# ON RESONANCES AND THE FORMATION OF GAPS IN THE SPECTRUM OF QUASI-PERIODIC SCHRÖDINGER EQUATIONS

MICHAEL GOLDSTEIN AND WILHELM SCHLAG

Abstract. We consider one-dimensional difference Schrödinger equations

$$\big[H(x,\omega)\varphi\big](n) \equiv -\varphi(n-1) - \varphi(n+1) + V(x+n\omega)\varphi(n) = E\varphi(n)\ ,$$

$n \in \mathbb{Z}$, $x, \omega \in [0,1]$ with real-analytic potential function $V(x)$. If $L(E,\omega_0) > 0$ for all $E \in (E', E'')$ and some Diophantine $\omega_0$, then the integrated density of states is absolutely continuous for almost every $\omega$ close to $\omega_0$, see [GolSch2]. In this work we apply the methods and results of [GolSch2] to establish the formation of a dense set of gaps in $\bigcup_x \mathrm{sp}\ H(x,\omega) \cap (E', E'')$. Our approach is based on multi-scale arguments, and is therefore both constructive as well as quantitative. We show how resonances between eigenfunctions of one scale lead to "pre-gaps" at a larger scale. Then we show how these pre-gaps cannot be filled more than a finite (and uniformly bounded) number of times. To accomplish this, we relate a pre-gap to pairs of complex zeros of the Dirichlet determinants off the unite circle using the techniques of [GolSch2]. Of basic importance to our entire construction are the finite-volume description of Anderson localization as well as the separation of Dirichlet eigenvalues in a finite volume which were obtained in [GolSch2]. Another essential ingredient is the elimination of triple resonances from Chan [Cha], a special case of which is reproduced here.

## 1. Introduction and statement of the main results

The main goal of this work is to establish a multiscale description of the structure of the spectrum of quasi-periodic Schrödinger equations

$$\big[H(x,\omega)\varphi\big](n) \equiv -\varphi(n-1) - \varphi(n+1) + \lambda V(x+n\omega)\varphi(n) = E\varphi(n) \tag{1.1}$$

in the regime of exponentially localized eigenfunctions. We assume that $V(x)$ is a 1-periodic, real-analytic function, and that $\omega \in [0,1]$. Let $H_N(x,\omega)$ be the restriction of $H(x,\omega)$ to the finite interval $[1,N]$ with zero boundary conditions. Consider the union $\mathcal{S}_N = \bigcup_x \mathrm{sp}\ H_N(x,\omega)$, where sp $H_N(x,\omega)$ stands for the spectrum of $H_N(x,\omega)$. The set $\mathcal{S}_N$ is closed, so

$$\mathcal{S}_N = \big[\underline{E}(N), \overline{E}(N)\big] \setminus \bigcup_k \big(\underline{E}(N,k), \overline{E}(N,k)\big),$$

where $\underline{E}(N) = \min_{\mathcal{S}_N} E$, $\overline{E}(N) = \max_{\mathcal{S}_N} E$, and $\big(\underline{E}(N,k), \overline{E}(N,k)\big)$ are the maximal intervals of $\big[\underline{E}(N), \overline{E}(N)\big] \setminus \mathcal{S}_N$. More specifically, the goals of this work are as follows:

(a) To relate the intervals $\big(\underline{E}(N,k), \overline{E}(N,k)\big)$ and $\big(\underline{E}(N',k'), \overline{E}(N',k')\big)$ for "consecutive scales" $N \gg N'$.

(b) To "label" the interval $\big(\underline{E}(N,k), \overline{E}(N,k)\big)$ in accordance with its relation to intervals $\big(\underline{E}(m,\ell), \overline{E}(m,\ell)\big)$ of the previous scales.

(c) To describe the mechanism responsible for the formation of intervals $\big(\underline{E}(N,k), \overline{E}(N,k)\big)$ inside the set $\mathcal{S}_{N'}$, $N' \ll N$, independently of any $\big(\underline{E}(N',k'), \overline{E}(N',k')\big)$.

Our interest in these properties is largely motivated by possible applications to inverse spectral problems for the quasi-periodic Schrödinger equation and the Toda lattice with quasi-periodic initial data [Tod]. To

The first author was partially supported by an NSERC grant. The second author was partially supported by the NSF, DMS-0300081. The first author is very grateful to Galina Perelman for a number of suggestions during her visit at the University of Toronto in July of 2005.

establish these facts we use most of the methods developed in the recent work [GolSch2]. For the convenience of the reader, we recall – and expand upon – some of the material of that paper in Sections 2-5.

By a result of Rellich (see Reed, Simon [ReeSim4] page 4), the eigenvalues

$$E_1^{(N)}(x) < E_2^{(N)}(x) < \cdots < E_N^{(N)}(x)$$

of $H_N(x, \omega)$ are real analytic functions of $x \in [0, 1]$. Although the graphs of the functions $E_j^{(N)}(x)$ can be very complicated, the following was proved in [GolSch2]: There exist intervals $\left(E'_{N,k}, E''_{N,k}\right)$, $k = 1, 2, \dots, k_N$, with $\max_k \left(E''_{N,k} - E'_{N,k}\right) \le \exp\bigl(-\bigl(\log N\bigr)^A\bigr)$, $k_N \le \exp\left(\bigl(\log N\bigr)^B\right)$, with constants $1 \ll B \ll A$, such that if $E_j^{(N)}(x) \notin \mathcal{E}_N = \bigcup_k \left(E'_{N,k}, E''_{N,k}\right)$, for some $j$ and $x$, then $\left|\partial_x E_j^{(N)}(x)\right| > \exp\bigl(-N^\delta\bigr)$. Here $0 < \delta \ll 1$ is an arbitrary but fixed small parameter. In other words, the portions of the graphs of $E_j(x)$ have controlled slopes off a small set $\mathcal{E}_N$. The reader should note that by our estimates $\limsup_{N\to\infty} \mathcal{E}_N$ is a set of Hausdorff dimension zero.

The segments of the graph where $E_j^{(N)}(x) \in I$ and $I = (\underline{E}, \overline{E})$ is an interval disjoint from $\mathcal{E}_N$, are called *$I$–segments*. It is convenient to denote the $I$–segments as $\left\{E_j^{(N)}(x), \underline{x}, \bar{x}\right\}$, where $E_j^{(N)}(\underline{x}) = \underline{E}$, $E_j^{(N)}(\bar{x}) = \overline{E}$. The $I$–segments are important for our purposes, because they allow us to locate the resonances and to describe the graphs of the functions $E_j^{(\overline{N})}(x)$ for $\overline{N} \gg N$ in the region where the resonance occurs. A possible definition of a resonance is as follows: With $A \gg 1$ fixed,

$$\tau = \left|E_{j_1}^{(N)}(x, \omega) - E_{j_2}^{(N)}(x + m\omega, \omega)\right| < m^{-A} \tag{1.2}$$

for some $x \in \mathbb{T}$, $1 \le j_1$, $j_2 \le N$ and $m > N$.

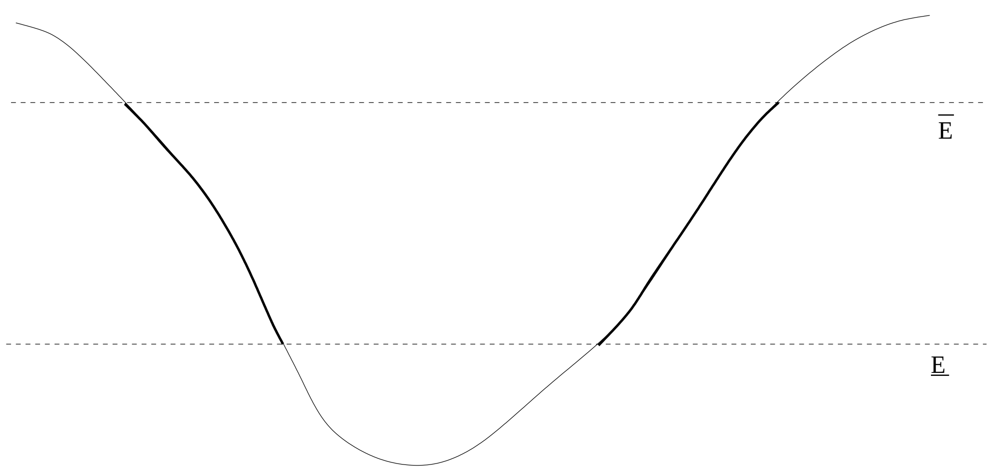


Figure 1. $I$–segments

The significance of such resonances was explained in the work by Sinai [Sin] on quasi-periodic Anderson localization in the regime of large $|\lambda|$ and $V = \cos$ (or cosine-like), see (1.1). Sinai developed a KAM-type scheme to analyze the functions $E_j^{(N)}(x)$ and the corresponding eigenvectors. The critical points of $E_j^{(\overline{N})}(x)$ with $\overline{N} \gg N$ were proved to be closely related to resonances as in (1.2). It is very important for the analysis of the resonances (1.2) in [Sin] that given $x \in \mathbb{T}$ and $j_1$ there exist at most one $j_2$ and $m \le \overline{N}$ so that (1.2) holds. *For that reason the function $V(x)$ in* [Sin] *is assumed to have two monotonicity intervals with non-degenerate critical points.* That allows one to reduce the analysis of $E_j^{(\overline{N})}(x)$ to an eigenvalue problem for a $2 \times 2$ matrix function of the form

$$A(x) = \begin{bmatrix} E_1(x - x_0) & \varepsilon(x) \\ \varepsilon(x) & E_2(x - x_0) \end{bmatrix} , \tag{1.3}$$

where $E_1(0) = E_2(0)$, $\partial_x E_1 < 0$, $\partial_x E_2 > 0$ locally around zero, and $\varepsilon(x)$ is small together with its derivatives. It is easy to check that the eigenvalues $E^+(x)$, $E^-(x)$ of $A(x)$ plotted against $x$ are as in Figure 2, at least locally around $x_0$.

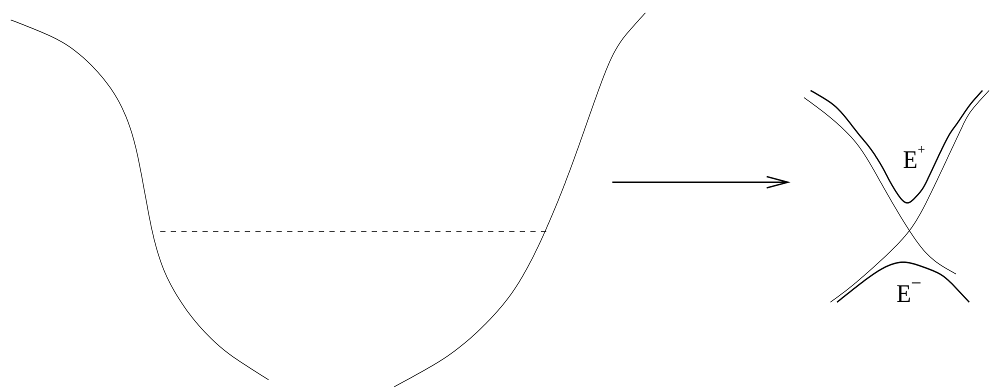


FIGURE 2. Formation of the resonant eigenvalues

We would like to emphasize that some of the conclusions which we reach in this paper are similar in spirit to those of Sinai [Sin]. This is particularly true in regards to the main result involving gaps and the aforementioned pictures describing the splitting of eigenvalues. At the same time, we stress that we use entirely nonperturbative methods (i.e., we are only assuming positive Lyapunov exponent rather than large $|\lambda|$) and we work with more general potentials than cosine. In this respect we would like to mention the recent breakthrough by Puig [Pui], who established the Cantor structure of the spectrum for the almost Mathieu case (cosine potential) and Diophantine $\omega$. Earlier, Choi, Elliott, and Yui [ChoEllYui] had obtained gaps for the case of Liouville rotation numbers $\omega$. The remaining cases of irrational rotation numbers (i.e., those with behavior intermediate to Diophantine and Liouville) was settled by Avila and Jitomirskaya [AviJit] (but this again only applies to the cosine).

The main objective in this work is to locate those segments of the graphs of some $E_{k_1}^{(\overline{N})}(x)$, $E_{k_2}^{(\overline{N})}(x)$ which look like $E^+(x)$ and $E^-(x)$ in Figure 2. Ultimately, such regions give rise to gaps in the spectrum. Before we state the main result of this work let us recall the central notions involved in it.

It is convenient to replace $V(x)$ in (1.1) by $V\big(e(x)\big)$ (with $e(x) = e^{2\pi i x}$), where $V(z)$ is an analytic function in the annulus $\mathcal{A}_{\rho_0} = \{z \in \mathbb{C} : 1 - \rho_0 < |z| < 1 + \rho_0\}$ which assumes only real values for $|z| = 1$. The monodromy matrices are as follows

$$\begin{aligned} M_{[a,b]}(z, \omega, E) &= \prod_{k=b}^{a} A\big(ze(k\omega), \omega, E\big) \\ A(z, \omega, E) &= \begin{bmatrix} V(z) - E & -1 \\ 1 & 0 \end{bmatrix} \end{aligned} \tag{1.4}$$

$a, b \in \mathbb{Z}$, $a < b$, $E \in \mathbb{C}$. For $M_{[1,N]}(z, \omega, E)$ we reserve the notation $M_N(z, \omega, E)$. For almost all $z = e(x + iy) \in \mathcal{A}_{\rho_0}$ the limit

$$\lim_{N \to \infty} N^{-1} \log \big\| M_N(z, \omega, E) \big\| \tag{1.5}$$

exists; if $\omega$ is irrational, then the limit does not depend on $x$ a.s. and it is denoted by $L(y, \omega, E)$. The most important case is $y = 0$, and we reserve the notation $L(\omega, E)$ for the Lyapunov exponents $L(0, \omega, E)$. We always assume that the frequency $\omega$ satisfies the same Diophantine condition as in [GolSch2], namely

$$\|n\omega\| \geq \tfrac{c}{n(\log n)^a} \qquad \text{for all } n \geq 1 \tag{1.6}$$

and some $a > 1$. We denote the class of $\omega$ satisfying (1.6) by $\mathbb{T}_{c,a}$.

**Theorem 1.1.** *Assume that $L(E, \omega_0) \geq \gamma_0 > 0$ for some $\omega_0 \in \mathbb{T}_{c,a}$ and any $E \in (E', E'')$ and fix $\delta > 0$ small. There exist $\rho^{(0)} = \rho^{(0)}(\lambda, V, \omega_0, \gamma_0, \delta) > 0$ such that for almost all $\omega \in \big(\omega_0 - \rho^{(0)}, \omega_0 + \rho^{(0)}\big)$ the following assertion holds: Let $N^{(t)}$, $t = 0, 1, \dots$ be an arbitrary sequence of integers such that $N^{(0)} \geq N_0(\lambda, V, \omega, \gamma_0, \delta)$ and*

$$N^{(t-1)} \asymp \Big(\log N^{(t)}\Big)^K \tag{1.7}$$

*$K = K(\lambda, V, \omega, \gamma_0, \delta)$, and let $\mathcal{S}^{(t)} = \bigcup_{x \in \mathbb{T}} \mathrm{sp}\; H_{N^{(t)}}(x, \omega)$. Then for each $t = 0, 1, \dots$ the set $(E', E'') \setminus \mathcal{S}^{(t)}$ contains a collection of intervals $\big(\underline{E}_j(s,t), \overline{E}_j(s,t)\big)$, $j = 1, 2, \dots, j(s,t)$, $s = 1, 2, \dots, t$, such that*

1. *$\exp\Big(-\big(N^{(s)}\big)^\delta\Big) \leq \overline{E}_j(s,t) - \underline{E}_j(s,t) \leq \exp\Big(-\big(N^{(s-1)}\big)^\delta\Big)$ for all $j, s$,*
2. *$\big(\underline{E}_j(s,t), \overline{E}_j(s,t)\big) \subset \mathcal{S}^{(s-1)} \setminus \mathcal{S}^{(s)}$ for each $1 \leq s \leq t$,*
3. *for each interval $(\underline{E}, \overline{E}) \subset (E', E'')$ with $\overline{E} - \underline{E} > \exp\Big(-\big(N^{(s)}\big)^{\delta/2}\Big)$, there exists $j$ such that $\big(\underline{E}_j(s+1,t), \overline{E}_j(s+1,t)\big) \subset (\underline{E}, \overline{E})$,*
4. *$\big(\underline{E}_j(s,t), \overline{E}_j(s,t)\big) \subset (E', E'') \setminus \bigcup_x \mathrm{sp}\; H(x, \omega)$.*

The strategy behind the derivation of Theorem 1.1 is as follows: We show that segments $E_1(x), E_2(x)$ as in the matrix (1.3) do exist. Then we use the estimates for the separation of the Dirichlet eigenvalues and the zeros of the Dirichlet determinants established in [GolSch2] to prove that this resonance defined by $E_1, E_2$ leads to two new eigenvalues $E^+(x), E^-(x)$ of the "next scale" which look exactly as in Figure 2. We call the interval

$$\Big(\max_{x \in J} E^-(x), \min_{x \in J} E^+(x)\Big)$$

a pre-gap at scale $\overline{N}$. The interval $J$ here is the common domain of $E_1$ and $E_2$. At this point we face the following difficulty: showing that for typical $\omega$ there is no room for so-called triple resonances. The resonance defined by (1.2) is called a *double resonance* if

$$\Big|E_{j_1}^{(N)}(x, \omega) - E_{j_3}^{(N)}(x + m'\omega, \omega)\Big| > {m'}^{-A} \tag{1.8}$$

for any pair $(j_3, m') \neq (j_2, m)$, with $m' \leq \overline{N}$, where $\overline{N} \asymp \exp\big(N^{\delta'}\big)$ is the "next scale". Otherwise, it is called a triple (or higher) resonance. As we have already mentioned, this issue is very important in Sinai's perturbative method. In fact, by the choice of a cosine-like potential and for large $|\lambda|$ this type of resonance is excluded in [Sin].

For general potentials, it was shown in recent work by Jackson Chan [Cha] that if $\big|\partial_x E_j^{(N)}\big| + \big|\partial_{xx} E_j^{(N)}\big| > c_0 > 0$, then triple resonances do not occur for most $\omega$. In Section 8 we bring a complete proof of a special case (namely, if $E_j^{(N)}(x)$ have controlled slopes) of the result by Chan.

Once again, since the graphs of the functions $E_j^{(N)}(x)$ are rather complicated, it is not clear how the pre-gaps develop into gaps at higher scales. Complex zeros of the Dirichlet determinants are very effective for the description of this mechanism. The latter are the characteristic determinants of $H_N(x,\omega)$. So, using complexified notations, these determinants are as follows:

$$f_N(z,\omega,E) = \det\left(H_N(z,\omega) - E\right)$$

$$(1.9)\qquad = \begin{vmatrix} \lambda V\big(ze(\omega)\big) - E & -1 & 0 & \cdots & \cdots & 0 \\ -1 & \lambda V\big(ze(2\omega)\big) - E & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & -1 \\ 0 & \dots\dots\dots\dots\dots & 0 & -1 & & \lambda V\big(ze(N\omega)\big) - E \end{vmatrix}$$

where

$$(1.10)\qquad f_{[a,b]}(z,\omega,E) = f_{b-a+1}\big(ze(a\omega),\omega,E\big)\ .$$

At the same time, these functions are closely related to the monodromies (1.4). Namely,

$$(1.11)\qquad M_N(z,\omega,E) = \begin{bmatrix} f_N(z,\omega,E) & -f_{N-1}\big(ze(\omega),\omega,E\big) \\ f_{N-1}(z,\omega,E) & -f_{N-2}\big(ze(\omega),\omega,E\big) \end{bmatrix}\ .$$

By means of this relation, large deviation estimates and an avalanche principle expansion for $\log\big|f_N(z,\omega,E)\big|$ were developed in [GolSch2]. In Section 2 we give the statements of these results as well as some corollaries. These corollaries, combined with some version of the Jensen formula (see (e) in Section 2) enable one to locate and count the zeros of $f_N(\cdot,\omega,E)$ in the annulus $\mathcal{A}_{\rho_0}$ and its subdomains. In particular, this technique allows one to claim that if

$$E \in \Big(\max E^-(x) + \exp\big(-\overline{N}^{1/2}\big), \min E^+(x) - \exp\big(-\overline{N}^{1/2}\big)\Big)\,,$$

where $(\max E^-(x), \min E^+(x))$ is a pre-gap at scale $\overline{N}$, then $f_{\overline{N}}(\cdot,\omega,E)$ has two complex zeros $\zeta_\ell = e(x_\ell + iy_\ell)$, with $\exp\big(-N^\delta\big) > |y_\ell| > \exp\big(-\overline{N}^\delta\big)$, $\ell = 1,2$. This is due to the double resonance and the stability of the number of zeros of $f_{\overline{N}}(\cdot,\omega,E)$ under small perturbations of $E$. The most effective form of the last property consists of the Weierstrass preparation theorem for $f_N(\cdot,\omega,E)$, which is described in (f) of Section 2. To complete the description of the formation of a gap from a pre-gap we use the translations of the segments $\Big\{E_{j_1}^{(N)}(x), \underline{x}, \bar{x}\Big\}$ under the shifts $x \to x + k\omega$. Using the localization property of eigenfunctions on a finite interval (see Section 3), we show that if a double resonance (1.2) occurs then the same is true for a sequence of segments which are "almost" identical with the shifts $E_{j_1}(x+k\omega), E_{j_2}(x+k\omega)$, $1 \le k \le \overline{N}\big(1-0(1)\big)$. That gives rise to a sequence of complex zeros $\zeta_{k,\ell} \cong e\big(x_\ell + k\omega + iy_\ell\big)$ of $f_N(\cdot,\omega,E)$. So, the numbers

$$\mathcal{M}_N(E) = N^{-1}\#\left\{z : 1-\rho_N < |z| < 1+\rho_N, f_N(z,\omega,E) = 0\right\}$$

$\rho_N = \exp\big(-N^\delta\big)$ decrease at least by $2 - o(1)$ when we go from scale $N$ to scale $\overline{N}$, provided $E$ is in the pre-gap. After a finite number of rescalings one can locate a gap and complete the proof of Theorem 1.1.

## 2. A review of the basic tools

In this section we give a sketch of the main ingredients of the method developed in [GolSch2]. We of course do not reproduce all the material from that paper in full detail, and refer the reader for most proofs to [GolSch2]. Nevertheless, the statements in this section are essential for the analysis of the spectrum in Sections 6-9.

We start our discussion with the classical Cartan estimate for analytic functions.

**(a) Cartan Estimate**

**Definition 2.1.** *Let $H \gg 1$. For an arbitrary subset $\mathcal{B} \subset \mathcal{D}(z_0, 1) \subset \mathbb{C}$ we say that $\mathcal{B} \in \mathrm{Car}_1(H, K)$ if $\mathcal{B} \subset \bigcup_{j=1}^{j_0} \mathcal{D}(z_j, r_j)$ with $j_0 \le K$, and*

$$\sum_j r_j < e^{-H} \ . \tag{2.1}$$

*If $d$ is a positive integer greater than one and $\mathcal{B} \subset \prod_{i=1}^{d} \mathcal{D}(z_{i,0}, 1) \subset \mathbb{C}^d$ then we define inductively that $\mathcal{B} \in \mathrm{Car}_d(H, K)$ if for any $1 \le j \le d$ there exists $\mathcal{B}_j \subset \mathcal{D}(z_{j,0}, 1) \subset \mathbb{C}, \mathcal{B}_j \in \mathrm{Car}_1(H, K)$ so that $\mathcal{B}_z^{(j)} \in \mathrm{Car}_{d-1}(H, K)$ for any $z \in \mathbb{C} \setminus \mathcal{B}_j$, here $\mathcal{B}_z^{(j)} = \{(z_1, \dots, z_d) \in \mathcal{B} : z_j = z\}$.*

**Remark 2.2.** *(a) This definition is consistent with the notation of Theorem 4 in Levin's book* [Lev]*, p. 79.*
*(b) It is important in the definition of $\mathrm{Car}_d(H, K)$ for $d > 1$ that we control both the measure and the complexity of each slice $\mathcal{B}_z^{(j)}$, $1 \le j \le d$.*

The following lemma is a straightforward consequence of this definition.

**Lemma 2.3.**

(1) *Let $\mathcal{B}_j \in \mathrm{Car}_d(H, K)$, $\mathcal{B}_j \subset \prod_{j=1}^{d} \mathcal{D}(z_{j,0}, 1)$, $j = 1, 2, \dots, T$. Then $\mathcal{B} = \bigcup_j \mathcal{B}_j \in \mathrm{Car}_d\big(H - \log T, TK\big)$.*

(2) *Let $\mathcal{B} \in \mathrm{Car}_d(H, K)$, $\mathcal{B} \subset \prod_{j=1}^{d} \mathcal{D}\big(z_{j,0}, 1\big)$. Then there exists $\mathcal{B}' \in \mathrm{Car}_{d-1}(H, K)$, $\mathcal{B}' \subset \prod_{j=2}^{d} \mathcal{D}\big(z_{j,0}, 1\big)$, such that $\mathcal{B}_{(w_2, \dots, w_d)} \in \mathrm{Car}_1(H, K)$, for any $(w_2, \dots, w_d) \in \mathcal{B}'$.*

Next, we generalize the usual Cartan estimate to several variables.

**Lemma 2.4.** *Let $\varphi(z_1, \dots, z_d)$ be an analytic function defined in a polydisk $\mathcal{P} = \prod_{j=1}^{d} \mathcal{D}(z_{j,0}, 1)$, $z_{j,0} \in \mathbb{C}$. Let $M \ge \sup_{\underline{z} \in \mathcal{P}} \log|\varphi(\underline{z})|$, $m \le \log\big|\varphi(\underline{z}_0)\big|$, $\underline{z}_0 = (z_{1,0}, \dots, z_{d,0})$. Given $H \gg 1$ there exists a set $\mathcal{B} \subset \mathcal{P}$, $\mathcal{B} \in \mathrm{Car}_d\left(H^{1/d}, K\right)$, $K = C_d H(M - m)$, such that*

$$\log\big|\varphi(z)\big| > M - C_d H(M - m) \tag{2.2}$$

*for any $z \in \prod_{j=1}^{d} \mathcal{D}(z_{j,0}, 1/6) \setminus \mathcal{B}$.*

*Proof.* The proof goes by induction over $d$. For $d = 1$ the assertion is Cartan's estimate for analytic functions. Indeed, Theorem 4 on page 79 in [Lev] applied to $f(z) = e^{-m}\varphi(z)$ yields that

$$\log\big|\varphi(z)\big| > m - CH(M - m) = M - (CH + 1)(M - m)$$

holds outside of a collection of disks $\{\mathcal{D}(a_k, r_k)\}_{k=1}^{K}$ with $\sum_{k=1}^{K} r_k \lesssim \exp(-H)$. Increasing the constant $C$ leads to (2.2). Moreover, $K/5$ cannot exceed the number of zeros of the function $\varphi(z)$ in the disk $\mathcal{D}(z_{1,0}, 1)$, which is in turn estimated by Jensen's formula, see next section, as $\lesssim M - m$. Although this bound on $K$ is not explicitly stated in Theorem 4 in [Lev], it can be deduced from the proofs of Theorems 3 and 4 in [Lev]. Indeed, one can assume that each of the disks $\mathcal{D}(a_k, r_k)$ contains a zero of $\varphi$, and it is shown in the proof of Theorem 3 in [Lev] that no point is contained in more than five of these disks. Hence we have proved the $d = 1$ case with a bad set $\mathcal{B} \in \mathrm{Car}_1(H, C(M - m))$, which is slightly better than stated above (the $H$ dependence of $K$ appears if $d > 1$ and we will ignore some slight improvements that are possible to the statement of the lemma due to this issue).

