Interacting Fermi liquid in two dimensions at finite temperature Part I: Convergent Attributions

M. Disertori and V. Rivasseau
Centre de Physique Théorique, CNRS UPR 14
Ecole Polytechnique
91128 Palaiseau Cedex, FRANCE

Abstract

Using the method of continuous renormalization group around the Fermi surface, we prove that a two-dimensional interacting system of Fermions at low temperature T remains a Fermi liquid (analytic in the coupling constant λ) for $\lambda \leq c/|\log T|$ where c is some numerical constant. This bound is a step in the program of rigorous (non-perturbative) study of the BCS phase transition for many Fermions systems; it proves in particular that in dimension two the transition temperature (if any) must be non-perturbative in the coupling constant. The proof is organized into two parts: the present paper deals with the convergent contributions, and a companion paper (Part II) deals with the renormalization of dangerous two point subgraphs and achieves the proof.

I Introduction

Conducting electrons in a metal at low temperature are well described by Fermi liquid theory. However we know that the Fermi liquid theory is not valid down to 0 temperature. Indeed below the BCS critical temperature the dressed electrons or holes which are the excitations of the Fermi liquid bound into Cooper pairs and the metal becomes superconducting.

During the last ten years a program has been designed to investigate rigorously this phenomenon by means of field theoretic methods [BG][FT1-2][FMRT1-3][S1]. In particular the renormalization group of Wilson and followers has been extended to models with surface singularities such as the Fermi surface. The ultimate goal is to create a mathematically rigorous theory of the BCS transition and of similar phenomena of solid state physics. This is a long and difficult program which requires to glue together several ingredients in particular renormalization group around the Fermi surface and spontaneous symmetry breaking.

A more accessible task is to precise the mathematical status of Fermi liquid theory itself. Fermi liquid theory is not valid at zero temperature because of the BCS instability. Even when the dominant electron interaction is repulsive, the Kohn-Luttinger instabilities prevent the Fermi liquid theory to be generically valid down to zero temperature. There are nevertheless two proposals for a mathematically rigorous Fermi liquid theory:

- one can block the BCS and Kohn-Luttinger instabilities by considering models in which the Fermi surface is not invariant under $p \to -p$ [FKLT]. In two dimensions it is possible to prove (even non perturbatively) that in this case the Fermi liquid theory remains valid at zero temperature, and the corresponding program is well under way [FKST]. However this program requires to control rigorously the stability of a non-spherical Fermi surface under the renormalization group flow, a difficult technical issue [FST];

- one can study the Fermi liquid theory at finite temperature above the BCS transition temperature. A system of weakly interacting electrons has an obviously stable thermodynamic limit at high enough temperature, since the temperature acts as an infrared cutoff on the propagator in the field theory description of the model. In this point of view, advocated by [S2], the non trivial theorem consists in showing that stability (i.e. summability of perturbation theory) holds for all temperatures higher than a certain critical temperature whose dependence in terms of the initial interaction should be as precise as possible.

This is what we do here, by proving an upper bound on any critical temperature for two dimensional systems of Fermions which is *exponentially small* in the coupling constant, hence invisible in perturbation theory. Our analysis relies on a renormalization group analysis around the Fermi surface. Renormalization group flows were studied perturbatively in the context of a spherical Fermi surface in [FT2]. A non perturbative study in 2 dimension was performed in [FMRT1], but it was limited to so called “completely convergent graphs”. In this paper we rely heavily on the ideas introduced in [FMRT1], but we extend them to include non perturbative renormalization of the two point functions which allow the rigorous exponentially small upper bound. This extension is not trivial since renormalization in phase space in this context is complicated by the need for anisotropic sectors. Also we use (in contrast with [FMRT1]) a *continuous* renormalization group scheme around the Fermi surface (an other idea advocated in [S2]). This scheme has been tested first in the simpler case of the Gross Neveu model (a field theory where there is no Fermi surface) in [DR1].

The next natural step in this program is to add the computation of coupling constants flows (i.e. renormalization of four point functions). This should be a rather straightforward extension of the methods of this paper. It would allow to compute the optimal expected value c_o of the constant c in our upper bound on the critical temperature of Fermions systems. A more difficult step is to glue this analysis to a kind of $1/N$ expansion and to a bosonic analysis to control the region at distance $\Delta_{BCS} \simeq e^{-c_o/\lambda}$ of the Fermi surface [FMRT2]. In two dimensions and finite temperature we cannot expect true symmetry breaking by the Mermin-Wagner theorem, but we can expect a Kosterlitz-Thouless phase for a two dimensional bosonic field in a rotation invariant effective potential. Finally at 0 temperature we have effectively a three-dimensional theory (two dimensions for space, one for imaginary time). Continuous symmetry breaking can then occur, with the associated Goldstone boson. The last part of the analysis consists therefore in the non-perturbative control of the infrared divergences associated to this Goldstone boson, using Ward identities at the constructive level [FMRT3].

Our result has quite a long proof, which we organized therefore in two main parts. In this paper we introduce the model and prove the analyticity of the "convergent contributions" to the vertex functions, hence we reproduce the results of [FMRT1], but with the continuous renormalization group technique. In a companion paper [DR2] we consider the complete sum of all graphs, perform renormalization of the two point subgraphs and obtain our main theorem.

II Model and Notations

The simplest free continuum model for interacting Fermions is the isotropic jellium model with a continuous rotation invariant ultraviolet cutoff. This model is rotation invariant, a feature which simplifies considerably the study of the renormalization group flows after introducing the interaction. In particular it has a spherical Fermi surface. It is a realistic model for instance in solid state physics in the limit of weak electrons densities (where the Fermi surface becomes approximately spherical).

The simplest Fermion interaction perturbing this free model is a local four body interaction. This is a realistic interaction for instance in a solid where the dominant interaction is not the Coulomb interaction but the electron-phonon interaction. After integrating out the phonons modes an effective four body interaction is obtained, which is not strictly local due to the non local phonon propagator. However at long distances it is well approximated by a local interaction. [1]

We use the formalism of non-relativistic field theory at imaginary (periodic) time of [FT1-2][BG] to describe the interacting fermions at finite temperature. Our model is therefore similar to the Gross-Neveu model, but with a different, not relativistic propagator[2].

II.1 Propagator without ultraviolet cutoff

Using the Matsubara formalism, the propagator at temperature T, $C(x_0, \vec{x})$, is antiperiodic in the variable x_0 with antiperiod $\frac{1}{T}$. This means that the Fourier transform defined by

$$\hat{C}(k) = \frac{1}{2} \int_{-\frac{1}{T}}^{\frac{1}{T}} dx_0 \int d^2x \, e^{-ikx} \, C(x) \tag{II.1}$$

is not zero only for discrete values (called the Matsubara frequencies) :

$$k_0 = \frac{2n+1}{\beta}\pi \ , \quad n \in \mathbb{Z} , \tag{II.2}$$

[1]Interaction with non-local but well-decaying kernels can be added without much cost to our analysis.

[2] However there are some important differences:

- in GN the infrared singularity lies at $k = 0$. Renormalization subtracts divergent functions at this point. In the Fermi liquid the singularity lies on the surface $k_0 = 0$, $|\vec{k}| = 1$, so renormalization is more complicated;
- in GN a natural infrared cut-off is given by the mass, in Fermi liquid it is given by the temperature;
- in GN we are interested in the ultraviolet limit, the low energy (renormalized) parameters being kept fixed; in the Fermi liquid we fix the ultraviolet cut-off and we want to deduce the long range properties from the microscopic theory.

where $\beta = 1/T$ (we take $\hbar = k = 1$). Remark that only odd frequencies appear, because of antiperiodicity.

Our convention is that a three dimensional vector is denoted by $x = (x_0, \vec{x})$ where $\vec{x}$ is the two dimensional spatial component. The scalar product is defined as $kx := -k_0 x_0 + \vec{k}\vec{x}$. By some slight abuse of notations we may write either $C(x - \bar{x})$ or $C(x, \bar{x})$, where the first point corresponds to the field and the second one to the antifield (using translation invariance of the corresponding kernel).

Actually $\hat{C}(k)$ is obtained from the real time propagator by changing k_0 in ik_0 and is equal to:

$$\hat{C}_{ab}(k) = \delta_{ab} \frac{1}{ik_0 - e(\vec{k})}, \qquad e(\vec{k}) = \frac{\vec{k}^2}{2m} - \mu \ , \tag{II.3}$$

where $a, b \in \{1, 2\}$ are the spin indices. The vector $\vec{k}$ is two-dimensional. Since our theory has two spatial dimensions and one time dimension, there are really three dimensions. The parameters m and μ correspond to the effective mass and to the chemical potential (which fixes the Fermi energy). To simplify notation we put $2m = \mu = 1$, so that $e(\vec{k}) = \vec{k}^2 - 1$. Hence,

$$C_{ab}(x) = \frac{1}{(2\pi)^2 \beta} \sum_{k_0} \int d^2k \; e^{ikx} \; \hat{C}_{ab}(ik_0, \vec{k}) \ . \tag{II.4}$$

The notation $\sum_{k_0}$ means really the discrete sum over the integer n in (II.2). When $T \to 0$ (which means $\beta \to \infty$) k_0 becomes a continuous variable, the corresponding discrete sum becomes an integral, and the corresponding propagator $C_0(x)$ becomes singular on the Fermi surface defined by $k_0 = 0$ and $|\vec{k}| = 1$. In the following to simplify notations we will write:

$$\int d^3k \ \equiv \ \frac{1}{\beta} \sum_{k_0} \int d^2k \quad , \quad \int d^3x \ \equiv \ \frac{1}{2} \int_{-\beta}^{\beta} dx_0 \int d^2x \ . \tag{II.5}$$

In determining the spatial decay we will need the following lemma

Lemma 1 *The function C defined in (II.4) can also be written as*

$$C(x) = f(x_0, \vec{x}) := \sum_{m \in \mathbb{Z}} (-1)^m \, C_0 \left(x_0 + \frac{m}{T}, \vec{x} \right) \ . \tag{II.6}$$

where C_0 is the propagator at $T = 0$.

Proof To prove this lemma we first prove that the function f is antiperiodic on $\frac{1}{T}$. Since $\hat{f}(k) = \hat{C}(k)\ \forall k$, the Lemma holds. ∎

In this paper we do not perform yet any renormalization, hence we do not introduce any counterterm, and the interaction is simply:

$$S_V = \frac{\lambda}{2} \int_V d^3x\ (\sum_a \bar{\psi}\psi)^2 \tag{II.7}$$

where $V := [-\beta, \beta] \times V'$ and V' is an auxiliary volume cutoff in two dimensional space, that will be soon sent to infinity. Remark that in (II.2) $|k_0| \geq \pi/\beta \neq 0$ hence the denominator in $C(k)$ can never be 0 at non zero temperature. This is why the temperature provides a natural infrared cut-off.

II.2 Propagator with an ultraviolet cutoff

It is convenient to add a continuous ultraviolet cut-off (at a fixed scale Λ_0) to the propagator (II.3) for two reasons: first because it makes its Fourier transformed kernel in position space well defined, and second because a non relativistic theory does not make sense anyway at high energies. To preserve physical (or Osterwalder-Schrader) positivity one should introduce this ultraviolet cutoff only on spatial frequencies [FT2]. However for convenience we introduce this cutoff both on spatial and on Matsubara frequencies as in [FMRT1]; indeed the Matsubara cutoff could be lifted with little additional work.

For technical reasons it is also convenient to introduce, as in [DR1], an auxiliary infrared cut-off at scale Λ, whose variation controls the renormalization group flow. At the end the limit $\Lambda \to 0$ is taken (we recall that the true infrared cutoff is the temperature, which is not taken to 0 in this paper). The propagator (II.3) equipped with these two cutoffs is called $C_\Lambda^{\Lambda_0}$. It is defined as:

$$C_\Lambda^{\Lambda_0}(k) := C(k)\ \Big[u(r/\Lambda_0^2) - u(r/\Lambda^2)\Big]\Big|_{r=k_0^2+e^2(\vec{k})} \tag{II.8}$$

where we fixed $\Lambda_0 = 1$ (for simplicity), $0 \leq \Lambda \leq 1$ and the compact support function $u(r) \in \mathcal{C}_0^\infty(\mathrm{R})$ satisfies:

$$u(r) = 0 \quad \text{for } |r| > 1/2\ ;\ u(r) = 1 \quad \text{for } |r| < 1/4\ ;\ \int u(r)dr = 3/4\ . \tag{II.9}$$

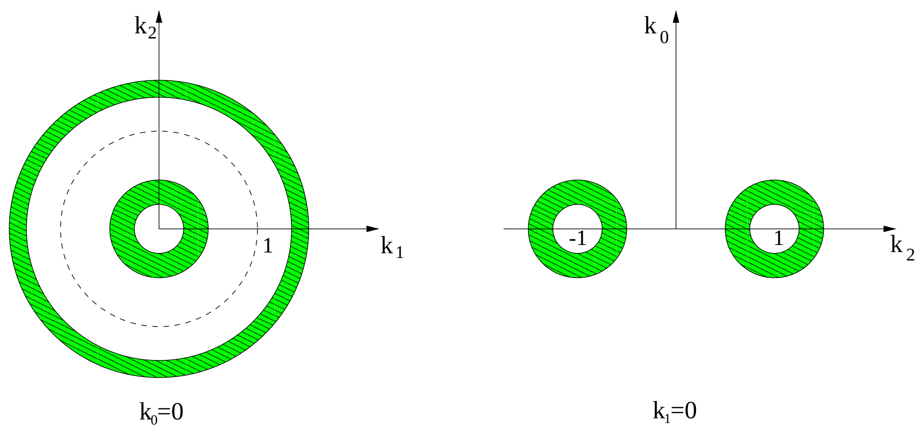

Figure 1: support of C_α

For later calculations it is useful to choose u to be a Gevrey function[3]. The propagator can be parametrized as:

$$C_\Lambda^{\Lambda_0}(k) = \int_{\Lambda_0^{-2}}^{\Lambda^{-2}} d\alpha\; C_\alpha(k) \tag{II.12}$$

where

$$C_\alpha(k) = C(k)\; \eta[\alpha\; r]|_{r=k_0^2+e^2(\vec{k})} \qquad \eta(\alpha\; r) = -ru'(\alpha\; r)\; . \tag{II.13}$$

As $u'(\alpha\; r) \neq 0$ only for $r \simeq 1/\alpha$ the propagator $C_\alpha(k)$ is non zero only for $1/2\sqrt{\alpha} \leq \sqrt{k_0^2 + e^2(\vec{k})} \leq 1/\sqrt{2\alpha}$, hence for momenta in the volume between two tori in R^3 centered on the critical circle $|\vec{k}| = 1, k_0 = 0$ (see Fig.1):

[3] A function $f \in \mathcal{C}^\infty(\mathrm{R}^d)$ with compact support is in the Gevrey class of order s if there exist two constants A and μ such that

$$\forall n \geq 0, \quad ||f^{(n)}||_1 \leq A\mu^{-n} \left(\frac{n}{e}\right)^{ns} \tag{II.10}$$

and its Fourier transform satisfies (see [G]):

$$\forall k \in \mathrm{R}^d \quad |\hat{f}(k)| \leq Ae^{-s\left(\frac{\mu}{\sqrt{d}}|k|\right)^{1/s}} \tag{II.11}$$

In short in the support of $C_\alpha(k)$ we have $||\vec{k}|-1| \simeq \frac{1}{\sqrt{\alpha}}$ and $k_0 \simeq 1/\sqrt{\alpha}$, but they cannot be *simultaneously* much smaller. Remark that the temperature cut-off implies that $C_\alpha = 0$ if $1/\sqrt{2\alpha} < \pi/\beta$, hence the real non zero propagator is

$$\begin{aligned} C_\Lambda^{\Lambda_0}(k) &:= \int_{\Lambda_0^{-2}}^{\Lambda_T^{-2}} d\alpha \; C_\alpha(k) \\ &= C(k) \left[u(r/\Lambda_0^2) - u(r/\Lambda_T^2) \right]\Big|_{r=k_0^2+e^2(\vec{k})} \end{aligned} \tag{II.14}$$

where we defined

$$\Lambda_T := \max\left[\Lambda \,,\; \sqrt{2}\,\pi T\right] \;. \tag{II.15}$$

II.3 Vertex functions

The vertex functions are defined through the partition function:

$$\begin{aligned} Z_V^{\Lambda\Lambda_0}(\xi,\bar\xi) &= \int d\mu_{C_\Lambda^{\Lambda_0}}(\psi,\bar\psi) e^{-\mathcal{S}_V(\psi,\bar\psi)+<\psi,\xi>+<\xi,\psi>} \\ <\psi,\xi> &=: \int_V d^3x \; \bar\psi(x)\xi(x). \end{aligned} \tag{II.16}$$

where ξ is an external field. The $2p$-point vertex function is defined as:

$$\begin{aligned} \Gamma^{\Lambda\Lambda_0}(\{y\},\{z\}) &:= \Gamma^{\Lambda\Lambda_0}(y_1,...,y_p,z_1,...,z_p) \qquad \text{(II.17)} \\ &= \lim_{V'\to\infty} \frac{\delta^{2p}}{\delta\xi(z_1)..\delta\xi(z_p)\delta\bar\xi(y_1)..\delta\bar\xi(y_p)} \left((\ln Z_V^{\Lambda\Lambda_0} - F)(C_\Lambda^{\Lambda_0})^{-1}(\xi) \right)\Big|_{\xi=0} \end{aligned}$$

where $F(\xi) =< \xi, C_\Lambda^{\Lambda_0}\xi >$ is the bare propagator. These functions are the coefficients of the effective action (expanded in powers of the external fields) at energy Λ. They are in fact distributions (as easily seen because there are graphs for which several external arguments hook to the same vertex, hence create δ functions). Therefore we will later smear the vertex functions Γ with smooth test functions $\phi_1(y_1)$,... $\phi_p(y_p)$, $\phi_{p+1}(z_1)$, ... $\phi_{2p}(z_p)$ that are L_∞ and L_1 in position space. Actually, as we work at finite temperature, we can treat test functions as propagators, that is introduce them at $T=0$ and then define the corresponding functions at $T \neq 0$.

Expanding the exponential in Z we have:

$$
\begin{aligned}
Z_V^{\Lambda\Lambda_0}(\xi) &= \sum_{p=0}^{\infty} \frac{1}{p!^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \lambda^n \int_V d^3y_1...d^3y_p d^3z_1...d^3z_p d^3x_1...d^3x_n \\
& \prod_{i=1}^{p} \xi(z_i)\bar{\xi}(y_i) \left\{ \begin{array}{cccccccc} y_1 & ... & y_p & x_1 & x_1 & ... & x_n & x_n \\ z_1 & ... & z_p & x_1 & x_1 & ... & x_n & x_n \end{array} \right\} \qquad \text{(II.18)}
\end{aligned}
$$

where we used Cayley's notation for determinants:

$$
\left\{ \begin{array}{c} u_{i,a} \\ v_{j,b} \end{array} \right\} = \det(C^{\Lambda_0}_{\Lambda,ab}(u_i - v_j)) \ . \qquad \text{(II.19)}
$$

The determinant is the sum over all Feynman graphs amplitudes, and the logarithm selects the sum over connected graphs. To obtain $\log Z$ without expanding completely the determinant we use a forest formula. Forest formulas are Taylor expansions with integral remainders which test links (here the propagators) between $n \geq 1$ points (here the vertices) and stop as soon as the final connected components are built. The result is a sum over forests, a forest being a set of disjoint trees.

