}$$

where u is given by

$$u(l,i) = \left(v_l + z_i^2 - z_i^{-2}\right)^{-1}. \tag{6}$$

The Ansatz (1) and the Eqs. (3)-(6) can be generalized to include chemical potentials to control the tile densities in the ensemble.

3 Entropy

The entropy can now be calculated from the eigenvalue (2). Some finite size results for the entropy per lattice site and extrapolation are given in Table 1.

Table 1: Numerical data for the entropy

N	σ_N
8	0.19482
13	0.17779
21	0.17088
34	0.16853
55	0.16763
89	0.16730
∞	0.1671 (1)

According to the random tiling hypothesis the entropy per area has its maximum at the point where the densities match those of the quasiperiodic tiling. For the tenfold tiling the expression for the entropy in terms of the phason strain near the quasicrystalline point is given by

$$\sigma_a(\mathbf{E}) = \sigma_0 - \frac{1}{4}K_1\left((E_{1x}+E_{2y})^2+(E_{1y}-E_{2x})^2\right) - \frac{1}{4}K_2\left((E_{1x}-E_{2y})^2+(E_{1y}+E_{2x})^2\right) - \frac{1}{2}K_3\left(E_{3x}^2+E_{3y}^2\right). \quad (7)$$

The last term in (7) is present if the height component in the third commensurate direction is 'rough'. This can be checked by calculating the finite size corrections to the entropy. For a critical system these go as

$$\sigma_\infty = \sigma_N - \frac{\pi c v_s}{6N^2} + \mathcal{O}\left(\frac{1}{N^2}\right), \quad (8)$$

where c is the central charge and v_s is a geometrical factor. The central charge measures the number of independent height components in these kinds of models. Indeed, we find $c = 3$.

Although Eqs. (3)-(6) in principle can be solved numerically for very large values of N we found that the structure of the solution for the groundstate is numerically complicated, which is why we as yet cannot push the systemsize N larger than 89. This also has prevented us so far to give estimates for the elastic constants.

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