In the general case take $1 \le j \le d$ and consider $\psi(z) = \varphi\left(z_{1,0}, \dots, z_{j-1,0}, z, z_{j+1,0}, \dots, z_{d,0}\right)$. Due to the $d = 1$ case there exists $\mathcal{B}^{(j)} \in \mathrm{Car}_1\left(H^{1/d}, C_1(M - m)\right)$, such that

$$\log\big|\psi(z)\big| > M - C_1 H^{1/d}(M - m)$$

for any $z \in \mathcal{D}(z_{j,0}, 1/6) \setminus \mathcal{B}^{(j)}$. Take arbitrary $z_{j,1} \in \mathcal{D}(z_{j,0}, 1/6) \setminus \mathcal{B}^{(j)}$ and consider the function

$$\chi(z_1, z_2, \dots, z_{j-1}, z_{j+1}, \dots, z_d) = \varphi(z_1, \dots, z_{j-1}, z_{j1}, z_{j+1}, \dots, z_d)$$

in the polydisk $\prod_{i \neq j} \mathcal{D}(z_{i,0}, 1)$. Then

$$\sup \log\bigl|\chi(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_d)\bigr| \le M,$$
$$\log\bigl|\chi(z_{1,0}, \dots, z_{j-1,0}, z_{j,1}, z_{j+1,0}, \dots, z_{d,0})\bigr| > M - CH^{1/d}(M-m).$$

Thus $\chi$ satisfies the conditions of the lemma with the same $M$ and with $m$ replaced with

$$M - CH^{1/d}(M-m).$$

We now apply the inductive assumption for $d-1$ and with $H$ replaced with $H^{\frac{d-1}{d}}$ to finish the proof. □

Later we will need the following general assertion which is a combination of the Cartan-type estimate of the previous lemma and Jensen's formula on the zeros of analytic functions, see (e) of the present section.

**Lemma 2.5.** *Fix some $\underline{w}_0 = (w_{1,0}, w_{2,0}, \dots, w_{d,0}) \in \mathbb{C}^d$ and suppose that $f(\underline{w})$ is an analytic function in $\mathcal{P} = \prod_{j=1}^{d} D(w_{j,0}, 1)$. Assume that $M \ge \sup_{\underline{w} \in \mathcal{P}} \log|f(\underline{w})|$, and let $m \le \log|f(\underline{w}_1)|$ for some $\underline{w}_1 = (w_{1,1}, w_{2,1}, \dots, w_{d,1}) \in \prod_{j=1}^{d} \mathcal{D}(w_{j,0}, 1/2)$. Given $H \gg 1$ there exists $\mathcal{B}'_H \subset \mathcal{P}' = \prod_{j=2}^{d} \mathcal{D}(w_{j,0}, 3/4)$, $\mathcal{B}'_H \in \mathrm{Car}_{d-1}\left(H^{1/d}, K\right)$, $K = CH(M-m)$ such that for any $\underline{w}' = (w_2, \dots, w_d) \in \mathcal{P}' \setminus \mathcal{B}'_H$ the following holds: if*

$$\log|f(\tilde{w}_1, \underline{w}')| < M - C_d H(M-m) \quad \text{for some} \quad \tilde{w}_1 \in \mathcal{D}(w_{1,0}, 1/2),$$

*then there exists $\hat{w}_1$ with $|\hat{w}_1 - \tilde{w}_1| \lesssim e^{-H^{\frac{1}{d}}}$ such that $f(\hat{w}_1, \underline{w}') = 0$.*

*Proof.* Due to Lemma 2.4, there exists $\mathcal{B}_H \subset \mathcal{P}$, $\mathcal{B}_H \in \mathrm{Car}_d\bigl(H^{1/d}, K\bigr)$, $K = C_d H(M-m)$ such that for any $\underline{w} \in \prod_{j=1}^{d} \mathcal{D}(w_{j,0}, 3/4) \setminus \mathcal{B}_H$ one has

$$\log\bigl|f(\underline{w})\bigr| > M - C_d H(M-m)\ . \tag{2.3}$$

By Lemma 2.3, part (2), there exists $\mathcal{B}'_H \subset \prod_{j=2}^{d} \mathcal{D}\bigl(w_{j,0}, 1\bigr)$, $\mathcal{B}'_H \in \mathrm{Car}_{d-1}(H^{\frac{1}{d}}, K)$ such that $\bigl(\mathcal{B}_H\bigr)_{\underline{w}'} \in \mathrm{Car}_1(H^{\frac{1}{d}}, K)$ for any $\underline{w}' = (w_2, \dots, w_d) \in \mathcal{B}'_H$. Here $(\mathcal{B})_{\underline{w}'}$ stands for the $\underline{w}'$–section of $\mathcal{B}$. Assume

$$\log\bigl|f(\tilde{w}_1, \underline{w}')\bigr| < M - C_d H(M-m)$$

for some $\tilde{w}_1 \in \mathcal{D}(w_{1,0}, 1/2)$, and $\underline{w}' \in \mathcal{P}' \setminus \mathcal{B}'_H$. Since $\bigl(\mathcal{B}_H\bigr)_{\underline{w}'} \in \mathrm{Car}_1(H^{\frac{1}{d}}, K)$ there exists $r \lesssim \exp\bigl(-H^{1/d}\bigr)$ such that

$$\{z : |z - \tilde{w}_1| = r\} \cap \bigl(\mathcal{B}_H\bigr)_{\underline{w}'} = \emptyset\ .$$

Then in view of (2.3),

$$\log\bigl|f(z, \underline{w}')\bigr| > M - C_d H(M-m)$$

for any $|z - \tilde{w}_1| = r$. It follows from Jensen's formula, see (e) in the present section, that $f(\cdot, \underline{w}')$ has at least one zero in the disk $\mathcal{D}(\tilde{w}_1, r)$, as claimed. □

### (b) Large deviation theorem for the monodromies and their entries

Let $M_n(z, \omega, E)$ be the monodromies defined as in (1.4). The entries of $M_n(z, \omega, E)$ are the determinants $f_{[1+a, N-b]}(z, \omega, E)$, $a, b \in \{0, 1\}$, see (1.9), (1.10). Let $L(y, \omega, E)$ be the Lyapunov exponents defined as in (1.5).

**Theorem 2.6.** *Assume that $\gamma = L(y,\omega,E) > 0$ for some $y \in (-y_0, y_0)$, $\omega \in \mathbb{T}_{c,a}$, $E \in \mathbb{C}$. Then there exists $N_0 = N_0(\lambda, V, \omega, \gamma)$ such that for any $N > N_0$, $H > \big(\log N\big)^A$ one has*

$$\text{mes}\,\big\{x : \big|\log\big\|M_N\big(e(x+iy),\omega,E\big)\big\| - NL(y,\omega,E)\big| > H\big\} \le C\exp\Big(-H/\big(\log N\big)^A\Big)\ , \tag{2.4}$$

$$\text{mes}\,\big\{x : \big|\log\big|f_N\big(e(x+iy),\omega,E\big)\big| - NL(y,\omega,E)\big| > H\big\} \le C\exp\Big(-H/\big(\log N\big)^A\Big)\ , \tag{2.5}$$

*where $A, C > 1$ are constants.*

The estimate (2.4) for the monodromies follows from (2.5). However, the proof of the second half of Theorem 2.6 is more involved. There is a way to pass from (2.4) to (2.5), see Sections 2, 3 in [GolSch2].

**Remark 2.7.** *The estimates of Theorem 2.6 imply the following via Fubini: Assume that $L(y,\omega,E) \ge \gamma > 0$ for some $y \in (-y_0, y_0)$, $\omega \in \mathbb{T}_{c,a}$, and any $E \in \mathcal{D}$, where $\mathcal{D} \subset \mathbb{C}$ is some subset. There exists $N_0 = N_0(\lambda, V, \omega, \gamma)$ such that for any $N > N_0$, $H > (\log N)^A$ there exsts a subset $\mathcal{B}_{N,y,\omega,H} \subset \mathbb{T}$ with $\text{mes}\,(\mathcal{B}_{N,\omega}) \lesssim \exp(-H/2(\log N)^A)$ and*

$$\text{mes}\,\{E \in \mathcal{D} \,:\, |\log\|M_N(e(x+iy),\omega,E)\| - NL(y,\omega,E)| > H\} \le C\exp(-H/2(\log N)^A)\,\text{mes}\,(\mathcal{D}) \tag{2.6}$$

$$\text{mes}\,\{E \in \mathcal{D} \,:\, |\log|f_N(e(x+iy),\omega,E)| - NL(y,\omega,E)| > H\} \le C\exp(-H/2(\log N)^A)\,\text{mes}\,(\mathcal{D}) \tag{2.7}$$

*for any $x \in \mathbb{T}\setminus\mathcal{B}_{N,y,\omega,H}$. We will refer to these estimates as large deviation theorem in the $E$-variable.*

### (c) The avalanche principle expansion for the Dirichlet determinants

Let $f_N(z,\omega,E)$ be the determinants defined as in (1.9), and let $L(\omega,E)$ be the Lyapunov exponent.

**Theorem 2.8.** *Assume that $\gamma = L(\omega,E) > 0$ for some $\omega \in \mathbb{T}_{c,a}$, $E \in \mathbb{C}$. There exists $N_0 = N_0(\lambda, V, \omega, \gamma)$, $\rho^{(0)} = \rho^{(0)}(\lambda, V, \omega, \gamma) > 0$ such that for any $N > N_0(\lambda, V, \omega, \gamma)$ and any integers $\ell_1, \dots, \ell_n$, $\big(\log N\big)^A < \ell_j < cN$, $\sum\limits_j \ell_j = N$ the following expansion is valid:*

$$\log\big|f_N\big(e(x+iy),\omega,E\big)\big| = \sum_{j=1}^{n-1}\log\big\|A_{j+1}(z)\,A_j(z)\big\| - \sum_{j=2}^{n-1}\log\big\|A_j(z)\big\| + O\Big(\exp\big(-\underline{\ell}^{1/2}\big)\Big) \tag{2.8}$$

*for any $z = e(x+iy) \in \mathcal{A}_{\rho_0}\setminus\mathcal{B}_{N,\omega,E}$, where $\mathcal{B}_{N,\omega,E} = \bigcup\limits_{k=1}^{k_0}\mathcal{D}\Big(\zeta_k, \exp\big(-\underline{\ell}^{1/2}\big)\Big)$ , $\underline{\ell} = \min_j \ell_j$, $k_0 \lesssim N$, $A_m(z) = M_{\ell_m}\big(ze(s_m\omega),\omega,E\big)$, $m = 2,\dots,n-1$, $A_1(z) = M_{\ell_1}(z,\omega,E)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, $A_n(z) = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} M_{\ell_n}\big(ze(s_n\omega),\omega,E\big)$, $s_m = \sum\limits_{j<m}\ell_j$.*

A detailed derivation of this theorem can be found in Sections 2 and 3 of [GolSch2].

### (d) Uniform upper estimates on the norms of monodromy matrices

The proof of the uniform upper estimate is based on an application of the avalanche principle expansion in combination with the following useful general property of averages of subharmonic functions.

**Lemma 2.9.** *Let $1 > \rho > 0$ and suppose $u$ is subharmonic on $\mathcal{A}_\rho$ such that $\sup_{z\in\mathcal{A}_\rho} u(z) \le 1$ and $\int_{\mathbb{T}} u(e(x))\,dx \ge 0$. Then for any $r_1, r_2$ so that $1 - \frac{\rho}{2} < r_1, r_2 < 1 + \frac{\rho}{2}$ one has*

$$|\langle u(r_1 e(\cdot))\rangle - \langle u(r_2 e(\cdot))\rangle| \le C_\rho\,|r_1 - r_2|,$$

*here $\langle v(\cdot)\rangle = \int_0^1 v(\xi)d\xi$.*

For the proof see Lemma 4.1 in [GolSch2]. This assertion immediately implies the following corollary regarding the continuity of $L_N$ in $y$.

**Corollary 2.10.** *Let $L_N(y,\omega,E)$ and $L(y,\omega,E)$ be defined as above. Then with some constant $\rho>0$ that is determined by the potential,*

$$|L_N(y_1,\omega,E)-L_N(y_2,\omega,E)|\le C|y_1-y_2| \quad \text{for all} \quad |y_1|,|y_2|<\rho$$

*uniformly in $N$. In particular, the same bound holds for $L$ instead of $L_N$ so that*

$$\inf_E L(\omega,E)>\gamma>0$$

*implies that*

$$\inf_{E,|y|\ll\gamma} L(y,\omega,E)>\frac{\gamma}{2}.$$

The following result improves on the uniform upper bound on the monodromy matrices from [BouGol] and [GolSch1]. The $(\log N)^A$ error here (rather than $N^\sigma$, say, as in [BouGol] and [GolSch1]) is crucial for the study of the distribution of the zeros of the determinants and eigenvalues, see Proposition 4.3 in [GolSch2].

**Proposition 2.11.** *Assume $L(\omega,E)>0$, $\omega\in\mathbb{T}_{c,a}$. Then for all large integers $N$,*

$$\sup_{x\in\mathbb{T}}\log\|M_N(x,\omega,E)\|\le NL_N(\omega,E)+C(\log N)^A\ ,$$

*for some constants $C$ and $A$.*

We now list some straightforward applications of this upper bound. See Section 4 of [GolSch2].

**Corollary 2.12.** *Fix $\omega_1\in\mathbb{T}_{c,a}$ and $E_1\in\mathbb{C}$, $|y|<\rho_0$. Assume that $L(y,\omega_1,E_1)>0$. Then*

$$\begin{aligned}\sup\big\{\big\|M_N\left(e(x+iy),\omega,E\right)\big\| : |E-E_1|+|\omega-\omega_1|<\exp\left(-(\log N)^C\right), x\in\mathbb{T}\big\}\\ \lesssim \exp\left(NL_N(y,\omega_1,E_1)+(\log N)^A\right)\end{aligned}$$

*for all $|y|<\rho_0$.*

**Corollary 2.13.** *Fix $\omega_1\in\mathbb{T}_{c,a}$ and $E_1\in\mathbb{C}$, $|y|<\rho_0$. Assume that $L(y,\omega_1,E_1)>0$. Let $\partial$ denote any of the partial derivatives $\partial_x,\partial_y,\partial_E$ or $\partial_\omega$. Then*

$$\begin{aligned}\sup\left\{\big\|\partial M_N\left(e(x+iy),\omega,E\right)\big\| : |E-E_1|+|\omega-\omega_1|<e^{-(\log N)^C}, x\in\mathbb{T}\right\}\\ \lesssim \exp\left(NL_N(y,\omega_1,E_1)+(\log N)^A\right)\end{aligned}$$

*for all $|y|<\rho_0$.*

*Proof.* Clearly, for all $x,y,\omega,E$,

$$\partial M_N\left(e(x+iy),\omega,E\right)=\sum_{n=1}^{N} M_{N-n}\left(e(x+n\omega+iy),\omega,E\right)\partial\begin{bmatrix}\lambda V-E & -1\\ 1 & 0\end{bmatrix} M_{n-1}\left(e(x+iy),\omega,E\right)\ .$$

Since $|E-E_1|+|\omega-\omega_1|<e^{-(\log N)^C}$, the statement now follows from Corollary 2.12 and Corollary 2.10. □

**Corollary 2.14.** *Under the assumptions of the previous corollary,*

$$\begin{aligned}&\|M_N\left(e(x+iy),\omega,E\right)-M_N\left(e(x_1+iy_1),\omega_1,E_1\right)\|\\ &\lesssim\left(|E-E_1|+|\omega-\omega_1|+|x-x_1|+|y-y_1|\right)\cdot\exp\left(NL_N\left(y_1,\omega_1,E_1\right)+(\log N)^A\right)\end{aligned}$$

*provided $|E-E_1|+|\omega-\omega_1|+|x-x_1|<e^{-(\log N)^A}$, $|y_1|<\rho_0/2$, $|y-y_1|<N^{-1}$. In particular*

$$\left|\log\frac{\left|f_N\big(e(x+iy),\omega,E\big)\right|}{\left|f_N\big(e(x_1+iy_1),\omega_1,E_1\big)\right|}\right|\lesssim\left(|E-E_1|+|\omega-\omega_1|+|x-x_1|+|y-y_1|\right)\frac{\exp\left(NL(y_1,\omega_1,E_1)+(\log N)^A\right)}{\left|f_N\big(e(x_1+iy_1),\omega_1,E_1\big)\right|}\ , \tag{2.9}$$

*provided the right-hand side of (2.9) is less than $1/2$.*

**Corollary 2.15.** *Using the notation of the previous corollary one has*

$$\left|\log \frac{\big\|M_N\left(e(x+iy),\omega,E\right)\big\|}{\big\|M_N\left(e(x_1+iy_1),\omega_1,E_1\right)\big\|}\right| < \exp\left(-(\log N)^A\right) \tag{2.10}$$

$$\left|\log \frac{\big|f_N\left(e(x+iy),\omega,E\right)\big|}{\big|f_N\left(e(x_1+iy_1),\omega_1,E_1\right)\big|}\right| < \exp\left(-(\log N)^A\right) \tag{2.11}$$

*for any* $|E-E_1|+|\omega-\omega_1|+|x-x_1|+|y-y_1| < \exp\left(-(\log N)^{2A}\right)$, $e(x_1+iy_1)\in\mathcal{A}_{\rho_0/2}\setminus\mathcal{B}_{\omega_1,E_1}$, *where* $\mathrm{mes}\,\mathcal{B}_{\omega_1,E_1} < \exp\left(-(\log N)^{A/2}\right)$, $\mathrm{compl}(\mathcal{B}_{\omega_1,E_1})\lesssim N$.

**Proposition 2.16.** *Let* $\omega\in\mathbb{T}_{c,a}$. *Then for any* $x_0\in\mathbb{T}$, $E_0\in\mathbb{R}$ *one has*

$$\#\left\{E\in\mathbb{R}\,:\, f_N\big(e(x_0),\omega,E\big)=0,\ |E-E_0|<\exp\left(-(\log N)^A\right)\right\}\le(\log N)^{A_1} \tag{2.12}$$

$$\#\left\{z\in\mathbb{C}\,:\, f_N(z,\omega,E_0)=0,\ |z-e(x_0)|<N^{-1}\right\}\le(\log N)^{A_1} \tag{2.13}$$

*for all sufficiently large* $N$.

Another application of the uniform upper bounds is the following analogue of Wegner's estimate from the random case. We provide the proof here just to demonstrate how the previous corollaries can be applied.

**Lemma 2.17.** *Suppose* $\omega\in\mathbb{T}_{c,a}$. *Then for any* $N\gg1$, $E\in\mathbb{R}$, $H\ge(\log N)^A$ *one has*

$$\mathrm{mes}\,\{x\in\mathbb{T}\,:\,\mathrm{dist}\,(\mathrm{sp}\ H_N(x,\omega),E)<\exp(-H)\}\le\exp\left(-H/(\log N)^A\right)\ . \tag{2.14}$$

*Moreover, the set on the left-hand side is contained in the union of* $\lesssim N$ *intervals each of which does not exceed the bound stated in* (2.14) *in length.*

*Proof.* By Cramer's rule

$$\left|\Big(H_N(x,\omega)-E\Big)^{-1}(k,m)\right| = \frac{\big|f_{[1,k]}\big(e(x),\omega,E\big)\big|\,\big|f_{[m+1,N]}\big(e(x),\omega,E\big)\big|}{\big|f_N\big(e(x),\omega,E\big)\big|}\ .$$

By Proposition 2.11

$$\log\big|f_{[1,k]}\big(e(x),\omega,E\big)\big|+\log\big|f_{[m+1,N]}\big(e(x),\omega,E\big)\big|\le NL(\omega,E)+(\log N)^{A_1}$$

for any $x\in\mathbb{T}$. Therefore,

$$\big\|\,(H_N(x,\omega)-E)^{-1}\,\big\|\le N^2\ \frac{\exp\big(NL(\omega,E)+(\log N)^A\big)}{\big|f_N\big(e(x),\omega,E\big)\big|}$$

for any $x\in\mathbb{T}$. Since

$$\mathrm{dist}\,\Big(\mathrm{sp}\,\big(H_N(x,\omega),E\big)\Big)^{-1}=\big\|\,(H_N(x,\omega)-E)^{-1}\,\big\|\ ,$$

the lemma follows from Theorem 2.6. ∎

We conclude this subsection with an important application of Lemma 2.5 and Proposition 2.11 to the Dirichlet determinants $f_N$.

**Corollary 2.18.** *Suppose* $\omega\in\mathbb{T}_{c,a}$. *Given* $E_0\in\mathbb{C}$ *and* $H>(\log N)^A$, *there exists*

$$\mathcal{B}_{N,E_0,\omega}(H)\subset\mathbb{C},\qquad \mathcal{B}_{N,E_0,\omega}(H)\in\mathrm{Car}_1(\sqrt{H},HN^2)$$

*such that for any* $x\in\mathbb{T}\setminus\mathcal{B}_{N,E_0,\omega}(H)$, *and large* $N$ *the following holds: If*

$$\log\big|f_N\left(e(x),\omega,E_1\right)\big|<NL(\omega,E_1)-H(\log N)^A,\quad |E_0-E_1|<\exp(-(\log N)^C),$$

*then* $f_N\left(e(x),\omega,E\right)=0$ *for some* $|E-E_1|\lesssim\exp(-\sqrt{H})$. *Similarly, given* $x_0\in\mathbb{T}$ *and* $|y_0|<N^{-1}$, *let* $z_0=e(x_0+iy_0)$. *Then for any* $H>(\log N)^A$, *there exists*

$$\mathcal{E}_{N,z_0,\omega}(H)\subset\mathbb{C},\qquad \mathcal{E}_{N,z_0,\omega}(H)\in\mathrm{Car}_1\left(\sqrt{H},H\exp((\log N)^A)\right) \tag{2.15}$$

*such that for any* $E \in \mathcal{D}(0, 2 + |\lambda|\|V\|_\infty) \setminus \mathcal{E}_{N,z_0,\omega}(H)$*, the following assertion holds: If*

$$\log\big|f_N\big(z_1, \omega, E\big)\big| < NL(\omega, E) - H(\log N)^A, \quad |z_0 - z_1| < \exp(-(\log N)^C),$$

*then* $f_N\big(z, \omega, E\big) = 0$ *for some* $|z - z_1| \lesssim \exp(-\sqrt{H})$*.*

*Proof.* Set $r_0 = \exp(-(\log N)^C)$ with some large constant $C$. Fix any $z_0$ with $|z_0| = 1$ and consider the analytic function

$$f(z, E) = f_N(z_0 + (z - z_0)N^{-1}, E_0 + (E - E_0)r_0, \omega)$$

on the polydisk $\mathcal{P} = \mathcal{D}(z_0, 1) \times \mathcal{D}(E_0, 1)$. Then, by Proposition 2.11,

$$\sup_{\mathcal{P}} \log|f(z, E)| \le NL(E_0, \omega) + (\log N)^{2C} = M$$

and by the large deviation theorem,

$$\log|f(z_1, E_0)| > NL(E_0, \omega) - (\log N)^{2C} = m$$

for some $|z_0 - z_1| < 1/100$, say. By Lemma 2.5 there exists

$$\mathcal{B}_{z_0,E_0,\omega}(H) \subset \mathbb{C}, \qquad \mathcal{B}_{z_0,E_0,\omega}(H) \in \mathrm{Car}_1(\sqrt{H}, H(\log N)^{3C})$$

so that for any $z \in \mathcal{D}(z_0, 1/2) \setminus \mathcal{B}_{z_0,E_0,\omega}(H)$ the following holds: If

$$\log|f(z, E_1)| < NL(E_0, \omega) - H(\log N)^{3C}$$

for some $|E_1 - E_0| < 1/2$, then there is $E$ with $|E_1 - E| \lesssim \exp(-\sqrt{H})$ such that $f(z, E) = 0$. Now let $z_0$ run over a $N^{-\frac{3}{2}}$-net on $|z| = 1$ and define $\mathcal{B}_{N,E_0,\omega}(H)$ to be the union of the sets $z_0 + N^{-1}\mathcal{B}_{z_0,E_0,\omega}(H)$. The first half of the lemma now follows by taking $A$ sufficiently large and by absorbing some powers of $\log N$ into $H$ if needed. The second half of the lemma dealing with zeros in the $z$ variable can be shown analogously. □

**Remark 2.19.** *We can draw the following conclusion from the preceding corollary: Let* $\omega \in \mathbb{T}_{c,a}$ *be fixed, and define*

$$\mathcal{E}_{N,\omega}(H) = \bigcup_{x_0} \mathcal{E}_{N,e(x_0),\omega}(H)$$

*where the union runs over an* $N^{-1}$*-net of points* $x_0 \in \mathbb{T}$*. Then, for any* $x \in \mathbb{T}$*, if*

$$\log\big|f_N\big(x, \omega, E\big)\big| < NL(\omega, E) - H(\log N)^A, \quad E \in \mathcal{D}(0, 2 + |\lambda|\|V\|_\infty) \setminus \mathcal{E}_{N,\omega}(H)$$

*then* $f_N\big(z, \omega, E\big) = 0$ *for some* $|z - e(x)| \lesssim \exp(-\sqrt{H})$*. Moreover,* (2.15) *holds for* $\mathcal{E}_{N,\omega}(H)$*.*