Like in [DR1] we use the *ordered Brydges-Kennedy Taylor formula*, which states [AR1] that for any smooth function H of the $n(n-1)/2$ variables u_l, $l \in P_n = \{(i,j)|i,j \in \{1,..,n\}, i \neq j\}$,

$$
H|_{u_l=1} = \sum_{o-\mathcal{F}} \left(\int_{0 \leq w_1 \leq ... \leq w_k \leq 1} \prod_{q=1}^{k} dw_q \right) \left(\prod_{q=1}^{k} \frac{\partial}{\partial u_{l_q}} H \right) (w_l^{\mathcal{F}}(w_q), l \in P_n) \qquad \text{(II.20)}
$$

where $o-\mathcal{F}$ is any ordered forest, made of $0 \leq k \leq n-1$ links $l_1, ..., l_k$ over the n points. To each link l_q $q = 1, ..., k$ of $\mathcal{F}$ is associated the parameter w_q, and to each pair $l = (i,j)$ is associated the weakening factor $w_l^{\mathcal{F}}(w_q)$. These factors replace the variables u_l as arguments of the derived function $\prod_{q=1}^{k} \frac{\partial}{\partial u_{l_q}} H$ in (II.20). These weakening factors $w_l^{\mathcal{F}}(w)$ are themselves functions of the parameters w_q, $q = 1, ..., k$ through the formulas

$$
\begin{aligned}
w_{i,i}^{\mathcal{F}}(w) &= 1 \\
w_{i,j}^{\mathcal{F}}(w) &= \inf_{l_q \in P_{i,j}^{\mathcal{F}}} w_q, \qquad \text{if } i \text{ and } j \text{ are connected by } \mathcal{F} \\
& \quad \text{where } P_{i,j}^{\mathcal{F}} \text{ is the unique path in the forest } \mathcal{F} \text{ connecting } i \text{ to } j \\
w_{i,j}^{\mathcal{F}}(w) &= 0 \qquad \text{if } i \text{ and } j \text{ are not connected by } \mathcal{F}. \qquad \text{(II.21)}
\end{aligned}
$$

We apply this formula to the determinant in (II.18), inserting the interpolation parameter u_l in the UV cut-off Λ_0 of the covariance $C_\Lambda^{\Lambda(u)}(x_i, x_j)$, when $i \neq j$. We define $\Lambda(u)$ by:

$$\Lambda^2(u) = u(\Lambda_0^2 - \Lambda^2) + \Lambda^2 \ ; \ \Lambda(0) = \Lambda \ ; \ \Lambda(1) = \Lambda_0 \ . \tag{II.22}$$

Now the product in (II.20) becomes:

$$\left(\prod_{q=1}^{k} \frac{\partial}{\partial u_{l_q}} H\right)(w_l^{\mathcal{F}}(w_q), l \in P_n) = \left(\prod_{q=1}^{k} \frac{\partial}{\partial w_q} C_\Lambda^{\Lambda(w_q)}(k)(x_{l_q}, y_{l_q})\right) \det \mathcal{M} \tag{II.23}$$

which is the product of the forest line propagators and a remaining determinant which contains all possible contractions of loop lines. Actually, the elements of the matrix $\mathcal{M}$ are the loop line propagators weakened by the forest formula.

Now, taking the logarithm of Z and including (as announced above) the smearing of external arguments by test functions we obtain a tree expansion for the vertex function similar to the one of [DR1]:

$$\Gamma_{2p}^{\Lambda\Lambda_0}(\phi_1, ...\phi_{2p}) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{o-\mathcal{T}} \sum_{E} \sum_{\Omega} \varepsilon(\mathcal{T}, \Omega) \int d^3x_1 ... d^3x_n \phi_1^{\Lambda_T}(x_{i_1}) ... \phi_{2p}^{\Lambda_T}(x_{j_p})$$
$$\int_{w_T \leq w_1 \leq ... \leq w_{n-1} \leq 1} \left[\prod_{q=1}^{n-1} \frac{\partial}{\partial w_q} C_\Lambda^{\Lambda(w_q)}(x_{l_q}, \bar{x}_{l_q}) dw_q\right] \det \mathcal{M}(E) \tag{II.24}$$

where $o - \mathcal{T}$ is the set of ordered trees over n vertices, and E is the set of pairs (ϕ_j, v_j) which specifies which test function ϕ_j is hooked to which internal vertex v_j for $j = 1, ..., 2p$ (see [DR1]). Ω specifies for each tree line whether it comes from a $\psi\bar{\psi}$ or $\bar{\psi}\psi$ contraction. $\varepsilon(\mathcal{T}, \Omega)$ is a global $\pm$ sign whose exact (inessential) value is given in [AR2]. Finally w_T is defined by $\Lambda(w_T) = \Lambda_T$. Remark that $w_T = 0$ if $\Lambda_T = \Lambda$, i.e. if $\Lambda \geq \sqrt{2}\pi T$, and $w_T > 0$ otherwise. The bound $\Lambda(w_i) \geq \Lambda_T \ \forall i$ is due to (II.14-II.15).

In the following, as we are interested in the effective theory at the energy Λ_T, we consider only external impulsions below this energy. Therefore instead of ϕ we use the test function with UV cut-off ϕ^{Λ_T} defined by

$$\hat{\phi}^{\Lambda_T}(k) := \hat{\phi}(k) \left[u(r/\Lambda_T^2)\right]\Big|_{r=k_0^2+e^2(\vec{k})} \ . \tag{II.25}$$

II.4 Bands

The strategy to analyze (II.24) is similar to the one of [DR1]. The determinant is bounded by a Gram inequality (which gives no factorial)[4]. Spatial integrals are performed using the spatial decay of the tree propagators $|\frac{\partial}{\partial w} C_\Lambda^{\Lambda(w)}|$. To send the IR cut-off to zero without generating unwanted factorials, we need to perform some renormalization. These renormalizations, although more complicated than in the field theory case, still involve only two and four point subgraphs [FT1-2]. Therefore as in [DR1] we need to distinguish the so called dangerous subgraphs, which means four-point and two-point quasi-local subgraphs. Remark that a subgraph is called quasi-local if all internal lines have energy higher than all external lines [R]. These contributions are decomposed into a renormalized part with improved power counting, and a localized part which in turn is absorbed into a flow of effective constants.

To implement this renormalization group program, the first tool is to cut the momentum space into bands, which form a partition of unity. The ordering of the tree in the previous section cuts in a natural way the space of momenta into n bands [DR1]. Indeed:

$$w_T \leq w_1 \leq w_2 \leq ... \leq w_{n-1} \leq 1 \rightarrow \Lambda_T \leq \Lambda(w_1) \leq \Lambda(w_2).. \leq \Lambda(w_{n-1}) \leq \Lambda_0\,. \tag{II.26}$$

The set of bands is called $B = \{1, ..., n\}$. The q-th band corresponds to scales between $\Lambda(w_{q-1})$ and $\Lambda(w_q)$, where we adopt the convention $w_n = 1$ and $w_0 = w_T$ (hence $\Lambda(w_0) = \Lambda_T$ and $\Lambda(w_n) = \Lambda_0$). Then we can attribute each loop line to a well defined band.

On the other hand to external lines are associated the test functions $\phi_1^{\Lambda_T}(x_{i_1})$, $...\phi_{2p}^{\Lambda_T}(x_{j_p})$, with a UV cutoff at Λ_T, hence with impulsions lower than the first band. We can say that external lines belong to a first band with index 0, that contains all impulsions at a distance at most Λ_T from the Fermi surface.

Let's see how propagators for tree and loop lines look like.

---

[4]The first example of combining a tree expansion with a Gram bound appears in [L]. We thank G. Gallavotti and C. Wieczerkowski for pointing out this reference to us.

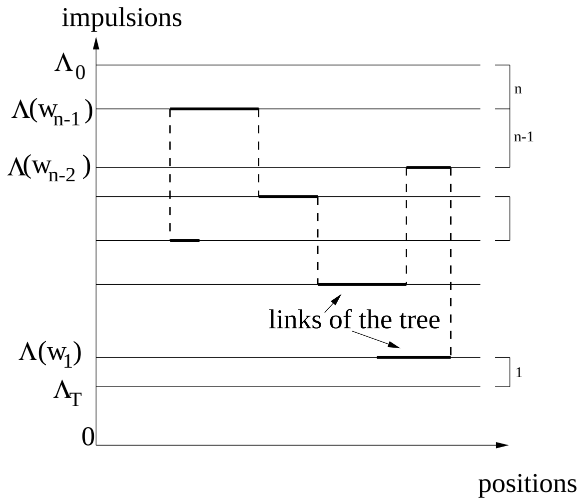

Figure 2: Band structure

II.4.1 Tree propagators

The q-th tree line propagator is given by:

$$\begin{aligned} C^{w_q}(k) &= \frac{\partial}{\partial w_q} \int_{\Lambda^{-2}(w_q)}^{\Lambda^{-2}} d\alpha \, C_\alpha(k) = \frac{(\Lambda_0^2 - \Lambda^2)}{\Lambda^4(w_q)} \, C_\alpha(k)|_{\alpha=\Lambda^{-2}(w_q)} \\ &= \frac{(\Lambda_0^2 - \Lambda^2)}{\Lambda^4(w_q)} \, [ik_0 + e(\vec{k})] \, u' \left(\frac{k_0^2 + e^2(\vec{k})}{\Lambda^2(w_q)} \right) \; . \end{aligned} \tag{II.27}$$

The derivative with respect to w_q fixes the α parameter of the line on the top of the band b_q, and this tree line propagator is considered by convention to belong to the q-th band. In this way we have one tree line in each band, except the last one b_n (see Fig.2).

II.4.2 Loop lines

Loop line propagators are the elements of the $(n+1-p) \times (n+1-p)$ matrix $\mathcal{M}(E)$:

$$\mathcal{M}_{fg} = C_\Lambda^{\Lambda(w^{\mathcal{T}}_{f,g}(w))}(x_f, x_g) \; . \tag{II.28}$$

The corresponding loop fields (respectively antifields) are labeled by the index f (respectively g). Altogether they form a set L labeled by an index $a \in$

$L =: \{1, ..., 2n+2-2p\}$, hence a indexes both the rows and columns of the determinant in (II.24): $a(f_1) = 1, ..., a(f_{n+1-p}) = n+1-p, a(g_1) = n+2-p, ..., a(g_{n+1-p}) = 2n+2-2p$. Similarly to each tree line l_i there corresponds two half tree lines called f_i and g_i. Each loop propagator can be written as a sum of propagators restricted to single bands:

$$C_\Lambda^{\Lambda(w^{\mathcal{T}}_{f,g}(w))}(k) = \sum_{j=1}^{i^{\mathcal{T}}_{f,g}} \int_{\Lambda^{-2}(w_j)}^{\Lambda^{-2}(w_{j-1})} d\alpha\, C_\alpha(k) = C(k) \sum_{j=1}^{i^{\mathcal{T}}_{f,g}} u^j(k) \tag{II.29}$$

where we define $i^{\mathcal{T}}_{f,g}$ as the lowest index in the path $P^{\mathcal{T}}_{f,g}$ (defined in equation (II.21))

$$i^{\mathcal{T}}_{f,g} = \inf \{q \mid l_q \in P^{\mathcal{T}}_{f,g}\}, \tag{II.30}$$

and the function u^j is the cutoff for the j-th band

$$u^j(k) := \Big[u[r\ \Lambda^{-2}(w_j)] - u[r\ \Lambda^{-2}(w_{j-1})]\Big]\Big|_{r=[k_0^2+e^2(\vec{k})]} . \tag{II.31}$$

By multi-linearity one can expand the determinant in (II.24) according to the different bands in the sum (II.29) for each row and column:

$$\det \mathcal{M}(E) = \sum_\mu \det \mathcal{M}(\mu, E) \tag{II.32}$$

where an attribution μ is a collection of band indices for each loop field $a \in L$:

$$\mu = \{\mu(f_1), ..\mu(f_{n+1-p}), \mu(g_1), ..\mu(g_{n+1-p})\}\ ,\ \mu(a) \in B \ \text{ for } a = 1...2n+2-2p. \tag{II.33}$$

Now, for each attribution μ we need to exploit power counting. This requires notations for the various types of fields or half-lines which form the analogs of the quasi local subgraphs of [R] in our formalism (that is subgraphs with all internal lines higher than the external ones). For a loop half line (with index a) or an external line (with index j) we call v_a or v_j the vertex to which it hooks. Similarly, for tree half lines f_i and g_i, we call v_{f_i} or v_{g_i} the vertex to which they hook. We define as i_v the band index of the highest tree line hooked to the vertex v, and, for each $k \geq 1$:

$$T_k = \{l_i \in \mathcal{T} |\ i \geq k\}\ . \tag{II.34}$$

In particular we define t_k as the unique connected component of T_k containing the tree line l_k. We say that a vertex $v \in t_k$ if $i_v \geq k$ and $l_{i_v} \in t_k$. The matrix element of the determinant in (II.32) is then

$$\mathcal{M}_{fg}(\mu)(x_f, x_g) = \delta_{\mu(f),\mu(g)} \frac{1}{(2\pi)^2} \int d^3k \; e^{ik(x_f - x_g)} C(k) \; u^{\mu(f)}(k) W^{\mu(f)}_{v_f, v_g} \; , \tag{II.35}$$

where

$$\begin{aligned} W^k_{v,v'} &= 1 \quad \text{if } v \text{ and } v' \text{are connected by } T_k \\ &= 0 \quad \text{otherwise .} \end{aligned} \tag{II.36}$$

Now we define the quasi-local subgraph at level k g_k as

$$\begin{aligned} g_k &= t_k \cup il_k \\ et_k &= \{l_i \in \mathcal{T} | v_i \in t_k, i < k\} \\ il_k &= \{a \in L | v_a \in t_k, \mu(a) > \mathcal{A}(k)\} \\ el_k &= \{a \in L | v_a \in t_k, \mu(a) \leq \mathcal{A}(k)\} \\ ee_k &= \{(\phi_j^{\Lambda_T}, v_j) \in E | v_j \in t_k\} \\ eg_k &= et_k \cup el_k \cup ee_k \\ V_k &= \{v | v \in t_k\} \end{aligned} \tag{II.37}$$

where et_k, el_k and ee_k are the tree, loop and real external external half lines respectively, and il_k are the internal loop half lines, and we denoted by $\mathcal{A}(k)$ the index of the highest tree external line of g_k.

In defining internal and external loop half-lines we have observed that no new line connects to t_k in the interval between k and $\mathcal{A}(k)$. Hence all loop half-lines connected to the vertices of t_k with attributions between k and $\mathcal{A}(k)$ are in fact internal lines for the subgraph g_k as they must contract between themselves. Therefore we have considered as external loop half lines only the ones with attributions $\mu(a) \leq \mathcal{A}(k)$. In the following, we will note by $|A|$ the number of elements in some set A.

Tadpoles Remark that $\mu(a) \leq i_{v_a}$ always. Indeed we could have $\mu(a) > i_{v_a}$ only if the line a belongs to a tadpole. But the contribution of a tadpole is zero[5], as proved by the following lemma:

Lemma 2 *The amplitude of a tadpole with loop line in some band i is zero $\forall i$.*

Proof The loop integral is:

$$\frac{1}{(2\pi)^2}\int d^3k\; C^{\Lambda(w_i)}_{\Lambda(w_{i-1})}(k) = -\frac{1}{(2\pi)^2\beta}\sum_{k_0}\int d^2k\;\frac{ik_0+e(\vec{k})}{k_0^2+e^2(\vec{k})}U\left[k_0^2,e^2(\vec{k})\right] \tag{II.38}$$

where

$$U\left[k_0,e^2(\vec{k})\right] = \left[u\left(\frac{k_0^2+e^2}{\Lambda^2(w_i)}\right) - u\left(\frac{k_0^2+e^2}{\Lambda^2(w_{i-1})}\right)\right]. \tag{II.39}$$

By the properties of u, $U\neq 0$ only for $\Lambda^2(w_{i-1})/4 \leq k_0^2+e^2 \leq \Lambda^2(w_i)/2$. The integral reduces to

$$-\frac{1}{(2\pi)^2\beta}\sum_{k_0}\int d^2k\frac{e(\vec{k})}{k_0^2+e^2(\vec{k})}U\left[k_0^2,e^2(\vec{k})\right] \tag{II.40}$$

as the other term in odd under k_0. Performing the change of variables $t=|\vec{k}|^2-1$ the spatial integral (for any k_0 fixed) becomes

$$\int_0^{2\pi}d\theta\int_{-1}^{\infty}\frac{dt}{2}\frac{t}{k_0^2+t^2}U(k_0^2,t^2) = \pi\int_{-1}^{1}dt\frac{t}{k_0^2+t^2}U(k_0^2,t^2) = 0 \tag{II.41}$$

by parity. Remark that the domain of t can be reduced to $[-1,1]$ since, for $t\geq 1$, $k_0^2+t^2\geq 1 > \Lambda^2(w_i)/2$, hence $U=0$. ∎

II.4.3 Analyticity of convergent attributions

We call an attribution μ convergent if it satisfies $eg_k \geq 6$ for any $k>1$. Remark that for $k=1$, $eg_1=2p$, and for $p\leq 2$ we cannot require that this last subgraph has more than 4 external legs.

[5] Tadpoles are exactly zero because we choose our ultraviolet cutoff small enough. Otherwise the tadpole would simply be very small, which would add some inessential complications.

The convergent part of the theory is defined by the functions

$$\Gamma^{\Lambda\Lambda_0}_{2p,\mathrm{conv.}}(\phi_1^{\Lambda_T},...\phi_{2p}^{\Lambda_T}) = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\sum_{o-\mathcal{T}}\sum_{E\,\Omega}\varepsilon(\mathcal{T},\Omega)\int d^3x_1...d^3x_n\;\phi_1^{\Lambda_T}(x_{i_1})...\phi_{2p}^{\Lambda_T}(x_{j_p}) \int_{w_T\le w_1\le...\le w_{n-1}\le 1}\left[\prod_{q=1}^{n-1}\frac{\partial}{\partial w_q}C_\Lambda^{\Lambda(w_q)}(x_{l_q},\bar{x}_{l_q})dw_q\right]\sum_{\mu\ \mathrm{conv.}}\det\mathcal{M}(\mu,E)\ . \tag{II.42}$$

We start with a first theorem which essentially reproduces the result of [FMRT1] in our framework of continuous cutoffs. This theorem states that the infrared limit (i.e the zero temperature limit) of the convergent part of the theory exists and is analytic in the bare coupling constant.

The full theorem on the Fermi liquid, which includes renormalization and requires a finite temperature cutoff is postponed to the companion paper (Part II).

Theorem 1 *For fixed Λ_0 and $T \geq 0$, the limit $\Lambda \to 0$ of the function $\Gamma^{\Lambda\Lambda_0}_{2p,\mathrm{conv.}}(\phi_1^{\Lambda_T},...\phi_{2p}^{\Lambda_T})$ exists and is analytic in λ for any $|\lambda| \leq c$ where c is the convergence radius.*

This partial result is interesting because it isolates the constructive arguments from the computation of the renormalization group flow. We conjecture that the same theorem holds in three dimensions but have no proof until now (see however [MR] for a partial result in that direction). The rest of the paper is devoted to a proof of Theorem 1.

III Further Expansion Steps

III.1 Chains

The decomposition into bands has a price, that is we have to perform the additional sum over convergent attributions μ. As in [DR1] this sum might develop a factorial. In other words fixing the band index for each single half-line develops too much the determinant. To overcome this difficulty we remark that the attributions contain much more information than necessary, hence we can group attributions into packets to reduce the number of determinants to bound. This operation is based on four remarks. For each band index i we analyze the subgraph g_i:

- for each g_i nothing happens in the interval between i and $\mathcal{A}(i)$, as it contains just loop internal half lines that contract between themselves. Therefore we can regroup all the attributions in this interval;
- if $|eg_i| \leq 10$ we want to know exactly which loop fields are external and which ones are internal;
- if $|eg_i| \geq 11$ and $|et_i|+|ee_i| < 11$ we just want to fix the attributions for $11 - |et_i| - |ee_i|$ loop fields, but we do not need to fix the attributions for the remaining loop fields;
- if $|eg_i| \geq 11$ and $|et_i| + |ee_i| \geq 11$ we do not fix the attributions for any loop field.

Remark that a subgraph is potentially divergent when it has two or four external lines. For this reason in [DR1] we selected at most five external lines to ensure convergence. Here we select at most eleven external lines because of additional technical difficulties due to the sector counting and renormalization, that will be explained in the following. As seen below this does not develop too much the determinant.

Hence, instead of expanding the loop determinant over lines and columns as a sum over all attributions

$$\det \mathcal{M} = \sum_{\mu} \det \mathcal{M}(\mu) \tag{III.1}$$

we write it as a sum over a smaller set $\mathcal{P}$ (called the set of packets). These packets are defined by means of a function

$$\begin{array}{rcl} \phi : \{\mu\} & \longrightarrow & \mathcal{P} \\ \mu & \mapsto & \mathcal{C} = \phi(\mu) \end{array} \tag{III.2}$$

which to each attribution μ associates a class $\mathcal{C} = \phi(\mu)$ element of $\mathcal{P}$. For our resummation purpose, the function ϕ must have two crucial properties:

- $\#\{\mathcal{P}\} \leq K^n$ (this is critical for summation over packets);

- there exists a matrix $\mathcal{M}'$ such that

$$\sum_{\mu\in\phi^{-1}(\mathcal{C})} \det \mathcal{M}(\mu) = \det \mathcal{M}'(\mathcal{C}) \tag{III.3}$$

and some form of Gram's inequality applies to $\det \mathcal{M}'(\mathcal{C})$.

The construction of a function ϕ with these properties is developed in detail in [DR1] [6]. We just recall the result: for each class $\mathcal{C}$, each loop field a belongs no longer to a single band $\mu(a)$, but to a set of bands:

$$J_a(\mathcal{C}) = \{\mu(a) | m(a,\mathcal{C}) \leq \mu(a) \leq M(a,\mathcal{C}) \leq i_{v_a}\} \tag{III.4}$$

and the new matrix elements are

$$\mathcal{M}'_{x_f,x_g}(\mathcal{C}) = \frac{1}{(2\pi)^2} \int d^3k \; e^{ik(x_f-x_g)} C(k) \sum_{q=1}^{n} \eta^q_{a(f)} \eta^q_{a(g)} u^q(k) W^q_{v_f,v_g} \tag{III.5}$$

where $\mathcal{M}'$ is a function of $\mathcal{C}$, and η_a is the characteristic function of the set of bands attributed by $\mathcal{C}$ to the loop field a:

$$\eta_a(\mathcal{C}) : B \to \{0,1\} \quad \eta^q_a(\mathcal{C}) = \begin{cases} 0 & \text{if } q \notin J_a(\mathcal{C}) \\ 1 & \text{if } q \in J_a(\mathcal{C}) \; . \end{cases} \tag{III.6}$$

Finally we remark that the construction of [DR1] groups convergent attributions μ into *convergent classes* $\mathcal{C}$ which form a subset of the set $\mathcal{P}$. Therefore the convergent functions $\Gamma^{\Lambda\Lambda_0}_{2p,\,\text{conv.}}(\phi^{\Lambda_T}_1, ...\phi^{\Lambda_T}_{2p})$ can be rewritten as:

$$\begin{aligned} &\Gamma^{\Lambda\Lambda_0}_{2p,\,\text{conv.}}(\phi^{\Lambda_T}_1, ...\phi^{\Lambda_T}_{2p}) = \\ &\quad \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{o-\mathcal{T}} \sum_{E} \sum_{\Omega} \sum_{\mathcal{C}_c} \varepsilon(\mathcal{T},\Omega) \int d^3x_1 ... d^3x_n \phi^{\Lambda_T}_1(x_{i_1}) ... \phi^{\Lambda_T}_{2p}(x_{j_p}) \\ &\quad \int_{w_T \leq w_1 \leq ... \leq w_{n-1} \leq 1} \left[\prod_{q=1}^{n-1} C^{w_q}(\bar{x}_{l_q}, x_{l_q}) dw_q \right] \det \mathcal{M}'(\mathcal{C}, E) \end{aligned} \tag{III.7}$$

[6] We need only to modify ϕ slightly to accommodate the expansion up to eleven external lines instead of five external lines. This has no other consequences than a larger constant K for the first condition (the number 3^5 in [DR,(IV.13)] is replaced by 3^{11}).

and the definitions of internal and external lines for each subgraph g_i can be generalized:

$$\begin{aligned} il_i(\mathcal{C}) &:= \{a \in L | v_a \in t_i, M(a,\mathcal{C}) > \mathcal{A}(i)\} \\ el_i(\mathcal{C}) &:= \{a \in L | v_a \in t_i, M(a,\mathcal{C}) \le \mathcal{A}(i)\} \\ eg_i(\mathcal{C}) &:= et_i \cup el_i(\mathcal{C}) \cup ee_i \ . \end{aligned} \tag{III.8}$$

III.2 Partial ordering

We have seen that attributions contain much more information than necessary and that this affects the convergence of the series. Hence we have regrouped attributions into packets preserving only the information to perform power counting.

Similarly the total ordering over tree line energies contains unnecessary information that make power counting more complicated and less transparent. Indeed we are not interested in the relative ordering of tree lines that belong to mutually disjoint connected components g_i. Hence we reorganize the scale analysis according to a structure that we call Clustering Tree Structure (CTS), that contains the desired scale information and no more. This structure is closely related to the "Gallavotti-Nicolo" trees.

Definition A clustering Tree Structure CTS is an unlabeled rooted tree, with $2n-2$ lines and $2n-1$ vertices of two different types : $n-1$ crosses and n dots, such that the root is a cross with coordination 2, each other cross has coordination 3 and each dot coordination 1 (see Fig.3).

Obviously

Lemma 3 *The number of CTS at order n is at most* 3^{n-1}.

Proof: We start from the cross root and climb in the structure. At each cross there are at most three choices for the two vertices immediately above : two dots, one dot and one cross, or two crosses. Hence the number of crosses being $n-1$ the total number of choices is bounded by 3^{n-1} (this is only an upper bound because some choices may not lead to a structure made of $n-1$ crosses and n dots).

■

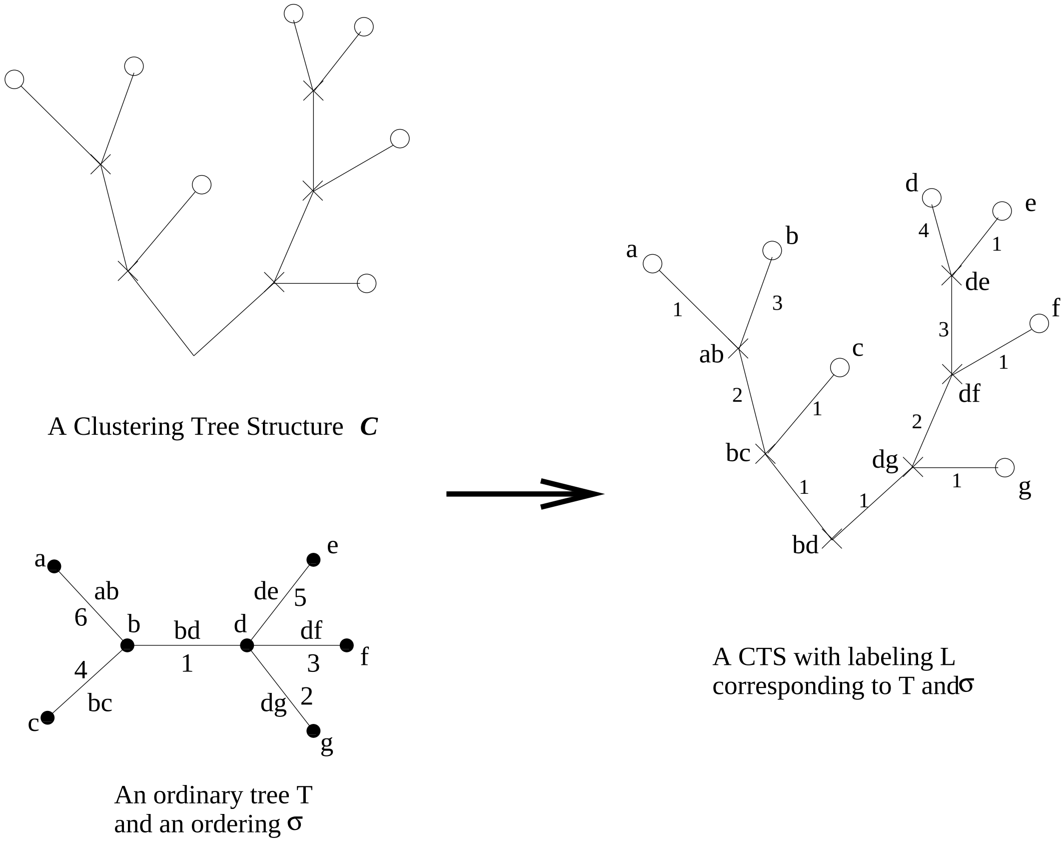

Figure 3: left: A CTS and a tree, with an ordering; right: The associated CTS with labeling induced. The vertices of the tree are named as a,b,c,d,e,f,g; the lines are named by the pair of vertices they join; the ordering is indicated by numbers 1,2,3,4,5,6 on the lines of $\mathcal{T}$. Finally on the right, the numbers $N_\ell(\mathcal{T}, \mathcal{L})$ are shown on each line ℓ.

III.2.1 Labeling

We want to relate a CTS at order n to an ordinary tree $\mathcal{T}$ with n vertices. The $n-1$ lines of $\mathcal{T}$ are labeled by an index l and the $2n-2$ lines of the CTS are labeled by an index ℓ (they should not be confused). A labeling $\mathcal{L}$ of the CTS is a one to one map between the set of vertices (crosses and dots) of the CTS and the vertices and lines of $\mathcal{T}$, so that each cross of the CTS is labeled by a particular line of $\mathcal{T}$, and each dot of the CTS by a particular vertex of $\mathcal{T}$, satisfying a further constraint. For each ℓ, let $T_\ell(\mathcal{L})$ be the subset of $\mathcal{T}$ made of all lines and vertices of $\mathcal{T}$ corresponding to all crosses and dots "above ℓ" (that is such that the unique path in CTS joining this cross or dot to the root passes through ℓ). The constraint on the labeling $\mathcal{L}$ is that $T_\ell(\mathcal{L})$ has to be connected for all ℓ. We call $N_\ell(\mathcal{T},\mathcal{L})$ the number of external lines of T hooked to $T_\ell(\mathcal{L})$.

$$\mathcal{L}\{\times,\circ\} \longrightarrow \{l,v\} \quad \mathcal{L}(\times)=l \quad \mathcal{L}(\circ)=v\ .$$

We consider only in what follows trees $\mathcal{T}$ with coordination N_v at each vertex v bounded by 4 (since other trees cannot appear as subgraphs in the model we consider). Remark that a tree can be considered as the list $V=\{N_v\}$ of its coordination numbers plus the set of Wick contractions W which associates together two by two the half lines or "fields" hooked to each vertex, subject to the constraint that the resulting graph is a tree.

Let $\mathcal{T}$ be a tree with n vertices, and σ_T a total ordering of its lines. In [DR1] it is shown how to construct an associated CTS and a labeling $\mathcal{L}$. We recall the rule : the first line in the ordering is the cross root. When cut, it separates $\mathcal{T}$ into two ordered trees $\mathcal{T}_1$ and $\mathcal{T}_2$ (possibly reduced to a single vertex). The process is iterated in each subtree: in $\mathcal{T}_1$ and $\mathcal{T}_2$ the lowest lines give the label of the crosses immediately above the root and so on (see Fig.3). When subtrees reduced to a single vertex are met, a dot appears instead of a cross.

Conversely for a given tree $\mathcal{T}$, the same CTS and labeling $\mathcal{L}$ can be obtained from many total orderings σ_T. Indeed CTS and $\mathcal{L}$ induce only a *partial* ordering σ_P on the lines of $\mathcal{T}$: $l_i \geq_P l_j$ if the path from the cross with label l_i to the root passes through the cross with label l_j. Every total ordering σ_T compatible with this partial ordering gives the same CTS and labeling $\mathcal{L}$. This is somehow a defect. Our new point of view resums all these total orderings to retain only the partial ordering σ_P (which is the one relevant for scale analysis).

Hence the sum over ordered trees can be written as

$$\sum_{o-\mathcal{T}} = \sum_{u-\mathcal{T}} \sum_{\sigma_T} = \sum_{u-\mathcal{T}} \sum_{CTS} \sum_{\mathcal{L}} \sum_{\sigma_T \to (CTS,\mathcal{L})} = \sum_{CTS} \sum_{u-\mathcal{T}} \sum_{\mathcal{L}} \sum_{\sigma_T \to (CTS,\mathcal{L})}$$

where $u-\mathcal{T}$ is an unordered tree and $\sum_{\sigma_T \to (CTS,\mathcal{L})}$ is the sum over the set of total orderings that give the same couple $(CTS, \mathcal{L})$, for $u-\mathcal{T}$ fixed. Now we observe that

$$\sum_{\sigma_T \to (CTS,\mathcal{L})} \int_{w_T \leq w_1 \leq \ldots \leq w_{n-1} \leq 1} = \int_{w_T \leq w_{\mathcal{A}(i)} \leq w_i \leq 1,\ \forall i}$$

where the integration is now on the region of the w's parameters satisfying the partial ordering relations associated to σ_P. We call $w_r := \min_i w_i$ the parameter associated to the lowest tree line, that is the root of the CTS, and by convention we put $w_{\mathcal{A}(r)} := w_T$. Remark that now for any w_i we only know that

$$\min[w_{i'}, w_{i''}] \geq w_i \geq w_{\mathcal{A}(i)} \tag{III.9}$$

where $w_{i'}$ and $w_{i''}$ are the parameters associated to the two crosses above i (if there is a dot instead we assume $w_{i'} = 1$). In this new point of view the band q corresponds to the energy interval $[\Lambda(w_q), \Lambda(w_{\mathcal{A}(q)})]$ instead of $[\Lambda(w_q), \Lambda(w_{q-1})]$ and in (III.5), the new matrix element $W^q_{v_f,v_q}$ selects only the vertices connected by t_q, hence in (II.36) T_k has to be replaced by t_k. The expression (III.7) for the vertex function becomes

$$\Gamma^{\Lambda\Lambda_0}_{2p,\,\text{conv.}}(\phi_1^{\Lambda_T}, \ldots \phi_{2p}^{\Lambda_T}) = \\ \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{CTS} \sum_{u-\mathcal{T}} \sum_{\mathcal{L}} \sum_{E} \sum_{\Omega} \sum_{\mathcal{C}_c} \varepsilon(\mathcal{T}, \Omega) \int d^3x_1 \ldots d^3x_n \phi_1^{\Lambda_T}(x_{i_1}) \ldots \phi_{2p}^{\Lambda_T}(x_{j_p}) \\ \int_{w_T \leq w_{\mathcal{A}(i)} \leq w_i \leq 1} \left[\prod_{q=1}^{n-1} C^{w_q}(x_{l_q}, \bar{x}_{l_q}) dw_q \right] \det \mathcal{M}'(\mathcal{C}, E)\ . \tag{III.10}$$

III.3 Sectors

Band decoupling is not enough to obtain correct power counting. Roughly speaking, this happens for two reasons.

1. The partition of unity for internal lines (tree and loop lines) is not fine enough, as the volume in phase space $\Delta x \Delta k$ depends on α. Actually Δx

is given by the rate of spatial decay which is $1/\sqrt{\alpha}$ in all three directions. On the other hand Δk is given by the band volume, proportional to $1/\alpha$. Then $\Delta x \Delta k \simeq \sqrt{\alpha}$. To obtain a phase space volume independent from α we must take a smaller volume in the momentum space. For that, adapting to our continuous formalism the idea of [FMRT1], we cut the two dimensional Fermi surface $|\vec{k}| = 1$ into angular sectors of size $1/\alpha_s^{1/4}$ [7]. Now the volume in phase space of a single angular sector is $1/(\alpha\alpha_s^{1/4})$. The spatial decay rate is $1/\sqrt{\alpha}$ on two directions, and $1/\alpha_s^{1/4}$ on the third one, tangential direction (provided α_s is not bigger than α, as explained in [FMRT1]). Then the phase space volume becomes a constant independent from α and α_s, as it should for a single "degree of freedom" of the theory.

2. When $2p > 0$ we need to cut the support of $\hat{\phi}^{\Lambda_T}$ into angular sectors in order to exploit momentum conservation, at least for subgraphs g_i with $|eg_i(\mathcal{C})| \leq 10$ (as in this case we know all the external lines of the subgraph).

III.3.1 Sector Cutoffs

To introduce the angular sectors we insert in

$$C_\alpha(x) = \frac{1}{(2\pi)^2\beta} \sum_{k_0} \int_0^\infty d|k|\ |k| \ \int_0^{2\pi} d\theta\ e^{ikx} C_\alpha(k)$$

and in

$$\phi_i^{\Lambda_T}(x) = \frac{1}{(2\pi)^2\beta} \sum_{k_0} \int_0^\infty d|k|\ |k| \int_0^{2\pi} d\theta\ e^{ikx}\ u(r/\Lambda_T)\ \hat{\phi}_i(k)$$

the unitary integral

$$\frac{4}{3}\alpha_s^{1/4} \int_0^{2\pi} d\theta_s\ \chi_{\alpha_s}^{\theta}(\theta_s) = 1\ , \tag{III.11}$$

where $\chi_{\alpha_s}^{\theta}(\theta_s) = \chi_{\alpha_s}^{\theta_s}(\theta)$ selects a small angular sector centered on θ_s. The factor $\frac{4}{3}\alpha_s^{1/4}$ is needed to normalize properly the integral (see (II.9)). Indeed to define χ we use again the Gevrey function $u : \mathbb{R} \to \mathbb{R}$ of the previous section:

$$\chi_{\alpha_s}^{\theta}(\theta_s) := u_p^{\alpha_s}[\alpha_s^{1/4}(\theta - \theta_s)]\ , \tag{III.12}$$

[7] α_s is not necessarily equal to α, since we need to exploit momentum conservation of sectors at various intermediate scales between α and the ultraviolet scale. The power $1/4$ is chosen as in [FMRT1], to avoid a logarithmic divergence related to "almost collapsed rhombuses".

where $u_p^{\alpha_s}$ is the periodic function of period $\tau = 2\pi\alpha_s^{1/4}$, obtained from u by: $u_p^{\alpha_s}(y) = u(x)$ when $y = x + n\tau$ for some $x \in [-1/2, 1/2[$ and $n \in \mathbb{Z}$, and $u_p^{\alpha_s}(y) = 0$ otherwise. This definition satisfies the condition (III.11).

A *sector* is *defined* as a couple (α_s, θ_s). For a given sector (α_s, θ_s), we define the support $\Sigma(\alpha_s, \theta_s)$ to be the support of the function $\chi_{\alpha_s}^{\theta_s}(\theta)$. Inside this support $|\theta - \theta_s| \leq (1/2)\alpha_s^{-1/4}$.

Now, in order to exploit momentum conservation at each vertex and subgraph, we need to decompose each half-line (either loop, tree or external) a certain number of times into sectors with different values of α_s, starting from larger sizes (hence smaller α_s) and then refining them into smaller ones.

This process requires to define a sequence of scales for each line. These scales roughly speaking represent all scales i for which the half line is external to the subgraph g_i and $|eg_i(\mathcal{C})| \leq 10$ (as we can exploit momentum conservation only in this case), plus a last scale, characteristic of the line and the class $\mathcal{C}$. The subgraph g_r requires a particular treatment: its external lines are the only real external lines of the whole graph, hence we can always exploit momentum conservation, even if $2p = |eg_r| > 10$.

III.4 Choice of scales α_s for each half-line

Let us introduce an index h which parametrizes loop, tree and external half-lines. The sum over sector choices will be done inductively, from the root towards the leaves. We then choose as root vertex the external vertex x_{e_1}, and as root the test function ϕ_{e_1}. Now we denote the two half-lines belonging to the tree line l_i as h_i^L (h left) and h_i^R (h right) in such a way that $h_i^R \to h_i^L$ is oriented towards the root vertex. Hence we define $\mathcal{T}_L$ and $\mathcal{T}_R$ as the set of tree half-lines of left and right type respectively.