### (e) A corollary of the Jensen formula

The Jensen formula states that for any function $f$ analytic on a neighborhood of $\mathcal{D}(z_0, R)$, see [Lev],

$$\int_0^1 \log|f(z_0 + Re(\theta))|\, d\theta - \log|f(z_0)| = \sum_{\zeta: f(\zeta)=0} \log\frac{R}{|\zeta - z_0|} \tag{2.16}$$

provided $f(z_0) \ne 0$. In the previous section, we showed how to combine this fact with the large deviation theorem and the uniform upper bounds to bound the number of zeros of $f_N$ which fall into small disks, in both the $z$ and $E$ variables. In what follows, we will refine this approach further. For this purpose, it will be convenient to average over $z_0$ in (2.16). Henceforth, we shall use the notation

$$\nu_f(z_0, r) = \#\{z \in \mathcal{D}(z_0, r) : f(z) = 0\} \tag{2.17}$$

$$\mathcal{J}(u, z_0, r_1, r_2) = \fint_{\mathcal{D}(z_0,r_1)} dx\, dy \fint_{\mathcal{D}(z,r_2)} d\xi d\eta\, [u(\zeta) - u(z)]. \tag{2.18}$$

**Lemma 2.20.** *Let $f(z)$ be analytic in $\mathcal{D}(z_0, R_0)$. Then for any* $0 < r_2 < r_1 < R_0 - r_2$

$$\nu_f(z_0, r_1 - r_2) \le 4\frac{r_1^2}{r_2^2}\mathcal{J}(\log|f|, z_0, r_1, r_2) \le \nu_f(z_0, r_1 + r_2)$$

*Proof.* Jensen's formula yields

$$\begin{aligned}\mathcal{J}(\log|f|, z_0, r_1, r_2) &= \fint_{\mathcal{D}(z_0,r_1)} dx\, dy \left[\frac{2}{r_2^2}\int_0^{r_2} dr\left(r \sum_{f(\zeta)=0, \zeta\in\mathcal{D}(z,r)} \log(\frac{r}{|\zeta - z|})\right)\right] \\ &\le \sum_{f(\zeta)=0,\zeta\in\mathcal{D}(z_0,r_1+r_2)} (\frac{1}{\pi r_1^2})\left[\frac{2}{r_2^2}\int_0^{r_2} dr\left(r\int_{\mathcal{D}(\zeta,r)} \log(\frac{r}{|z-\zeta|}) dx\, dy\right)\right] \\ &= \frac{1}{4}(\frac{r_2^2}{r_1^2})\nu_f(z_0, r_1 + r_2),\end{aligned}$$

which proves the upper estimate for $\mathcal{J}(\log|f|, z_0, r_1, r_2)$. The proof of the lower estimates is similar. □

**Corollary 2.21.** *Let $f$ be analytic in $\mathcal{D}(z_0, R_0)$, $0 < r_2 < r_1 < R_0 - r_2$. Assume that $f$ has no zeros in the annulus $\mathcal{A} = \{r_1 - r_2 \le |z - z_0| \le r_1 + r_2\}$. Then*

$$\nu_f(z_0, r_1) = 4\ \frac{r_1^2}{r_2^2}\ \mathcal{J}\left(\log|f|, z_0, r_1, r_2\right)\ .$$

**Corollary 2.22.** *Let $f(z), g(z)$ be analytic in $\mathcal{D}(z_0, R_0)$. Assume that for some* $0 < r_2 < r_1 < R_0 - r_2$

$$|\mathcal{J}(\log|f|, z_0, r_1, r_2) - \mathcal{J}(\log|g|, z_0, r_1, r_2)| < \frac{r_2^2}{4r_1^2}$$

*Then*

$$\nu_f(z_0, r_1 - r_2) \le \nu_g(z_0, r_1 + r_2), \qquad \nu_g(z_0, r_1 - r_2) \le \nu_f(z_0, r_1 + r_2).$$

We shall also need a simple generalization of these estimates to averages over general domains. More precisely, set

$$\begin{aligned}&\nu_f(\mathcal{D}) = \#\{z \in \mathcal{D} : f(z) = 0\} \\ &\mathcal{J}(u, \mathcal{D}, r_2) = \fint_{\mathcal{D}} dx\, dy \fint_{\mathcal{D}(z,r_2)} d\xi d\eta\, [u(\zeta) - u(z)].\end{aligned} \tag{2.19}$$

Given a domain $\mathcal{D}$ and $r > 0$ , set $\mathcal{D}(r) = \{z : \mathrm{dist}(z, \mathcal{D}) < r\}$. Let $f(z)$ be analytic in $\mathcal{D}(R)$. Then for any $0 < r_2 < r_1 < R - r_2$

$$\nu_f(\mathcal{D}(r_1 - r_2)) \le 2\frac{\mathrm{mes}\,(\mathcal{D})}{\pi r_2^2}\mathcal{J}(\log|f|, \mathcal{D}(r_1), r_2) \le \nu_f(\mathcal{D}(r_1 + r_2)) \tag{2.20}$$

Let $\mathcal{A}_{R_1,R_2} := \{z \in \mathbb{C}\ :\ R_1 < |z| < R_2\}$.

**Lemma 2.23.**

$$\begin{aligned}&N^{-1}\mathcal{J}\Big(\log\big|f_N(\cdot, \omega, E)\big|, \mathcal{A}_{R_1,R_2}, r_2\Big) = \\ &2(R_2^2 - R_1^2)^{-1} r_2^{-2}\int_{R_1}^{R_2} \rho\, d\rho \int_0^{r_2} r\, dr \int_0^1 dy\Big[L_N(\xi(\rho, r, y), \omega, E) - L_N(\xi(\rho), \omega, E)\Big]\end{aligned} \tag{2.21}$$

*where $\xi(\rho, r, y) = \log|\rho + re(y)|$, $\xi(\rho) = \log\rho$.*

*Proof.* Due to the definition of $\mathcal{J}(u,\mathcal{D},r_2)$ one has

$$
\begin{aligned}
&N^{-1}\mathcal{J}(\log|f_N(\cdot,\omega,E)|,\mathcal{A}_{R_1,R_2},r_2)\\
&=N^{-1}\frac{4\pi}{|\mathcal{A}_{R_1,R_2}|r_2^2}\int_{R_1}^{R_2}\rho\,d\rho\int_0^{r_2}r\,dr\Big\{\int_0^1 dx\int_0^1 dy\Big[\log|f_N(\rho e(x)+re(y),\omega,E)|-\log|f_N(\rho e(x),\omega,E)|\Big]\Big\}\\
&=N^{-1}\frac{4\pi}{|\mathcal{A}_{R_1,R_2}|r_2^2}\int_{R_1}^{R_2}\rho\,d\rho\int_0^{r_2}r\,dr\Big\{\int_0^1 dx\int_0^1 dy\Big[\log|f_N(|\rho+re(y)|e(x),\omega,E)|-\log|f_N(\rho e(x),\omega,E)|\Big]\Big\}\\
&=2(R_2^2-R_1^2)^{-1}r_2^{-2}\int_{R_1}^{R_2}\rho\,d\rho\int_0^{r_2}r\,dr\int_0^1 dy\Big[L_N(\xi(\rho,r,y),\omega,E)-L_N(\xi(\rho),\omega,E)\Big]
\end{aligned}
$$

as claimed. □

Set

$$\mathcal{M}_N(\omega,E,R_1,R_2)=\tfrac{1}{N}\#\{z\in\mathcal{A}_{R_1,R_2}:f_N(z,\omega,E)=0\}\ . \tag{2.22}$$

**Lemma 2.24.** *Assume $\gamma=L(\omega,E)>0$ and fix some small $0<\sigma\ll 1$. There exist $N_0=N_0(\lambda,V,\omega,\gamma,\sigma)$, $\rho^{(0)}=\rho^{(0)}(\lambda,V,\omega,\gamma)>0$ such that for any $n>N_0$, $N>\exp(\gamma_1 n^\sigma)$, $1-\rho^{(0)}<R_1<R_2<1+\rho^{(0)}$ one has*

$$
\begin{aligned}
&\mathcal{M}_N(\omega,E,R_1+r_2,R_2-r_2)\le\mathcal{M}_n(\omega,E,R_1-r_2,R_2+r_2)+n^{-1/4}\\
&\mathcal{M}_n(\omega,E,R_1+r_2,R_2-r_2)\le\mathcal{M}_N(\omega,E,R_1-r_2,R_2+r_2)+n^{-1/4}
\end{aligned}
\tag{2.23}
$$

*where $r_2=n^{-1/4}(R_2-R_1)$ and provided $r_2>\exp(-\gamma_4 n^\sigma)$.*

*Proof.* Recall that due to avalanche principle expansion one has

$$
\begin{aligned}
&\Bigg|\log\frac{\big\|M_n\big(e(x+n\omega+iy),\omega,E\big)\big\|\ \big\|M_n\big(e(x+iy),\omega,E\big)\big\|}{\big\|M_{2n}\big(e(x+iy),\omega,E\big)\big\|}-\\
&\log\frac{\big\|M_\ell\big(e(x+n\omega+iy)\big)\big\|\ \big\|M_\ell\big(e(x+(n-\ell)\omega+iy),\omega,E\big)\big\|}{\big\|M_{2\ell}\big(e(x+(n-\ell)\omega+iy)\big\|}\Bigg|\le\exp\left(-\gamma_1 n^{1/2}\right)
\end{aligned}
$$

for any $|y|<\rho_0/2$, $x\in\mathbb{T}\setminus\mathcal{B}_y$, mes $\mathcal{B}_y<\exp\left(-\gamma_1 n^{1/2}\right)$ where $\ell=\left[n^{1/2}\right]$, $\gamma_k=L(\omega,E)/2^k$.

That implies in particular

$$
\begin{aligned}
&L_n(y,\omega,E)-L_{2n}(y,\omega,E)=\\
&\frac{\ell}{n}(L_\ell(y,\omega,E)-L_{2\ell}(y,\omega,E))+O\Big(\exp\big(-\gamma_2 n^{1/2}\big)\Big)
\end{aligned}
\tag{2.24}
$$

Let $\xi(\rho)=\log\rho$, $\xi(\rho,r,y)=\log|\rho+re(y)|$, $R_1<\rho<R_2$, $0<r<r_2$, $0\le y\le 1$, as in Lemma 2.23. Then, by Lemma 2.9

$$|L_{j\ell}\left(\xi(\rho,r,y),\omega,E\right)-L_{j\ell}\left(\xi(\rho),\omega,E\right)|\le CR_1^{-1}r \qquad j=1,2 \tag{2.25}$$

Recall that for any $N>\exp(\gamma_1 n^\sigma)$ one has

$$|L_N(y,\omega,E)-2L_{2n}(y,\omega,E)+L_n(y,\omega,E)|<\exp\left(-\gamma_2 n^\sigma\right)\ ,$$

see [GolSch1]. Hence, due to (2.20) and Lemma 2.23

$$
\begin{aligned}
&\mathcal{M}_N(\omega,E,R_1+r_2,R_2-r_2)\le\frac{4|\mathcal{A}_{R_1,R_2}|}{r_2^2 N}\,\mathcal{J}\Big(\log\big|f_N(\cdot,\omega,E)\big|,\mathcal{A}_{R_1,R_2},r_2\Big)\\
&=\frac{4|\mathcal{A}_{R_1,R_2}|}{r_2^2}\,\mathcal{J}\Big(n^{-1}[\log\big|f_{2n}(\cdot,\omega,E)\big|-\log\big|f_n(\cdot,\omega,E)\big|],\mathcal{A}_{R_1,R_2},r_2\Big) && (2.26)\\
&+O\Big((R_2-R_1)r_2^{-2}\exp(-\gamma_2 n^\sigma)\Big) && (2.27)
\end{aligned}
$$

Next, we rewrite the Jensen average in (2.26) using Lemma 2.23

$$\begin{aligned}&\mathcal{J}\Big(n^{-1}[\log\big|f_{2n}(\cdot,\omega,E)\big|-\log\big|f_n(\cdot,\omega,E)\big|],\mathcal{A}_{R_1,R_2},r_2\Big)\\ &=2\mathcal{J}\Big(\frac{1}{2n}\log\big|f_{2n}(\cdot,\omega,E)\big|-\frac{1}{n}\log\big|f_n(\cdot,\omega,E)\big|,\mathcal{A}_{R_1,R_2},r_2\Big) &(2.28)\\ &+\mathcal{J}\Big(n^{-1}\log\big|f_n(\cdot,\omega,E)\big|,\mathcal{A}_{R_1,R_2},r_2\Big) &(2.29)\end{aligned}$$

Inserting (2.29) into (2.26) leads to the main term on the right-hand side of (2.23). It is bounded above by $\mathcal{M}_n(\omega,E,R_1-r_2,R_2+r_2)$ in view of (2.20). It remains to bound the error term (2.28). We introduce the short-hand notation

$$\begin{aligned}&\mathcal{S}[L_n(\xi(\rho,r,y),\omega,E)-L_n(\xi(\rho),\omega,E)]\\ &=\frac{4\pi}{(R_2^2-R_1^2)r_2^2}\int_{R_1}^{R_2}\rho d\rho\int_0^{r_2}rdr\int_0^1 dy\,\big[L_n(\xi(\rho,r,y),\omega,E)-L_n(\xi(\rho),\omega,E)\big]\end{aligned}$$

Hence, the Jensen-average in (2.28) equals, see (2.24),

$$\begin{aligned}&\mathcal{S}[L_{2n}(\xi(\rho,r,y),\omega,E)-L_n(\xi(\rho,r,y),\omega,E)]-\mathcal{S}[L_{2n}(\xi(\rho),\omega,E)-L_n(\xi(\rho),\omega,E)]\\ &=\frac{\ell}{n}\mathcal{S}[L_{2\ell}(\xi(\rho,r,y),\omega,E)-L_\ell(\xi(\rho,r,y),\omega,E)]-\frac{\ell}{n}\mathcal{S}[L_{2\ell}(\xi(\rho),\omega,E)-L_\ell(\xi(\rho),\omega,E)]\\ &\quad+O\Big(\exp\big(-\gamma_2 n^{1/2}\big)\Big)\end{aligned}$$

By the Lipschitz bound (2.25), we can further estimate the absolute value here by

$$\begin{aligned}&\lesssim\Big|\frac{\ell}{n}\mathcal{S}[L_{2\ell}(\xi(\rho,r,y),\omega,E)-L_{2\ell}(\xi(\rho),\omega,E)]\Big|+\Big|\frac{\ell}{n}\mathcal{S}[L_\ell(\xi(\rho,r,y),\omega,E)-L_\ell(\xi(\rho),\omega,E)]\Big|\\ &\quad+O\Big(\exp\big(-\gamma_2 n^{1/2}\big)\Big)\\ &\lesssim n^{-1/2}r_2+O\Big(\exp\big(-\gamma_2 n^{1/2}\big)\Big)\end{aligned}$$

So the total error is the sum of this term times $\frac{4|\mathcal{A}_{R_1,R_2}|}{r_2^2}$ plus the error in (2.27). In view of our assumptions on $r_2$ the lemma is proved. ∎

### (f) The Weierstrass preparation theorem for Dirichlet determinants

Recall the Weierstrass preparation theorem for an analytic function $f(z,w_1,\dots,w_d)$ defined in a polydisk

$$\mathcal{P}=\mathcal{D}(z_0,R_0)\times\prod_{j=1}^d\mathcal{D}(w_{j,0},R_0),\quad z_0,\ w_{j,0}\in\mathbb{C}\qquad\frac12\ge R_0>0\ . \tag{2.30}$$

**Theorem 2.25.** *Assume that $f(\cdot,w_1,\dots,w_d)$ has no zeros on some circle $\{z:|z-z_0|=\rho_0\}$, $0<\rho_0<R_0/2$, for any $\underline{w}=(w_1,\dots,w_d)\in\mathcal{P}_1=\prod_{j=1}^d\mathcal{D}(w_{j,0},r_1)$ where $0<r_1<R_0$. Then there exist a polynomial $P(z,\underline{w})=z^k+a_{k-1}(\underline{w})z^{k-1}+\dots+a_0(\underline{w})$ with $a_j(\underline{w})$ analytic in $\mathcal{P}_1$ and an analytic function $g(z,\underline{w}),(z,\underline{w})\in\mathcal{D}(z_0,\rho_0)\times\mathcal{P}_1$ so that the following properties hold:*

(a) *$f(z,\underline{w})=P(z,\underline{w})g(z,\underline{w})$ for any $(z,\underline{w})\in\mathcal{D}(z_0,\rho_0)\times\mathcal{P}_1$.*
(b) *$g(z,\underline{w})\neq0$ for any $(z,\underline{w})\in\mathcal{D}(z_0,\rho)\times\mathcal{P}_1$*
(c) *For any $\underline{w}\in\mathcal{P}_1$, $P(\cdot,\underline{w})$ has no zeros in $\mathbb{C}\setminus\mathcal{D}(z_0,\rho_0)$.*

*Proof.* By the classical Weierstrass argument,

$$b_p(\underline{w}) := \sum_{j=1}^{k} \zeta_j^p(\underline{w}) = \frac{1}{2\pi i} \oint_{|z-z_0|=\rho_0} z^p \, \frac{\partial_z f(z,\underline{w})}{f(z,\underline{w})} \, dz$$

are analytic in $\underline{w} \in \mathcal{P}_1$. Here $\zeta_j(\underline{w})$ are the zeros of $f(\cdot,\underline{w})$ in $\mathcal{D}(z_0,\rho_0)$. Since the coefficients $a_j(\underline{w})$ are linear combinations of the $b_p$, they are analytic in $\underline{w}$. Analyticity of $g$ follows by standard arguments. $\square$

Since there is an estimate for the local number of the zeros of the Dirichlet determinant and also the local number of the Dirichlet eigenvalues, one can apply Theorem 2.25 to $f_N(z,\omega,E)$. We need to do this in both the $z$ and the $E$ variables. See Section 6 of [GolSch2] for more details.

**Proposition 2.26.** *Given $z_0 \in \mathcal{A}_{\rho_0/2}$, $E_0 \in \mathbb{C}$, and $\omega_0 \in \mathbb{T}_{c,a}$, there exist a polynomial*

$$P_N(z,\omega,E) = z^k + a_{k-1}(\omega,E) z^{k-1} + \cdots + a_0(E,\omega)$$

*with $a_j(\omega,E)$ analytic in $\mathcal{D}(E_0,r_1) \times \mathcal{D}(\omega_0,r_1)$, $r_1 \asymp \exp\left(-(\log N)^{A_1}\right)$ and an analytic function*

$$g_N(z,\omega,E), \quad (z,\omega,E) \in \mathcal{P} = \mathcal{D}(z_0,r_0) \times \mathcal{D}(E_0,r_1) \times \mathcal{D}(\omega_0,r_1)$$

*with $r_0 \asymp N^{-1}$ such that:*

(a) $f_N(z,\omega,E) = P_N(z,\omega,E) g_N(z,\omega,E)$
(b) $g_N(z,\omega,E) \neq 0$ *for any* $(z,\omega,E) \in \mathcal{P}$
(c) *For any* $(\omega,E) \in \mathcal{D}(\omega_0,r_1) \times \mathcal{D}(E_0,r_1)$*, the polynomial* $P_N(\cdot,\omega,E)$ *has no zeros in* $\mathbb{C} \setminus \mathcal{D}(z_0,r_0)$
(d) $k = \deg P_N(\cdot,\omega,E) \leq (\log N)^A$.

The preparation theorem relative to $E$ is easier since we need it only in the neighborhood of the unit circle, i.e., in the neighborhood of points $e(x_0)$ with $x_0 \in \mathbb{T}$. In this case, one can use the fact that $H_N(e(x_0),\omega)$ is self-adjoint.

**Proposition 2.27.** *Given $x_0 \in \mathbb{T}$, $E_0 \in \mathbb{C}$, and $\omega_0 \in \mathbb{T}_{c,a}$, there exist a polynomial*

$$P_N(z,\omega,E) = E^k + a_{k-1}(z,\omega) E^{k-1} + \cdots + a_0(z,\omega)$$

*with $a_j(z,\omega)$ analytic in $\mathcal{D}(z_0,r_1) \times \mathcal{D}(\omega_0,r_1)$, $z_0 = e(x_0)$, $r_1 \asymp \exp\left(-(\log N)^{A_1}\right)$ and an analytic function $g_N(z,\omega,E)$, $(z,\omega,E) \in \mathcal{P} = \mathcal{D}(z_0,r_1) \times \mathcal{D}(\omega_0,r_1) \times \mathcal{D}(E_0,r_1)$ such that*

(a) $f_N(z,\omega,E) = P_N(z,\omega,E) g_N(z,\omega,E)$
(b) $g_N(z,\omega,E) \neq 0$ *for any* $(z,\omega,E) \in \mathcal{P}$
(c) *For any* $(z,\omega) \in \mathcal{D}(z_0,r_1) \times \mathcal{D}(\omega_0,r_1)$, $P_N(z,\cdot,\omega)$ *has no zeros in* $\mathbb{C} \setminus \mathcal{D}(E_0,r_0)$, $r_0 \asymp \exp\left(-(\log N)^{A_0}\right)$
(d) $k = \deg P_N(z,\cdot,\omega) \leq (\log N)^{A_2}$

*Proof.* Recall that due to Proposition 2.16 one has

$$\#\left\{E \in \mathbb{C} : f_N(z_0,\omega_0,E) = 0, \quad |E - E_0| < \exp\left(-(\log N)^A\right)\right\} \leq (\log N)^{A_2}$$

Find $r_0 \asymp \exp\left(-(\log N)^{A_0}\right)$ such that $f_N(z_0,\omega_0,\cdot)$ has no zeros in the annulus

$$\left\{r_0(1 - 2N^{-2}) < |E - E_0| < r_0(1 + 2N^{-2})\right\}.$$

Since $H_N(z_0,\omega_0)$ is self adjoint, $f_N(z,\omega,\cdot)$ has no zeros in the annulus

$$\left\{r_0(1 - N^{-2}) < |E - E_0| < r_0(1 + N^{-2})\right\},$$

provided $|z - z_0| \ll r_1 = r_0 N^{-4}$, $|\omega - \omega_0| \ll r_1$. The proposition now follows from Theorem 2.25. $\square$

**(g) Eliminating close zeros using resultants**

Let $f(z) = z^k + a_{k-1}z^{k-1} + \cdots + a_0$, $g(z) = z^m + b_{m-1}z^{m-1} + \cdots + b_0$ be polynomials, $a_i, b_j \in \mathbb{C}$. Let $\zeta_i$, $1 \le i \le k$ and $\eta_j$, $1 \le j \le m$ be the zeros of $f(z)$ and $g(z)$, respectively. The resultant of $f$ and $g$ is defined as follows:

$$\mathrm{Res}(f,g) = \prod_{i,j}(\zeta_i - \eta_j)$$

The discriminant of the polynomial $f$ is defined as

$$\mathrm{disc}\, f = \prod_{i \ne j}(\zeta_i - \zeta_j) \ .$$

One has also

$$\mathrm{disc}\, f = (-1)^{n(n-1)/2}\, \mathrm{Res}(f, f') \ .$$

The resultant $\mathrm{Res}(f,g)$ can be found explicitly in terms of the coefficients, see [Lan], page 200:

$$\mathrm{Res}(f,g) = \begin{vmatrix} \overbrace{\begin{matrix} 1 & 0 & \cdots \\ a_{k-1} & 1 & \cdots \\ a_{k-2} & a_{k-1} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_0 & a_1 & \\ 0 & a_0 & \end{matrix}}^{m} & \overbrace{\begin{matrix} 1 & 0 & \cdots & 0 \\ b_{m-1} & 1 & \cdots & \cdots \\ b_{m-2} & b_{m-1} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ & & & \\ & & & \end{matrix}}^{k} \end{vmatrix} \tag{2.31}$$

In particular, one has the following property:

**Lemma 2.28.** *Let $f(z;w) = z^k + a_{k-1}(w)z^{k-1} + \cdots + a_0(w)$, $g(z;w) = z^m + b_{m-1}(w)z^{m-1} + \cdots + b_0(w)$ be polynomials whose coefficients $a_i(w)$, $b_j(w)$ are analytic functions defined in a domain $G \subset \mathbb{C}^d$. Then $\mathrm{Res}(f(\cdot,w), g(\cdot,w))$ is analytic in $G$.*

Our goal here is to separate the zeros of two analytic functions using the resultants by means of shifts in the argument, see Section 7 of [GolSch2], in particular Lemma 7.4. This can be reduced to the same question for polynomials due to the Weierstrass preparation theorem. Here is a simple observation regarding the resultant of a polynomial and a shifted version of another polynomial.

**Lemma 2.29.** *Let $f(z) = z^k + a_{k-1}z^{k-1} + \cdots + a_0$, $g(z) = z^m + b_{m-1}z^{m-1} + \cdots + b_0$ be polynomials. Then*

$$\mathrm{Res}\,(f(\cdot + w), g(\cdot)) = (-w)^n + c_{n-1}w^{n-1} + \cdots + c_0 \tag{2.32}$$

*where $n = km$, and $c_0, c_1 \cdots$ are some coefficients.*

*Proof.* Let $\zeta_j$, $1 \le j \le k$ (resp. $\eta_i, 1 \le i \le m$) be the zeros of $f(\cdot)$ (resp. $g(\cdot)$). The zeros of $f(\cdot + w)$ are $\zeta_j - w$, $1 \le j \le k$. Hence

$$\mathrm{Res}\,(f(\cdot + w), g(\cdot)) = \prod_{i,j}(\zeta_j - w - \eta_i)$$

and (2.32) follows. □

Due to the basic definition of the resultant, one has

$$\chi(\eta, \underline{w}) = \prod_{i,j} |\zeta_{i,1}(\underline{w}) - \zeta_{j,2}(\eta, \underline{w})|$$

where $\zeta_{i,1}(\underline{w})$, $\zeta_{j,2}(\eta, \underline{w})$ are the zeros of $P_1(\cdot, \underline{w})$ and $P_2(\cdot + \eta, \underline{w})$, respectively. Therefore, if $\big|\zeta_{i,1}(\underline{w}) - \zeta_{j,2}(\underline{w})\big| < \exp(-kH)$, then $\big|\chi(\eta, \underline{w})\big| < \exp(-kH)$. That allows one to separate the zeros $\zeta_{i,1}(\underline{w})$ from the zeros $\zeta_{j,2}(\underline{w})$ provided $\underline{w}$ falls outside of a set whose measure and complexity is controlled by Cartan's estimate.