Remark that, for any subgraph g_k with $k \neq r$, (as $eg_r = 2p$ then there is no tree external line) there is at most one tree half-line $h_i \in et_k \cap \mathcal{T}_R$ (that we call h_k^{root}) going towards the root. If $e_1 \in ee_i$ all tree external half-lines belong to $\mathcal{T}_L$ and we put $h_k^{root} = e_1$. The sector of this half-line is kept fixed in the sum over sector choices until scale 0. In the same way the sector of each tree right half-line h_i^R is kept fixed in the sum over sector choices until scale i; by momentum conservation along the tree line l_i this sector is then equal to that of h_i^L. Therefore for each tree line l_i we perform sector decoupling and sector sums only for h_i^L (as h_i^R is automatically fixed by h_i^L). In the following, h^R will appear only as $h_i^{\rm root}$ for some subgraph g_i, hence to simplify notation we write simply h_i for h_i^L.

Given the class $\mathcal{C}$ we define a natural scale $i(h)$ associated to each $h \in L \cup \mathcal{T}_L \cup E$

- For the left half-line belonging to the tree line l_i obviously $i(h_i) = i$.
- For a loop half-line $h = a$ we choose $i(h) = M(a, \mathcal{C})$ (this choice avoids the "logarithmic divergence" associated to momentum conservation in 2 dimensions, see [FMRT1], lemma 2).
- For all external lines we choose $i(e) = 0$ which is the band to which they belong. This means that we cut them in sectors of size $\alpha_0^{-\frac{1}{4}} := \Lambda_T^{\frac{1}{2}}$.

We introduce then a growing sequence of indices $j_{h,1} = i(h), ..., j_{h,n_h} = i_{v_h}$ such that each scale $j_{h,r}$ of the sequence corresponds to a refining of that half-line in sectors of size $1/\alpha_{j_{h,r}}^{1/4} = \Lambda^{1/2}(w_{j_{h,r}})$. Remark that the lowest refining scale is $i(h)$.

The choice of these indices is the following: a half-line $h \in \mathcal{T}_L \cup L \cup E$ is refined at scale $j = i(h)$ and at all scales j such that $h \in eg_i(\mathcal{C})$ for some level i with $j = \mathcal{A}(i)$ and such that $|eg_i(\mathcal{C})| \leq 10$.

This multiple decomposition has to be adapted to the different bounds satisfied by tree, loop, and external lines.

III.4.1 Tree lines

As explained above, we introduce the multi-sector decomposition only for the left half-line of l_i, h_i. We must ensure that the spatial decay of the tree line l_i depends only on the finest sector (at level i), hence, we apply the identity (III.11) just one time, at the scale i.

We then decompose each tree left half-line on larger sectors introducing the identity

$$1 = \left[\frac{4}{3}\alpha_{j_{h_i,r}}^{1/4}\right] \int_0^{2\pi} d\theta_{h_i,r}\ \chi_{\alpha_{j_{h_i,r}}}^{\theta_{h_i,1}}(\theta_{h_i,r})\ . \tag{III.13}$$

This actually selects $\theta_{h_i,r}$ to be in a sector of size $\Lambda^{1/2}(w_{j_{h_i,r}})$ around $\theta_{h_i,1}$. Hence, for the half-tree line $h_i \in \mathcal{T}_L$ the complete decomposition is:

$$1 \ = \ \left[\tfrac{4}{3}\alpha_{j_{h_i,1}}^{1/4}\right] \int_0^{2\pi} d\theta_{h_i,1}\ \chi_{\alpha_{j_{h_i,1}}}^{\theta_i}(\theta_{h_i,1}) \left\{ \prod_{r=2}^{n_{h_i}} \left[\tfrac{4}{3}\alpha_{j_{h_i,r}}^{1/4}\right] \int_0^{2\pi} d\theta_{h_i,r}\ \chi_{\alpha_{j_{h_i,r}}}^{\theta_{h_i,1}}(\theta_{h_i,r}) \right\}$$

$$= \left[\tfrac{4}{3}\alpha^{1/4}_{j_{h_i,n_{h_i}}}\right]\int_0^{2\pi} d\theta_{h_i,n_{h_i}} \left[\tfrac{4}{3}\alpha^{1/4}_{j_{h_i,n_{h_i}-1}}\right]\int_{\Sigma_{j_{h_i,n_{h_i}}}} d\theta_{h_i,n_{h_i}-1} \;\cdots$$

$$\left[\tfrac{4}{3}\alpha^{1/4}_{j_{h_i,2}}\right]\int_{\Sigma_{j_{h_i,3}}} d\theta_{h_i,2}\left[\left[\tfrac{4}{3}\alpha^{1/4}_{j_{h_i,1}}\right]\int_{\Sigma_{j_{h_i,2}}} d\theta_{h_i,1}\left[\prod_{r=2}^{n_h}\chi^{\theta_{h_i,1}}_{\alpha_{j_{h_i,r}}}(\theta_{h_i,r})\right]\chi^{\theta_i}_{\alpha_{j_{h_i,1}}}(\theta_{h_i,1}) \tag{III.14}$$

where we defined sectors twice as large as the previous ones:

$$\Sigma_{j_{h,r}} := \Sigma(\alpha_{j_{h,r}}/2^4, \theta_{h,r}) \equiv \{\theta \mid |\theta_{h,r}-\theta| \le \Lambda^{1/2}(w_{j_{h,r}})\}\ . \tag{III.15}$$

Indeed the integration domain for $\theta_{h_i,r}$, $r \geq 2$, can be restricted to $\Sigma_{j_{h_i,r+1}}$ if we observe that the product $\chi^{\theta_{h_i,1}}_{\alpha_{j_{h_i,r}}}(\theta_{h_i,r})\chi^{\theta_{h_i,1}}_{\alpha_{j_{h_i,r+1}}}(\theta_{h_i,r+1})$ can be non zero only if $\theta_{h_i,r} \in \Sigma_{j_{h_i,r+1}}$. This is also true for $r = 1$ since the single function $\chi^{\theta_{h_i,1}}_{\alpha_{j_{h_i,2}}}(\theta_{h_i,2})$ is non zero only if $|\theta_{h_i,1} - \theta_{h_i,2}| \le \frac{1}{2}\Lambda^{1/2}(w_{j_{h,2}})$, which implies $\theta_{h_i,1} \in \Sigma_{j_{h_i,2}}$.

Finally we remark that for each $r \geq 1$, $\theta_i \in \Sigma_{j_{h_i,r}}$, where θ_i is the angular variable for the momentum of the propagator of line i.

III.4.2 External and loop half-lines

External test functions enter in spatial integration too, hence we perform the sector decomposition in the same way as for tree left half-lines. Loop lines are not used in spatial decay, and there is no sector conservation along the line, as we do not know exactly which loop fields are contracted. Hence we can decompose them as we want. To simplify notation, we treat them exactly in the same way as the tree left half-lines.

Hence the expression (III.10) for the convergent part of the vertex function becomes:

$$\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\sum_{CTS}\sum_{u-\mathcal{T}}\sum_{\mathcal{L}}\sum_{E}\sum_{\Omega}\sum_{\mathcal{C}_c}\varepsilon(\mathcal{T},\Omega) \tag{III.16}$$

$$\prod_{h\in L\cup\mathcal{T}_L\cup E}\Bigg\{\left[\tfrac{4}{3}\Lambda^{-\frac{1}{2}}(w_{j_{h,n_h}})\right]\int_0^{2\pi} d\theta_{h,n_h}\left[\tfrac{4}{3}\Lambda^{-\frac{1}{2}}(w_{j_{h,n_h-1}})\right]\int_{\Sigma_{j_{h,n_h}}} d\theta_{h,n_h-1}$$

$$\ldots\ \left[\tfrac{4}{3}\Lambda^{-\frac{1}{2}}(w_{j_{h,1}})\right]\int_{\Sigma_{j_{h,2}}} d\theta_{h,1}\left[\prod_{h\in\mathcal{T}_L\cup L\cup E}\prod_{r=2}^{n_h}\chi^{\theta_{h,r}}_{\alpha_{j_{h,r}}}(\theta_{h,1})\right]\Bigg\}$$

$$\int d^3x_1 ... d^3x_n \;\; \phi_1^{\Lambda_T}(x_{i_1},\theta_{e_1,1}) \; ... \; \phi_{2p}^{\Lambda_T}(x_{j_p},\theta_{e_{2p},1})$$
$$\int_{w_T\le w_{\mathcal{A}(i)}\le w_i\le 1}\Big[\prod_{q=1}^{n-1} C^{w_q}(x_q,\bar{x}_q,\theta_{h,1})\; dw_q\Big] \det \mathcal{M}'(\mathcal{C},E,\{\theta_{a,1}\}) \; ,$$

where for $1\le l\le 2p$

$$\phi_l^{\Lambda_T}(x,\theta_{e_l,1}) := \frac{1}{(2\pi)^2}\int d^3k\; e^{ikx}\;\chi_{\alpha_0}^{\theta_{e,1}}(\theta)\;\hat{\phi}(k)\;[u(r/\Lambda_T^2)]\Big|_{r=k_0^2+e^2(\vec{k})} \; , \quad \text{(III.17)}$$

$$C^{w_q}(\bar{x}_q,x_q,\theta_{h,1}) := \frac{1}{(2\pi)^2}\int d^3k\; e^{ik(x_q-\bar{x}_q)}C^{w_q}(k)\chi_{\alpha_{j_{h,1}}}^{\theta_{h,1}}(\theta)\; , \quad \text{(III.18)}$$

and the coefficients of the matrix $\mathcal{M}'(\mathcal{C},E,\{\theta_{a,1}\})$ are

$$\mathcal{M}'(\mathcal{C},E,\{\theta_{a,1}\})_{x_f,x_g} := \quad \text{(III.19)}$$
$$\frac{1}{(2\pi)^2}\int d^3k\; e^{ik(x_f-x_g)}C(k)\sum_{q=1}^{n}\eta_{a(f)}^q\eta_{a(g)}^q u^q(k) W_{v_f,v_g}^q\left[\chi_{\alpha_{j_{a(f),1}}}^{\theta_{a(f),1}}(\theta)\right]\left[\chi_{\alpha_{j_{a(g),1}}}^{\theta_{a(g),1}}(\theta)\right] .$$

Remark that the sums over sectors have been taken out of the determinant by multi-linearity, and that we used $\chi_{\alpha_1}^{\theta_1}(\theta)=\chi_{\alpha_1}^{\theta}(\theta_1)$.

Now we want to exploit momentum conservation. At each subgraph g_i with $i=r$ or $|eg_i(\mathcal{C})|\le 10$ we refine all external lines in sectors at the scale $\mathcal{A}(i)$, except for the half-line h_i^{root} which is fixed in a sector of size $\Lambda^{\frac{1}{2}}(w_j)\le\Lambda^{\frac{1}{2}}(w_{\mathcal{A}(i)})$ (for some $0\le j\le\mathcal{A}(i)$). Actually the volume of integration for the new sectors is restricted by momentum conservation. To take into account these effects we insert in the expression above

$$1=\Upsilon\left(\theta_{h_i^{root}},\{\theta_{h,r(i)}\}_{h\in eg_i^*}\right)+\left[1-\Upsilon\left(\theta_{h_i^{root}},\{\theta_{h,r(i)}\}_{h\in eg_i^*}\right)\right] \; , \quad \text{(III.20)}$$

where we defined $r(i)$ as the number of refinements we have done on the half-line h until $\mathcal{A}(i)$ (this means $j_{h,r(i)}=\mathcal{A}(i)$). We also set $eg_i^*:=eg_i\backslash\{h_i^{root}\}$ and define the function Υ to be 0 if the set of selected sectors is forbidden by momentum conservation at this subgraph, and we define Υ to be 1 otherwise. Therefore after insertion of (III.20) the term $1-\Upsilon$, forbidden by momentum conservation, gives a zero contribution. Hence we can insert freely in (III.16) the product

$$\prod_{\{g_i|i=1 \text{ or } (|eg_i(\mathcal{C})|)\le 10\}}\Upsilon\left(\theta_{h_i^{root}},\{\theta_{h,r(i)}\}_{h\in eg_i^*}\right) . \quad \text{(III.21)}$$

In this way we exploit momentum conservation at each subgraph, but we still have to exploit it at each vertex. For that we need some additional notation. We call $H(v)$ the set of half-lines hooked to v (and $|H(v)|$ its cardinal). We define $H^*(v) := H(v)\backslash h_v^{\rm root}$ where $h_v^{\rm root}$ is the half-line going towards the root. Remark that the scale i_v is the largest scale of refinement for each of the elements of $H^*(v)$: $i_v = j_{h,n_h}$, $\forall h \in H^*(v)$. Again we can insert the function $\Upsilon\left(\theta_{h_v^{root}}, \{\theta_{h,n_h}\}_{h\in H^*(v)}\right)$, which is zero when the sectors are not permitted by momentum conservation at vertex v. Hence, by the same argument as above, we can freely insert in (III.16)

$$\prod_v \Upsilon\left(\theta_{h_v^{root}}, \{\theta_{h,n_h}\}_{h\in H^*(v)}\right) \ . \tag{III.22}$$

IV Main result and Bounds

Now we have all the elements to perform the bounds. We insert absolute values inside the sums and integrals and obtain the inequality

$$\begin{aligned}
|\Gamma^{\Lambda\Lambda_0}_{2p\,\text{conv.}}| \leq & \sum_{n=0}^{\infty} \frac{|\lambda|^n}{n!} \sum_{CTS} \sum_{u-\mathcal{T}} \sum_{\mathcal{L}} \sum_{E} \sum_{\Omega} \sum_{\mathcal{C}_c} \qquad \text{(IV.1)}\\
& \prod_{h\in L\cup\mathcal{T}_L\cup E} \Bigg\{ \left[\tfrac{4}{3}\Lambda^{-\frac{1}{2}(w_{j_{h,n_h}})}\right] \int_0^{2\pi} d\theta_{h,n_h} \ \left[\tfrac{4}{3}\Lambda^{-\frac{1}{2}(w_{j_{h,n_h-1}})}\right] \int_{\Sigma_{j_{h,n_h}}} d\theta_{h,n_h-1} \\
& \dots \ \left[\tfrac{4}{3}\Lambda^{-\frac{1}{2}(w_{j_{h,1}})}\right] \int_{\Sigma_{j_{h,2}}} d\theta_{h,1} \Bigg\} \left[\prod_{h\in\mathcal{T}_L\cup L\cup E} \prod_{r=2}^{n_h} \chi^{\theta_{h,r}}_{\alpha_{j_{h,r}}}(\theta_{h,1}) \right] \\
& \prod_{\{g_i|i=r \text{ or } (|eg_i(\mathcal{C})|)\leq 10\}} \Upsilon\left(\theta_i^{root}, \{\theta_{h,r(i)}\}_{h\in eg_i^*}\right) \prod_v \Upsilon\left(\theta_{h_v^{root}}, \{\theta_{h,n_h}\}_{h\in H^*(v)}\right) \\
& \int d^3x_1 \dots d^3x_n \ \ |\phi_1^{\Lambda_T}(x_{i_1}, \theta_{e_1,1})| \dots |\phi_{2p}^{\Lambda_T}(x_{j_p}, \theta_{e_{2p},1})| \\
& \int_{w_T\leq w_{\mathcal{A}(i)}\leq w_i\leq 1} \left[\prod_{q=1}^{n-1} |C^{w_q}(x_q, \bar{x}_q, \theta_{h,1})| dw_q \right] |\det \mathcal{M}'(\mathcal{C}, E, \{\theta_{a,1}\})|
\end{aligned}$$

Actually we prove the following theorem (more precise than Theorem 1):

Theorem 2 *Let $\varepsilon > 0$, $\Lambda_0 = 1$ and $T \geq 0$ be fixed. The series (IV.1) is absolutely convergent for $|\lambda| \leq c$, c small enough. This convergence is* *uniform in* Λ*, then the IR limit* $\Gamma^{\Lambda_0}_{2p,conv} = \lim_{\Lambda\to\infty} \Gamma^{\Lambda\Lambda_0}_{2p,conv}$ *exists and satisfies the bound:*

$$|\Gamma^{\Lambda_0}_{2p>4,conv}(\phi_1^{\Lambda_T},...,\phi_{2p}^{\Lambda_T})| \leq \qquad \text{(IV.2)}$$

$$K_0 \; {}_{||\phi_1||_1 \prod_{i=2}^{2p} ||\hat\phi_i||_\infty} \frac{T^{\frac{7}{4}2p-\frac{1}{2}}}{2p-4} \; [K_1(\varepsilon)]^p \;\; (p!)^2 \;\; K(c) \; e^{-(1-\varepsilon)\Lambda_T^{\frac{1}{s}} d_{\mathcal{T}}^{\frac{1}{s}}(\Omega_1,...\Omega_{2p})}$$

$$|\Gamma^{\Lambda_0}_{4,conv}(\phi_1^{\Lambda_T},..,\phi_4^{\Lambda_T})| \leq K_0' \; {}_{||\phi_1||_1 \prod_{i=2}^{2p} ||\hat\phi_i||_\infty} T^{\frac{13}{2}} |\log T| K(c) \; e^{-(1-\varepsilon)\Lambda_T^{\frac{1}{s}} d_{\mathcal{T}}^{\frac{1}{s}}(\Omega_1,...\Omega_4)}$$

$$|\Gamma^{\Lambda_0}_{2,conv}(\phi_1^{\Lambda_T},\phi_2^{\Lambda_T})| \leq K_0'' \; {}_{||\phi_1||_1 \prod_{i=2}^{2p} ||\hat\phi_i||_\infty} T^2 \;\; K(c) \; e^{-(1-\varepsilon)\Lambda_T^{\frac{1}{s}} d_{\mathcal{T}}^{\frac{1}{s}}(\Omega_1,\Omega_2)}$$

where Ω_i *is the compact support of* ϕ_i*,* $K_1(\varepsilon)$ *is a constant dependent from* ε*,* $K(c)$ *is a function of* c *that tends to one when* c *tends to zero, and* s *is the Gevrey index of our cutoff function* u *(we assume that* $1 < s < 2$*). Finally we defined*

$$\begin{aligned} d_{\mathcal{T}}(\Omega_1,...\Omega_{2p}) &:= \inf_{x_i\in\Omega_i} d_{\mathcal{T}}(x_1,...,x_{2p}) \ , \\ d_{\mathcal{T}}(x_1,...,x_{2p}) &:= \inf_{u-\mathcal{T}} \sum_{l\in\mathcal{T}} |\bar x_l - x_l| \ , \end{aligned} \qquad \text{(IV.3)}$$

where in the definition of $d_{\mathcal{T}}(x_1,...,x_{2p})$ *(called the tree distance of* $x_1,...x_{2p}$*) the infimum over* $u-\mathcal{T}$ *is taken over all unordered trees (with any number of vertices) connecting* $x_1,...x_{2p}$*.*

IV.1 Loop determinant

To bound the loop determinant we apply Gram's inequality, which states that if M is a $n\times n$ matrix whose elements $M_{ij} =< f_i, g_j >$ are scalar products of vectors f_i, g_j in a Hilbert space, then $|\det M| \leq \prod_{i=1}^n ||f_i|| \; \prod_{j=1}^n ||g_j||$.