Due to the Weierstrass preparation theorem this method can be applied to the Dirichlet determinants $f_{\ell_1}(\cdot,\omega,E)$ and $f_{\ell_2}(\cdot e(t\omega),\omega,E)$. We now state a result in this direction, see Section 8 of [GolSch2]. We shall use the following notation

$$\mathcal{Z}(f,\Omega)=\{z\in\Omega: f(z)=0\}$$

where $\Omega\subset\mathbb{C}$ and

$$\mathcal{Z}(f,z_0,r_0)=\mathcal{Z}(f,\mathcal{D}(z_0,r_0))$$

**Proposition 2.30.** *Assume that $\gamma_0=L(\omega_0,E_0)>0$ for some $\omega_0\in\mathbb{T}_{c,a}$, $E_0\in\mathbb{R}$. There exist $N_0=N_0(\lambda_0,V,\omega_0,\gamma_0)$, $\rho^{(0)}=\rho^{(0)}(\lambda,V,\omega_0,\gamma_0)$ such that for any $\ell_1\ge\ell_2>N_0$ the following holds: Given $t>\exp\Big(\big(\log\ell_1\big)^A\Big)$, $H>1$, there exists a set $\Omega_{\ell_1,\ell_2,t,H}\subset\mathbb{T}$, with*

$$\operatorname{mes}\,(\Omega_{\ell_1,\ell_2,t,H})<C(\lambda,V,\omega_0,\gamma_0)e^{-\sqrt{H}}$$

$$\operatorname{compl}(\Omega_{\ell_1,\ell_2,t,H})<C(\lambda,V,\omega_0,\gamma_0)t^2H$$

*such that for any $\omega\in(\mathbb{T}_{c,a}\setminus\Omega_{\ell_1,\ell_2,t,H})\cap\big(\omega_0-\rho^{(0)},\omega_0+\rho^{(0)}\big)$ there exists a set $\mathcal{E}_{\ell_1,\ell_2,t,H,\omega}$ with*

$$\operatorname{mes}\,(\mathcal{E}_{\ell_1,\ell_2,t,H,\omega})<C(\lambda,V,\omega_0,\gamma_0)te^{-\sqrt{H}}$$

$$\operatorname{compl}(\mathcal{E}_{\ell_1,\ell_2,t,H,\omega})<C(\lambda,V,\omega_0,\gamma_0)t^2H^2$$

*such that for any $E\in\big(E_0-\rho^{(0)},E_0+\rho^{(0)}\big)\setminus\mathcal{E}_{\ell_1,\ell_2,t,H,\omega}$ one has*

$$\operatorname{dist}\big(\mathcal{Z}\big(f_{\ell_1}(\cdot,\omega,E),\mathcal{A}_{\rho^{(0)}}\big),\mathcal{Z}\big(f_{\ell_2}(\cdot e(t\omega),\omega,E),\mathcal{A}_{\rho^{(0)}}\big)\big)>e^{-H(\log\ell_1)^B}\ .$$

*Here $\mathcal{A}_{\rho^{(0)}}=\big\{z\in\mathbb{C}\,:\,1-\rho^{(0)}<|z|<1+\rho^{(0)}\big\}$, and $A,B$ are large constants.*

For the proof see Section 8 of [GolSch2].

### (h) Harnack's inequality, Jensen's formula for the logarithm of the norms of monodromy matrices, and counting zeros of Dirichlet determinants

The logarithm of the norm of an analytic matrix-function is a subharmonic function. Harnack's estimate in this context is not as sharp as for the logarithm of the modulus of an analytic function. The same comment applies to Jensen's averages.

We now describe how these technical issues were addressed in [GolSch2] for the monodromy matrices. The reader should not be distracted by technicalities, but rather notice how the norms of the matrices mimic the behavior of the entries. For the latter the crucial piece of information is the number of zeros in various disks. The results of this section can be found in Sections 12, and 13 of [GolSch2].

**Proposition 2.31.** (i) *Suppose that one of the Dirichlet determinants*

$$f_{[1,N]}(\cdot,\omega,E),\ f_{[1,N-1]}(\cdot,\omega,E),\ f_{[2,N]}(\cdot,\omega,E),\ f_{[2,N-1]}(\cdot,\omega,E)$$

*has no zeros in $\mathcal{D}(z_0,r_1)$, $\exp(-\sqrt{N})\le r_1\le\exp\big(-(\log N)^C\big)$. Then*

$$\Big|\log\frac{\big\|M_N(z,\omega,E)\big\|}{\big\|M_N(z_0,\omega,E)\big\|}-\log\big|1+a_0(z-z_0)\big|\Big|\le|z-z_0|^2r_2^{-2}\tag{2.33}$$

*for any $z\in\mathcal{D}(z_0,r_2)$, $r_2=r_1\exp\big(-(\log N)^{2C}\big)$, and with $|a_0|\lesssim r_2^{-1}$.*

(ii) *Assume that the following conditions are valid*

(a) *each of the determinants $f_{[a,N-b]}(\cdot,\omega,E)$, $a=1,2$; $b=0,1$ has at least one zero in $\mathcal{D}(\zeta_0,\rho_0)$, where $e^{-\sqrt{N}}\le\rho_0\le\exp\big(-(\log N)^{B_0}\big)$*

(b) *no determinant $f_{[a,N-b]}(\cdot,\omega,E)$ has a zero in $\mathcal{D}(\zeta_0,\rho_1)\setminus\mathcal{D}(\zeta_0,\rho_0)$, $\rho_1\ge\exp\big((\log N)^{B_1}\big)\rho_0$, $B_0\gg B_1+A$.*

*Let* $k_0 = \min_{a,b} \mathcal{Z}(f_{[a,N-b]}(\cdot,\omega,E),\zeta_0,\rho_0)$. *Then for any*

$$z,\zeta \in \mathcal{D}(\zeta_0,\rho_1') \setminus \mathcal{D}(\zeta_0,\rho_2),\ \rho_1' = \exp\bigl(-(\log N)^{B_2}\bigr)\rho_1,\ \rho_2 = \exp\bigl((\log N)^{B_2}\bigr)\rho_0, \quad B_1 \gg B_2 \gg 1$$

*one has*

$$\Bigl|\log\frac{\|M_N(\zeta,\omega,E)\|}{\|M_N(z,\omega,E)\|} - k_0\log\frac{|\zeta-\zeta_0|}{|z-\zeta_0|}\Bigr| \le \exp\bigl(-(\log N)^C\bigr)$$

Next, we discuss Jensen averages.

**Proposition 2.32.** (i) *Assume that one of the Dirichlet determinants* $f_{[a,N-b]}(\cdot,\omega,E)$, $a=1,2$, $b=0,1$ *has no zeros in* $\mathcal{D}(z_0,r_1)$, $\exp(-\sqrt{N}) \le r_1 \le \exp\bigl(-(\log N)^{C_1}\bigr)$. *Then*

$$4\frac{\rho_1^2}{\rho_2^2}\mathcal{J}\left(\log\|M_N(\cdot,\omega,E)\|,z_0,\rho_1,\rho_2\right) \le \rho_1^2 r_1^{-2}\exp\bigl((\log N)^B\bigr) \tag{2.34}$$

*for any* $r_1\exp(-\sqrt{N}) \le \rho_1 \le r_1\exp\bigl(-(\log N)^A\bigr)$, $\rho_2 = c\rho_1$

(ii) *Assume that for some* $\zeta_0$ *the following conditions are valid*

(a) *each of the determinants* $f_{[a,N-b]}(\cdot,\omega,E)$, $a=1,2$; $b=0,1$ *has at least one zero in* $\mathcal{D}(\zeta_0,\rho_0)$, $\exp(-\sqrt{N}) < \rho_0 \le \exp\bigl(-(\log N)^{B_0}\bigr)$.

(b) *no determinant* $f_{[a,N-b]}(\cdot,\omega,E)$ *has a zero in* $\mathcal{D}(\zeta_0,\rho_1)\setminus\mathcal{D}(\zeta_0,\rho_0)$, $\rho_1 \ge \exp\bigl((\log N)^{B_1}\bigr)\rho_0$, $B_0 > B_1$.

*Let* $k_0 = \min_{a,b} \#\mathcal{Z}(f_{[a,N-b]}(\cdot,\omega,E),\zeta_0,\rho_0)$. *Then for any*

$$z_1 \in \mathcal{D}(\zeta_0,\rho_1')\setminus\mathcal{D}(\zeta_0,\rho_2),\ \rho_1' = \exp\bigl(-(\log N)^{B_2}\bigr)\rho_1,\ \rho_2 \asymp \exp\bigl((\log N)^{B_2}\bigr)\rho_0,$$

$B_1 > B_2$, *one has*

$$\Bigl|4\frac{r_1^2}{r_2^2}\mathcal{J}\left(\log\|M_N(\cdot,\omega,E)\|,z_1,r_1,r_2\right) - k_0\Bigr| \le \exp\bigl(-(\log N)^C\bigr)$$

*where* $|z_1-\zeta_0|(1+2c) < r_1 < \rho_1'$, $r_2 = cr_1$, *and* $0 < c \ll 1$ *is some constant.*

**Remark 2.33.** *The estimates of Propositions 2.30 and 2.31 are established in* [GolSch2] *not just for the mondromies* $M_N(z,\omega,E)$ *but for general analytic matrix functions which obey a certain abstract form of the large deviation theorem, see condition (I) in Sections 12,13 of that paper. Due to Remark 2.7 the matrix function* $E \mapsto M_N(e(x),\omega,E)$ *obeys estimates* (2.6), *provided* $x \notin \mathcal{B}_{N,\omega,H}$, $\mathrm{mes}\,(\mathcal{B}_{N,\omega,H}) < \exp(-H/2(\log N)^A)$ *for any* $H \gg (\log N)^A$. *That allows one to establish estimates for* $\log\|M_N(e(x),\omega,\cdot)\|$ *which are analogous to those of Propositions 2.31 and 2.32 provided* $x \notin \mathcal{B}_{N,\omega,H}$.

Now we describe how to combine the last proposition with the avalanche principle expansion to count precisely the number of the zeros of Dirichlet determinants. The following definition is very important in this regard.

**Definition 2.34.** *Let* $\ell \gg 1$ *be some integer, and* $s \in \mathbb{Z}$. *We say that* $s$ *is* adjusted *to a disk* $\mathcal{D}(z_0,r_0)$ *at scale* $\ell$ *if for all* $k \asymp \ell$

$$\mathcal{Z}(f_k(\cdot e((s+m)\omega),\omega,E),z_0,r_0) = \emptyset \qquad \forall\,|m| \le C\ell.$$

Consider the avalanche principle expansion of $\log\bigl|f_N(z,\omega,E)\bigr|$:

$$\log\bigl|f_N(z,\omega,E)\bigr| = \sum_{m=1}^{n-1}\log\bigl\|A_{m+1}(z)A_m(z)\bigr\| - \sum_{m=2}^{n-1}\log\bigl\|A_m(z)\bigr\| + O\left(\exp\bigl(-\ell^{1/2}\bigr)\right)\ , \tag{2.35}$$

for any $z \in \mathcal{A}_{\rho_0/2}\setminus\mathcal{B}_{N,\omega,E}$, $\mathrm{mes}\,\mathcal{B}_{N,\omega,E} \le \exp\bigl(-\ell^{1/2}\bigr)$, where $A_m(z) = M_{\ell_m}\bigl(ze(s_m\omega),\omega,E+i\eta\bigr)$, $m = 2,\dots,n-1$, $A_1(z) = M_{\ell_1}(z,\omega,E)\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $A_n(z) = \begin{bmatrix}1&0\\0&0\end{bmatrix}M_{\ell_n}\bigl(ze(s_n\omega),\omega,E\bigr)$, $\ell_m \asymp \ell$, $m=1,2,\dots,n$, $\sum_{m=1}^n \ell_m = N$, $\ell \asymp (\log N)^A$, $s_m = \sum_{j<m}\ell_j$.

This expansion allows us to control the number of zeros of the large scale object (in this case $f_N$) by means of the number of zeros (or rather, the Jensen averages) of the small-scale objects (here $w_j$, see below) and vice versa.

**Lemma 2.35.** *Assume that $\{s_{m_j}\}_{j=1}^{j_0}$ is adjusted to $\mathcal{D}(z_0, r_0)$ at scale $\ell$. Set $m_0 = 0$, $m_{j_0+1} = n$, and*

$$w_j(z) = \log\Big\| \prod_{m=m_{j+1}}^{m_j+1} A_m(z)\Big\| \quad \text{for any} \quad 0 \le j \le j_0$$

*Then*

$$4\frac{r_1^2}{r_2^2}\Big|\mathcal{J}\left(\log\left|f_N(\cdot,\omega,E)\right|, z_0, r_1, r_2\right) - \sum_{j=0}^{j_0}\mathcal{J}(w_j(\cdot), z_0, r_1, r_2)\Big| \le N\exp\big((\log\ell)^C\big)\, r_1^2 r_0^{-2} \tag{2.36}$$

*for any $e^{-\sqrt{\ell}} < r_1 \lesssim \exp(-\ell^\delta)r_0$, and $r_2 = cr_1$. Here $0 < \delta \ll 1$ is arbitrary but fixed. In particular,*

$$4\frac{r_1^2}{r_2^2}\,\mathcal{J}\left(\log\left|f_N(\cdot,\omega,E)\right|, z_0, r_1, r_2\right) \ge 4\frac{r_1^2}{r_2^2}\sum_{j\in\mathcal{J}}\,\mathcal{J}\big(w_j(\cdot), z_0, r_1, r_2\big) - N\exp\big((\log\ell)^C\big)\, r_1^2 r_0^{-2} \tag{2.37}$$

*for any $\mathcal{J} \subset [0, j_0]$.*

**Corollary 2.36.** *Using the notations of Lemma 2.35 assume in addition that*

$$\mathcal{Z}\left(f_k\big(\cdot e(s_j\omega),\omega,E\big), \mathcal{D}(z_0, 2r_1)\setminus\mathcal{D}(z_0, r_1/2)\right) = \emptyset$$

$$\mathcal{Z}\left(f_{t_j}\big(\cdot e(s_{m_j}\omega),\omega,E\big), \mathcal{D}(z_0, 2r_1)\setminus\mathcal{D}(z_0, r_1/2)\right) = \emptyset, \quad t_j = s_{m_{j+1}} - s_{m_j}$$

*for all $s_j$, $j = 1, 2, \dots,$ and $k \asymp \ell$. Then*

$$\nu_{f_N(\cdot,\omega,E)}(z_0, r_1) = \sum_{j=0}^{j_0}\nu_{f_{t_j}\left(\cdot e(s_{m_j}\omega),\omega,E\right)}(z_0, r_1)$$

*In particular, if every $1 \le s \le N$ is adjusted to $\mathcal{D}(z_0, r_0)$ at scale $\ell$, then $\nu_{f_N(\cdot,\omega,E)}(z_0, r_1) = 0$.*

*Proof.* Applying the avalanche principle expansion one obtains

$$\log|f_{t_j}\big(ze(s_{m_j}\omega),\omega,E\big)| = \sum_{m=m_j+1}^{m_{j+1}-1}\log\|\tilde{A}_{m+1}(z)\tilde{A}_m(z)\| - \sum_{m=m_j+2}^{m_{j+1}-1}\log\|\tilde{A}_m(z)\| + 0\big(\exp(-\ell^{1/2})\big),$$

as well as

$$w_j(z) = \sum_{m=m_j+1}^{m_{j+1}-1}\log\|A_{m+1}(z)A_m(z)\| - \sum_{m=m_j+2}^{m_{j+1}-1}\log\|A_m(z)\| + 0\big(\exp(-\ell^{1/2})\big)$$

for any $z \in \mathcal{A}_{\rho_0/2}\setminus\mathcal{B}_{N,\omega,E}$, where $A_m(z), m = 1, 2, \cdots, n$ and $\mathcal{B}_{N,\omega,E}$ are the same as in (2.35), $\tilde{A}_m(z) = A_m(z), m_j + 1 < m < m_{j+1}, \tilde{A}_{m_j+1}(z) = A_m(z)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \tilde{A}_{m_{j+1}}(z) = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}A_m(z)$.

Subtracting these expansion and evaluating the Jensen averages with the use of Proposition 2.32 one obtains

$$4\frac{r_1^2}{r_2^2}\Big|\mathcal{J}\left(\log\left|f_{t_j}\big(\cdot e(s_{m_j}\omega),\omega,E\big)\right|, z_0, r_1, r_2\right) - \mathcal{J}\big(w_j(\cdot), z_0, r_1, r_2\big)\Big| \le \exp\big((\log\ell)^C\big)\, r_1^2 r_0^{-2}$$

for $j = 0, 1, \cdots, j_0$. Hence,

$$4\frac{r_1^2}{r_2^2}\Big|\mathcal{J}\left(\log\left|f_n(\cdot,\omega,E)\right|, z_0, r_1, r_2\right) - \mathcal{J}\big(\log\big|\prod_{j=0}^{j_0} f_{t_j}\big(\cdot e(s_{m_j}\omega),\omega,E\big)\big|, z_0, r_1, r_2\big)\Big| \le 2N\exp\big((\log\ell)^C\big) r_1^2 r_0^{-2}$$

due to the additivity of the Jensen's averages $\mathcal{J}(\cdot, z_0, r_1, r_2)$. We can adjust the parameters in such a way that the right-hand side here is much less than one. Hence, by Corollary 2.21

$$\nu_{f_n(\cdot,\omega,E)}(z_0, r_1 - r_2) \le \nu_g(z_0, r_1 + r_2),\ \nu_g(z_0, r_1 - r_2) \le \nu_{f_n(\cdot,\omega,E)}(z_0, r_1 + r_2),$$

where

$$g(z) = \prod_{j=0}^{j_0} f_{t_j}\big(ze(s_{m_j}\omega), w, E\big)$$

Replacing $r_1$ by $(r_1 \pm r_2)$, one obtains similarly

$$\nu_{f_N(\cdot,\omega,E)}(z_0, r_1) \le \nu_g(z_0, r_1 + 2r_2), \qquad \nu_g(z_0, r_1 - 2r_2) \le \nu_{f_N(\cdot,\omega,E)}(z_0, r_1)$$

Due to the assumption of the lemma

$$\nu_g(z_0, r_1 + 2r_2) = \nu_g(z_0, r_1 - 2r_2)$$

and the assertion follows. □

One can establish similar results in regards to counting the zeros of $f_N(e(x), \omega, E)$ in the $E$-variable.

**Definition 2.37.** *Let $\ell \gg 1$ be some integer, and $s \in \mathbb{Z}$. We say that $s$ is adjusted to a polydisk $\mathcal{D}(e(x_0), \underline{r}_0) \times \mathcal{D}(E_0, r_0)$ at scale $\ell$ if for all $k \asymp \ell$*

$$\mathcal{Z}(f_k(e(x + (s + m)\omega), \omega, \cdot), E_0, r_0) = \emptyset \quad \forall\, |m| \le C\ell$$

*for any $z \in \mathcal{D}(z_0, \underline{r}_0)$. Here $z_0 = e(x_0)$, $x_0 \in \mathbb{T}$, $E_0 \in \mathbb{R}$.*

Consider the avalanche principle expansion (2.35) for arbitrary $E \in \mathcal{D}(E_0, r_0)$. Then, there exists $\mathcal{B}_{N,\omega} \subset \mathbb{T}$ with $\operatorname{mes}(\mathcal{B}_{N,\omega}) < \exp(-\sqrt{\ell}/2)$ such that for any $z = e(x) \in \mathcal{D}(z_0, r_0) \setminus \mathcal{B}_{N,\omega}$ there exists $\mathcal{E}_{N,x,\omega}$ with $\operatorname{mes}(\mathcal{E}_{N,x,\omega}) < \exp(-\sqrt{\ell}/2)$ such that the expansion (2.35) is valid for any $E \in \mathcal{D}(E_0, r_0) \setminus \mathcal{E}_{N,x,\omega}$. Due to Remark 2.33 one can evaluate the Jensen averages in (2.35) with respect to $E$ as in Lemma 2.35 and Corollary 2.36, provided $z = e(x) \in \mathcal{D}(z_0, r_0) \setminus \mathcal{B}_{N,\omega}$ is fixed. That leads to the following result:

**Lemma 2.38.** *Assume that $\{s_{m_j}\}_{j=1}^{j_0}$ are adjusted to $\mathcal{D}(e(x_0), \underline{r}_0) \times \mathcal{D}(E_0, r_0)$ at scale $\ell$. Assume also that*

$$\mathcal{Z}\left(f_{t_j}\big(e(x + m_{s_j}\omega), \omega, \cdot\big), \mathcal{D}(E_0, 2r_1) \setminus \mathcal{D}(E_0, r_1/2)\right) = \emptyset, \quad t_j = s_{m_{j+1}} - s_{m_j}$$

*for any $e(x) \in \mathcal{D}(e(x_0), \underline{r}_0)$, where $e^{-\sqrt{\ell}/4} < r_1 \lesssim \exp\left(-\ell^\delta\right) r_0$. Then*

$$\nu_{f_N\left(e(x),\omega,\cdot\right)}(E_0, r_1) = \sum_{j=0}^{j_0} \nu_{f_{\ell_{m_j}}\left(e(x+s_{m_j}\omega),\omega,\cdot\right)}(E_0, r_1) \tag{2.38}$$

*for any $e(x) \in \mathcal{D}(e(x_0), \underline{r}_0) \setminus \mathcal{B}_{N,\omega}$, $\operatorname{mes} \mathcal{B}_{N,\omega} < \exp(-\sqrt{\ell}/2)$. Moreover, if $\underline{r}_0 < cr_1$, then (2.38) for an $e(x) \in \mathcal{D}(e(x_0), \underline{r}_0)$.*

*In particular, if every $1 \le s \le N$ is adjusted to $\mathcal{D}(e(x_0), \underline{r}_0) \times \mathcal{D}(E_0, r_0)$ at scale $\ell$, then*

$$\nu_{f_N\left(e(x),\omega,\cdot\right)}(E_0, r_1) = 0$$

*for any $e(x) \in \mathcal{D}(e(x_0), \underline{r}_0) \setminus \mathcal{B}_{N,\omega}$ as above.*

## 3. Localized eigenfunctions of the problem on a finite interval

In this section we apply the results of the previous sections to the study of the eigenfunctions of the Hamiltonian restricted to intervals on the integer lattice. More precisely, we shall obtain a finite-volume version of Anderson localization (albeit, at the expense of removing a small set of energies). This section corresponds to Section 9 of [GolSch2].

**Lemma 3.1.** *Let $\omega \in \mathbb{T}_{c,a}$. Suppose $L(\omega, E_0) > 0$,*

$$\log \big| f_N(z_0, \omega, E_0)\big| > NL(\omega, E_0) - K/2 \tag{3.1}$$

*for some $z_0 = e(x_0)$, $x_0 \in \mathbb{T}$, $E_0 \in \mathbb{R}$, $N \gg 1$, $K > (\log N)^A$. Then*

$$\big|\mathcal{G}_{[1,N]}(z_0, \omega, E)(j,k)\big| \le \exp\Big(-\frac{\gamma}{2}(k-j) + K\Big) \tag{3.2}$$

$$\big\|\mathcal{G}_{[1,N]}(z_0, \omega, E)\big\| \le \exp(K) \tag{3.3}$$

*where $\mathcal{G}_{[1,N]}(z_0, \omega, E_0) = (H(z_0, \omega) - E_0)^{-1}$ is the Green function, $\gamma = L(\omega, E_0)$, $1 \le j \le k \le N$.*

*Proof.* By Cramer's rule and the uniform upper bound of Proposition 2.11 as well as the rate of convergence estimate (2.24),

$$\begin{aligned} \big|\mathcal{G}_{[1,N]}(z_0, \omega, E)(j,k)\big| = \big|f_{j-1}(z_0, \omega, E_0)\big| \cdot \big|f_{N-k}\big(z_0 e(k\omega), \omega, E_0\big)\big| \cdot \\ \big|f_N(z_0, \omega, E_0)\big|^{-1} \le \big|f_N(z_0, \omega, E_0)\big|^{-1} \\ \exp\big(NL(\omega, E_0) - (k-j)L(\omega, E_0) + (\log N)^C\big) \end{aligned} \tag{3.4}$$

Therefore, (3.2) follows from condition (3.1). Estimate (3.3) follows from (3.2). □

Any solution of the equation

$$-\psi(n+1) - \psi(n-1) + v(n)\psi(n) = E\psi(n)\ , \quad n \in \mathbb{Z}\ , \tag{3.5}$$

obeys the relation

$$\psi(m) = \mathcal{G}_{[a,b]}(E)(m, a-1)\psi(a-1) + \mathcal{G}_{[a,b]}(E)(m, b+1)\psi(b+1), \quad m \in [a,b]. \tag{3.6}$$

where $\mathcal{G}_{[a,b]}(E) = \big(H_{[a,b]} - E\big)^{-1}$ is the Green function, $H_{[a,b]}$ is the linear operator defined by (3.5) for $n \in [a,b]$ with zero boundary conditions. In particular, if $\psi$ is a solution of equation (3.5), which satisfies a zero boundary condition at the left (right) edge, i.e.,

$$\psi(a-1) = 0 \quad (\text{resp. } \psi(b+1) = 0)\ ,$$

then

$$\psi(m) = \mathcal{G}_{[a,b]}(m, b+1)\psi(b+1)$$
$$\big(\text{resp. } \psi(m) = \mathcal{G}_{[a,b]}(m, a-1)\psi(a-1)\ \big)$$

If, for instance, in addition $\big|\mathcal{G}(m, b+1)\big| < 1$, then $|\psi(m)| < |\psi(b+1)|$.

The following lemma states that after removal of certain rotation numbers $\omega$ and energies $E$, but uniformly in $x \in \mathbb{T}$, only one choice of $n \in [1,N]$ can lead to a determinant $f_\ell(x + n\omega, \omega, E)$ with $\ell \asymp (\log n)^C$ which is not large. This relies on the elimination results, see (g) in Section 2, and is of crucial importance for all our work.

**Lemma 3.2.** *Given $N$, there exists $\Omega_N \subset \mathbb{T}$ with*

$$\mathrm{mes}\,(\Omega_N) < \exp\big(-(\log N)^{C_2}\big)\ , \quad \mathrm{compl}(\Omega_N) < \exp\big((\log N)^{C_1}\big)\ ,$$

*$C_1 \ll C_2$ such that for all $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N$ there is $\mathcal{E}_{N,\omega} \subset \mathbb{R}$, $\mathrm{mes}\,(\mathcal{E}_{N,\omega}) < e^{-(\log N)^{C_2}}$, $\mathrm{compl}(\mathcal{E}_{N,\omega}) < e^{(\log N)^{C_1}}$, with the following property: For any $x \in \mathbb{T}$ and any $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N$, $E \in \mathbb{R} \setminus \mathcal{E}_{N,\omega}$ either*

$$\log \big| f_\ell\big(e(x + n\omega), \omega, E\big)\big| > \ell L(\omega, E) - \sqrt{\ell} \tag{3.7}$$

*for all $\ell \asymp (\log N)^C$ and all $1 \le n \le N$, or there exists $n_1 = n_1(x, \omega, E) \in [1,N]$ such that (3.7) holds for all $n \in [1,N] \setminus [n_1 - L, n_1 + L]$, $L \asymp \exp\big((\log\log N)^A\big)$, but not for $n = n_1$. However, in this case*

$$\big|f_{[1,n]}\big(e(x), \omega, E\big)\big| > \exp\big(nL(\omega, E) - (\log N)^C\big) \tag{3.8}$$

*for each $1 \le n \le n_1 - L$ and*

$$\big|f_{[n,N]}\big(e(x), \omega, E\big)\big| > \exp\big((N-n)L(\omega, E) - (\log N)^C\big) \tag{3.9}$$

*for each* $n_1 + L \le n \le N$*.*

*Proof.* Define $\Omega_N = \bigcup \Omega_{\ell_1,\ell_2,t,H}$ where the union runs over $\ell_1, \ell_2 \asymp (\log N)^C$, $N > t > \exp\big((\log\log N)^A\big)$ with fixed $H \asymp (\log N)^{C/100}$. Here $\Omega_{\ell_1,\ell_2,t,H}$ is as in Proposition 2.30. Similarly, for any $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N$ set

$$\mathcal{E}_{N,\omega} = \bigcup_{(\log N)^C \le k \le N} \mathcal{E}_{k,\omega}(H) \cup \bigcup \mathcal{E}_{\ell_1,\ell_2,t,H,\omega}$$

where the second union is the same as before, and where $\mathcal{E}_{k,\omega}(H)$ are as in Remark 2.19. The measure and complexity estimates follow from Corollary 2.18. Now suppose (3.7) does not hold. Then

$$\log\big|f_{\ell_1}\big(e(x+n_1\omega),\omega,E\big)\big| < \ell_1 L(\omega,E) - \sqrt{\ell_1}$$

for some $1 \le n_1 \le N$ and $\ell_1 \asymp (\log N)^C$. By Corollary 2.18 there exists $z_1$ with $|z_1 - e(x+n_1\omega)| < e^{-\ell_1^{\frac14}}$ and

$$f_{\ell_1}(z_1,\omega,E) = 0\ .$$

If

$$\log\big|f_{\ell_2}\big(e(x+n_2\omega),\omega,E\big)\big| < \ell_2 L(\omega,E) - \sqrt{\ell_2}$$

for some $\ell_2 \asymp (\log N)^C$ and $|n_2 - n_1| > \exp\big((\log\log N)^A\big)$, then for some $z_2$, and $t = n_1 - n_2$

$$f_{\ell_2}\left(z_2 e(t\omega),\omega,E\right) = 0$$

with $|z_1 - z_2| < e^{-(\log N)^C}$, which contradicts our choice of $(\omega,E)$, see Proposition 2.30. Thus (3.7) holds for all $\ell \asymp (\log N)^C$ and $1 \le n \le N$, $|n - n_1| > \exp\big((\log\log N)^A\big)$, as claimed. This allows one to apply the avalanche principle at scale $\ell \asymp (\log N)^C$ to $f_{[1,n]}\big(e(x),\omega,E\big)$ with $(\log N)^C \ll n \le n_1 - L$. It yields that

$$\log\big|f_{[1,n]}\big(e(x),\omega,E\big)\big| \ge nL(\omega,E) - C\frac{n}{(\log N)^C} > 0\ .$$

Note that by Corollary 2.14, if (3.7) holds at $x$, then also for all $z \in \mathcal{D}\big(e(x), e^{-\ell}\big)$. Thus,

$$f_{[1,n]}(z,\omega,E) \neq 0 \tag{3.10}$$

for those $z$ by the avalanche principle. Now suppose

$$\log\big|f_{[1,n]}\big(e(x),\omega,E\big)\big| \le nL(\omega,E) - (\log N)^B$$

for some large constant B. By our choice of $E$,

$$f_{[1,n]}(z,\omega,E) = 0$$

for some $|z - e(x)| < \exp\big(-(\log N)^{B/2}\big)$. This contradicts (3.10) provided $B$ is sufficiently large. Hence, (3.8) holds and (3.9) follows from a similar argument. □

**Remark 3.3.** *It follows from Corollary 2.14 that* (3.7) *is stable under perturbations of* $E$ *by an amount* $< e^{-C\ell}$*. More precisely, if* (3.7) *holds for* $E$*, then*

$$\log\big|f_\ell\big(e(x+n\omega),\omega,E'\big)\big| > \ell L(E',\omega) - 2\sqrt{\ell}$$

*for any* $E'$ *with* $|E'-E| < e^{-C\ell}$*. Inspection of the previous proof now shows that* (3.8) *and* (3.9) *are also stable under such perturbations.*

The previous lemma yields the following finite volume version of Anderson localization.

**Proposition 3.4.** *For any* $x, \omega \in \mathbb{T}$*, let* $\Big\{E_j^{(N)}(x,\omega)\Big\}_{j=1}^N$ *and* $\Big\{\psi_j^{(N)}(x,\omega,\cdot)\Big\}_{j=1}^N$ *denote the eigenvalues and normalized eigenvectors of* $H_{[1,N]}(x,\omega)$*, respectively. Let* $\Omega_N$ *and* $\mathcal{E}_{N,\omega}$ *be as in the previous lemma. If* $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N$ *and for some* $j$*,* $E_j^{(N)}(x,\omega) \notin \mathcal{E}_{N,\omega}$*, then there exists a point* $\nu_j^{(N)}(x,\omega) \in [1,N]$ *(which we call the center of localization) so that for any* $\exp\big((\log\log N)^A\big) \le Q \le N$ *and with* $\Lambda_Q := [1,N] \cap \Big[\nu_j^{(N)}(x,\omega) - Q, \nu_j^{(N)}(x,\omega) + Q\Big)$ *one has*

(i) $\operatorname{dist}\Big(E_j^{(N)}(x,\omega), \operatorname{spec}\big(H_{\Lambda_Q}(x,\omega)\big)\Big) < e^{-(\log N)^C}$

(ii) $\sum_{k\in[1,N]\setminus\Lambda_Q} \big|\psi_j^{(N)}(x,\omega;k)\big|^2 < e^{-Q\gamma/4}$, *where* $\gamma > 0$ *is a lower bound for the Lyapunov exponents.*

*Proof.* Fix $N$, $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N$ and $E_j^{(N)}(x,\omega) \notin \mathcal{E}_{N,\omega}$. Let $n_1 = \nu_j^{(N)}(x,\omega)$ be such that

$$\big|\psi_j^{(N)}(x,\omega;n_1)\big| = \max_{1\le n\le N} \big|\psi_j^{(N)}(x,\omega;n)\big| \ .$$

Fix some $\ell \asymp (\log N)^C$ and suppose that, with $E = E_j^{(N)}(x,\omega)$, and $\Lambda_0 := [1,N]\cap[n_1-\ell, n_1+\ell]$,

$$\log\big|f_{\Lambda_0}(x,\omega,E)\big| > |\Lambda_0|L(\omega,E) - \sqrt{\ell} \tag{3.11}$$

By Lemma 3.2

$$\big|G_{\Lambda_0}(x,\omega,E)(k,j)\big| < \exp\Big(-\frac{\gamma}{2}|k-j| + C\sqrt{\ell}\Big)$$

for all $k,j\in\Lambda_0$. But this contradicts the maximality of $\big|\psi_j^{(N)}(x,\omega;n_1)\big|$ due to (3.6). Hence (3.11) above fails, and we conclude from Lemma 3.2 that

$$\log\big|f_{\Lambda_1}(x,\omega,E)\big| > |\Lambda_1|L(\omega,E) - \sqrt{\ell}$$

for every $\Lambda_1 = [k-\ell, k+\ell]\cap[1,N]$ provided $|k-n_1| > \exp\big((\log\log N)^A\big)$. Since (3.11) fails, we conclude that $f_{\Lambda_0}(z_0,\omega,E) = 0$ for some $z_0$ with $|z_0 - e(x)| < e^{-\ell^{1/4}}$. By self-adjointness of $H_{\Lambda_0}(x,\omega,E)$ we obtain

$$\operatorname{dist}\big(E, \operatorname{spec}\big(H_{\Lambda_0}(x,\omega)\big)\big) < e^{-\ell^{1/4}} \ ,$$

as claimed (the same arguments applies to the larger intervals $\Lambda_Q$ around $n_0$). From (3.8) of the previous lemma with $n = n_1 - Q/2$ (if $n_1 - Q/2 < 1$, then proceed to the next case) one concludes that

$$\big|G_{[1,n_1-\frac12 Q]}(x,\omega,E)(k,m)\big| < \exp\big(-\gamma|k-m| + (\log N)^C\big) \tag{3.12}$$

for all $1\le k,m\le n_1 - \frac12 Q$. In particular,

$$\big|\psi_j^{(N)}(x,\omega;k)\big| < e^{-\frac{\gamma}{2}|n_1-\frac12 Q-k|}$$

for all $1\le k\le n_1 - Q$. Finally, the same reasoning applies to

$$G_{[n_1+\frac12 Q,N]}(x,\omega,E)$$

via (3.9) of the previous lemma, and (ii) follows. □

The following corollary deals with the stability of the localization statement of Proposition 3.4 with respect to the energy. As in previous stability results of this type in this paper, the most important issue is the relatively large size of the perturbation, i.e., $\exp(-(\log N)^B)$ instead of $e^{-N}$, say.

**Corollary 3.5.** *Let* $\Omega_N$, $\mathcal{E}_{N,\omega}$, $\Big\{E_j^{(N)}(x,\omega)\Big\}_{j=1}^N$, *and* $\Big\{\psi_j^{(N)}(x,\omega;\cdot)\Big\}_{j=1}^N$, *be as in the previous proposition. Then for any* $\omega\in\mathbb{T}_{c,a}\setminus\Omega_N$, *any* $x\in\mathbb{T}$, $E_j^{(N)}(x,\omega)\notin\mathcal{E}_{N,\omega}$ *let* $\nu_j^{(N)}(x,\omega)$ *be as in the previous proposition. For such* $\omega, E_j^{(N)}(x,\omega)$, *if* $|E - E_j^{(N)}(x,\omega)| < e^{-(\log N)^B}$ *with* $B$ *sufficiently large, then*

$$\sum_{n=1}^{\nu_j^{(N)}(x,\omega)-Q} \big|f_{[1,n]}\big(e(x),\omega,E\big)\big|^2 < e^{-c\gamma Q} \sum_{n\in\Lambda_Q} \big|f_{[1,n]}\big(e(x),\omega,E\big)\big|^2 \tag{3.13}$$

*where* $\Lambda_Q = \Big[\nu_j^{(N)}(x,\omega) - Q, \nu_j^{(N)}(x,\omega)+Q\Big]\cap[1,N]$. *Similarly,*

$$\sum_{n=\nu_j^{(N)}(x,\omega)+Q}^{N} \big|f_{[n,N]}(x,\omega,E)\big|^2 < e^{-c\gamma Q} \sum_{n\in\Lambda_Q} \big|f_{[n,N]}(x,\omega,E)\big|^2 \tag{3.14}$$

*Finally, under the same assumptions one has*

$$\big|f_{[1,n]}\left(e(x),\omega,E\right)-f_{[1,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\big|\le \exp\left((\log N)^C\right)\big|E-E_j^{(N)}(x,\omega)\big|\,\big|f_{[1,n]}(e(x),\omega,E_j^{(N)}(x,\omega))\big| \tag{3.15}$$

*provided* $1\le n\le \nu_j^{(N)}(x,\omega)-Q$*, and similarly for* $f_{[n,N]}$.

*Proof.* For each $j$ there exists a constant $\mu_j(x,\omega)$ so that

$$\psi_j^{(N)}(x,\omega;n)=\mu_j(x,\omega)f_{[1,n-1]}\left(x,\omega;E_j^{(N)}(x,\omega)\right)$$

for all $1\le n\le N$ (with the convention that $f_{[1,0]}=1$). A similar formula holds for

$$f_{[n+1,N]}\left(e(x),\omega,E_j^{(N)}(x,\omega)\right).$$

As in the previous proof, one obtains estimate (3.12) with $E=E_j^{(N)}(x,\omega)$. Thus, for $1\le n\le \nu_j^{(N)}(x,\omega)-Q$

$$\left|f_{[1,n]}\left(e(x),\omega,E_j^{(N)}(x,\omega)\right)\right|<e^{-c\gamma|\nu_j^{(N)}(x,\omega)-n|}\left|f_{[1,\nu_j^{(N)}(x,\omega)]}\left(e(x),\omega,E_j^{(N)}(x,\omega)\right)\right|,$$

which implies (3.13) for $E=E_j^{(N)}(x,\omega)$, and (3.14) follows by a similar argument for this $E$. Corollary 2.14 implies that

$$\begin{aligned}&\left|f_{[1,n]}\big(e(x),\omega,E\big)-f_{[1,n]}\left(e(x),\omega,E_j^{(N)}(x,\omega)\right)\right|\\&\le \exp\left((\log N)^C\right)\left|E-E_j^{(N)}(x,\omega)\right|\left|f_{[1,n]}\left(e(x),\omega,E_j^{(N)}(x,\omega)\right)\right|\end{aligned}$$

for all $1\le n\le \nu_j^{(N)}(x,\omega)-Q$, and (3.15) follows for all $\big|E-E_j^{(N)}(x,\omega)\big|<\exp\big(-(\log N)^B\big)$. □

## 4. Minimal distance between the Dirichlet eigenvalues on a finite interval

In this section it will be convenient for us to work with the operators $H_{[-N,N]}(x,\omega)$ instead of $H_{[1,N]}(x,\omega)$. Abusing our notation somewhat, we use the symbols $E_j^{(N)},\psi_j^{(N)}$ to denote the eigenvalues and normalized eigenfunctions of $H_{[-N,N]}(x,\omega)$, rather than the eigenvalues and normalized eigenfunctions of $H_{[1,N]}(x,\omega)$, as in the previous section. A similar comment applies to $\Omega_N,\mathcal{E}_{N,\omega}$.

The following proposition states that the eigenvalues $\{E_j^{(N)}(x,\omega)\}_{j=1}^{2N+1}$ are separated from each other by at least $e^{-N^\delta}$ provided $\omega\notin\Omega_N$ and provided we delete those eigenvalues that fall into a bad set $\mathcal{E}_{N,\omega}$ of energies. We remind the reader that

$$\mathrm{mes}\,(\mathcal{E}_{N,\omega})\lesssim\exp(-(\log N)^{A_2}),\quad \mathrm{compl}(\mathcal{E}_{N,\omega})\lesssim\exp((\log N)^{A_1}),$$

where $A_2\gg A_1$, and the same for $\Omega_N$, see Lemma 3.2. This section corresponds to Section 11 of [GolSch2].

**Proposition 4.1.** *For any* $\omega\in\mathbb{T}_{c,a}\setminus\Omega_N$ *and all* $x$ *one has for all* $j,k$ *and any small* $\delta>0$

$$\big|E_j^{(N)}(x,\omega)-E_k^{(N)}(x,\omega)\big|>e^{-N^\delta} \tag{4.1}$$

*provided* $E_j^{(N)}(x,\omega)\notin\mathcal{E}_{N,\omega}$ *and* $N\ge N_0(\delta)$.

*Proof.* Fix $x\in\mathbb{T},E_j^{(N)}(x,\omega)\notin\mathcal{E}_{N,\omega}$. Let $Q\asymp\exp\left((\log\log N)^C\right)$. By Proposition 3.4 there exists

$$\Lambda_Q:=\left[\nu_j^{(N)}(x,\omega)-Q,\nu_j^{(N)}(x,\omega)+Q\right]\cap[-N,N]$$

so that

$$\begin{aligned}&\sum_{n\in[-N,N]\setminus\Lambda_Q}\big|f_{[-N,n]}\big(e(x),\omega;E_j^{(N)}(x,\omega)\big)\big|^2\\&<e^{-2Q\gamma}\sum_{n=-N}^{N}\big|f_{[-N,n]}\big(e(x),\omega;E_j^{(N)}(x,\omega)\big)\big|^2.\end{aligned}\tag{4.2}$$

Here we used that with some $\mu = \text{const}$

$$\psi_j^{(N)}(x,\omega;n) = \mu \cdot f_{[-N,n-1]}\Big(e(x),\omega; E_j^{(N)}(x,\omega)\Big)$$

for $-N \le n \le N$. We use the convention that

$$f_{[-N,-N-1]} = 0, \quad f_{[-N,-N]} = 1.$$

One can assume $\nu_j^{(N)}(x,\omega) \ge 0$ by symmetry. Using Corollary 2.14 and (3.15), we conclude that

$$\begin{aligned}
&\sum_{n=-N}^{\nu_j^{(N)}(x,\omega)-Q} \big|f_{[-N,n]}(e(x),\omega,E) - f_{[-N,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\big|^2 \\
&\le e^{-2\gamma Q}\big|E - E_j^{(N)}(x,\omega)\big|^2 e^{(\log N)^C} \sum_{n\in\Lambda_Q} \big|f_{[-N,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\big|^2
\end{aligned} \tag{4.3}$$

Let $n_1 = \nu_j^{(N)}(x,\omega) - Q - 1$. Furthermore,

$$\begin{aligned}
&\left\| \begin{pmatrix} f_{[-N,n+1]}(e(x),\omega,E) \\ f_{[-N,n]}(e(x),\omega,E) \end{pmatrix} - \begin{pmatrix} f_{[-N,n+1]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big) \\ f_{[-N,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big) \end{pmatrix} \right\| \\
&= \Bigg\| M_{[n_1+1,n]}(e(x),\omega,E)\begin{pmatrix} f_{[-N,n_1+1]}(e(x),\omega,E) \\ f_{[-N,n_1]}(e(x),\omega,E) \end{pmatrix} \\
&\qquad - M_{[n_1+1,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\begin{pmatrix} f_{[-N,n_1+1]}\big(e(x),\omega,E_j^{(N)}\big) \\ f_{[-N,n_1]}\big(e(x),\omega,E_j^{(N)}\big) \end{pmatrix} \Bigg\| \\
&\le e^{C(n-n_1)} e^{-\gamma Q}\big|E - E_j^{(N)}(x,\omega)\big| e^{(\log N)^C} \Bigg( \sum_{n\in\Lambda_Q} \big|f_{[-N,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\big|^2 \Bigg)^{\frac12} .
\end{aligned} \tag{4.4}$$

Now suppose there is $E_k^{(N)}(x,\omega)$ with $\big|E_k^{(N)}(e(x),\omega) - E_j^{(N)}(x,\omega)\big| < e^{-N^\delta}$ for some small $\delta > 0$. Then (4.3), (4.4) imply that

$$\begin{aligned}
&\sum_{n=-N}^{\nu_j^{(N)}(x,\omega)+Q} \big|f_{[-N,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big) - f_{[-N,n]}\big(e(x),\omega,E_k^{(N)}(x,\omega)\big)\big|^2 \\
&< e^{-\frac12 N^\delta} \sum_{n\in\Lambda_Q} \big|f_{[-N,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\big|^2 ,
\end{aligned} \tag{4.5}$$

provided $N^\delta > \exp\big((\log\log N)^A\big)$. Let us estimate the contributions of $\Big[\nu_j^{(N)}(x,\omega)+Q, N\Big]$ to the sum terms in the left-hand side of (4.5).

For both $E = E_j^{(N)}$ and $E_k^{(N)}$ one has

$$f_{[-N,n]}(e(x),\omega,E) = G_{[\nu_j^{(N)}(x,\omega)+\frac{Q}{2},N]}(e(x),\omega,E)\big(n, \nu_j^{(N)}(x,\omega)+\frac{Q}{2}\big) f_{[-N,\nu_j^{(N)}(x,\omega)+\frac{Q}{2}-1]}(e(x),\omega,E)$$

due to the zero boundary condition at $N+1$, i.e.,

$$f_{[-N,N]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big) = f_{[-N,N]}\big(e(x),\omega,E_k^{(N)}(x,\omega)\big) = 0 .$$

Therefore,

$$\sum_{n=\nu_j^{(N)}+Q}^{N} \big|f_{[-N,n]}(e(x),\omega,E)\big|^2 \le e^{-\frac{\gamma Q}{4}} \sum_{k\in\Lambda_Q} \big|f_{[-N,k]}(e(x),\omega,E)\big|^2 \tag{4.6}$$

again for both $E = E_j^{(N)}(x,\omega)$ and $E = E_k^{(N)}(x,\omega)$. Finally, in view of (4.5) and (4.6),

$$\begin{aligned}\sum_{n=-N}^{N} &\big|f_{[-N,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big) - f_{[-N,n]}\big(e(x),\omega,E_k^{(N)}(x,\omega)\big)\big|^2 \\ &< e^{-\frac{\delta Q}{4}}\Big[\sum_{n\in\Lambda_Q}\big|f_{[-N,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\big|^2 \\ &\quad + \sum_{n\in\Lambda_Q}\big|f_{[-N,n]}\big(e(x),\omega,E_k^{(N)}(x,\omega)\big)\big|^2\Big]\end{aligned} \tag{4.7}$$

By orthogonality of $\Big\{f_{[-N,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\Big\}_{n=-N}^{N}$ and $\Big\{f_{[-N,n]}\big(e(x),\omega,E_k^{(N)}(x,\omega)\big)\Big\}_{n=-N}^{N}$, we obtain a contradiction from (4.7). □

**Remark 4.2.** *Later in this paper we will need to refer also to the proof of this proposition and not just to its statement. More precisely, we shall encounter two normalized eigenfunctions $\psi^{\pm}$ of the Dirichlet problem*

$$H_{[-N,N]}(x_0,\omega)\psi^{\pm} = E^{\pm}\psi^{\pm}$$

*where $\psi^{\pm}$ are exponentially decaying outside of some "window" $\Lambda = [N', N''] \subset [-N,N]$ of size $|\Lambda| \le N^{\varepsilon}$. Furthermore, we know (and need) that $f_k(ze(s\omega),\omega,E) \neq 0$ for any $z \in \mathcal{D}(e(x_0),r_0)$, $E \in \mathcal{D}(E_0,r_0)$ with $r_0 = \exp(-(\log\ell)^A)$ and $|E_0 - E^{\pm}| < r_0/2$, $k \asymp \ell$, $s \in [1,N'] \cup [N'',N]$ where $C\ell < N' < N'' < N - C\ell$, $N'' - N' < N^{\varepsilon}$. Under these assumptions, it follows from the proof of the previous proposition that*

$$|E^+ - E^-| > e^{-N^{\delta}}$$

*provided that $\delta > \varepsilon$ and $N$ is sufficiently large.*

*Note that here we do not need to remove energies. Indeed, the removal of the energies is only needed to ensure the existence of the window of localization, whereas in this remark we are dealing with functions $\psi^{\pm}$ that already have this structure. Also, note that in the proof of Proposition 4.1 the window has size $\exp((\log\log N)^C)$. However, this only entered into the proof via the estimate $\exp((\log\log N)^C) < N^{\varepsilon}$. Furthermore, we remark that under the conditions on $f_k$ stated in the previous paragraph the proof of Proposition 3.4 assures that $\psi^{\pm}$ decay exponentially outside of the window $\Lambda$. In fact, one has the bound*

$$\big|\psi^{\pm}(x_0,n)\big| \le \exp\left(-\gamma\,\mathrm{dist}(n,\Lambda)/2\right)$$

*for all $n \in [-N,N]$.*

By the well-known Rellich theorem, the eigenvalues $E_j^{(N)}(x,\omega)$ of the Dirichlet problem on $[-N,N]$ are analytic functions of $x$ and can therefore be extended analytically to a complex neighborhood of $\mathbb{T}$. Moreover, by simplicity of the eigenvalues of the Dirichlet problem, the graphs of these functions of $x$ do not cross. Proposition 4.1 makes this non-crossing quantitative, up to certain sections of the graphs where we loose control. These are the portions of the graph that intersect horizontal strips corresponding to those energies in $\mathcal{E}_{N,\omega}$. The quantitative control provided by (4.1) allows us to give lower bounds on the radii of the disks to which the functions $E_j^{(N)}(x,\omega)$ extend analytically.

**Corollary 4.3.** *Let $\Omega_N$, $\mathcal{E}_{N,\omega}$ be as above. Take arbitrary $x_0 \in \mathbb{T}$. Assume $f_N(x_0,\omega_0,E_0) = 0$ for some $\omega_0 \in \mathbb{T}_{c,a} \setminus \Omega_N$ and $E_0 \notin \mathcal{E}_{N,\omega_0}$. Then there exist $r_0, r_1$, $r_1 = e^{-N^{\delta}}$, $r_0 = cr_1$, such that (with $\omega_0$ fixed)*

$$f_N(z,\omega_0,E) = \big(E - b_0(z)\big)\chi(z,E) \tag{4.8}$$

*for all $z \in \mathcal{D}(x_0,r_0)$, $E \in \mathcal{D}(E_0,r_1)$. Moreover, $b_0(z)$ is analytic on $\mathcal{D}(x_0,r_0)$, $\chi(z,E)$ is analytic and nonzero on $\mathcal{D}(x_0,r_0) \times \mathcal{D}(E_0,r_1)$, $b_0(x_0) = E_0$.*

*Proof.* By Proposition 4.1, $f_N(x_0,\omega_0,E) \neq 0$ if $E \in \mathcal{D}(E_0,r_1)$, $E \neq E_0$. Since $H_N(x_0,\omega_0)$ is self adjoint and

$$\big\|H_N(z,\omega_0) - H_N(x_0,\omega_0)\big\| \lesssim |z-x_0|,$$

it follows that $f_N(z,\omega_0,E) \neq 0$ for any $|z-x_0| \ll r_1$, $r_1/2 < |E-E_0| < \frac{3}{4}r_1$. The representation (4.8) is now obtained by the same arguments that lead to the Weierstrass preparation theorem, see Theorem 2.25. □

As an application of Proposition 3.4 combined with Proposition 4.1 we now illustrate how to relate the localized eigenfunctions of consecutive scales. Indeed, by Proposition 3.4 any eigenfunction $\psi_j^{(N)}(x,\omega,\cdot)$ is exponentially localized around some interval $\Lambda$ of size $N' \asymp \exp((\log\log N)^A)$ provided $E_j^{(N)}(x,\omega)$ is outside of some set $E_{N,\omega}$. Due to this fact and the separation of eigenvalues, the restriction of $\psi_j^{(N)}(x,\omega,\cdot)$ to $\Lambda$ closely resembles some eigenfunction $\psi_{j'}^{(N')}(x',\omega,\cdot)$. In particular, it is exponentially localized around some interval $\Lambda'$ of size $N'' = \exp((\log\log N')^A)$.