Lemma 4 *The matrix* $\mathcal{M}'(\mathcal{C})$ *satisfies the following Gram inequality:*

$$|\det \mathcal{M}'(\mathcal{C})| \leq \prod_f ||F_f||_{\mathcal{C}} \; \prod_g ||G_g||_{\mathcal{C}} \qquad \text{(IV.4)}$$

$$= \prod_f \left[\frac{1}{(2\pi)^2} \int d^3k \; u^f_{\mathcal{C}}(k) |F_f(k)|^2 \right]^{\frac{1}{2}} \prod_g \left[\frac{1}{(2\pi)^2} \int d^3k \; u^g_{\mathcal{C}}(k) |G_g(k)|^2 \right]^{\frac{1}{2}}$$

where the cut-off $u^a_{\mathcal{C}}(k)$ is defined by:

$$u^a_{\mathcal{C}}(k) := \left[u\left(\frac{k_0^2+e^2(\vec{k})}{\Lambda^2(w_{M(a,\mathcal{C})})}\right) - u\left(\frac{k_0^2+e^2(\vec{k})}{\Lambda^2(w_{\mathcal{A}(m(a,\mathcal{C}))})}\right)\right] . \tag{IV.5}$$

Proof The proof is identical to that of Lemma 4 in [DR1]. The only difference is that here we have partial order instead of the total order in [DR1]. We just resume it for completeness. We observe that the matrix element (III.19) can be written as

$$\frac{1}{(2\pi)^2}\int d^3k\ F_f(k)\ G^*_g(k)\ \sum_{q=1}^{n} W^q_{v_f,v_g}\ u^q(k)\ \eta^q_{a(f)}\ \eta^q_{a(g)} \tag{IV.6}$$

where we defined

$$F_f(k) = e^{ix_f k}\chi^{\theta_{f,1}}_{\alpha_{j_{f,1}}}(\theta)\frac{1}{(k_0^2+e^2(\vec{k}))^{\frac{1}{4}}} \quad G_g(k) = e^{ix_g k}\chi^{\theta_{g,1}}_{\alpha_{j_{g,1}}}(\theta)\frac{(ik_0+e(\vec{k}))}{(k_0^2+e^2(\vec{k}))^{\frac{3}{4}}} . \tag{IV.7}$$

We introduce the matrix

$$\mathcal{W}^q_{v,a;v',b}\ :=\ R^q_{a,b}\ W^q_{v,v'} := \eta^q_a\ \eta^q_b\ W^q_{v,v'} \tag{IV.8}$$

for v, v' belonging to the set of n vertices, a, b to the set of $2n+2-2p$ loop half-lines (fields and anti-fields). Both $R^q_{a,b}$ and $W^q_{v,v'}$ can be written (modulo permutation of field and vertex indices) as block diagonal positive matrices or sums of matrices of the type

$$\begin{pmatrix} 1_k & 0 \\ 0 & 0 \end{pmatrix} \tag{IV.9}$$

where 1_k is a $k\times k$ matrix with all elements equal to 1. Then $\mathcal{W}^q$ is positive, $\sum_q u^q\ \mathcal{W}^q$ is positive too and there exists a positive matrix U defined by

$$\sum_{w,c} U_{v,a;w,c}\ U_{w,c;v',b}\ :=\ \sum_q\ u^q\ \mathcal{W}^q_{v,a;v',b}\ . \tag{IV.10}$$

The determinant can be written as the scalar product of two functions

$$\mathcal{M}'_{fg} = \frac{1}{(2\pi)^2}\int d^3k\ \sum_{v',s}\mathcal{F}^f_{v's}\ \mathcal{G}^{g*}_{v's}\ = < \vec{\mathcal{F}}^f, \vec{\mathcal{G}}^g > , \tag{IV.11}$$

where we defined

$$\mathcal{F}^f_{v's}(k) = F_f(k)\; U_{v',s;v(f),a(f)} \quad , \quad \mathcal{G}^g_{v's}(k) = G_g(k)\; U_{v',s;v(g),a(g)} \; . \qquad \text{(IV.12)}$$

Applying Gram inequality we obtain (IV.4). ∎

With these definitions, the norms of F_f and G_g satisfy the bounds

$$\begin{aligned} ||F_f||_{\mathcal{C}} &\leq K\; \Lambda^{\frac{1}{4}}(w_{M(f,\mathcal{C})})\; [\Lambda(w_{M(f,\mathcal{C})}) - \Lambda(w_{\mathcal{A}(m(f,\mathcal{C}))})]^{\frac{1}{2}} \\ ||G_g||_{\mathcal{C}} &\leq K\; \Lambda^{\frac{1}{4}}(w_{M(f,\mathcal{C})})[\Lambda(w_{M(g,\mathcal{C})}) - \Lambda(w_{\mathcal{A}(m(g,\mathcal{C}))})]^{\frac{1}{2}} \; . \end{aligned} \qquad \text{(IV.13)}$$

Indeed let us bound for instance the norm of F_f:

$$\begin{aligned} ||F_f||^2_{\mathcal{C}} &= \frac{1}{(2\pi)^2} \int d^3k\; \frac{[\chi^{\theta_{f,1}}_{\alpha_{j_{f,1}}}(\theta)]^2}{[k_0^2 + e^2(\vec{k})]^{\frac{1}{2}}} \left[u\left(\frac{k_0^2 + e^2(\vec{k})}{\Lambda^2(w_{M(f,\mathcal{C})})} \right) - u\left(\frac{k_0^2 + e^2(\vec{k})}{\Lambda^2(w_{\mathcal{A}(m(f,\mathcal{C}))})} \right) \right] \\ &= \int_{\Lambda^{-2}(w_{M(f,\mathcal{C})})}^{\Lambda^{-2}(w_{\mathcal{A}(m(f,\mathcal{C}))})} d\alpha \frac{1}{(2\pi)^2} \int d^3k\; [\chi^{\theta_{f,1}}_{\alpha_{j_{f,1}}}(\theta)]^2\; \left[-x^{\frac{1}{2}}\; u'[\alpha x] \right]\Big|_{x=k_0^2+e^2(\vec{k})} \\ &\leq \int_{\Lambda^{-2}(w_{M(f,\mathcal{C})})}^{\Lambda^{-2}(w_{\mathcal{A}(m(f,\mathcal{C}))})} d\alpha\; \frac{1}{\beta} |\mathrm{S}|\; \sup_{\mathrm{S}} \left[\chi^{\theta_{f,1}}_{\alpha_{j_{f,1}}}(\theta) \left(-x^{\frac{1}{2}} u'[\alpha x] \right)\Big|_{x=k_0^2+e^2(\vec{k})} \right] \\ &\leq K\Lambda^{\frac{1}{2}}(w_{M(f,\mathcal{C})}) \int_{\Lambda^{-2}(w_{M(f,\mathcal{C})})}^{\Lambda^{-2}(w_{\mathcal{A}(m(f,\mathcal{C}))})} d\alpha\; \alpha^{-\frac{3}{2}} \\ &\leq K\; \Lambda^{\frac{1}{2}}(w_{M(f,\mathcal{C})})\; [\Lambda(w_{M(f,\mathcal{C})}) - \Lambda(w_{\mathcal{A}(m(f,\mathcal{C}))})] \; , \end{aligned} \qquad \text{(IV.14)}$$

where K is some constant, S is the set in momentum space selected by the cut-offs χ and u', and we applied the bounds:

$$\begin{aligned} & [\chi^{\theta_{f,1}}_{\alpha_{j_{f,1}}}(\theta)]^2 \leq \chi^{\theta_{f,1}}_{\alpha_{j_{f,1}}}(\theta) \\ & \sup_{\mathrm{S}} \left[\chi^{\theta_{f,1}}_{\alpha_{j_{f,1}}}(\theta) \left(-x^{\frac{1}{2}} u'[\alpha x] \right)\Big|_{x=k_0^2+e^2(\vec{k})} \right] \leq\; K\alpha^{-\frac{1}{2}} \\ & |\mathrm{S}| \leq\; \beta\Lambda^{\frac{1}{2}}(w_{M(f,\mathcal{C})})\; \alpha^{-1} \; . \end{aligned} \qquad \text{(IV.15)}$$

Finally the loop determinant is bounded by

$$|\det \mathcal{M}'(\mathcal{C}, E, \{\theta_{a,1}\})| \leq K^n \prod_{a \in L}\; \Lambda^{\frac{1}{4}}(w_{M(a,\mathcal{C})})\; [\Lambda(w_{M(a,\mathcal{C})}) - \Lambda(w_{\mathcal{A}(m(a,\mathcal{C}))})]^{\frac{1}{2}} \; . \qquad \text{(IV.16)}$$

This bound no longer depends from $\{\theta_{a,r}\}$ or E.

IV.2 Spatial integrals

To perform spatial integration we use the decay of tree lines. The test functions are taken out of the integral and bounded by their L_∞ norm, except $\phi_1^{\Lambda_T}$ which is used to perform the integration over the root x_1.

$$\begin{aligned}
&\int d^3x_1...d^3x_n|\phi_1^{\Lambda_T}(x_{i_1},\theta_{e_1,1})|...|\phi_{2p}^{\Lambda_T}(x_{j_p},\theta_{e_{2p},1})|\prod_{q=1}^{n-1}|C^{w_q}(x_q,\bar{x}_q,\theta_{h_q,1})| \\
&\le ||\phi_1^{\Lambda_T}(\theta_{e_1,1})||_1 \prod_{i=2}^{2p}||\phi_i^{\Lambda_T}(\theta_{e_i,1})||_\infty \int d^3x_2...d^3x_n \prod_{q=1}^{n-1}|C^{w_q}(x_q,\bar{x}_q,\theta_{h_q,1})| \ .
\end{aligned} \tag{IV.17}$$

We now bound the norms of the test functions and the spatial decay of the tree propagators.

IV.2.1 Test functions

Each test function $\phi_i^{\Lambda_T}$ $(i=1,...2p)$ obeys the bound

$$\begin{aligned}
||\phi_i^{\Lambda_T}(\theta_{e_i,1})||_\infty &= \sup_x \left| \frac{1}{(2\pi)^2}\int d^3k\ e^{ikx}\left[\chi_{\theta_{e_i,1}}(\theta)\right]\hat{\phi}_i(k)\ [u(r/\Lambda_T^2)]\Big|_{r=k_0^2+e^2(|k|)} \right| \\
&\le \frac{1}{(2\pi)^2}\int d^3k\ \chi_{\theta_{e_i,1}}(\theta)\ \ \left|\hat{\phi}_i(k)\right|\ [u(r/\Lambda_T^2)]\Big|_{r=k_0^2+e^2(|k|)} \\
&\le ||\hat{\phi}_i||_\infty \frac{1}{(2\pi)^2}\int d^3k\ \chi_{\theta_{e_i,1}}(\theta)\ [u(r/\Lambda_T^2)]\Big|_{r=k_0^2+e^2(|k|)} \le K\ \Lambda_T^{\frac{5}{2}}\ ||\hat{\phi}_i||_\infty\ ,
\end{aligned} \tag{IV.18}$$

where in the third line we used $1/\alpha_{j_{e,1}} = \Lambda^2(w_{j_{e,1}}) = \Lambda_T^2\ \forall e$. For the test function hooked to the root, we need to perform a different bound. We write (recalling our convention (II.5) of integration, which includes that the imaginary time variable is integrated on a circle):

$$||\phi_1^{\Lambda_T}(\theta_{e_1,1})||_1 \ = \ \int d^3xd^3y\ \left|\phi_1(x,\theta_{e_1,1})\eta_{\theta_{e_1,1}}(x-y)\right| \tag{IV.19}$$

where we defined $\eta_{\theta_{e_1,1}}(x)$ as the Fourier transform of

$$\hat{\eta}_{\theta_{e_1,1}}(k) = \chi_{\theta_{e_1,1}}(\theta)\ [u(r/\Lambda_T^2)]\Big|_{r=k_0^2+e^2(|k|)} \tag{IV.20}$$

Lemma 5 *$\eta_{\theta_{e_1,1}}(z)$ decays as:*

$$|\eta_{\theta_{e_1,1}}(z)| \le K \sum_m \frac{\Lambda_T^{5/2}}{[1+\Lambda_T^2|z_0+2m\beta|^2+\Lambda_T^2|z_r|^2+\Lambda_T|z_t|^2]^2} \tag{IV.21}$$

where z_r and z_t are the radial and tangential components of $\vec{z}$ relative to the sector center $\theta_{e_1,1}$.

Proof This is a standard duality between direct and momentum space. However since the time variable is periodic we cannot immediately derive with respect to the 0-th component of the momentum. Instead we can derive first the decay of the $T=0$ analog $\eta^0_{\theta_{e_1,1}}(z)$ of the function $\eta_{\theta_{e_1,1}}(z)$. We write

$$\begin{aligned} F &= [1+\Lambda_T^2|z_0|^2+\Lambda_T^2|z_r|^2+\Lambda_T|z_t|^2]^2\ |\eta^0_{\theta_{e_1,1}}(z)| \\ &= \frac{1}{(2\pi)^2}\left|\int d^3k\ [1+\Lambda_T^2|z_0|^2+\Lambda_T^2|z_r|^2+\Lambda_T|z_t|^2]^2\ e^{ikz}\ \ \eta_{\theta_{e_1,1}}(k)\right| \\ &\le K\int d^3k\ \left|[1-\Lambda_T^2\partial_{k_0}^2-\Lambda_T^2\partial_{k_r}^2-\Lambda_T\partial_{k_t}^2]^2\ \ \eta_{\theta_{e_1,1}}(k)\right| \\ &\le K\ \Lambda_T^{\frac{5}{2}}\ . \end{aligned} \tag{IV.22}$$

Therefore

$$|\eta^0_{\theta_{e_1,1}}(z)| \le K\ \frac{\Lambda_T^{\frac{5}{2}}}{[1+\Lambda_T^2|z_0|^2+\Lambda_T^2|z_r|^2+\Lambda_T|z_t|^2]^2}\ .$$

Now applying (II.6) we can end the proof. ■

We introduce the spatial decay (IV.21) in (IV.19) to obtain:

$$\begin{aligned} &||\phi_1^{\Lambda_T}(\theta_{e_1,1})||_1 \le ||\phi_1^0(y)||_1 \\ &\sum_m \int_{-\frac{1}{T}}^{\frac{1}{T}} dz_0 \int d^2z\ \ \frac{K\Lambda_T^{\frac{5}{2}}}{(1+\Lambda_T^2|z_0+\frac{2m}{T}|^2+\Lambda_T^2|z_r|^2+\Lambda_T|z_t|^2)^2} \\ &= ||\phi_1^0(y)||_1 \int dz_0 \int d^2z\ \ \frac{K\Lambda_T^{\frac{5}{2}}}{(1+\Lambda_T^2|z_0|^2+\Lambda_T^2|z_r|^2+\Lambda_T|z_t|^2)^2} \le K||\phi_1(y)||_1. \end{aligned} \tag{IV.23}$$

where we performed the change of variable $z_0+\frac{m}{T}\to z_0$.

IV.2.2 Spatial decay of tree lines

We consider now tree line propagators and prove that they decay as Gevrey functions of class s where s is the Gevrey index of our initial cutoff u.

$$|C^{w_q}(\delta x_q, 0, \theta_{h_q,1})| \leq \qquad \text{(IV.24)}$$
$$K \, \frac{\Lambda_0^2 - \Lambda^2}{\Lambda^4(w_q)} \, \Lambda^{\frac{1}{2}}(w_q) \, \Lambda^3(w_q) \, e^{-a\left[|(\delta x_q)_0 \Lambda(w_q)|^{\frac{1}{s}} + |(\delta x_q)_r \Lambda(w_q)|^{\frac{1}{s}} + |(\delta x_q)_t \Lambda^{\frac{1}{2}}(w_q)|^{\frac{1}{s}}\right]}$$

where we applied translational invariance, $\delta x_q := x_q - \bar{x}_q$, $(\delta x_q)_r$ and $(\delta x_q)_t$ are the radial and tangential components of $\vec{x}$ relative to the sector center $\theta_{h,1}$, K and a are some positive constants. Remark that the smallest sector governs the spatial decay rate.

To prove this formula we study, as for the test function ϕ_1, the propagator at $T = 0$ $C_0^{w_q}$. Using the properties of Gevrey functions with compact support, $C_c^{w_q}$ satisfies (IV.24) too (see Appendix A). Then applying (II.6) achieves the proof of (IV.24).

IV.2.3 Bound

Now we can complete the bound on (IV.17). But before that, in order to extract the exponential decay between the test functions supports of Theorem 3, we take out a fraction $(1-\varepsilon)$ of the exponential decay of each tree line in (IV.24). This factor is bounded by

$$\prod_{q=1}^{n-1} e^{-a(1-\varepsilon)\left(|(\delta x_q)_0 \Lambda(w_q)|^{\frac{1}{s}} + |(\delta x_q)_r \Lambda(w_q)|^{\frac{1}{s}} + |(\delta x_q)_t \Lambda^{\frac{1}{2}}(w_q)|^{\frac{1}{s}}\right)} \leq e^{-a\,(1-\varepsilon)\,\Lambda_T^{\frac{1}{s}}\, d_T^{\frac{1}{s}}(\Omega_1,\ldots,\Omega_{2p})}. \qquad \text{(IV.25)}$$

We keep the remaining fraction ε of the decay to perform spatial integration:

$$\begin{aligned} & \int d^3x_2 \ldots d^3x_n \prod_{q=1}^{n-1} e^{-a\varepsilon\left(|(\delta x_q)_0 \Lambda(w_q)|^{\frac{1}{s}} + |(\delta x_q)_r \Lambda(w_q)|^{\frac{1}{s}} + |(\delta x_q)_t \Lambda^{\frac{1}{2}}(w_q)|^{\frac{1}{s}}\right)} \\ \leq \; & \prod_{q=1}^{n-1} \left[\int d^3x \; e^{-a\varepsilon\left(|x_0 \Lambda(w_q)|^{\frac{1}{s}} + |x_r \Lambda(w_q)|^{\frac{1}{s}} + |x_t \Lambda^{\frac{1}{2}}(w_q)|^{\frac{1}{s}}\right)} \right] \\ \leq \; & \prod_{q=1}^{n-1} \frac{1}{\Lambda^{\frac{5}{2}}(w_q)} \int d^3u \; e^{-a\varepsilon\left[u_0^{\frac{1}{s}} + u_1^{\frac{1}{s}} + u_2^{\frac{1}{s}}\right]} \; \leq \; K \prod_{q=1}^{n-1} \frac{1}{\Lambda^{\frac{5}{2}}(w_q)} \end{aligned} \qquad \text{(IV.26)}$$

and eq(IV.17) is bounded by

$$K_{\|\phi_1\|_1}\left[\prod_{i=2}^{2p}\|\hat\phi_i\|_\infty\right]\left(\Lambda_T^{\frac{5}{2}}\right)^{(2p-1)} e^{-a\,(1-\varepsilon)\,\Lambda_T^{\frac{1}{s}}\,d_T^{\frac{1}{s}}(\Omega_1,\dots,\Omega_{2p})}.\prod_{q=1}^{n-1}\frac{1}{\Lambda^3(w_q)}\ . \qquad \text{(IV.27)}$$

IV.3 Sector sum

We still have to perform the sums over sector choices:

$$\prod_{h\in L\cup\mathcal{T}_L\cup E}\left[\tfrac{4}{3}\Lambda^{-\frac{1}{2}}\left(w_{j_{h,n_h}}\right)\right]\int_0^{2\pi} d\theta_{h,n_h}\quad \left[\tfrac{4}{3}\Lambda^{-\frac{1}{2}}\left(w_{j_{h,n_h-1}}\right)\right]\int_{\Sigma_{j_{h,n_h}}} d\theta_{h,n_h-1}$$
$$\dots\quad\left[\tfrac{4}{3}\Lambda^{-\frac{1}{2}}\left(w_{j_{h,1}}\right)\right]\int_{\Sigma_{j_{h,2}}} d\theta_{h,1}$$
$$\prod_{\{g_i|i=r\ \text{ or }\ |eg_i(\mathcal{C})|\le 10\}}\Upsilon\left(\theta_i^{root}\{\theta_{h,r(i)}\}_{h\in eg_i^*}\right)\prod_v\Upsilon\left(\theta_{h_v^{root}},\{\theta_{h,n_h}\}_{h\in H^*(v)}\right)\ , \qquad \text{(IV.28)}$$

where the products $\left[\prod_{h\in\mathcal{T}_L\cup L\cup E}\prod_{r=2}^{n_h}\chi^{\theta_{h,1}}_{\alpha_{j_{h,r}}}(\theta_{h,r})\right]$ have been bounded by one.

We perform the sums for each half-line starting from the lowest scale $i(h)$ and going up towards the leaves (that means the vertices). The sum over the root sector is bounded by $\Lambda_T^{-\frac{1}{2}}$. The sums for different half-lines are mixed by the Υ function.