**Lemma 4.4.** *Using the notations of Proposition 4.1 assume that $\omega \in \mathbb{T}_{c,a} \setminus (\Omega_N \cup \Omega_{N'})$, where $N' \asymp \exp\big((\log\log N)^{C_1}\big)$, $C_1 \gg C$, and with $Q = \exp\big((\log\log N)^C\big)$. If*

$$E_j^{(N)}(x,\omega) \notin \mathcal{E}_{N,\omega}, \qquad \operatorname{dist}\left(E_j^{(N)}(x,\omega), \mathcal{E}_{N',\omega}\right) > \exp\left(-(N')^{1/2}\right),$$

*then there exists $\nu \in \mathbb{Z}$, $|\nu - \nu_j^{(N)}(x,\omega)| \le Q$ and*

$$E_{j'}^{(N')}(x+\nu\omega,\omega) \in \left(E_j^{(N)}(x,\omega) - \exp(-\gamma_1 N'), E_j^{(N)}(x,\omega) + \exp(-\gamma_1 N')\right) ,$$

*where $\gamma_1 = c\gamma$, $\gamma = \inf L(E,\omega)$. Moreover, the corresponding normalized eigenfunctions*

$$\psi_j^{(N)}(x,\omega,k), \qquad \psi_{j'}^{(N')}(x+\nu\omega,\omega,k-\nu)$$

*satisfy*

$$\sum_{k\in[\nu-N',\nu+N']} \left|\psi_j^{(N)}(x,\omega,k) - \psi_{j'}^{(N')}(x+\nu\omega,\omega,k-\nu)\right|^2 \le \exp(-\gamma_1 N') \ . \tag{4.9}$$

*Proof.* Assume first $-N+N' < \nu_j^{(N)}(x,\omega) < N-N'$. Then with $\nu = \nu_j^{(N)}(x,\omega)$ one has:

$$\left\|\left(H_{[\nu-N',\nu+N']}(x,\omega) - E_j^{(N)}(x,\omega)\right)\psi_j^{(N)}(x,\omega,\cdot)\right\| \le \exp(-\gamma N'/4) \ , \tag{4.10}$$

$$1 - \sum_{k\in[\nu-N',\nu+N']} \left|\psi_j^{(N)}(x,\omega,k)\right|^2 < \exp(-\gamma N'/4) \tag{4.11}$$

due to Proposition 4.1. Hence, there exists

$$E_{j'}^{(N')}(x+\nu\omega,\omega) \in \left(E_j^{(N)}(x,\omega) - \exp(-\gamma_1 N'), E_j^{(N)}(x,\omega) + \exp(-\gamma_1 N')\right).$$

Moreover, due to assumptions on $E_j^{(N)}(x,\omega)$, one has $E_{j'}^{(N')}(x+\nu\omega,\omega) \notin \mathcal{E}_{N',\omega}$. Hence,

$$\left|E_{j'}^{(N')}(x+\nu\omega,\omega) - E_k^{(N')}(x+\nu\omega,\omega)\right| > \exp\left(-(N')^\delta\right) \tag{4.12}$$

for any $k \neq j'$. Then (4.10)–(4.12) combined imply (4.9) (expand in the orthonormal basis $\{\psi_k^{(N')}\}_k$). If $\nu_j^{(N)}(x,\omega) \le -N+N'$ (resp $\nu_j^{(N)}(x,\omega) \ge N-N'$), then (4.10)–(4.12) are valid with $\nu_j^{(N)}(x,\omega) = -N+N'$ (respectively, with $\nu_j^{(N)}(x,\omega) = N-N'$). □

Next, we iterate the construction of the previous lemma to obtain the following.

**Corollary 4.5.** *Given integers $m^{(1)}, m^{(2)}, \dots, m^{(t)}$ such that*

$$\log m^{(s+1)} \asymp \exp\left(\left(\log m^{(s)}\right)^{\delta}\right) \ , \quad s = 1, 2, \dots, t-1 \tag{4.13}$$

*there exist subsets $\Omega^{(s)} \subset \mathbb{T}$, $s = 1, 2, \dots$,*

$$\mathrm{mes}\ \left(\Omega^{(s)}\right) < \exp\left(-\left(\log m^{(s)}\right)^{A_2}\right), \qquad \mathrm{compl}\ \left(\Omega^{(s)}\right) < \exp\left(\left(\log m^{(s)}\right)^{A_1}\right),$$

*$1 \ll A_1 \ll A_2$, such that for any $\omega \in \mathbb{T}_{c,a} \setminus \bigcup_s \Omega^{(s)}$ there exist subsets $\mathcal{E}_\omega^{(s)} \subset \mathbb{R}$ with*

$$\mathrm{mes}\ \left(\mathcal{E}_\omega^{(s)}\right) < \exp\left(-\left(\log m^{(s)}\right)^{A_3}\right), \qquad \mathrm{compl}\ \left(\mathcal{E}_\omega^{(s)}\right) < \exp\left(\left(\log m^{(s)}\right)^{A_4}\right)$$

*with $A_4 \ll A_3$ such that for any $x \in \mathbb{T}$ and any $E \in \mathrm{sp}\ (H_{m^{(t)}}(x,\omega)) \setminus \bigcup_s \mathcal{E}_\omega^{(s)}$, the corresponding eigenfunction $\psi(n)$, $1 \le n \le m^{(t)}$ of $H_{m^{(t)}}(x,\omega)$ has the following property: there exists an integer $\nu^{(t)}(x,\omega) \in \left[1, m^{(t)}\right]$ such that*

$$\sum_{|n-\nu^{(t)}(x,\omega)|>Q} |\psi(n)|^2 \lesssim \exp(-\gamma' Q)$$

*where $Q = \exp\left(\left(\log\log m^{(1)}\right)^{A}\right)$ and $\gamma' = c\gamma > 0$.*

*Proof.* The proof goes by induction over $t = 1, 2, \dots$. For $t = 1$, the assertion is valid due to Proposition 3.4. So, assume that it is valid for $H_{m^{(t-1)}}(\tilde{x},\omega)$ for any $\widetilde{E} \notin \bigcup_{s=1}^{t-1} \widetilde{\mathcal{E}}_\omega^{(s)}$, $\tilde{x} \in \mathbb{T}$. Let $E, \psi$ be as in the statement. By the previous lemma there exist an interval $\Lambda = [a, a+\widetilde{N}]$, $\widetilde{N} = m^{(t-1)}$, $1 \le a \le N - \widetilde{N}$ and a normalized eigenfunction $\widetilde{\psi}$ such that $H_{[a,a+\widetilde{N}]}(x,\omega)\widetilde{\psi} = \widetilde{E}\,\widetilde{\psi}$, $|\widetilde{E} - E| < \exp\left(-\gamma\widetilde{N}\right)$, $\left|\widetilde{\psi}(n) - \psi(n)\right| < \exp\left(-\gamma\widetilde{N}\right)$, $n \in \Lambda$. Applying now the inductive assumption to $H_{m^{(t-1)}}(x + a\omega, \omega)$ one obtains the assertion. □

The arguments used in Lemma 4.4, based on combination of Propositions 3.4 and 4.1 enable one to define the "translations" of the eigenfunctions $\psi_j^{(N)}(.\,,.)$ under the shift $x \to x + \omega$.

**Lemma 4.6.** *Using the notations of Proposition 4.1 assume that (i) $\mathrm{dist}\left(E_j^{(N)}(x,\omega), \mathcal{E}_{N,\omega}\right) > 2\exp\left(-N^{(\delta)}\right)$, $-N + N^{1/2} < \nu_j^{(N)}(x,\omega) < N - N^{1/2}$, where $\nu_j^{(N)}(x,\omega)$ is the same as in Proposition 3.4. Then for any $k$ such that $-N + N^{1/2}/2 < \nu_j^{(N)}(x,\omega) + k < N - N^{1/2}/2$ there exists a unique $E_{j_k}^{(N)}(x + k\omega, \omega) \in \mathrm{sp}\ H_{[-N,N]}(x + k\omega, \omega)$ such that*

$$\left|E_j^{(N)}(x,\omega) - E_{j_k}^{(N)}(x + k\omega, \omega)\right| < \exp\left(-\gamma_2 N^{1/2}\right) \ , \tag{4.14}$$

$$\left|\partial_x E_j^{(N)}(x,\omega) - \partial_x E_{j_k}^{(N)}(x + k\omega, \omega)\right| < \exp\left(-\gamma_2 N^{1/2}\right) \ , \tag{4.15}$$

$$E_{j_k}^{(N)}(x + k\omega, \omega) \notin \mathcal{E}_{N,\omega} \ , \tag{4.16}$$

$$\left|\nu_{j_k}^{(N)}(x + k\omega, \omega) - \left(\nu_j^{(N)}(x,\omega) + k\right)\right| \le N^{1/2}/4 \ , \tag{4.17}$$

$$-N + N^{1/2}/4 < \nu_{j_k}^{(N)}(x + k\omega, \omega) < N - N^{1/2}/4 \ , \tag{4.18}$$

$$\sum_{|m+k-\nu_j^{(N)}(x,\omega)| \le N^{1/2}/4} \left|\psi_{j_k}^{(N)}(x + k\omega, m) - \psi_j^{(N)}(x, m+k)\right|^2 < \exp\left(-\gamma_3 N^{1/2}\right) \tag{4.19}$$

*where $\gamma_t = 2^{-t+1}\gamma_1$.*

*Proof.* Note that

$$H_{[-N,N]}(x+k\omega,\omega)\left(\psi_j^{(N)}(x,\omega,\cdot+k)\right)(m) = H_{[-N,N]}(x,\omega)\left(\psi_j^{(N)}(x,\omega.)\right)(m+k) = E_j^{(N)}(x,\omega)\psi_j^{(N)}(x,\omega,k+m) \tag{4.20}$$

provided $-N < m+k < N$ and $-N < m < N$. Recall also that $\left|\psi_j^{(N)}(x,\omega,\pm N)\right| \le \exp\bigl(-\gamma_3 N^{1/2}\bigr)$, since $-N + N^{1/2} < \nu_j^{(N)}(x,\omega) < N - N^{1/2}$ Hence

$$\left\| \left(H_{[-N,N]}(x+k\omega,\omega) - E_j^{(N)}(x,\omega)\right)\psi_j^{(N)}(x,\omega,\cdot+k)\right\| < \exp\bigl(-\gamma_4 N^{1/2}\bigr)\ . \tag{4.21}$$

Therefore, there exists $E_{j_k}^{(N)}(x+k\omega,\omega) \in \left(E_j^{(N)}(x,\omega) - \exp\bigl(-\gamma_5 N^{1/2}\bigr), E_j^{(N)}(x,\omega) + \exp\bigl(-\gamma_5 N^{1/2}\bigr)\right)$. Moreover, due to our assumptions on $E_j^{(N)}(x,\omega)$, one has $E_{j_k}^{(N)}(x+k\omega,\omega) \notin \mathcal{E}_{N,\omega}$. Hence,

$$\left|E_{j_k}^{(N)}(x+k\omega,\omega) - E_{j'}^{(N)}(x+k\omega,\omega)\right| > \exp\left(-\bigl(\log N\bigr)^A\right) \tag{4.22}$$

for any $j' \ne j_k$. Then (4.20)–(4.21) combined imply (4.14). Relations (4.16), (4.17) follow from (4.14). The estimate (4.19) follows from (4.21) and (4.22) via the spectral theorem for Hermitian matrices. Finally, (4.15) follows from the well-known formula

$$\partial_x E_j^{(N)}(x,\omega) = \sum_{\ell=-N}^{N} V'(x+\ell\omega)\left|\psi_j^{(N)}(x,\omega,\ell)\right|^2$$

and the preceding estimates. □

## 5. Mobility of eigenvalues and the separation of zeros of $f_N$ in $z$

In this section, we will use the separation of the eigenvalues from Section 4 to obtain lower bounds on the derivatives of the Rellich functions off some small bad set of phases. In particular, this will use Corollary 4.3.

**Lemma 5.1.** *Let $\varphi(z)$ be analytic in some disk $\mathcal{D}(0,r)$, $r>0$. Then*

$$\mathrm{mes}\ \{w : w = \varphi(z),\ z \in \mathcal{D}(0,r),\ |\varphi'(z)| < \eta\} \le \pi r^2\eta^2 \tag{5.1}$$

*Proof.* Set $A = \{z = x+iy \in \mathcal{D}(0,r) : |\varphi'(z)| < \eta\}$. By the general change of variables formula, see Theorem 3.2.3 in [Fed],

$$\int_A \Bigl|\frac{\partial(u,v)}{\partial(x,y)}\Bigr| dx\,dy = \int_{\mathbb{R}^2} \#\left\{(x,y)\in A : \varphi(x+iy) = u+iv\right\} du\,dv \ge \mathrm{mes}\ \varphi(A)$$

where $\varphi(x+iy) = u(x,y) + iv(x,y)$. On the other hand,

$$\Bigl|\frac{\partial(u,v)}{\partial(x,y)}\Bigr| = |\varphi'(x+iy)|^2$$

since $\varphi$ is analytic. □

The following lemma will allow us to transform the separation of the eigenvalues into a lower bound on the derivative of the Rellich functions. The logic behind Lemma 5.2 is as follows: Let $b_0$ be as in (i). By Lemma 5.1, the measure of those $w$ which satisfy $w = b_0(z)$ with $b_0'(z)$ small, is small. However, we also require a bound on the complexity of this set of $w$ which is only logarithmic in $r_0$ and $r_1$. This is where property (ii) comes into play, and the complexity will be proportional to a power of the degree $k$ as well as to $\log[(r_0r_1)^{-1}]$.

**Lemma 5.2.** *Let $f(z,w)$ be an analytic function defined in $\mathcal{D}(0,1)\times\mathcal{D}(0,1)$. Assume that one has the following representations:*

(i) *$f(z,w) = (w - b_0(z))\chi(z,w)$, for any $z \in \mathcal{D}(0,r_0)$, $w \in \mathcal{D}(0,r_1)$, where $b_0(z)$ is analytic in $\mathcal{D}(0,r_0)$, $\sup|b_0(z)| \le 1$, $\chi(z,w)$ is analytic and non-vanishing on $\mathcal{D}(0,r_0)\times\mathcal{D}(0,r_1)$, where $0 < r_0, r_1 < \frac{1}{2}$*

(ii) *$f(z,w) = P(z,w)\theta(z,w)$, for any $z \in \mathcal{D}(0,r_0)$, $w \in \mathcal{D}(0,r_1)$ where*

$$P(z,w) = z^k + c_{k-1}(w)z^{k-1} + \cdots + c_0(w) \ ,$$

*$c_j(w)$ are analytic in $\mathcal{D}(0,r_0)$, and $\theta(z,w)$ is analytic and non-vanishing on $\mathcal{D}(0,r_0)\times\mathcal{D}(0,r_1)$, and all the zeros of $P(z,w)$ belong to $\mathcal{D}(0,1/2)$.*

*Then given $H \gg k^2\log[(r_0r_1)^{-1}]$ one can find a set $\mathcal{S}_H \subset \mathcal{D}(w_0,r_1)$ with the property that*

$$\mathrm{mes}\,(\mathcal{S}_H) \lesssim r_1^2\exp\left(-cH/k^2\log[(r_0r_1)^{-1}]\right), \quad \textit{and} \quad \mathrm{compl}(S_H) \lesssim k^2\log[(r_0r_1)^{-1}]$$

*such that for any $w \in \mathcal{D}(0,r_1/2)\setminus\mathcal{S}_H$ and $z \in \mathcal{D}(0,r_0)$ for which $w = b_0(z)$ one has*

$$\left|b_0'(z)\right| > e^{-kH}2^{-k}r_1 \ .$$

*Moreover, for those $w$ the distance between any two zeros of $P(\cdot,w)$ exceeds $e^{-H}$.*

*Proof.* Assume that $k\ge 2$ and set $\psi(w) = \mathrm{disc}\,P(\cdot,w)$. If $k=1$, then skip to (5.7). Then $\psi(w)$ is analytic in $\mathcal{D}(0,r_1)$. Assume that $|\psi(w)| < \tau$ for some $\tau>0$, $w\in\mathcal{D}(0,r_1)$. Recall that due to the basic property of the discriminant for any $w$

$$\psi(w) = \prod_{i\neq j}\left(\zeta_i(w)-\zeta_j(w)\right) \ , \tag{5.2}$$

where $\zeta_i(w)$, $i=1,2,\ldots,k$ are the zeros of $P(\cdot,w)$. Then $|\zeta_i(w)-\zeta_j(w)| < \tau^{2/k(k-1)}$ for some $i\neq j$. Set $\zeta_i = \zeta_i(w)$, $\zeta_j=\zeta_j(w)$. Assume first $\zeta_i\neq\zeta_j$. Then

$$f(\zeta_i,w)=0 \ \ f(\zeta_j,w)=0, \quad 0<|\zeta_i-\zeta_j|<\tau^{2/k(k-1)} \ .$$

Due to (i) one has $w=b_0(\zeta_i)=b_0(\zeta_j)$. Hence,

$$|b_0'(\zeta_i)| \le \frac{1}{2}|\zeta_i-\zeta_j|\max|b_0''(z)| \lesssim |\zeta_i-\zeta_j|r_0^{-2} < r_0^{-2}\tau^{2/k(k-1)} \ . \tag{5.3}$$

If $\zeta_i=\zeta_j$ then $P(\zeta_i,w)=0$, $\partial_zP(\zeta_i,w)=0$. Then $f(\zeta_i,w)=0$, $\partial_zf(\zeta_i,w)=0$ due to the representation (ii). Then $w-b_0(\zeta_i)=0$, $b'(\zeta_i)=0$ due to the representation (i). Thus (5.3) holds at any event. Combining that bound with the estimate (5.1) one obtains

$$\mathrm{mes}\,\{w\in\mathcal{D}(0,r_1) : |\psi(w)|<\tau\} \lesssim r_0^{-2}\tau^{2/k(k-1)}. \tag{5.4}$$

On the other hand, due to (5.2) one obtains

$$\sup\{|\psi(w)| : w\in\mathcal{D}(0,r_1)\} \le 1.$$

Take $\tau \ll (r_0r_1)^{k(k-1)/2}$. Then one obtains from (5.4) that

$$|\psi(w)| \ge \tau$$

for some $|w|<\frac{r_1}{2}$. By Cartan's estimate there exists a set $\mathcal{T}_H\subset\mathcal{D}\left(0,\frac{r_1}{2}\right)$ with

$$\mathrm{mes}\ \mathcal{T}_H \lesssim r_1^2\exp\left(-cH/k^2\log[(r_0r_1)^{-1}]\right)$$

and of complexity $\lesssim k^2\log[(r_0r_1)^{-1}]$ such that

$$\log|\psi(w)| > -H \tag{5.5}$$

for any $w\in\mathcal{D}(0,\frac{r_1}{2})\setminus\mathcal{T}_H$.

In particular, (5.5) implies that

$$|\zeta_i(w)-\zeta_j(w)| > e^{-H} \tag{5.6}$$

for any $w\in\mathcal{D}(0,\frac{r_1}{2})\setminus\mathcal{T}_H$, $i\neq j$. Take arbitrary $w_0$ such that $\mathrm{dist}(w_0,\mathcal{T}_H) > 2e^{-H}$, $w_0=b_0(z_0)$ for some $z_0\in\mathcal{D}(0,r_0)$. Then

$$|P(z,w_0)| \ge (2e^H)^{-k} \ \text{ for all } \ |z-z_0| = e^{-H}/2$$

by the separation of the zeros (5.6). By our assumption on the zeros of $P(z,w)$,

$$\sup_{z\in\mathcal{D}(0,r_0)}\ \sup_{w\in\mathcal{D}(0,r_1)} |\partial_wP(z,w)| \lesssim r_1^{-1}.$$

Thus,

$$|P(z,w)| > \frac{1}{2}2^{-k}e^{-kH} \quad \text{if} \quad |z-z_0| = e^{-H}/2, \quad |w-w_0| \ll 2^{-k}e^{-kH}r_1.$$

Then due to the Weierstrass preparation theorem, see Theorem 2.25,

$$P(z,w) = \big(z-\zeta(w)\big)\lambda(z,w) \tag{5.7}$$

for any $z \in \mathcal{D}(z_0,r_0')$, $w \in \mathcal{D}(w_0,r_1')$, where $r_0' = e^{-H}/2$, $r_1' \ll e^{-kH}2^{-k}r_1$, and $\zeta(w)$ is an analytic function in $\mathcal{D}(w_0,r_1')$, $\lambda(z,w)$ is analytic and non-vanishing on $\mathcal{D}(z_0,r_0') \times \mathcal{D}(w_0,r_0')$. Comparing the representation (i) and (5.7) one obtains

$$\begin{cases} w - b_0(z) = 0 \quad \text{iff} \\ z - \zeta(w) = 0 \end{cases} \tag{5.8}$$

for any $z \in \mathcal{D}(z_0,r_0')$, $w \in \mathcal{D}(w_0,r_1')$. It follows from (5.8) that

$$\big|b_0'\big(\zeta(w)\big)\big| \ge \big|\zeta'(w)\big|^{-1} \gtrsim r_1' \gtrsim e^{-kH}2^{-k}r_1,$$

as claimed. □

Now choose arbitrary $\omega_0 \in \mathbb{T}_{c,a} \setminus \Omega_N$, $E_0 \in (-C(V),C(V)) \setminus \widetilde{\mathcal{E}}_{N,\omega_0}$, where

$$\widetilde{\mathcal{E}}_{N,\omega_0} = \Big\{E : \operatorname{dist}(E,\mathcal{E}_{N,\omega_0}) < \exp(-N^{\delta/2})\Big\} \ ,$$

$\Omega_N$, $\mathcal{E}_{N,\omega_0}$ are the same as in Proposition 4.1. Then for any $x \in \mathbb{T}$ one has

$$\begin{aligned} &\min\Big\{\big|E_j^{(N)}(x,\omega_0) - E_i^{(N)}(x,\omega_0)\big| : E_j^{(N)}(x,\omega_0), E_i^{(N)}(x,\omega_0) \in \\ &\Big(E_0 - \exp(-N^{\delta/2}), E_0 + \exp(-N^{\delta/2})\Big), i \ne j\Big\} \ge \exp(-N^{\delta}) \end{aligned} \tag{5.9}$$

Here $E_j^{(N)}(x,\omega)$ stand for the eigenvalues of $H_N(x,\omega)$ as usual.

Now assume that there is $x_0 \in \mathbb{T}$ such that $E_{j_0}^{(N)}(x_0,\omega_0) \in \big(E_0 - \exp(-N^{\delta/2}), E_0 + \exp(-N^{\delta/2})\big)$ for some $j_0$. Then, as in Lemma 5.2,

$$f_N(z,\omega_0,E) = (E - b_0(z))\,\chi(z,E)$$

where $(z,E) \in \mathcal{P} = \mathcal{D}(x_0,r_0) \times \mathcal{D}(E_{j_0}^{(N)}(x_0,\omega_0),r_0)$, $r_0 = \exp(-N^{\delta_1})$ with $\delta_1 \gg \delta$, and the analytic functions $b_0(z)$, $\chi(z,E)$ satisfy the properties stated in Lemma 5.2. On the other hand, due to the Weierstrass preparation theorem in the $z$-variable, see Proposition 2.26,

$$f_N(z,\omega,E) = P_N(z,\omega,E)g_N(z,\omega,E) \tag{5.10}$$

$(z,\omega,E) \in \mathcal{P}_1 = \mathcal{D}(x_0,r_1) \times \mathcal{D}(\omega_0,r_1) \times \mathcal{D}(E_0,r_1)$, $r_1 \asymp \exp\big(-(\log N)^C\big)$, where $P_N, g_N$ satisfy conditions (a)–(d) of Proposition 2.26. Thus, all conditions needed to apply Lemma 5.2 are valid for $f_N(z,\omega_0,E)$. So, using the notations of the previous two paragraphs we obtain the following

**Corollary 5.3.** *There exist constants $\delta_1 \ll \delta_2 \ll 1$ with the following properties: Set $E_1 = E_{j_0}^{(N)}(x_0,\omega_0)$ where $\omega_0 \in \mathbb{T}_{a,c} \setminus \Omega_N$ and $x_0 \in \mathbb{T}$. There exists a subset $\mathcal{E}'_{N,\omega_0,x_0,j_0} \subset \mathbb{C}$, with*

$$\operatorname{mes}(\mathcal{E}'_{N,\omega_0,x_0,j_0}) \le \exp(-N^{\delta_2}), \quad \operatorname{compl}(\mathcal{E}'_{N,\omega_0,x_0,j_0}) \le N$$

*such that for any $E \in \mathcal{D}(E_1,r_1) \setminus \mathcal{E}'_{N,\omega_0,x_0,j_0}$ and $z \in D(x_0,r_1)$, $r_1 = \exp(-N^{\delta_1})$, for which $E = b_0(z,\omega_0)$ one has*

$$|\partial_z b_0(z)| > \exp(-N^{2\delta_2}) \ .$$

*Moreover, for any $E \in \mathcal{D}(E_1,r_1) \setminus \mathcal{E}'_{N,\omega_0,x_0,j_0}$ the distance between any two zeros of the polynomial $P_N(\cdot,\omega_0,E)$ which fall into the disk $\mathcal{D}(x_0,r_1)$ exceeds $\exp(-N^{2\delta_2})$.*

As usual, we can go from an exceptional set in the energies to one in the phases $x$ by means of the Wegner-type bound of Lemma 2.17.

**Corollary 5.4.** *Let us use the notations of the previous corollary. Let $\omega_0 \in \mathbb{T}_{a,c} \setminus \Omega_N$ and $x_0 \in \mathbb{T}$. Then there exists a subset $\mathcal{B}'_{N,\omega} \subset (x_0 - r_1, x_0 + r_1)$*

$$\operatorname{mes}(\mathcal{B}'_{N,\omega_0}) \le \exp(-N^{\delta_2}), \quad \operatorname{compl}(\mathcal{B}'_{N,\omega_0}) \le N^2$$

*such that for any $x \in (x_0 - r_1, x_0 + r_1) \setminus \mathcal{B}'_{N,\omega_0}$ one has*

$$\big|\partial_x b_0(x,\omega)\big| \gtrsim e^{-N^{2\delta_2}} \tag{5.11}$$

*Proof.* Let $E \in \mathcal{D}(E_1, r_1)$. Suppose $x \in (x_0 - r_1, x_0 + r_1)$. Then $E = b_0(x)$ iff $E_1 \in \operatorname{sp}(H_N(x,\omega_0))$. Due to Lemma 2.17 there exists $\mathcal{B}'_{N,\omega_0} \subset (x_0 - r_1, x_0 + r_1)$ with the stated measure and complexity bounds such that for any $x \in (x_0 - r_1, x_0 + r_1) \setminus \mathcal{B}'_{N,\omega_0}$ one has

$$\operatorname{sp}(H_N(x,\omega_0)) \cap \mathcal{E}'_{N,\omega_0,x_0,j_0} = \emptyset$$

Here $\mathcal{E}'_{N,\omega_0,x_0,j_0}$ is the same as in Corollary 5.3. □

With $\omega_0 \in \mathbb{T}_{c,a} \setminus \Omega_N$ fixed as above, we take the union of the sets $\mathcal{E}'_{N,\omega_0,x_0,j_0}$ in $x_0, j_0$ with $x_0 \in \mathbb{T}$ running over an appropriate net, to conclude the following assertions

**Corollary 5.5.** *There exists a set $\mathcal{E}''_{N,\omega_0} \subset \mathbb{R}$ with $\operatorname{mes}\big(\mathcal{E}''_{N,\omega_0}\big) \le \exp\big(-N^{2\delta_1}\big)$, $\operatorname{compl}\mathcal{E}''_{N,\omega_0} \le \exp\big(N^{\delta_1}\big)$ such that for each function $E_j(x,\omega_0)$ and any $x$ one has*

$$|\partial_x E_j(x,\omega_0)| > \exp\big(-N^{2\delta_2}\big)$$

*provided $E_j(x,\omega_0) \notin \mathcal{E}''_{N,\omega_0}$.*

**Corollary 5.6.** *There exists a set $\mathcal{E}''_{N,\omega_0} \subset \mathbb{R}$ with*

$$\operatorname{mes}(\mathcal{E}''_{N,\omega_0}) \le \exp\big(-N^{2\delta_1}\big), \qquad \operatorname{compl}(\mathcal{E}''_{N,\omega_0}) \le \exp\big(N^{\delta_1}\big),$$

*such that for any $E \in \big(-C(V), C(V)\big) \setminus \mathcal{E}''_{N,\omega_0}$ and any $|\eta| \le \exp(-N^{2\delta_1})$ one has*

*the distance between any two zeros of $f_N(\cdot,\omega_0, E+i\eta)$ exceeds $\exp(-N^{\delta_2})$*

*where $\delta_1 \ll \delta_2 \ll 1$.*

To simplify the notations we will suppress the double prime in $\mathcal{E}''_{N,\omega}$ when referring to Corollary 5.5.

## 6. Segments of Rellich's parametrization of the Dirichlet eigenvalues and their translations

We now start our discussion of gap formation. The material from here on does not appear in [GolSch2]. Given $N$, let $\Omega_N$ and $\mathcal{E}_{N,\omega}$ stand for the sets defined in Corollary 5.5. Fix some $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N$. Let $I = [\underline{E}, \overline{E}]$ be arbitrary interval $\subset \mathbb{R} \setminus \mathcal{E}_{N,\omega}$. Let $E_j^{(N)}(x)$, $j = 1, 2, \ldots, 2N+1$ be the Dirichlet eigenvalues on $[-N, N]$ parameterized by $x \in \mathbb{T}$; here we suppressed $\omega$ from the notations just for convenience. Due to Corollary 5.5 the following assertion is valid:

**Lemma 6.1.** *Assume $E_j^{(N)}(x') \in (\underline{E}, \overline{E})$ for some $x' \in \mathbb{T}$. Then there exist $\underline{x}, \bar{x}$ such that:*

$$E^{(N)}(\underline{x}) = \underline{E}, \quad E_j^{(N)}(\bar{x}) = \overline{E}, \tag{6.1}$$

$$\Big|\partial_x E_j^{(N)}(x)\Big| > \exp\big(-N^{\sigma}\big) \quad \text{for} \quad x \in \big(\min(\underline{x}, \bar{x}), \max(\underline{x}, \bar{x})\big), \tag{6.2}$$

$$x' \in \big(\min(\underline{x}, \bar{x}), \max(\underline{x}, \bar{x})\big), \tag{6.3}$$

$$C(\lambda, V)^{-1}(\overline{E} - \underline{E}) < |\underline{x} - \bar{x}| < \exp\big(N^{\sigma}\big)\big(\overline{E} - \underline{E}\big), \tag{6.4}$$

*Here $0 < \sigma \ll 1$ is an arbitrary small but fixed parameter, and $N > N_0(\sigma)$.*

**Definition 6.2.** *If conditions (6.1), (6.2) of Lemma 6.1 hold, then we call $\Big\{E_j^{(N)}(x), \underline{x}, \bar{x}\Big\}$ an $I$–segment of $E_j^{(N)}(x)$. If $\partial_x E_j^{(N)} > 0$ (resp. $\partial_x E_j^{(N)} < 0$), $x \in (\underline{x}, \bar{x})$, (resp. $\partial_x E_j < 0, x \in (\bar{x}, \underline{x})$), we call it positive-slope (resp. negative-slope) segment.*

**Remark 6.3.** *Let* $\left\{E^{(N)}_{j^{(s)}}(x), \underline{x}^{(s)}, \bar{x}^{(s)}\right\}$, $s = 1, 2$, *be* $I$*–segments. If* $\underline{x}^{(1)} = \underline{x}^{(2)}$, *then* $j^{(1)} = j^{(2)}$ *and* $\bar{x}^{(1)} = \bar{x}^{(2)}$, *i.e., these segments coincide. The same conclusion is true in regards to* $\bar{x}^{(1)}, \bar{x}^{(2)}$.

**Lemma 6.4.** *Let* $\left\{E^{(N)}_{j^{(s)}}, \underline{x}^{(s)}, \bar{x}^{(s)}\right\}$, $s = 1, 2$ *be two different* $I$*-segments. Then*

$$\left|\underline{x}^{(1)} - \underline{x}^{(2)}\right|, \left|\bar{x}^{(1)} - \bar{x}^{(2)}\right| > \exp\left(-N^{\delta}\right)$$

*provided* $1 \gg \delta > \sigma > 0$, *see* (6.4), *and* $N > N_0(\delta, \sigma)$.

*Proof.* Due to the definition $\underline{E} \in \mathrm{sp}\,\left(H_{[-N,N]}\big(\underline{x}^{(s)}, \omega\big)\right)$, i.e., $f_{[-N,N]}\left(e\big(\underline{x}^{(s)}\big), \omega, \underline{E}\right) = 0$, $s = 1, 2$. Moreover, $\underline{E} \notin \mathcal{E}_{N,\omega}$. Due to Corollary 5.6, $\left|e\big(\underline{x}^{(1)}\big) - e\big(\underline{x}^{(2)}\big)\right| > \exp\left(-N^{\delta}\right)$, since $\underline{x}^{(1)} \neq \underline{x}^{(2)}$. Similarly, $\left|e\big(\bar{x}^{(1)}\big) - e\big(\bar{x}^{(2)}\big)\right| > \exp\left(-N^{\delta}\right)$. □

Since $E^{(N)}_j(x)$ are continuous and one-periodic, one has

**Lemma 6.5.** *If* $\left\{E^{(N)}_j(x), \underline{x}, \bar{x}\right\}$ *is a positive-slope (respectively, negative slope)* $I$*–segment, then there is at least one negative-slope (respectively, positive-slope)* $I$*–segment* $\left\{E^{(N)}_j, \underline{x}', \bar{x}'\right\}$ *of* $E^{(N)}_j(x)$.

We now turn to the analysis of the translations of $I$–segments. One can define these translations with the use of Lemma 4.6. Let $\left\{E^{(N)}_j(x), \underline{x}, \bar{x}\right\}$ be an $I$–segment such that

$$I' = \big(\underline{E} - \exp\big(-N^{\delta}\big), \overline{E} + \exp\big(-N^{\delta}\big)\big) \subset \mathbb{R} \setminus \mathcal{E}_{N,\omega}\ .$$

Assume for instance that $\left\{E^{(N)}_j(x), \underline{x}, \bar{x}\right\}$ is a positive-slope segment and for some $x \in (\underline{x}, \bar{x})$ the eigenvalue $E^{(N)}_j(\tilde{x})$ satisfies the conditions of Lemma 4.6, i.e., $\mathrm{dist}\left(E^{(N)}_j(\tilde{x}), \mathcal{E}_{N,\omega}\right) > 2\exp\big(-N^{\delta}\big)$, $-N + N^{1/2} < \nu^{(N)}_j(\tilde{x}, \omega) < N - N^{1/2}$. Assume now that

$$\exp\big(-N^{3\delta}\big) < \overline{E} - \underline{E} < \exp\big(-N^{2\delta}\big).$$

Then, by Lemma 6.1, $(\bar{x} - \underline{x}) < \exp\big(-N^{\delta}\big)$. By Section 3, we can assume that $\nu^{(N)}_j(x, \omega) = \nu^{(N)}_j(\tilde{x}, \omega)$ and also $\nu^{(N)}_j(x, \omega) = \nu^{(N)}_j(\tilde{x})$ for all $x \in (\underline{x}, \bar{x})$. Therefore Lemma 4.6 is valid for all $x \in (\underline{x}, \bar{x})$. Moreover,

$$\left|\partial_x E^{(N)}_j(x) - \partial_x E^{(N)}_{j_k}(x + k\omega)\right| < \exp\big(-\gamma_5 N^{1/2}\big) \tag{6.5}$$

for any $x \in (\underline{x}, \bar{x})$, and all $k$ for which

$$-N + \sqrt{N} < \nu^{(N)}_j(x, \omega) + k < N - \sqrt{N} \tag{6.6}$$

Due to Lemma 6.1, for each such $k$ an $I$–segment $\left\{E^{(N)}_{j_k}, \underline{x}_k, \bar{x}_k\right\}$ is defined. That leads to the following statement.

**Lemma 6.6.** *let* $\Omega_N$ *and* $\mathcal{E}_{N,\omega}$ *be the sets defined in Corollary 5.5. Fix some* $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N$. *Let* $\left\{E^{(N)}_j(\cdot, \omega), \underline{x}, \bar{x}\right\}$ *be an* $I$*–segment and pick any* $\tilde{x} \in (\min(\underline{x}, \bar{x}), \max(\underline{x}, \bar{x}))$. *Assume that the following conditions are valid:*

(a) $\exp\big(-N^{3\delta}\big) < \overline{E} - \underline{E} < \exp\big(-N^{2\delta}\big)$
(b) $\big(\underline{E} - \exp\big(-N^{\delta}\big), \overline{E} + \exp\big(-N^{\delta}\big)\big) \subset \mathbb{R} \setminus \mathcal{E}_{N,\omega}$,
(c) $-N + N^{1/2} < \nu^{(N)}_j(\tilde{x}, \omega) < N - N^{1/2}$.

*Then for each* $k$ *as in* (6.6) *there exists a unique* $I$*–segment* $\left\{E^{(N)}_{j_k}(\cdot, \omega), \underline{x}_k, \bar{x}_k\right\}$ *such that*

$$\big\|\underline{x}_k - \underline{x} - k\omega\big\|,\ \big\|\bar{x}_k - \bar{x} - k\omega\big\| < \exp\left(-\gamma_3 N^{1/2}\right)\ . \tag{6.7}$$

*Moreover,*

$$\left|E_j^{(N)}(x) - E_{j_k}^{(N)}(x+k\omega)\right| \ , \ \left|\partial_x E_j^{(N)}(x) - \partial_x E_{j_k}^{(N)}(x+k\omega)\right| < \exp\left(-\gamma_4 N^{1/2}\right) \ . \tag{6.8}$$

*for any* $x \in (\min(\underline{x}, \bar{x}), \max(\underline{x}, \bar{x}))$.

*Proof.* The uniqueness of a segment satisfying (6.7) follows from Lemma 6.4. The only assertion which one has to prove is (6.7). Assume, for instance, that $\left\{E_j^{(N)}, \underline{x}, \bar{x}\right\}$ is a positive-slope $I$–segment. If $\underline{x}_k < \{\underline{x} + k\omega\}$, then

$$\begin{aligned}\{\underline{x} + k\omega\} - \underline{x}_k &\le 2\exp\big(N^\sigma\big)\left(E_{j_k}^{(N)}\big(\{\underline{x}+k\omega\}\big) - E_{j_k}^{(N)}(\underline{x}_k)\right)\\ &= 2\exp\big(N^\sigma\big)\left(E_{j_k}^{(N)}(\underline{x}+k\omega) - E_j^{(N)}(\underline{x})\right)\\ &\le 2\exp\big(N^\sigma\big)\cdot\exp\big(-\gamma_3 N^{1/2}\big) < \exp\big(-\gamma_4 N^{1/2}\big)\ ,\end{aligned}$$

since $\partial_x E_{j_k}^{(N)}(x) > \exp\big(-N^\sigma\big)$ for $x \in (\underline{x}_k, \bar{x}_k)$ due to Lemma 6.1. Hence, $\|x_k - x - k\omega\| < \exp\big(-\gamma_4 N^{1/2}\big)$ in this case. In a similar way one can validate (6.7) in each situation. □

## 7. Double resonances and the formation of pre-gaps

**Definition 7.1.** *Fix small* $\varepsilon, \sigma_1 \in (0,1)$ *and large constants* $A, C$*. Let* $N > N_0(\varepsilon, \sigma_1, A, C)$ *be large. One says that* $(x_0, E_0) \in \mathbb{T}\times\mathbb{R}^1$ *is a* point of a double resonance for $H_N(\cdot)$ *if the following conditions are valid:*

*There exist intervals* $\Lambda_k = \big[N_k', N_k''\big]$, $k = 1,2$, $(\log N)^A < N_1' < N_1'' < N_2' < N_2'' < N - (\log N)^A$ *such that*

*(a)* $\mathrm{sp}\ H_{\Lambda_k}(x_0) \bigcap (E_0 - \kappa, E_0 + \kappa) = \big\{E_0\big\}$, $\#\big\{z \in \mathcal{D}\big(e(x_0), \kappa\big) : f_{\Lambda_k}(z, \omega, E_0) = 0\big\} = 1$, $k = 1,2$, *where* $\kappa = \exp\big(-\underline{N}^\varepsilon\big)$, $\underline{N} = \min_k |\Lambda_k|$,

*(b)* $\big(\log N\big)^A < \underline{N} < N^{\sigma_1}$, $k = 1,2$, $\big(\log N\big)^A < N_2' - N_1'' < N^{\sigma_1}$,

*(c) for any interval* $\widetilde{\Lambda} = \big[N', N''\big] \subset \big[1, N\big]$, $|\widetilde{\Lambda}| \asymp \big(\log N\big)^{A_1}$, $A_1 = A/2$, *which does not overlap with* $\big[N_1' + C\ell, N_1'' - C\ell\big] \cup \big[N_2' + C\ell, N_2'' - C\ell\big]$, $\ell \asymp \big(\log N\big)^{A_1}$ *one has*

$$\mathrm{sp}\ H_{\widetilde{\Lambda}}(x_0, \omega) \cap \big(E_0 - \underline{\kappa}, E_0 + \underline{\kappa}\big) = \emptyset$$

*where* $\underline{\kappa} = \exp\left(-\big(\log N\big)^{\varepsilon A_1}\right)$.

We can draw the following conclusion from this definition.

**Lemma 7.2.** *Using the notation of Definition 7.1 one can arrange the avalanche principle expansion* (2.35) *in such a way that the following conditions are valid:*

- $N_1' = s_{m_1}$, $N_1'' = s_{m_2}$, $N_2' = s_{m_3}$, $N_2'' = s_{m_4}$ *for some* $m_1 < m_2 < m_3 < m_4$
- $f_k(ze((s+m)\omega), \omega, E) \neq 0$ *for any* $(z, E) \in \mathcal{D}(e(x_0), \underline{r}_0) \times \mathcal{D}(E_0, r_0)$, $s \in [1, s_1] \cup [s_2, s_3] \cup [s_4, N]$ *and any* $|m| \le C\ell$*. In particular,* $s_{m_k}$, $1 \le k \le 4$ *are adjusted to the polydisk* $\mathcal{D}(e(x_0), \underline{r}_0) \times \mathcal{D}(E_0, r_0)$ *at scale* $\ell$ *where* $r_0 = c\kappa$, $\underline{r}_0 = \exp(-\sqrt{\ell}/2)$
- $f_{t_j}(ze(s_{j-1}\omega), \omega, E) \neq 0$ *for any* $(z, E) \in \mathcal{D}(e(x_0), \underline{r}_0) \times \mathcal{D}(E_0, r_0)$, $j = 1,3,5$*. Furthermore,* $f_{t_j}(ze(s_{j-1}\omega), \omega, E) \neq 0$ *for any*

  $$(z, E) \in \Big[(\mathcal{D}(z_0, 2\underline{r}_0) \setminus \mathcal{D}(z_0, r_0/2)) \times \{E_0\}\Big] \bigcup \Big[\mathcal{D}(z_0, \underline{r}_0) \times \mathcal{D}(E_0, 2r_0) \setminus \mathcal{D}(E_0, r_0/2)\Big]$$

  $j = 2,4$*. Here* $s_0 = 1$, $t_j = s_{m_j} - s_{m_{j-1}}$.
- $\nu_{f_{t_j}(\cdot e(s_{j-1}\omega), \omega, E_0)}(e(x_0), \underline{r}) = 1$ *for any* $0 < r \le \underline{r}_0$ *and* $\nu_{f_{t_j}(e(x+s_{j-1}\omega), \omega, \cdot)}(E_0, r_1) = 1$ *for any* $e(x) \in \mathcal{D}(e(x_0), \underline{r}_0)$*. Here* $\exp(-\sqrt{\ell}/4) < r_1 < \exp(-(\log \ell)^A) r_0$.
- $\nu_{f_N(\cdot, \omega, E_0)}(e(x_0), \underline{r}_0) = 2$ *and* $\nu_{f_N(e(x), \omega, \cdot)}(E_0, r_1) = 2$ *for any* $e(x) \in \mathcal{D}(e(x_0), \underline{r}_0)$.

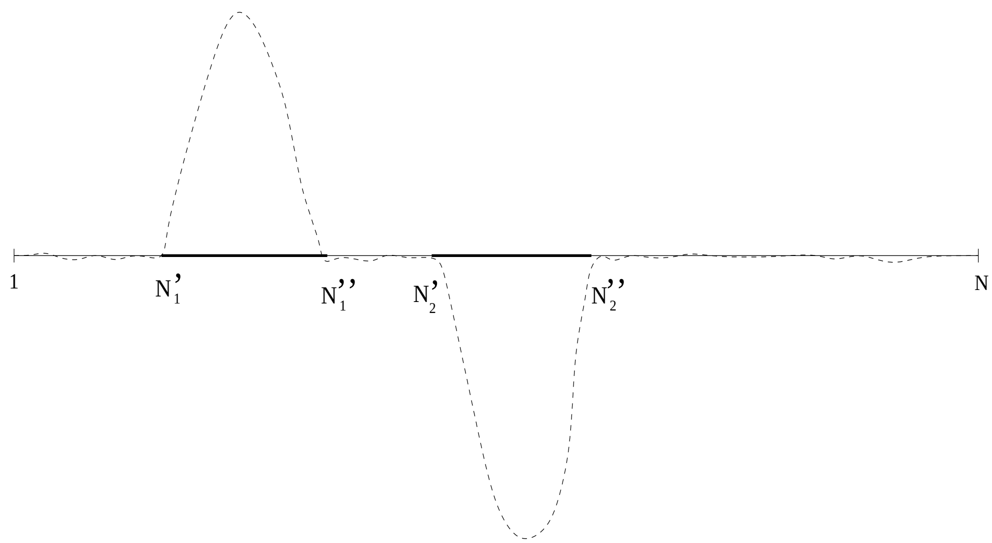


FIGURE 3. A double resonance and an associated eigenfunction

*Proof.* Clearly, one can choose $\ell_1, \dots, \ell_n$ in (2.35) so that the first property holds. It follows from property (c) in Definition 7.1 that $f_k(ze((s+m)\omega), \omega, E) \neq 0$ for any

$$(z, E) \in \mathcal{D}(e(x_0), \underline{r}_0) \times \mathcal{D}(E_0, r_0), \quad s \in [1, s_1] \cup [s_2, s_3] \cup [s_4, N]$$

and any $|m| \leq C\ell$. Therefore, we can apply Corollary 2.36 and Lemma 2.38 to verify that

$$f_{t_j}(ze(s_{j-1}\omega), \omega, E) \neq 0$$

for any $(z, E) \in \mathcal{D}(e(x_0), r_0) \times \mathcal{D}(E_0, r_0)$ and $j = 1, 3, 5$. It follows from property (a) of Definition 7.1 that $e(x_0)$ is the only zero of $f_{t_j}(\cdot e(s_{m_j}\omega), \omega, E_0)$ in $\mathcal{D}(e(x_0), \underline{r}_0)$ and that $E_0$ is the only zero of $f_{t_j}(e(x_0 + s_{m_j}\omega), \omega, \cdot)$ in $\mathcal{D}(E_0, r_0)$, $j = 2, 4$. That proves the third and fourth properties. The final claim follows by means of another application of Corollary 2.36 and Lemma 2.38. ☐

In particular, we emphasize that

$$\#\big(\big(\text{sp } H_N(x, \omega)\big) \cap \big(E_0 - \kappa, E_0 + \kappa\big)\big) = 2$$

for any $x \in (x_0 - \rho, x_0 + \rho)$, where $\rho \asymp \kappa$. By Rellich's theorem on analytic matrix functions there exist real analytic functions $E^{\pm}(x)$ and analytic vector functions $\{\psi^{\pm}(x, n) : n \in [1, N]\}, x \in (x_0 - \rho, x_0 + \rho)$ such that

$$H_N(x, \omega)\psi^{\pm}(x, n) = E^{\pm}(x)\psi^{\pm}(x, n), \qquad \sum_n \left|\psi^{\pm}(x, n)\right|^2 = 1$$

for any $x \in (x_0 - \rho, x_0 + \rho)$. Note that due to (b), (c), all the conditions needed for Proposition 4.1 on the separation of the eigenvalues are valid for $H_N(x, \omega)$ and $E = E^+(x)$. More precisely, we are in the situation described in Remark 4.2. So, one has

**Lemma 7.3.** $\left|E^+(x) - E^-(x)\right| \geq \tau = \exp\big(-N^\delta\big)$ *for any* $x \in (x_0 - \rho, x_0 + \rho)$.

One can assume for instance that

$$E^+(x) \geq E^-(x) + \tau\ , \quad x \in (x_0 - \rho, x_0 + \rho)\ . \tag{7.1}$$

The goal is to show that as a matter of fact

$$\min_x E^+(x) \geq \max_x E^-(x) + \sigma\ , \quad \sigma = \tau^4\ . \tag{7.2}$$

The main additional assumption for that is as follows:

(d) *Let* $E_k(x)$ *be the only eigenvalue of* $H_{\Lambda_k}(x)$ *belonging to* $\big(E_0-\kappa, E_0+\kappa\big)$*. Then*

$$\partial_x E_1 > \beta, \qquad x \in (x_0-\rho, x_0+\rho) \tag{7.3}$$
$$\partial_x E_2 < -\beta, \qquad x \in (x_0-\rho, x_0+\rho)$$

$\beta \asymp \exp\big(-\underline{N}^{\delta_1}\big)$, $0<\delta<\delta_1$, in particular,

$$E_1(x_0+\xi), E_2(x_0-\xi) > E_0+\beta\xi \ , \quad \xi>0 \tag{7.4}$$
$$E_1(x_0-\xi), E_2(x_0+\xi) < E_0-\beta\xi \ , \quad \xi<0$$

Combining (7.1), (7.4) and property (c) in the definition of a double resonance one obtains

**Lemma 7.4.**

$$\big|E^+(x)-E_1(x)\big|<\eta, \quad \big|\partial_x E^+(x)-\partial_x E_1(x)\big|<\eta \tag{7.5}$$
$$\big|E^-(x)-E_2(x)\big|<\eta, \quad \big|\partial_x E^-(x)-\partial_x E_2(x)\big|<\eta$$

*for any* $x \in (x_0+\rho/2, x_0+\rho)$

$$\big|E^+(x)-E_2(x)\big|<\eta, \quad \big|\partial_x E^+(x)-\partial_x E_2(x)\big|<\eta \tag{7.6}$$
$$\big|E^-(x)-E_1(x)\big|<\eta, \quad \big|\partial_x E^-(x)-\partial_x E_1(x)\big|<\eta$$

*for any* $x\in(x_0-\rho, x_0-\rho/2)$*, where* $\eta=\exp\big(-\gamma_4\underline{N}^{1/2}\big)$*.*

This lemma implies, in particular, that $E^\pm(x)$ assumes its minimum (maximum) at some critical point $x^+(x^-)$, where $\partial_x E^\pm\big|_{x=x^\pm}=0$.

Assume now that

$$E^-(x^-) > E^+(x^+)-\sigma \ . \tag{7.7}$$

Consider first the case

$$E^+(x^+) \ge E^-(x^-) > E^+(x^+)-\sigma \ . \tag{7.8}$$

Recall that $E^\pm(x)$ are the solutions of the equation $f_N\big(e(x),E\big)=0$, $x\in(x_0-\rho,x_0+\rho)$, $E\in\big(E_0-\kappa,E_0+\kappa\big)$. Let $f_N\big(e(x),E\big)=P(x,E)g(x,E)$ be the factorization of $f_N\big(e(x),E\big)$ in the neighborhood of $z_0=x_0$, $E_0$, defined in Theorem 2.25. Due to Lemma 7.2

$$\#\big\{z\in\mathcal{D}\big(e(x_0),\rho\big) : f_N(z,\omega,E_0)=0\big\}=2.$$

Hence,

$$P(x,E)=(x-x_0)^2+b_1(E)(x-x_0)+b_0(E) \tag{7.9}$$

where $b_j(E)$ are analytic in $\mathcal{D}(E_0,\kappa), \sup_E |b_j(E)|\le 1$, $j=0,1$. Then $\partial_x E^\pm\big|_{x=x^\pm}=0$ implies

$$-\big(\partial_x P(x,E)/\partial_E P(x,E)\big)\Big|_{x=x^\pm, E=E^\pm(x^\pm)}=0 \ .$$

So,

$$2(x^\pm-x_0)+b_1\big(E^\pm(x^\pm)\big)=0 \ .$$

In particular, by Cauchy's inequalities and (7.8)

$$|x^+-x^-| \le \big|b_1\big(E^+(x^+)\big)-b_1\big(E^-(x^-)\big)\big| \le 4\sigma\kappa^{-1} < \sigma^{1/2}=\tau^2 \ . \tag{7.10}$$

In its turn (7.10) implies

$$E^+(x^+)-E^-(x^+) \le E^+(x^+)-E^-(x^-)+C|x^+-x^-| \le \sigma+C\tau^2 < C_1\tau^2$$

in contradiction to (7.1). Assume now that

$$E^-(x^-) > E^+(x^+) \tag{7.11}$$

Then there exist $x_1 < x^- < x_2$ such that

$$E^-(x_j) = E^+(x^+) \ , \quad j = 1, 2 \ .$$

Note that due to (7.1), $x_1 \neq x^+$, $x_2 \neq x^+$. But that means that the equation

$$P\left(x, E^+(x^+)\right) = 0$$

has three different roots $x_1, x_2, x^+$, in contradiction to (7.9). Thus (7.7) is impossible and (7.2) is valid.