For any band i we consider the subgraph g_i. If $|eg_i(\mathcal{C})|\ge 11$ and $i\ne r$ there is no Υ function for this subgraph and only lines with $i(h)=\mathcal{A}(i)$ are refined. Hence we have to perform

$$\prod_{\substack{h\in eg_i^*(\mathcal{C})\\ j_{h,1}=i(h)=\mathcal{A}(i)}}\left\{\left[\tfrac{4}{3}\Lambda^{-\frac{1}{2}}\left(w_{j_{h,1}}\right)\right]\int_{\Sigma_{j_{h,2}}} d\theta_{h,1}\right\}1\le K^{\#\left\{\substack{h\in eg_i^*(\mathcal{C})\\ j_{h,1}=i(h)=\mathcal{A}(i)}\right\}}\prod_{\substack{h\in eg_i^*(\mathcal{C})\\ j_{h,1}=i(h)=\mathcal{A}(i)}}\frac{\Lambda^{\frac{1}{2}}(w_{j_{h,2}})}{\Lambda^{\frac{1}{2}}(w_{j_{h,1}})} \qquad \text{(IV.29)}$$

If $|eg_i(\mathcal{C})|\le 10$, or $i=r$ we have an Υ function expressing the momentum conservation at this subgraph, and all external fields have been refined. Each field $h\in eg_i$ except h_i^{root} is refined at the scale $\mathcal{A}(i)=j_{h,r(i)}$. Hence we have to perform

$$\prod_{h\in eg_i^*(\mathcal{C})}\left[\tfrac{4}{3}\Lambda^{-\frac{1}{2}}\left(w_{j_{h,r(i)}}\right)\right]\int_{\Sigma_{j_{h,r(i)+1}}} d\theta_{h,r(i)}\,\Upsilon\left(\theta_i^{root},\{\theta_{h,r(i)}\}_{h\in eg_i^*(\mathcal{C})}\right)\ . \qquad \text{(IV.30)}$$

We know that the function Υ reduces the size of the integrals to perform. Actually we can apply Lemma 6 below, which states that once the sectors for $|eg_i(\mathcal{C})|-2$ external lines have been fixed, the last two sectors are automatically fixed. This means that, since the sector θ_i^{root} is always fixed, we have to perform the sector sum only for $|eg_i(\mathcal{C})|-3$ external lines.

Lemma 6 *Let $\Sigma_i := (\alpha^{-1/4}, \theta_i^s)$ for $i=1,..l$ be a set of $l \geq 4$ sectors on the Fermi surface centered on θ_i^s of size $\alpha^{-1/4}$. Let the sector center θ_1^s be fixed, and the other sector centers θ_i^s vary over intervals Ω_i of the Fermi surface: $\theta_i^s \in \Omega_i$, for $i=2,..l$. We assume $|\Omega_i| > \alpha^{-1/4}$. We define the function $\Upsilon(\{\theta_i^s\})$ to be zero, unless there exist some set of momenta $\vec{k}_1, ...\vec{k}_l$ satisfying*

$$\sum_{i=1}^{l} \vec{k}_i = 0, \quad ; \quad ||\vec{k}_i| - 1| \leq 1/\sqrt{\alpha} \quad \forall i \quad ; \quad \vec{k}_i \in \Sigma_i \quad \forall i,$$

($\vec{k}_i \in \Sigma_i$ in radial coordinates means $|\theta_i - \theta_i^s| \leq \alpha^{-1/4}$).

Then the integral over $\theta_i^s \in \Omega_i$ of the Υ constraint is bounded by

$$\prod_{i=2}^{l} \left\{ \left[\tfrac{4}{3}\alpha^{\frac{1}{4}}\right] \int_{\Omega_i} d\theta_i^s \right\} \Upsilon(\{\theta_i^s\}_{i=1,..l}) \leq K^l \prod_{i \in I} \frac{|\Omega_i|}{\alpha^{-\frac{1}{4}}} \tag{IV.31}$$

where I is the subset of indices of the $l-3$ largest intervals among $\Omega_2, ...\Omega_l$.

Proof The proof is almost identical to the one of **Lemma** 3' in [FMRT1], but we include it in Appendix B for completeness.

∎

With these results we can bound the sum (IV.30) by

$$K^{|eg_i|} \prod_{h \in I(i)} \left(\frac{\Lambda^{\frac{1}{2}}\left(w_{j_{h,r(i)+1}}\right)}{\Lambda^{\frac{1}{2}}\left(w_{j_{h,r(i)}}\right)} \right) \tag{IV.32}$$

where we define $I(i)$ as the set of $|eg_i(\mathcal{C})|-3$ half-lines $h \in eg_i$, different from h_i^{root}, that have the largest sectors $\Sigma_{j_{h,r(i)+1}}$. For the particular case of g_r we have the bound

$$K^{2p} \prod_{e \in I(1)} \left(\frac{\Lambda^{\frac{1}{2}}\left(w_{j_{e,r(1)+1}}\right)}{\Lambda^{\frac{1}{2}}\left(w_{j_{e,r(1)}}\right)} \right) \tag{IV.33}$$

We still have to consider the sums over the largest sectors: they correspond to the vertices. Each vertex $v \in V$ can be treated as a subgraph with $|eg| \leq 10$, hence we can apply lemma 6 with $\Omega_i = [0, 2\pi]\ \forall i$, and obtain:

$$\prod_{v \in V} \prod_{h \in H^*(v)} \left\{ \left[\tfrac{4}{3} \Lambda^{-\frac{1}{2}} \left(w_{j_{h,n_h}} \right) \right] \int_0^{2\pi} d\theta_{h,n_h} \right\} \Upsilon \left(\theta_v^{root}, \{\theta_{h,n_h}\}_{h \in H^*(v)} \right)$$
$$\leq K^4 \Lambda^{-\frac{1}{2}} \left(w_{i_v} \right) \tag{IV.34}$$

(where inessential constants such as $|\Omega_i| = 2\pi$ are absorbed in a redefinition of K).

Remark that the refinement operations and the counting lemmas, also cost some constants. Hence we must check that:

Lemma 7 *The refinement and counting operations for tree and loop half-lines altogether at most cost K^n for some constant K.*

Proof At each band $b = i$ with $i \geq 1$ we consider the subgraph g_i (there is just one per band).

If $|eg_i(\mathcal{C})| \leq 10$ or $i = r$ we refine all external fields (tree, loop and real external), and we get a factor $K^{|eg_i(\mathcal{C})|} \leq K^{10}$.

If $|eg_i(\mathcal{C})| \geq 11$ and $i \neq r$ we just refine fields with $i(h) = \mathcal{A}(i)$ (there is no external field with $i(e) > 0$, hence they are are never refined in this case). On the whole we have to pay at most

$$\Big(\prod_{\substack{g_i | i = r \text{ or} \\ |eg_i(\mathcal{C})| \leq 10}} K^{10} \Big) K^{4n} \ , \tag{IV.35}$$

where the last factor comes from the finest refinement for each internal and external field (there are at most $4n$ such fields). Now $\#\{g_i : \ |eg_i(\mathcal{C})| \leq 10\} \leq n$. This ends the proof. ∎

IV.4 Main bound

With all these elements, we can bound the sum (IV.1):

$$|\Gamma^{\Lambda\Lambda_0}_{2p \text{ conv.}}| \leq e^{-a\ (1-\varepsilon)\ \Lambda_T^{\frac{1}{s}}\ d_T^{\frac{1}{s}}(\Omega_1, \ldots, \Omega_{2p})}$$

$$K_0\ ||\phi_1||_1 \prod_{i=2}^{2p} ||\hat\phi_i||_\infty \sum_{n=0}^{\infty} \frac{c^n}{n!} K^n \sum_{CTS} \sum_{u-\mathcal{T}} \sum_{\mathcal{L}} \sum_{\Omega\ E} \sum_{\mathcal{C}_c} \int_{w_T \le w_{\mathcal{A}(i)} \le w_i \le 1} \prod_{i=1}^{n-1} dw_i$$

$$\left[\Lambda^{\frac{5}{2}}(w_T)\right]^{(2p-1)} \prod_{j=1}^{n-1} \frac{1}{\Lambda^3(w_i)} \prod_{a\in L} \Lambda^{\frac{3}{4}}\left(w_{M(a,\mathcal{C})}\right) \left[1 - \frac{\Lambda\left(w_{\mathcal{A}(m(a,\mathcal{C}))}\right)}{\Lambda\left(w_{M(a,\mathcal{C})}\right)}\right]^{\frac{1}{2}}$$

$$\prod_{\substack{g_i,\\ i\neq r, |eg_i(\mathcal{C})|\ge 11}} \left[\prod_{\substack{h\in eg_i^*\\ j_{h,1}=i(h)=\mathcal{A}(i)}} \frac{\Lambda^{\frac{1}{2}}\left(w_{j_{h,2}}\right)}{\Lambda^{\frac{1}{2}}\left(w_{j_{h,1}}\right)}\right] \prod_{\substack{g_i|, i=r \text{ or}\\ |eg_i(\mathcal{C})|\le 10}} \left[\prod_{h\in I(i)} \frac{\Lambda^{\frac{1}{2}}\left(w_{j_{h,r(i)+1}}\right)}{\Lambda^{\frac{1}{2}}\left(w_{j_{h,r(i)}}\right)}\right]$$

$$\Lambda_T^{-\frac{1}{2}} \prod_{v\in V} \Lambda^{-\frac{1}{2}}\left(w_{i_v}\right) \tag{IV.36}$$

where we have bounded $|\lambda| \le c$. Now we can send Λ to zero, hence $\Lambda(w) = \sqrt{w}$ as $\Lambda_0 = 1$. The equation becomes

$$|\Gamma^{\Lambda_0}_{2p\ \text{conv.}}| \le K_0\ e^{-a\ (1-\varepsilon)\ \Lambda_T^{\frac{1}{s}}\ d_T^{\frac{1}{s}}(\Omega_1,\dots,\Omega_{2p})} ||\phi_1||_1 \prod_{i=2}^{2p} ||\hat\phi_i||_\infty \sum_{n=0}^{\infty} \frac{c^n}{n!} K^n \tag{IV.37}$$

$$\sum_{CTS} \sum_{u-\mathcal{T}} \sum_{\mathcal{L}} \sum_{\Omega\ E} \sum_{\mathcal{C}_c} \int_{w_T \le w_{\mathcal{A}(i)} \le w_i \le 1} \prod_{i=1}^{n-1} dw_i \prod_{i=1}^{n-1} w_i^{-\frac{3}{2}} \prod_{a\in L} w^{\frac{3}{8}}_{M(a,\mathcal{C})} \prod_{v\in V} w_{i_v}^{-\frac{1}{4}}$$

$$\prod_{\substack{g_i\\ i\neq r||eg_i(\mathcal{C})|\ge 11}} \left[\prod_{\substack{h\in et_i^*\cup el_i\\ j_{h,1}=i(h)=\mathcal{A}(i)}} \frac{w^{\frac{1}{4}}_{j_{h,2}}}{w^{\frac{1}{4}}_{j_{h,1}}}\right] \prod_{\substack{g_i|, i=r \text{ or}\\ |eg_i(\mathcal{C})|\le 10}} \left[\prod_{h\in I(i)} \frac{w^{\frac{1}{4}}_{j_{h,r(i)+1}}}{w^{\frac{1}{4}}_{j_{h,r(i)}}}\right] w_T^{\frac{5p}{2}-\frac{3}{2}}$$

where we have bounded $\left[1 - \frac{\Lambda(w_{\mathcal{A}(m(a,\mathcal{C}))})}{\Lambda(w_{M(a,\mathcal{C})})}\right]^{\frac{1}{2}}$ by one.

To factorize the integrals we perform the change of variable:

$$w_i = \frac{1}{\beta_i} w_{\mathcal{A}(i)} \qquad 1 \le i \le n-1. \tag{IV.38}$$

By (III.9) we have the following bound for β_i

$$\beta_i \in \left[\frac{w_{\mathcal{A}(i)}}{\min[w_{i'}, w_{i''}]}, 1\right] \tag{IV.39}$$

Now each w_i can be written

$$w_i = \left[\prod_{j\in C_i} \frac{1}{\beta_j}\right] w_T \tag{IV.40}$$

where we defined C_i as the set of crosses on the chain joining the cross i to the root. The Jacobian of this transformation is the determinant of the matrix

$$M_{ij} = \frac{\partial w_i}{\partial \beta_j} = -\frac{1}{\beta_j} w_i \ \chi(j \in C_i)$$

where $\chi(j \in C_i) = 1$ if $j \in C_i$ and 0 otherwise. If we order the rows and columns of M_{ij} putting the root first, then the first layer of the CTS and so on, we see that M_{ij} is a triangular matrix, hence its determinant is given by:

$$|Jac| = \left| \prod_{i=1}^{n-1} \frac{\partial w_i}{\partial \beta_i} \right| = w_T^{n-1} \prod_{i=1}^{n-1} \left[\frac{1}{\beta_i} \prod_{j \in C_i} \frac{1}{\beta_j} \right] = w_T^{n-1} \prod_{i=1}^{n-1} \left[\frac{1}{\beta_i} \left(\frac{1}{\beta_i} \right)^{n_i - 1} \right] \tag{IV.41}$$

where n_i is the number of vertices in the subgraph g_i. Indeed β_i appears in the chain C_j exactly for all $j \geq_P i$, hence its exponent is the number of crosses above i, which is the number of tree lines in g_i, hence $n_i - 1$ if we denote the number of vertices in g_i by n_i. In these new coordinates we have:

$$|\Gamma^{\Lambda_0}_{2p\,\mathrm{conv.}}| \leq K_0 \ ||\phi_1||_1 \prod_{i=2}^{2p} ||\hat{\phi}_i||_\infty e^{-a\,(1-\varepsilon)\,\Lambda_T^{\frac{1}{s}}\, d_T^{\frac{1}{s}}(\Omega_1,\dots,\Omega_{2p})} \sum_{n=0}^{\infty} \frac{c^n}{n!} K^n$$
$$\sum_{CTS} \sum_{u-\mathcal{T}} \sum_{\mathcal{L}} \sum_{\Omega} \sum_{E} \sum_{\mathcal{C}_c} \int_{w_T}^{1} \prod_{i=1}^{n-1} d\beta_i \ w_T^{n-1} \prod_{i=1}^{n-1} \beta_i^{-1+(1-n_i)} \prod_{i=1}^{n-1} \left[\left(\prod_{j \in C_i} \beta_i^{\frac{3}{2}} \right) w_T^{-\frac{3}{2}} \right]$$
$$\prod_{a \in L} \left[\left(\prod_{j \in C_{M(a,\mathcal{C})}} \beta_j^{-\frac{3}{8}} \right) w_T^{\frac{3}{8}} \right] \prod_{\substack{g_i\, i \neq r \\ |eg_i(\mathcal{C})| \geq 11}} \left[\prod_{\substack{h \in eg_i^* \\ j_{h,1} = i(h) = \mathcal{A}(i)}} \left(\prod_{j \in C_{r(i)+1} \setminus C_{r(i)}} \frac{1}{\beta_j^{\frac{1}{4}}} \right) \right]$$
$$\prod_{\substack{g_i|, i=r \text{ or} \\ |eg_i(\mathcal{C})| \leq 10}} \left[\prod_{h \in I(i)} \left(\prod_{j \in C_{r(i)+1} \setminus C_{r(i)}} \frac{1}{\beta_j} \right)^{\frac{1}{4}} \right] \prod_{v \in V} \left[\left(\prod_{j \in C_{i_v}} \beta_j \right)^{\frac{1}{4}} w_T^{-\frac{1}{4}} \right] w_T^{\frac{5p}{2} - \frac{3}{2}} \ , \tag{IV.42}$$

where we have taken as integration domain for all β_i the interval $[w_T, 1]$, that contains the exact integration domain, since $\frac{w_{\mathcal{A}(i)}}{\min[w_{i'}, w_{i''}]} \geq \frac{w_{\mathcal{A}(i)}}{1} \geq w_T$. We write the integrals over the different β_i as a product $\prod_{i=1}^{n-1} \int_{w_T}^{1} d\beta_i \ \beta_i^{-1+x_i}$. We have to find out the expression for x_i. We observe that

$$\prod_{i=1}^{n-1} \left(\prod_{j \in C_i} \beta_i^{\frac{3}{2}} \right) = \prod_{i=1}^{n-1} \beta_i^{\frac{3}{2}(n_i - 1)}$$

$$\prod_{a\in L}\left(\prod_{j\in C_{M(a,\mathcal{C})}}\beta_j^{-\frac{3}{8}}\right)=\prod_{i=1}^{n-1}\beta_i^{-\frac{3}{8}\ \#\{a\in L|M(a,\mathcal{C})\geq_P i\}}=\prod_{i=1}^{n-1}\beta_i^{-\frac{3}{8}|il_i(\mathcal{C})|}$$
$$\prod_{v\in V}\left(\prod_{j\in C_{i_v}}\beta_j\right)^{\frac{1}{4}}=\prod_{i=1}^{n-1}\beta_i^{\frac{1}{4}\#\{v\in V|i_v\geq_P i\}}=\prod_{i=1}^{n-1}\beta_i^{\frac{1}{4}n_i}\ ,$$

and the remaining products over sector attributions are equal to $\prod_{i=1}^{n-1}\beta_i^{-y_i}$ where

$$\begin{aligned} y_i &= \frac{1}{4}(|eg_i(\mathcal{C})|-3) \quad \text{if} \quad |eg_i(\mathcal{C})|\leq 10 \text{ or } i=r \\ &\leq \frac{1}{4}(|eg_i(\mathcal{C})|-1) \quad \text{if} \quad |eg_i(\mathcal{C})|>10\ . \end{aligned} \tag{IV.43}$$

To obtain this bound we observe that the factor β_i appears in the product with a power $-1/4$ each time there is a half-line $h\in\mathcal{T}_L\cup L\cup E$ with

$$i\in C_{r(i)+1}\backslash C_{r(i)}$$

for some r and the corresponding factor appears in the sector counting. Now, for each subgraph g_i we have three situations

- $|eg_i(\mathcal{C})|=2$: then the factor β_i does not appear, i.e. $y_i=0$.
- $4\leq|eg_i(\mathcal{C})|\leq 10$, hence all external half-lines except h^{root} are refined and the factor β_i appears with power $-1/4(|eg_i(\mathcal{C})|-3)$.
- $|eg_i(\mathcal{C})|>10$: only some of the external lines of g_i (other than h^{root}) are refined; therefore the factor β_i appears with power $-\frac{1}{4}a_i$ where $a_i\leq(|eg_i(\mathcal{C})|-1)$ is the number of external half-lines refined. This is why (IV.43) is a bound and not an equality.

Now we can bound (IV.42) (using that $|L|=2(n+1-p)$):

$$\begin{aligned} |\Gamma^{\Lambda_0}_{2p\,\text{conv.}}| \ &\leq\ K_0\ ||\phi_1||_1\prod_{i=2}^{2p}||\hat\phi_i||_\infty e^{-a\,(1-\varepsilon)\,\Lambda_T^{\frac{1}{s}}\,d_T^{\frac{1}{s}}(\Omega_1,\dots,\Omega_{2p})}w_T^{\frac{7p}{4}-\frac{1}{4}} \\ &\quad \sum_{n=0}^{\infty}\frac{c^n}{n!}K^n\sum_{CTS}\sum_{u-\mathcal{T}}\sum_{\mathcal{L}}\sum_{\Omega}\sum_{E}\sum_{\mathcal{C}_c}\prod_{i=1}^{n-1}\int_{w_T}^1 d\beta_i\ \beta_i^{-1+x_i} \end{aligned} \tag{IV.44}$$

where

$$x_i = \frac{1}{2}(n_i - 1) - \frac{3}{8}|il_i(\mathcal{C})| + \frac{1}{4}n_i - \frac{1}{4}(|eg_i(\mathcal{C})| - 3) \tag{IV.45}$$

when $i = 1$, or $4 \leq |eg_i(\mathcal{C})| \leq 10$ and

$$x_i \geq \frac{1}{2}(n_i - 1) - \frac{3}{8}|il_i(\mathcal{C})| + \frac{1}{4}n_i - \frac{1}{4}(|eg_i(\mathcal{C})| - 1)) \tag{IV.46}$$

when $|eg_i(\mathcal{C})| > 10$. The integrals over β_i are well defined only if $x_i > 0\ \forall i$. To check that it is true, we observe that

$$\frac{1}{2}(n_i - 1) - \frac{3}{8}|il_i(\mathcal{C})| + \frac{1}{4}n_i = \frac{1}{8}(3|eg_i(\mathcal{C})| - 10) \tag{IV.47}$$

where we applied the relation

$$|il_i(\mathcal{C})| = 2n_i + 2 - |eg_i(\mathcal{C})|.$$

Hence, for $i = r$, or $4 \leq |eg_i(\mathcal{C})| \leq 10$ we have

$$x_i = \frac{1}{8}(3|eg_i(\mathcal{C})| - 10) - \frac{1}{4}(|eg_i(\mathcal{C})| - 3) = \frac{1}{8}(|eg_i(\mathcal{C})| - 4) \tag{IV.48}$$

and when $|eg_i(\mathcal{C})| > 10$ (and $i \neq r$) we have

$$x_i \geq \frac{1}{8}(3|eg_i(\mathcal{C})| - 10) - \frac{1}{4}(|eg_i(\mathcal{C})| - 1) = \frac{1}{8}(|eg_i(\mathcal{C})| - 8) \geq \frac{1}{2} \tag{IV.49}$$

by construction. Remark that since the lowest subgraph g_r has no tree external line we can compute explicitly

$$x_r = \frac{1}{8}(|eg_r(\mathcal{C})| - 4) = \frac{1}{8}(2p - 4)\ . \tag{IV.50}$$

If $|eg_i(\mathcal{C})| = 4$, $x_i = 0$ and the graph is logarithmic in the temperature:

$$\int_{w_T}^{1} d\beta_i\ \beta_i^{-1} = -\log w_T = 2\ \left|\log(\sqrt{2}\pi T)\right| \tag{IV.51}$$

Finally if $|eg_i(\mathcal{C})| = 2$, then

$$x_i = \frac{1}{8}(3|eg_i(\mathcal{C})| - 10) = -\frac{1}{2} \tag{IV.52}$$

and the integral over β_i is linearly divergent with the temperature T:

$$\int_{w_T}^{1} d\beta_i \, \beta_i^{-1} = 2 \, (w_T^{-\frac{1}{2}} - 1) = 2 \, \left(\frac{1}{\sqrt{2}\pi T} - 1 \right) . \tag{IV.53}$$

Hence we have recovered the well known fact that the only divergent subgraphs are the four points and two points subgraphs [FT1-2]-[FMRT1]. In this paper we restrict ourselves to convergent attributions, for which x_i is always positive. However it is important (in order to bound later the sum over labelings) that we check that we have a lower bound on x_i which is proportional to the number of external tree lines of g_i:

Lemma 8 *For any subgraph g_i ($i \neq r$) we have*

$$x_i \geq \frac{|et_i|}{72} > 0 \; . \tag{IV.54}$$

Proof We distinguish several cases:

- if $|eg_i| \leq 10$

$$\frac{1}{8}(|eg_i(\mathcal{C})| - 4) \geq \frac{1}{4} > 0 \tag{IV.55}$$

  as for convergent attributions $|eg_i(\mathcal{C})| \geq 6$ (we cannot have $|eg_i(\mathcal{C})| = 5$ by parity). Now, if $|et_i| \geq 5$ we have

$$\frac{1}{8}(|eg_i(\mathcal{C})| - 4) \geq \frac{1}{8}(|et_i| - 4) \geq \frac{1}{5 \cdot 8}|et_i| \; . \tag{IV.56}$$

  If $|et_i| \leq 4$ we can write

$$\frac{1}{8}(|eg_i(\mathcal{C})| - 4) \geq \frac{1}{4} \geq \frac{1}{16}|et_i| \; . \tag{IV.57}$$

- if $|eg_i| > 10$ we have

$$\frac{1}{8}(|eg_i(\mathcal{C})| - 8) \geq \frac{1}{2} > 0 \; . \tag{IV.58}$$

  Repeating the same arguments as before for the case $|et_i| \geq 9$ and $|et_i| < 9$ we obtain

$$\frac{1}{8}(|eg_i(\mathcal{C})| - 8) \geq \frac{1}{8 \cdot 9}|et_i| \; . \tag{IV.59}$$

This completes the proof of the Lemma ■

Now we can perform the integrals on the β_i, to obtain

$$\begin{aligned} |\Gamma^{\Lambda_0}_{2p>4\text{ conv.}}| &\leq K_0 \; ||\phi_1||_1 \prod_{i=2}^{2p} ||\hat\phi_i||_\infty \; e^{-a\,(1-\varepsilon)\,\Lambda_T^{\frac{1}{s}}\; d_T^{\frac{1}{s}}(\Omega_1,\dots,\Omega_{2p})} \\ & w_T^{\frac{7p}{4}-\frac{1}{4}} \frac{1}{2p-4} \sum_{n=0}^{\infty} \frac{c^n}{n!} K^n \sum_{CTS} \sum_{u-\mathcal{T}} \sum_{\mathcal{L}} \sum_{\Omega} \sum_{E} \sum_{\mathcal{C}_c} \prod_{i\neq r} \frac{1}{|et_i|} \;, \end{aligned} \quad \text{(IV.60)}$$

where the factor $\prod_i (1 - w_T^{x_i})$ coming from the integrals over the variables β_i has been bounded by one. For the particular case of four point and two point vertex functions we have

$$\begin{aligned} |\Gamma^{\Lambda_0}_{4\text{ conv.}}| &\leq K_0 \; ||\phi_1||_1 \prod_{i=2}^{2p} ||\hat\phi_i||_\infty \; e^{-a\,(1-\varepsilon)\,\Lambda_T^{\frac{1}{s}}\; d_T^{\frac{1}{s}}(\Omega_1,\dots,\Omega_{2p})} \\ & w_T^{\frac{13}{4}} |\log w_T| \sum_{n=0}^{\infty} \frac{c^n}{n!} K^n \sum_{CTS} \sum_{u-\mathcal{T}} \sum_{\mathcal{L}} \sum_{\Omega} \sum_{E} \sum_{\mathcal{C}_c} \prod_{i\neq r} \frac{1}{|et_i|} \;. \end{aligned} \quad \text{(IV.61)}$$

$$\begin{aligned} |\Gamma^{\Lambda_0}_{2\text{ conv.}}| &\leq K_0 \; ||\phi_1||_1 \prod_{i=2}^{2p} ||\hat\phi_i||_\infty \; e^{-a\,(1-\varepsilon)\,\Lambda_T^{\frac{1}{s}}\; d_T^{\frac{1}{s}}(\Omega_1,\dots,\Omega_{2p})} \\ & w_T \sum_{n=0}^{\infty} \frac{c^n}{n!} K^n \sum_{CTS} \sum_{u-\mathcal{T}} \sum_{\mathcal{L}} \sum_{\Omega} \sum_{E} \sum_{\mathcal{C}_c} \prod_{i\neq r} \frac{1}{|et_i|} \;. \end{aligned} \quad \text{(IV.62)}$$

The sum $\sum_{\mathcal{C}_c}$ is over a set whose cardinal is bounded by K^n so we can bound it with the supremum over the set. The sum over Ω runs over a set of at most 2^{n-1} elements. The sum over E to attribute the $2p$ external lines to particular vertices runs over a set of at most n^{2p} (this is an overestimate!). Hence

$$\sum_{\mathcal{C}_c} \sum_{\Omega} \sum_{E} |F(\mathcal{C}_c, \Omega, E)| \leq (p!)^2 K^n \sup_{\mathcal{C}_c,\Omega,E} |F(\mathcal{C}_c, \Omega, E)| \;,$$

where we applied the bound

$$n^{2p} \leq (2p)! e^n \leq K^p \; (p!)^2 e^n \quad \forall n \geq 0 \;.$$

We still have to perform the sum over the CTS and $\mathcal{L}$. For each cross x of the CTS different from the root, there is one line ℓ_x^0 going down (towards the root), and two lines ℓ_x^1 and ℓ_x^2 going up (see Fig.4).

Lemma 9 *For any cross x different from the root :*

$$N_{\ell_x^0}(\mathcal{T},\mathcal{L}) = N_{\ell_x^1}(\mathcal{T},\mathcal{L}) + N_{\ell_x^2}(\mathcal{T},\mathcal{L}) - 2 \quad \text{(IV.63)}$$

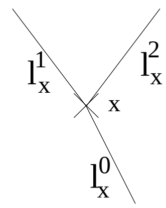

Figure 4: definition of ℓ_x^0,ℓ_x^1, ℓ_x^2

Proof: The clusters $\mathcal{T}_{\ell_x^1}(\mathcal{L})$ and $\mathcal{T}_{\ell_x^2}(\mathcal{L})$ are joined by a single line in the tree $\mathcal{T}$, which is the label of the cross x. This line is counted once as external line of $\mathcal{T}_{\ell_x^1}(\mathcal{L})$ and once as external line of $\mathcal{T}_{\ell_x^2}(\mathcal{L})$, and is no longer an external line of $\mathcal{T}_{\ell_x^0}(\mathcal{L})$. This proves the lemma.

■

The following Lemma is an improved version of Lemmas B4-B5 in [CR] (see also Lemma III.6), adapted to this formalism of relative rather than total orderings.

Lemma 10 *Let CTS be a fixed Clustering Tree Structure of order n. We have*

$$\sum_{\mathcal{T}} \sum_{\mathcal{L}} \frac{1}{n\ !} \prod_{\ell} \frac{1}{N_\ell(\mathcal{T},\mathcal{L})} \leq 4^n \ . \qquad \text{(IV.64)}$$

Proof: We decompose the sum over $\mathcal{T}$ and $\mathcal{L}$ into subsums. We call $\mathcal{L}_o$ the map which associates the dots of CTS to the vertices of $\mathcal{T}$ and $\mathcal{L}_x$ the map which associates the crosses of CTS to the lines of $\mathcal{T}$. By the previous lemma, once $\mathcal{L}_o$ and the collection $V = \{N_v\}$ of coordination numbers for each vertex v of $\mathcal{T}$ is given, the numbers $N_\ell(\mathcal{T},\mathcal{L})$ are all fixed, hence they do not depend on the particular contractions W and on $\mathcal{L}_x$. This suggests to split the sum over $\mathcal{T}$ and $\mathcal{L}$ as a sum over W and $\mathcal{L}_x$ followed by a sum over V and $\mathcal{L}_o$:

$$\sum_{\mathcal{T}} \sum_{\mathcal{L}} \frac{1}{n!} \prod_{\ell} \frac{1}{N_\ell(\mathcal{T},\mathcal{L})} = \sum_{V} \sum_{\mathcal{L}_o} \frac{1}{n!} \prod_{\ell} \frac{1}{N_\ell(V,\mathcal{L}_o)} \sum_{\mathcal{L}_x} \sum_{W} 1 \ . \qquad \text{(IV.65)}$$

But the number of labelings $\mathcal{L}_x$ and contractions W compatible with given V and $\mathcal{L}_o$ is precisely $\prod_\ell N_\ell(V,\mathcal{L}_o)$. Indeed starting from the n dots in CTS with their N_v hooked fields, and going down towards the root we can inductively build the contractions corresponding to each cross of CTS (this builds at the same time W and $\mathcal{L}_x$). To count the possible contractions for a cross x, we have to choose one external field in $T_{\ell_x^1}$ and one in $T_{\ell_x^2}$, hence the number of choices is *exactly* $N_{\ell_x^1}(V,\mathcal{L}_o)N_{\ell_x^2}(V,\mathcal{L}_o)$, where ℓ_x^1 and ℓ_x^2 were introduced in the previous lemma. Multiplying over all crosses, we get :

$$\sum_{\mathcal{T}}\sum_{\mathcal{L}}\frac{1}{n\,!}\prod_{\ell}\frac{1}{N_\ell(V,\mathcal{L}_o)}=\sum_{V}\sum_{\mathcal{L}_o}\frac{1}{n\,!}=\sum_{V}1\leq 4^n\ . \tag{IV.66}$$

Indeed $n!$ is exactly the number of labelings $\mathcal{L}_o$ of the dots of CTS, and for each vertex v N_v is an integer between 1 and 4, hence the sum over V is bounded by 4^n (this is an upper bound since we do not take into account the constraint $\sum_v N_v = 2n-2$). ■

Applying the lemma above, we bound

$$\frac{1}{n!}\sum_{u-\mathcal{T}}\sum_{\mathcal{L}}\prod_{i\neq r}\frac{1}{|et_i|}\leq 4^n\ . \tag{IV.67}$$

Hence the vertex function is bounded by

$$\begin{aligned}|\Gamma^{\Lambda_0}_{2p>4\text{ conv.}}| \quad\leq\quad & K_0\ ||\phi_1||_1\prod_{i=2}^{2p}||\hat\phi_i||_\infty e^{-a\,(1-\varepsilon)\,\Lambda_T^{\frac{1}{s}}\,d_T^{\frac{1}{s}}(\Omega_1,\ldots,\Omega_{2p})}\\ & \frac{w_T^{\frac{7p}{4}-\frac{1}{4}}}{2p-4}\ K_1^p\ \ (p!)^2\sum_{n=0}^{\infty}c^nK_2^n\end{aligned} \tag{IV.68}$$

$$|\Gamma^{\Lambda_0}_{4\text{ conv.}}|\leq K_0'\ ||\phi_1||_1\prod_{i=2}^{2p}||\hat\phi_i||_\infty\ e^{-a\,(1-\varepsilon)\,\Lambda_T^{\frac{1}{s}}\,d_T^{\frac{1}{s}}(\Omega_1,\ldots,\Omega_{2p})}\ w_T^{\frac{13}{4}}\ \sum_{n=0}^{\infty}c^nK_2^n \tag{IV.69}$$

$$|\Gamma^{\Lambda_0}_{2\text{ conv.}}|\leq K_0''\ ||\phi_1||_1\prod_{i=2}^{2p}||\hat\phi_i||_\infty\ e^{-a\,(1-\varepsilon)\,\Lambda_T^{\frac{1}{s}}\,d_T^{\frac{1}{s}}(\Omega_1,\ldots,\Omega_{2p})}\ w_T\ \sum_{n=0}^{\infty}c^nK_2^n \tag{IV.70}$$

for some constant K_2. This is convergent for $c<\frac{1}{K_2}$ and achieves the proof of Theorem 1 and 2. (Remark that we did not try to optimize the dependence of this bound in $2p$, the number of external points).

A Spatial decay

We prove that the $T=0$ propagator C_0 decays as

$$\begin{aligned}&|C_0^{w_q}(x,0,\theta_{h_q,1})|\leq\\ &\quad K\ \frac{\Lambda_0^2-\Lambda^2}{\Lambda^4(w_q)}\ \Lambda^{\frac{1}{2}}(w_q)\ \Lambda^3(w_q)\ e^{-a\left[|x_0\Lambda(w_q)|^{\frac{1}{s}}+|x_r\Lambda(w_q)|^{\frac{1}{s}}+|x_t\Lambda^{\frac{1}{2}}(w_q)|^{\frac{1}{s}}\right]}\ .\end{aligned} \tag{A.1}$$

Lemma 11 *Let $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ be such that its Fourier transform $\hat{f}$ has compact support of volume V_f and satisfies*

$$||\hat{f}^{(n_1,\dots,n_d)}||_\infty := \left|\left|\frac{\partial^{n_1}}{\partial p_1^{n_1}}\cdots\frac{\partial^{n_d}}{\partial p_d^{n_d}}\hat{f}\right|\right|_\infty \leq A_0 \prod_{i=1}^{d} \left[(\alpha_i C)^{n_i}\ \ (n_i!)^s\right], \tag{A.2}$$

where A_0, C, α_1,...α_d are some constants and $s \geq 1$ is some constant.

Then for some constants K, μ and a, one has

$$|f(x)| \leq K\ A_0\ V_f\ e^{-a\sum_{i=1}^{d}\left|\frac{x_i}{\alpha_i}\right|^{1/s}} \quad \forall x \in \mathbb{R}^d\ . \tag{A.3}$$

Proof. By Stirling's formula the first equation can be written

$$||\hat{f}^{(n_1,\dots,n_d)}||_\infty \leq A_0\ K\ \prod_{i=1}^{d}\left[\left(\frac{\alpha_i}{\mu}\right)^{n_i}\ \left(\frac{n_i}{e}\right)^{n_i s}\right] \tag{A.4}$$

where K and μ are some constants (eventually dependent from d). Hence, for any x we have:

$$\begin{aligned} |f(x)| &= \left|\frac{1}{(ix_1)^{n_1}\dots(ix_d)^{n_d}}\int e^{-ipx}\ \hat{f}^{(n_1,\dots,n_d)}(p)\right| \\ &\leq \frac{||\hat{f}^{(n_1,\dots,n_d)}||_\infty\ V_f}{|x_1|^{n_1}\dots|x_d|^{n_d}} \leq V_f\ K\ A_0\ \prod_{i=1}^{d}\left[\left|\frac{\alpha_i}{\mu x_i}\right|^{n_i}\ \left(\frac{n_i}{e}\right)^{n_i s}\right]\ . \end{aligned} \tag{A.5}$$

Optimizing to $n_i = \left|\frac{\mu x_i}{\alpha_i}\right|^{\frac{1}{s}}$, we obtain

$$|f(x)| \leq V_f\ A_0\ K\ \prod_{i=1}^{d} e^{-s\left|\frac{\mu x_i}{\alpha_i}\right|^{\frac{1}{s}}} \tag{A.6}$$

which ends the proof of the lemma, with $a = s\mu^{\frac{1}{s}}$. ∎

Lemma 12 $C_0^{w_q}$ *satisfies (A.1).*

Proof To prove (A.1) we write in momentum space:

$$|C_0^{w_q}(k_r,k_t,\theta_{h_q,1})| = \left|C_0^{w_q}(k)\chi^{\theta_{h_q,1}}_{\alpha_{j_{h_q,1}}}[\theta(k_r,k_t)]\right| = \frac{(\Lambda_0^2-\Lambda^2)}{\Lambda^4(w_q)}\left|C_0^{w_q,\theta_{h_q,1}}(k_0,k_r,k_t)\right| \tag{A.7}$$

where the radial and tangential variables k_r and k_t are defined by:

$$k_r = |\vec{k}|\cos(\theta-\theta_{h_q,1}) - 1 \quad ; \quad k_t = |\vec{k}|\sin(\theta-\theta_{h_q,1}) \ . \tag{A.8}$$

The function to study is (since $j_{h_q,1}=q$):

$$C_0^{w_q,\theta_{h_q,1}}(k_0,k_r,k_t) = u_p\left[\alpha_q^{1/4}(\theta-\theta_{h_q,1})\right] \ \ [ik_0+e(|\vec{k}|)]\ u'[\alpha_q(k_0^2+e^2(|\vec{k}|)], \tag{A.9}$$

and

$$\begin{aligned}
\theta-\theta_{h_q,1} &= f_1(k_r,k_t) = \arctan\frac{k_t}{1+k_r} \\
e(|\vec{k}|) &= |\vec{k}|^2-1 = f_2(k_r,k_t) = k_r^2+k_t^2+2k_r
\end{aligned} \tag{A.10}$$

The propagator $C_0^{w_q,\theta_{h_q,1}}$ can be written as the product of three functions

$$C_0^{w_q,\theta_{h_q,1}}(k_0,k_r,k_t) = F_1(k_r,k_t)\ F_2(k_0,k_r,k_t)\ F_3(k_0,k_r,k_t) \tag{A.11}$$

where

$$\begin{aligned}
F_1(k_r,k_t) &:= u_p\left(\alpha_q^{\frac{1}{4}}f_1(k_r,k_t)\right) \\
F_2(k_0,k_r,k_t) &:= [ik_0+f_2(k_r,k_t))] \\
F_3(k_0,k_r,k_t) &:= u'[\alpha_q(k_0^2+f_2^2(k_r,k_t))] \ ,
\end{aligned} \tag{A.12}$$

f_1, f_2 being defined in (A.10).

Now we know that $u(x)$ is a Gevrey function of class s, with compact support on $\left[-\frac{1}{2},\frac{1}{2}\right]$. The function f_1 takes values in the interval $\left[-\frac{\alpha_q^{-1/4}}{2},\frac{\alpha_q^{-1/4}}{2}\right]$, hence $k_r \in \left[-\frac{\alpha_q^{-\frac{1}{2}}}{2},\frac{\alpha_q^{-\frac{1}{2}}}{2}\right]$, $k_t \in \left[-\frac{\alpha_q^{-1/4}}{2},\frac{\alpha_q^{-1/4}}{2}\right]$. By hand or using the standard rules for derivation, product and composition of Gevrey functions (see

[G]) it is then easy to check that $C_0^{w_q,\theta_{h_q,1}}(k_0,k_r,k_t)$ is a Gevrey function with compact support of class s and satisfies the bound:

$$\left\|\frac{\partial^{n_0}}{\partial k_0^{n_0}}\frac{\partial^{n_r}}{\partial k_r^{n_r}}\frac{\partial^{n_t}}{\partial k_t^{n_t}}C_c^{w_q,\theta_s}\right\|_\infty \leq \frac{1}{\sqrt{\alpha_q}}C_0^{n_0+n_t+n_t}\ \left(\alpha_q^{\frac{1}{2}}\right)^{n_r+n_0}\ \left(\alpha_q^{\frac{1}{4}}\right)^{n_t}\ (n_0!n_r!n_t!)^s\ . \tag{A.13}$$

Hence, applying Lemma 2, with $A_0 = 1/\alpha_q^{\frac{1}{2}}$ and $V_f = \Lambda^{\frac{1}{2}}(w_q)\Lambda^2(w_q)$, proves (A.1). ■

B Proof of the Sector Counting Lemma 6

We define $\vec{k}_i'$ as the projection of $\vec{k}_i$ on the Fermi surface $\vec{k}_i' = \vec{k}_i/|\vec{k}_i|$ and $\vec{r}_i$ as the center of the sector Σ_i, with components $(1,\theta_i^s)$ in radial coordinates. Then, as in [FMRT1], we renumber $\vec{k}_2,...\vec{k}_l$ so that $|\vec{r}_l\cdot\vec{r}_{l-1}|$ is the minimum of the set $\{|\vec{r}_i\cdot\vec{r}_j||i,j>1\}$. This means that the angle between $\vec{k}_l'$ and $\vec{k}_{l-1}'$ $\phi := \angle(\vec{k}_{l-1}',\vec{k}_l')$ is as close as possible to $\pi/2$. All other angles $\angle(\vec{k}_i',\vec{k}_j')$ with $i,j\geq 2$ must be within $\phi+O(\alpha^{-1/4})$ of either 0 or π. The proof is performed in two steps.

1. When $2^{-i}\leq|\phi|\leq 2^{-i+1}$ or $2^{-i}\leq|\pi-\phi|\leq 2^{-i+1}$, for any i fixed, we have

$$N_l := \left[\tfrac{4}{3}\alpha^{\frac{1}{4}}\right]^2\int_{\Omega_l}d\theta_l^s\int_{\Omega_{l-1}}d\theta_{l-1}^s\Upsilon(\{\theta_i^s\}_{i=1,..l}) \leq K_0^l\ \left[\tfrac{4}{3}\alpha^{\frac{1}{4}}\right]^2\left(\alpha^{-\frac{1}{4}}\right)^2 \leq K^l \tag{B.1}$$

where K_0 and K are some constants and the sector centers $\theta_2,...\theta_{l-2}$, are not integrated yet. The proof is shown below.

2. We now have to perform the remaining integrals, then sum over all possible values of i. Assuming (B.1) true, the sum over all sectors is bounded by

$$\begin{aligned}\prod_{j=2}^{l}\left[\tfrac{4}{3}\alpha^{\frac{1}{4}}\right]\int_{\Omega_j}d\theta_j^s\ \Upsilon(\{\theta_j^s\}_{j=1,..l}) &\leq K^l\prod_{j\in J(i)}\left[\tfrac{4}{3}\alpha^{\frac{1}{4}}\right]\int_{2^{-i}}d\theta_j^s\prod_{j\notin J(i)}\left[\tfrac{4}{3}\alpha^{\frac{1}{4}}\right]\int_{\Omega_j}d\theta_j^s\,1\\ &= K'^l\prod_{j\in J(i)}\frac{2^{-i}}{\alpha^{-\frac{1}{4}}}\prod_{j\notin J(i)}\frac{|\Omega_j|}{\alpha^{-\frac{1}{4}}}\end{aligned} \tag{B.2}$$

where $J(i) := \{j | 2^{-i} \leq |\Omega_j|, 1 < j < l-1\}$. To perform the sum over all possible i we distinguish two situations, defining i_0 such that $2^{-i_0} \leq |\Omega_l| < 2^{-i_0+1}$:

- if $2^{-i} \leq 2^{-i_0}$, we have to perform

$$\sum_{i=i_0}^{\infty} \prod_{j\in J(i)} \frac{2^{-i}}{\alpha^{-\frac{1}{4}}} \prod_{j\notin J(i)} \frac{|\Omega_j|}{\alpha^{-\frac{1}{4}}} \leq \sum_{i=i_0}^{\infty} \frac{2^{-i}}{\alpha^{-\frac{1}{4}}} \prod_{j=3}^{l-2} \frac{|\Omega_j|}{\alpha^{-\frac{1}{4}}} = \frac{2^{-i_0+1}}{\alpha^{-\frac{1}{4}}} \prod_{j=3}^{l-2} \frac{|\Omega_j|}{\alpha^{-\frac{1}{4}}} \leq 2 \prod_{i\in I} \frac{|\Omega_i|}{\alpha^{-\frac{1}{4}}} \ . \tag{B.3}$$

- if $2^{-i} > 2^{-i_0}$, then, once fixed the sectors of all the $\vec{k}_i$ except the l^{th} there can be at most one i consistent with $\vec{k}_l$ falling in Ω_l. For this single value of i

$$\prod_{j\in J(i)} \frac{2^{-i}}{\alpha^{-\frac{1}{4}}} \prod_{j\notin J(i)} \frac{|\Omega_j|}{\alpha^{-\frac{1}{4}}} \leq \frac{\prod_{j=2}^{l-2} |\Omega_j|}{\alpha^{-\frac{l-3}{4}}} \leq \prod_{i\in I} \frac{|\Omega_i|}{\alpha^{-\frac{1}{4}}} \ . \tag{B.4}$$

This completes step 2 of the proof. We perform now step 1.

Proof of (B.1) (almost identical to [FMRT1], pg 701-704). We introduce the vectors $\vec{a}$ and $\vec{\varepsilon}$ defined as

$$\begin{aligned} \vec{a} &= -\vec{r}_1 - ... - \vec{r}_{l-2} = \vec{k}_{l-1} + \vec{k}_l - \sum_{j=1}^{l-2} (\vec{r}_j - \vec{k}_j) \\ \vec{a} + \vec{\varepsilon} &= \vec{k}'_l + \vec{k}'_{l-1} = -\vec{k}_1 - ... - \vec{k}_{l-2} + 2O(\alpha^{-\frac{1}{2}}) \end{aligned} \tag{B.5}$$

hence

$$\vec{\varepsilon} = \sum_{j=1}^{l-2} (\vec{r}_j - \vec{k}_j) + 2O(\alpha^{-\frac{1}{2}}) \ . \tag{B.6}$$

Remark that $\vec{a}$ is fixed, once fixed $\Sigma_1,...\Sigma_{l-2}$. We chose a coordinate system in which $\vec{r}_2 = (1,0)$. Then, since $\theta_j = \angle(\vec{r}_2, \vec{r}_j)$ satisfies $|\theta_j| = O(2^{-i})$ or $|\pi - \theta_j| = O(2^{-i})$ $\forall\ j \geq 2$ the x and y coordinates of every $\vec{k}_j$ $2 \leq j \leq l$, obey

$$\begin{aligned} \vec{k}_j &= \Big(\pm[1 + O(\alpha^{-\frac{1}{2}})] \cos O(2^{-i}), [1 + O(\alpha^{-\frac{1}{2}})] \sin O(2^{-i})\Big) \\ &= \Big([\pm 1 + O(2^{-2i})], O(2^{-i})\Big) \end{aligned} \tag{B.7}$$

where we assumed $2^{-i} \geq \alpha^{-1/4}$ (otherwise all sectors are automatically fixed). On the other hand, the differences $\vec{k}_j - \vec{r}_j$ can be written

$$\begin{aligned}
\vec{k}_j - \vec{r}_j &= \vec{k}'_j - \vec{r}_j + O(\alpha^{-\frac{1}{2}}) \\
&= \left(\cos\left(\theta_j + O(\alpha^{-\frac{1}{4}})\right), \sin\left(\theta_j + O(\alpha^{-\frac{1}{4}})\right)\right) - (\cos\theta_j, \sin\theta_j) + O(\alpha^{-\frac{1}{2}}) \\
&= \left(|\sin\theta_j| O(\alpha^{-\frac{1}{4}}), |\cos\theta_j| O(\alpha^{-\frac{1}{4}})\right) + O(\alpha^{-\frac{1}{2}}) \ . \qquad \text{(B.8)}
\end{aligned}$$

For any $j \geq 2$ we know that $|\sin\theta_j| = O(2^{-i})$ and $|\cos\theta_j| = O(1)$. For $j = 1$, since $k_1 = -\sum_{j=2}^{l} k_j$ we can check that $\max_{j\geq 2} \angle(\vec{k}'_1, \vec{k}'_j) \leq lO(2^{-i})$, hence $|\sin\theta_1| = lO(2^{-i})$. Therefore we have

$$\begin{aligned}
\vec{k}_j - \vec{r}_j &= \left(O(2^{-i}\alpha^{-\frac{1}{4}}), O(\alpha^{-\frac{1}{4}})\right) \quad \forall\, j > 1 \\
\vec{k}_1 - \vec{r}_1 &= l\left(O(2^{-i}\alpha^{-\frac{1}{4}}), O(\alpha^{-\frac{1}{4}})\right) \ . \qquad \text{(B.9)}
\end{aligned}$$

Inserting these results in the expressions for $\vec{a}$ and $\vec{\varepsilon}$ we have

$$\begin{aligned}
\vec{\varepsilon} &= l\, O\left(2^{-i}\alpha^{-\frac{1}{4}}, \alpha^{-\frac{1}{4}}\right) \\
\vec{a} &= N(2,0) + O\left(2^{-2i}, 2^{-i}\right) + l\, O\left(2^{-i}\alpha^{-\frac{1}{4}}, \alpha^{-\frac{1}{4}}\right) \\
&= N(2,0) + l\, O\left(2^{-2i}, 2^{-i}\right) \qquad \text{(B.10)}
\end{aligned}$$

where $N \in \{1, 0, -1\}$.

Now we can bound N_l in (B.1). We consider two cases. First, let $|N| = 1$. We rotate the coordinate system by $\pi\delta_{N,-1} + O(2^{-i})$ in such a way to make $\vec{a}$ run along the positive x axis (see Fig.5). In the new coordinate system the coordinates of $\vec{\varepsilon}$ obey, as before

$$\vec{\varepsilon} = l\, O\left(2^{-i}\alpha^{-\frac{1}{4}}, \alpha^{-\frac{1}{4}}\right) \ . \qquad \text{(B.11)}$$

Remark that, calling ψ the angle $\angle(\vec{k}'_{l-1}, \vec{a})$, we must have $\angle(\vec{k}'_l, \vec{a}) = \phi - \psi$.

Then the two components of the equation

$$\vec{k}'_{l-1} + \vec{k}'_l = (\cos\psi, \sin\psi) + (\cos(\phi-\psi), \sin(\phi-\psi)) = \vec{a} + \vec{\varepsilon} \qquad \text{(B.12)}$$

are

$$\cos\psi + \cos(\phi-\psi) = |\vec{a}| + l\, O(2^{-i}\alpha^{-\frac{1}{4}}) \ , \quad \sin\psi - \sin(\phi-\psi) = l\, O(\alpha^{-\frac{1}{4}}) \ . \qquad \text{(B.13)}$$

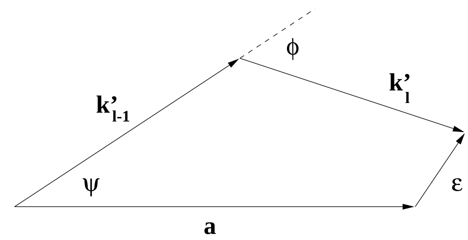

Figure 5: case 1, $|N| = 1$

The y component implies that

$$|2\psi - \phi| = l\ O(\alpha^{-\frac{1}{4}})\ , \qquad \psi = \frac{1}{2}\phi + l\ O(\alpha^{-\frac{1}{4}}) \tag{B.14}$$

then ϕ is determinated with precision $O(\alpha^{-1/4})$ once ψ has been fixed. Remark this was not obvious since the maximal variation for ϕ, without additional constraints, is 2^{-i} (remember $2^{-i} \le \phi \le 2^{-i+1}$). Therefore, for r_{l-1} fixed, θ_l^s is restricted to an interval of width $l\ O(\alpha^{-1/4})$. Finally, we consider the x component:

$$\begin{aligned}\cos\psi + \cos(\phi - \psi) &= \cos\psi + \cos\psi\cos(\phi - 2\psi) - \sin\psi\sin(\phi - 2\psi) \\ &= \left[2 + l^2 O(\alpha^{-\frac{1}{2}})\right]\cos\psi + lO(2^{-i}\alpha^{-\frac{1}{4}})\ .\end{aligned} \tag{B.15}$$

Then the angle ψ is

$$\psi = \cos^{-1}\left(\frac{|\vec{a}|}{2}\right) + lO\left(\frac{2^{-i}\alpha^{-\frac{1}{4}}}{2^{-i}}\right) \tag{B.16}$$

and θ_{l-1}^s must be integrated on an interval of width $lO(\alpha^{-1/4})$, instead of Ω_{l-1}. This completes the proof for $|N| = 1$. Finally we consider the case $|N| = 0$. This time we rotate the coordinate system by $O(2^{-i})$ or $\pi + O(2^{-i})$ so that $\vec{k}_{l-1}$ runs along the negative axis (see Fig.6).

Since $|\vec{a}|$ may be quite small we must perform a different estimate. The angle ϕ is determined by

$$\sin\left(\frac{\pi - \phi}{2}\right) = \frac{|\vec{a} + \vec{\varepsilon}|}{2} = \frac{|\vec{a}|}{2} + l\ O(\alpha^{-\frac{1}{4}}). \tag{B.17}$$

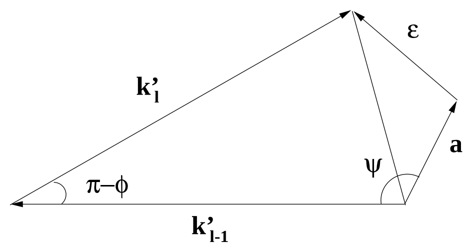

Figure 6: case 2, $|N| = 0$

Thus

$$\phi = \pi - 2\sin^{-1}\left(\frac{|\vec{a}|}{2}\right) + l\ O(\alpha^{-\frac{1}{4}}) \tag{B.18}$$

and θ_l^s is restricted to an interval of width $l\ O(\alpha^{-1/4})$, when $\vec{r}_{l-1}$ is held fixed. To evaluate ψ we apply the relation

$$\begin{aligned} \left|\sin\left(\psi - \frac{\phi}{2}\right)\right| &= \frac{\left|\vec{\varepsilon}\cdot\left(\cos\left(\frac{\pi-\phi}{2}\right), \sin\left(\frac{\pi-\phi}{2}\right)\right)\right|}{|\vec{a}|} \\ &\leq \frac{l\ O(2^{-i}\alpha^{-\frac{1}{4}})}{O(2^{-i}) - l\ O(\alpha^{-\frac{1}{4}})} \leq l\ O(\alpha^{-\frac{1}{4}})\ , \end{aligned} \tag{B.19}$$

where we applied the relation

$$|\vec{a} + \vec{\varepsilon}| = 2\sin\left(\frac{\pi - \phi}{2}\right) \geq O(2^{-i}) \tag{B.20}$$

that is proved with the hypothesis $2^{-i} \leq \phi \leq 2^{-i+1}$. Then

$$\psi = \frac{\phi}{2} + l\ O(\alpha^{-\frac{1}{4}}) \tag{B.21}$$

hence θ_l^s is restricted to an interval of width $l\ O(\alpha^{-1/4})$. This ends the proof.
■

Acknowledgements

We thank C. Kopper and J. Magnen for many interesting discussions; In particular the use of partial orderings (rather than total orderings as in [DR1]) came from common work with C. Kopper.

References

[AR1] A. Abdesselam and V. Rivasseau, Trees, forests and jungles: a botanical garden for cluster expansions, in Constructive Physics, ed by V. Rivasseau, Lecture Notes in Physics 446, Springer Verlag, 1995.

[AR2] A. Abdesselam and V. Rivasseau, Explicit Fermionic Cluster Expansion, Lett. Math. Phys. **44** 1998 77-88.

[BG] G. Benfatto and G. Gallavotti, Perturbation theory of the Fermi surface in a quantum liquid. A general quasi particle formalism and one dimensional systems, Journ. Stat. Phys. **59** (1990) 541.

[CR] C. de Calan and V. Rivasseau, Local existence of the Borel transform in Euclidean ϕ^4_4, Commun. Math. Phys. **82**, 69 (1981).

[DR1] M. Disertori and V. Rivasseau, Continuous Constructive Fermionic Renormalization, preprint (1998), to appear in Annales Henri Poincaré.

[DR2] M. Disertori and V. Rivasseau, Interacting Fermi liquid in two dimensions at finite temperature, Part II: Renormalization, to appear.

[FKLT] J. Feldman, H. Knörrer, D. Lehmann and E. Trubowitz, Fermi Liquids in Two Space Time Dimensions, in *Constructive Physics* ed. by V. Rivasseau, Springer Lectures Notes in Physics, Vol 446, 1995.

[FST] J. Feldman, M. Salmhofer and E. Trubowitz, Perturbation Theory around Non-nested Fermi Surfaces II. Regularity of the Moving Fermi Surface, RPA Contributions, Comm. Pure. Appl. Math. **51** (1998) 1133; Regularity of the Moving Fermi Surface, The Full Selfenergy, to appear in Comm. Pure. Appl. Math.

[FT1] J. Feldman and E. Trubowitz, Perturbation theory for Many Fermion Systems, Helv. Phys. Acta **63** (1991) 156.

[FT2] J. Feldman and E. Trubowitz, The flow of an Electron-Phonon System to the Superconducting State, Helv. Phys. Acta **64** (1991) 213.

[FMRT1] J. Feldman, J. Magnen, V. Rivasseau and E. Trubowitz, An infinite Volume Expansion for Many Fermion Green's Functions, Helv. Phys. Acta **65** (1992) 679.

[FMRT2] J. Feldman, J. Magnen, V. Rivasseau and E. Trubowitz, An Intrinsic 1/N Expansion for Many Fermion System, Europhys. Letters **24**, 437 (1993).

[FMRT3] J. Feldman, J. Magnen, V. Rivasseau and E. Trubowitz, Ward

Identities and a Perturbative Analysis of a U(1) Goldstone Boson in a Many Fermion System, Helv. Phys. Acta **66**, 498 (1993).

[G] M. Gevrey, Sur la nature analytique des solutions des équations aux dérivées partielles, (Ann. Scient. Ec. Norm. Sup., 3 série. t. 35, p. 129-190) in *Oeuvres de Maurice Gevrey* pp 243 , ed. CNRS (1970).

[L] A. Lesniewski, Effective Action for the Yukawa_2 Quantum Field Theory, Commun. Math. Phys. **108**, 437 (1987).

[MR] J. Magnen and V. Rivasseau, A Single Scale Infinite Volume Expansion for Three Dimensional Many Fermion Green's Functions, Math. Phys. Electronic Journal, Volume 1, 1995.

[R] V. Rivasseau, From perturbative to constructive renormalization, Princeton University Press (1991).

[S1] M. Salmhofer, Improved Power Counting and Fermi Surface Renormalization, Rev. Math. Phys. **10**, 553 (1998).

[S2] M. Salmhofer, Continuous renormalization for Fermions and Fermi liquid theory, Commun. Math. Phys.**194**, 249 (1998).