**Proposition 7.5.** *Let $(x_0, E_0)$ be a point of a double resonance, i.e., conditions (a) – (c) are valid. Assume also that the eigenvalue $E_k(x)$ of $H_{\Lambda_k}(x, \omega)$ falling into the interval $(E_0-\kappa, E_0+\kappa)$ satisfies condition (d), $k = 1, 2$. Then there exist real analytic functions $E^{(\pm)}(x)$ and analytic vector functions $\{\psi^{(\pm)}(x, n) : n \in [1, N]\}$, $x \in (x_0 - \rho, x_0 + \rho)$, $\rho \asymp \exp(-\underline{N}^{\delta_1})$, $0 < \delta_1 < \delta$*

(i) *$E^{(+)}(x), E^{(-)}(x)$ are the only eigenvalues of $H_N(x)$ falling into the interval $(E_0 - \kappa, E_0 + \kappa)$, and $\{\psi^{(\pm)}(x, n) : n \in [1, N]\}$ are the corresponding normalized eigenvectors, $|x - x_0| < \rho$*

(ii) *$\min E^{(+)}(x) > \max E^{(-)}(x) + \sigma$, $\sigma = \exp(-N^\delta)$*

(iii) *$|\psi^{(\pm)}(x, n)| \lesssim \exp\left(-\gamma \operatorname{dist}(n, [N_1', N_2''])\right)$.*

**Corollary 7.6.** *Set $\overline{E}' = \min_x E^{(+)}(x)$, $\underline{E}' = \max_x E^{(-)}(x)$. Then for any $E \in \left(\underline{E}' + \sigma/4, \overline{E}' - \sigma/4\right)$, the Dirichlet determinant $f_N(\cdot, E)$ has two zeros $\zeta_j = \zeta_j(E) = e(x_j+iy_j) \in \mathcal{D}(e(x_0), r_0)$, with $|y_j| > C(\lambda, V)^{-1}\sigma$, $j = 1, 2$.*

*Proof.* By Theorem 2.25 for any $E \in \mathcal{D}(E_0, \kappa)$, polynomial $P(\cdot, E)$ has two zeros $z_j = z_j(E) = x_j + iy_j$, $z_j \in \mathcal{D}(z_0, \ )$. On the other hand, for any $x \in (x_0 - \rho, x_0 + \rho)$, $E \in \left(\underline{E}' + \sigma/4, \overline{E}' - \sigma/4\right)$

$$\operatorname{dist}(\operatorname{sp}\ H_N(x), E) = \min\left(E^{(+)}(x) - E, E - E^{(-)}(x)\right) > \sigma/4 \ .$$

Since $H_N(x)$ is self adjoint and $\|H_N(x) - H_N(x+iy)\| < C(\lambda, V)|y|$, one has also

$$\operatorname{dist}(\operatorname{sp}\ H_N(x+iy), E) > \sigma/8 > 0$$

for any $x \in (x_0 - \rho, x_0 + \rho)$, $|y| < C(\lambda, V)_1^{-1}\sigma/8$, $E \in (\underline{E}', \overline{E}')$. In particular, $f_N\left(e(x+iy), E\right) \neq 0$ for such $x, y, E$. Hence $|y_j| \geq C(\lambda, V)^{-1}\sigma/8$, $j = 1, 2$. □

Due to Lemma 7.2 in the definition of the double resonance one can apply Proposition 3.4 (or more precisely, Remark 4.2) to the eigenfunctions $\psi^\pm(x, n)$. That implies the following estimate.

**Lemma 7.7.** *Using the notations of Proposition 7.5 one has*

$$|\psi^\pm(x, n)| \leq \exp\left(-\gamma_1 \operatorname{dist}(n, [N_1', N_2''])\right) \ .$$

**Proposition 7.8.** *One says that a double resonance point $(x_0, E_0)$ is* regular *if in addition to conditions (a)–(d) the following condition is valid*

*(e) $N^{1/2} < N_1'$, $N_2'' < N - N^{1/2}$.*

Let $(x_0, E_0)$ be a regular double resonant point. Conditions (b), (c), (e) combined with the previous lemma and Lemma 4.6 imply the following assertion.

**Proposition 7.9.** *Let $(x_0, E_0)$ be a regular double resonant point for $H_N(\cdot)$. Let $E^\pm(x)$ be the eigenvalues and $\psi^\pm(x, n)$, $n \in [1, N]$ be the corresponding eigenfunctions defined in Proposition 7.5. Then for each integer $k \in \left(2N^{1/2} - \nu_0, N - 2N^{1/2} - \nu_0\right)$, where $\nu_0 = (N_1' + N_2'')/2$, and any $x \in (x_0 - \rho, x_0 + \rho)$ there exist exactly two eigenvalues $E_k^\pm(x + k\omega)$ of $H_N(x + k\omega, \omega)$ falling into the interval $(E_0 - \kappa, E_0 + \kappa)$. Moreover,*

$$|E^\pm(x) - E_k^\pm(x + k\omega)| < \exp(-\gamma_4 N^{1/2})$$

$$|\partial_x E^\pm(x) - \partial_x E_k^\pm(x + k\omega)| < \exp(-\gamma_5 N^{1/2})$$

Now one can follow the exact same arguments as in Corollary 7.6 to validate the following

**Corollary 7.10.** *Let $(x_0, E_0)$ be a regular double resonant point for $H_N(\cdot)$. Set $\overline{E}' = \min_x E^+(x)$, $\underline{E}' = \max_x E^-(x)$, where $E^{\pm}(x)$ stand for the eigenvalues defined in Proposition 7.5. Then for any*

$$E \in \Big(\underline{E}' + \exp\big(-2N^\delta\big), \overline{E}' - \exp\big(-2N^\delta\big)\Big)$$

*the Dirichlet determinant $f_N(\cdot, \omega, E)$ has a sequence of zeros $\zeta_{k,j} = e(x_{kj} + iy_{kj})$, $k \in \big(2N^{1/2} - \nu_0, N - 2N^{1/2} - \nu_0\big)$, $j = 1, 2$, where $\|x_{kj} - x_0 - k\omega\| < \rho$, $C(\lambda, V)^{-1} \exp\big(-2N^\delta\big) < y_{kj} < \exp\Big(-\big(\log N\big)^A\Big)$, $N_1' < \nu_0 < N_1''$.*

Finally, we arrive at the following main conclusion of this section.

**Proposition 7.11.** *Let $(x_0, E_0)$ be a regular double resonance point. Then there exists an interval*

$$\big(\underline{E}^{(1)}, \overline{E}^{(1)}\big) \subset \Big(E_0 - \exp\Big(-\big(\log N\big)^A\Big), E_0 + \exp\Big(-\big(\log N\big)^A\Big)\Big),$$

*with $\overline{E}^{(1)} - \underline{E}^{(1)} > \exp\big(-2N^\delta\big)$, such that for any $E \in \big(\underline{E}^{(1)}, \overline{E}^{(1)}\big)$ one has*

$$\frac{1}{N} \# \big\{z \in \mathcal{A}_{\rho^{(1)}} : f_N(z, \omega, E) = 0\big\} \le \frac{1}{N} \# \{z \in \mathcal{A}_{\bar\rho} : f_N(z, \omega, E) = 0\} - 2 + 4N^{-1/2} \tag{7.12}$$

*where $\rho^{(1)} = \exp\big(-4N^\delta\big)$, $\bar\rho = \exp\Big(-\big(\log N\big)^A\Big)$.*

**Definition 7.12.** *If (7.12) is valid we say that $\big(\underline{E}^{(1)}, \overline{E}^{(1)}\big)$ is a pre-gap interval at scale $N$. We allow also in the definition the error term $4N^{-1/2}$ to be replaced by $\varepsilon(N)$, where $\varepsilon(N) \to 0$. For that we will specify each time the expression $\varepsilon(N)$ for different scales $N$.*

A very important feature of pre-gaps is that they are sustainable when the scale $N$ grows. That is due to Lemma 2.23. We will return to this important issue in Section 9. But first, we discuss the crucial topic of eliminating triple resonances, see Chan [Cha].

## 8. Elimination of triple resonances

**Lemma 8.1.** *Let $f(x, y)$ be a $C^1$-function defined on $\overline{\mathcal{R}} = [a, b] \times [c, d] \subset [0, 1]^2$. Assume that*

$$\mu = \min_{(x,y)\in\mathcal{R}} \partial_y f(x, y) > 0 \tag{8.1}$$

*If $f(x_0, y_0) = 0$ for some $(x_0, y_0) \in \mathcal{R} = (a, b) \times (c, d)$, then for any $x \in J_0 := (x_0 - \kappa_0, x_0 + \kappa_0) \cap (a, b)$ with $\kappa_0 = h_0\mu K^{-1}$, $K = \max_{x,y} |\partial_x f(x, y)|$, $h_0 = \min(y_0 - c, d - y_0)$ there exists a unique $y = \phi_0(x) \in (c, d)$ such that*

$$f(x, \phi_0(x)) = 0$$

*The function $\phi_0(x)$ is $C^1$ differentiable on $J_0$ and*

$$\sup_{x\in J_0} |\partial_x \phi_0(x)| \le K\mu^{-1}.$$

*Proof.* Note that for any $x \in [a, b]$ one has $|f(x, y_0)| \le K|x - x_0|$. In particular, for any $|x - x_0| < \kappa_0$

$$|f(x, y_0)| < h_0\mu$$

Given such $x$ consider the case $0 < f(x, y_0) < h_0\mu$. Since $c \le y_0 - h_0 < d$, we infer that

$$f(x, y_0 - h_0) < h_0\mu - h_0\mu = 0.$$

Hence, there exists a unique $y = \phi_0(x) \in (y_0 - h_0, y_0)$ such that $f(x, \phi_0(x)) = 0$. If instead $-h_0\mu < f(x, y_0) \le 0$ then there exists a unique $y = \phi_0(x) \in (y_0, y_0 + h_0)$ such that $f(x, \phi_0(x)) = 0$. It follows from the implicit function theorem and the chain rule that $|\partial_x \phi_0(x)| \le K\mu^{-1}$. □

**Lemma 8.2.** *Using the notations of the previous lemma, assume that $|f(x_1, y_1)| < \varepsilon$ for some $(x_1, y_1) \in \mathcal{R}$ and $0 < \varepsilon \le h_1\mu$ where $h_1 = \min(y_1 - c, d - y_1)/2$ and $\mu$ is as in (8.1). Then*

(1) *For any* $x \in J_1 := (x_1 - \kappa_1, x_1 + \kappa_1) \cap (a, b)$ *with* $\kappa_1 = h_1 \mu K^{-1}$ *there exists a unique* $y = \phi(x) \in (c, d)$ *such that* $f(x, \phi_1(x)) = 0$. *The function* $\phi_1(x)$ *is* $C^1$ *on* $J_1$ *and*

$$\sup_{x \in J_1} |\partial_x \phi_1(x)| \le K\mu^{-1}.$$

(2) *For any* $x \in J_1$ *and any* $y \in [c, d] \setminus (\phi_1(x) - \varepsilon\mu^{-1}, \phi_1(x) + \varepsilon\mu^{-1})$ *one has* $|f(x, y)| \ge \varepsilon$.

*Proof.* Assume for instance that $0 \le f(x_1, y_1) < \varepsilon$. Since $c \le y_1 - h_1 < d$, we conclude that $f(x_1, y_1 - h_1) < \varepsilon - \mu h_1 \le 0$. Hence, there exists a unique $\tilde{y}_1 \in (y_1 - h_1, y_1]$ such that $f(x_1, \tilde{y}_1) = 0$. By Lemma 8.1 applied to this point there exists a $C^1$-function $\phi_1(x)$ defined on the interval $J_2 := (x_1 - \tilde{\kappa}_1, x_1 + \tilde{\kappa}_1) \cap (a, b)$ with $\tilde{\kappa}_1 = \tilde{h}_1 \mu K^{-1}$, $\tilde{h}_1 = \min(\tilde{y}_1 - c, d - \tilde{y}_1)$ such that $f(x, \phi_1(x)) = 0$ for any $x \in J_2$. Moreover, $\sup_{x \in J_1} |\partial_x \phi_1(x)| \le K\mu^{-1}$. Note first that $\tilde{h}_1 \ge \min(y_1 - c, d - y_1) - h_1 = h_1$ by construction. So, $\phi_1(x)$ is defined on the interval $J_1$. Clearly, $|f(x, y)| \ge \varepsilon$ for any $x \in J_1$ and any $y \in [c, d] \setminus (\phi_1(x) - \varepsilon\mu^{-1}, \phi_1(x) + \varepsilon\mu^{-1})$. □

We can now combine these local lemmas with an obvious covering procedure to obtain the following global statement.

**Proposition 8.3.** *Let* $f(x, y)$ *be a* $C^1$ *function defined on* $\overline{\mathcal{R}} = [a, b] \times [c, d] \subset [0, 1]^2$. *Assume that*

$$\mu = \min_{(x,y) \in \mathcal{R}} \partial_y f(x, y) > 0$$

*and set* $\mu_1 := \min(1, \mu)$. *Given* $h_1 < (d - c)/4$, $\varepsilon < h_1 \mu$, *there exist intervals* $(x_j - \kappa_1, x_j + \kappa_1)$, $j = 1, 2, \ldots, k_1$, *with* $\kappa_1 = h_1 \mu_1 K^{-1}$, $K := 1 + \max_{(x,y) \in \mathcal{R}} |\partial_x f(x, y)|$, $k_1 \le \lceil 2(b - a)\kappa_1^{-1} \rceil$ *such that*

(1) *For each* $j$ *there exists a* $C^1$ *function* $\phi_j(x)$ *defined on* $(x_j - \kappa_1, x_1 + \kappa_1) \cap (a, b)$ *such that* $f(x, \phi_j(x)) = 0$, *and* $\sup_x |\partial_x \phi_j(x)| \le K\mu^{-1}$.

(2) *The set*

$$\mathcal{U}(h_1, \varepsilon) := \{(x, y) \in (a, b) \times (c + h_1, d - h_1) \,:\, |f(x, y)| < \varepsilon\}$$

*is covered by the union of the following sets*

$$\begin{aligned} \mathcal{S}_j &:= \mathcal{S}(\phi_j, \varepsilon\mu^{-1}) \\ &= \{(x, y) \,:\, x \in (x_j - \kappa_1, x_j + \kappa_1) \cap (a, b),\ y \in (\phi_j(x) - \varepsilon\mu^{-1}, \phi_j(x) + \varepsilon\mu^{-1})\} \end{aligned}$$

**Remark 8.4.** *Recall that if* $f(x, y)$ *is a* $C^2$*-function, then the implicit function* $x = \phi(y)$ *defined by the equation* $f(x, y) = 0$ *is also* $C^2$ *provided* $\partial_y f > 0$ *for all* $x, y$. *Moreover,*

$$|\partial_x^2 \phi| \le 4\frac{K^2}{\mu_1^3}$$

*where* $\mu_1 = \min(1, \inf \partial_y f(x, y)) > 0$, $K := 1 + \sup_{(x,y); 0 < |\alpha| \le 2} |\partial^\alpha f(x, y)|$

**Lemma 8.5.** *Let* $\phi(x)$ *be a* $C^2$ *function,* $x \in [\alpha, \beta]$,

$$B = \max\{|\partial^k \phi(x)| \,:\, x \in [\alpha, \beta],\ 1 \le k \le 2\}$$

*Given* $\delta \in (0, 1)$, $n > (\beta - \alpha)^{-1}$, *there exist intervals* $[\alpha_j', \alpha_j'']$, $1 \le j \le j_0$, $j_0 \le n$, *such that* $|\phi'(x)| \ge \delta$ *for any* $x \notin \bigcup_j [\alpha_j', \alpha_j'']$, *and such that*

$$\sum_j \Big( \max\{\phi(x) \,:\, x \in [\alpha_j', \alpha_j'']\} - \min\{\phi(x) \,:\, x \in [\alpha_j', \alpha_j'']\} \Big) \le \delta(\beta - \alpha) + B(\beta - \alpha)^2 n^{-1}$$

*Proof.* Set $\alpha_j = \alpha + j/n$, $0 \le j \le j_0$, $j_0 = \lfloor n(\beta - \alpha) \rfloor$, $\alpha_{j_0+1} = \beta$,

$$\tau_j = \min\{|\partial_x \phi(x)| \,:\, x \in [\alpha_j, \alpha_{j+1}]\}$$

If $\tau_j \le \delta$ for some $j$, then

$$\max\{|\partial_x \phi(x)| \,:\, x \in [\alpha_j, \alpha_{j+1}]\} \le \delta + B(\beta - \alpha)n^{-1}$$

and therefore also

$$\max\{|\phi(x) - \phi(y)| \,:\, x, y \in [\alpha_j, \alpha_{j+1}]\} \leq \Big(\delta + B(\beta-\alpha)n^{-1}\Big)(\beta-\alpha)n^{-1}$$

Summing this inequality over $j$ yields the statement of the lemma. □

**Proposition 8.6.** *Let $f(x,y)$ be a $C^2$ function in $\overline{\mathcal{R}} = [a,b]\times[c,d] \subset [0,1]^2$. Assume that*

$$\mu = \min_{(x,y)\in\mathcal{R}} \partial_y f(x,y) > 0$$

*Given $0 < \delta < \tau^2$,*

$$\begin{aligned}\mu_1 &= \min(1,\mu)\\ \tau &= \Big[\max(2, 4K^2\mu_1^{-3})\Big]^{-1}\\ K &= 1 + \max\{|\partial^\alpha f(x,y)| \,:\, (x,y)\in\mathcal{R},\ 0<|\alpha|\leq 2\}\end{aligned}$$

*and $\varepsilon < \delta\mu_1$, there exist intervals $(\underline{\xi}_{i,j}, \overline{\xi}_{i,j}) \subset [c,d]$, $1 \leq i \leq k_1$, $k_1 \leq 2(b-a)\delta^{-\frac{1}{2}}\mu_1^{-1}$, $j_i \leq \delta^{-2}$, and $C^2$ functions $\psi_{i,j}(y)$, $y \in (\underline{\xi}_{i,j}, \overline{\xi}_{i,j})$ such that the following conditions are valid:*

1. mes $\Big([c,d]\setminus \bigcup_{i,j}(\underline{\xi}_{i,j}, \overline{\xi}_{i,j})\Big) \leq 8\delta^{\frac{1}{2}}$
2. *the set $\mathcal{U}(f,\varepsilon) = \{(x,y)\in\overline{\mathcal{R}} \,:\, |f(x,y)|\leq\varepsilon\}$ satisfies*
$$\mathcal{U}(f,\varepsilon)\cap[a,b]\times\Big(\bigcup(\underline{\xi}_{i,j}, \overline{\xi}_{i,j})\Big) \subset \bigcup_{i,j}\mathcal{S}(\psi_{i,j}, \varepsilon\delta^{-1}\mu_1^{-1})$$
*where*
$$\mathcal{S}(\psi_{i,j},\varepsilon') = \{(x,y) \,:\, y\in(\underline{\xi}_{i,j}, \overline{\xi}_{i,j}),\ |\psi_{i,j}(y) - x| < \varepsilon'\}$$
3. $|\partial_y\psi_{i,j}| \leq \delta^{-1}$

*Proof.* We apply Proposition 8.3 to the function $f(x,y)$ and $h_1 = \delta^{\frac{1}{2}}$. By Remark 8.4 the functions $\phi_j(x)$ defined by Proposition 8.3 are $C^2$-smooth and $\sup_x |\partial_x^2\phi_j(x)| \leq 4K^2\mu_1^{-3} =: B$. By Lemma 8.5, for each $\phi_i(x)$ there exist intervals $[\alpha'_{i,j}, \alpha''_{i,j}]$, $1 \leq j \leq j(i)$ such that $|\partial_x\phi_i(x)| \geq \delta$ for any $x \notin \bigcup_j(\alpha'_{i,j}, \alpha''_{i,j})$, and

$$\sum_j (\overline{y}_{i,j} - \underline{y}_{i,j}) \leq 2\kappa_1\delta + 4B\kappa_1^2\delta^2 \leq 6\delta^{\frac{1}{2}}$$

where $\kappa_1 = h_1\mu_1K^{-1} \leq 1$ and

$$\underline{y}_{i,j} = \min\{\phi_i(x) \,:\, x\in[\alpha'_{i,j}, \alpha''_{i,j}]\}, \quad \overline{y}_{i,j} = \max\{\phi_i(x) \,:\, x\in[\alpha'_{i,j}, \alpha''_{i,j}]\}$$

Let $(\underline{\xi}_{i,j}, \overline{\xi}_{i,j})$ be the maximal intervals of

$$(c,d)\setminus\bigcup_j(\underline{y}_{i,j}, \overline{y}_{i,j})$$

On each interval $(\underline{\xi}_{i,j}, \overline{\xi}_{i,j})$ an inverse function $\psi_{i,j} = \phi_i^{-1}$ is defined, and moreover

$$\sup_y |\partial_y\psi_{i,j}| \leq \delta^{-1}$$

Note that if $y \in (\underline{\xi}_{i,j}, \overline{\xi}_{i,j})$ and $|y - \phi_i(x)| < \varepsilon\mu^{-1}$ for some $x \in (x_i - \kappa_1, x_i + \kappa_1)$, then $|x - \psi_{i,j}(y)| \leq \varepsilon\delta^{-1}\mu^{-1}$. In other words, in view of Proposition 8.3 one has

$$\mathcal{U}(h,\varepsilon)\bigcap\Big([(x_i-\kappa_1, x_i+\kappa_1)\cap(a,b)]\times(\underline{\xi}_{i,j}, \overline{\xi}_{i,j})\Big) \subset \mathcal{S}(\psi_{i,j}, \varepsilon\mu^{-1}\delta^{-1})$$

as claimed. □

**Theorem 8.7.** *Let $f(x,y), g(x,y)$ be $C^2$ functions in $\overline{\mathcal{R}} = [a,b] \times [c,d]$. Assume that*

$$\mu = \min_{(x,y)\in\mathcal{R}} \partial_y f(x,y) > 0$$
$$\overline{\mu} = \min_{(x,y)\in\mathcal{R}} \partial_y g(x,y) > \delta^{-2} K$$

*where $0 < \delta < \tau^2$, and $\tau, K, \mu_1$ are as in the previous proposition. Given $\varepsilon < \delta^7 \mu_1 T^{-1}$, $T = 1 + \max_{x,y} |\partial_x g(x,y)|$, there exist intervals $(\underline{\zeta}_j, \overline{\zeta}_j) \subset (c,d)$, $(\underline{\eta}_k, \overline{\eta}_k) \subset (c,d)$, $1 \le j \le j_0$, $1 \le k \le k_0$ such that*

1. $\sum_j (\overline{\zeta}_j - \underline{\zeta}_j) \le 8\delta^{\frac{1}{2}}$, $\sum_k (\overline{\eta}_k - \underline{\eta}_k) \le \varepsilon^{\frac{1}{2}}$
2. *the intervals $(\underline{\zeta}_j, \overline{\zeta}_j)$ only depend on the function $f$*
3. $j_0, k_0 \le \delta^{-3}$
4. *for any*
$$y \in (c,d) \setminus \Big[\bigcup_j (\underline{\zeta}_j, \overline{\zeta}_j) \bigcup \bigcup_k (\underline{\eta}_k, \overline{\eta}_k)\Big]$$
*and any $x \in (a,b)$ at least one of the inequalities*
$$|f(x,y)| < \varepsilon, \quad |g(x,y)| < \varepsilon$$
*fails.*

*Proof.* Let $(\underline{\xi}_{i,j}, \overline{\xi}_{i,j})$, $\psi_{i,j}$ be defined as in Proposition 8.6. Note that

$$\begin{aligned}\partial_y g(\psi_{i,j}(y), y) &\ge \partial_y g - |\partial_y \psi_{i,j}||\partial_x g| \\ &\ge \overline{\mu} - \delta^{-1} K > \overline{\mu}/2\end{aligned}$$

for any $y \in (\underline{\xi}_{i,j}, \overline{\xi}_{i,j})$. Hence, there exists $(\underline{\eta}_{i,j}, \overline{\eta}_{i,j}) \subset (\underline{\xi}_{i,j}, \overline{\xi}_{i,j})$ such that

$$\overline{\eta}_{i,j} - \underline{\eta}_{i,j} \le 4\varepsilon\delta^{-1}\mu_1^{-1} T \tag{8.2}$$

and

$$|g(\psi_{i,j}(y), y)| > 2\varepsilon\delta^{-1}\mu_1 T$$

for any $y \in (\underline{\xi}_{i,j}, \overline{\xi}_{i,j}) \setminus (\underline{\eta}_{i,j}, \overline{\eta}_{i,j})$. On the other hand, if $|f(x,y)| < \varepsilon$ and $y \in (\underline{\xi}_{i,j}, \overline{\xi}_{i,j})$, then $|x - \psi_{i,j}(y)| \le \varepsilon\delta^{-1}\mu_1$. Hence,

$$|g(x,y)| \ge |g(\psi_{i,j}(y), y)| - T\varepsilon\delta^{-1}\mu_1 > T\varepsilon\delta^{-1}\mu_1 > \varepsilon$$

provided $y \in (\underline{\xi}_{i,j}, \overline{\xi}_{i,j}) \setminus (\underline{\eta}_{i,j}, \overline{\eta}_{i,j})$. The set

$$[c,d] \setminus \bigcup_{i,j} (\underline{\xi}_{i,j}, \overline{\xi}_{i,j})$$

is the union of intervals $(\underline{\zeta}_j, \overline{\zeta}_j)$, whereas the intervals $(\underline{\eta}_k, \overline{\eta}_k)$ are merely a renumeration of the intervals $(\underline{\eta}_{i,j}, \overline{\eta}_{i,j})$. The estimates stated in the theorem now follow from Proposition 8.6 and by summing (8.2) over $i, j$. □

Let $E_i(x, \omega)$ be $C^2$ functions $(x, \omega) \in \mathcal{R}_i = (x_i - \tau, x_i + \tau) \times (\omega_i - \tau, \omega_i + \tau) \subset [0,1] \times [0,1]$, $i = 1,2,3$. Let $n_2, n_3$ be integers. Our ultimate goal in this section is to show that under some natural conditions on the functions $E_i(x,\omega)$ , $i = 1,2,3$, the system

$$|E_i(x + \{n_i \omega\}, \omega) - E| < \varepsilon$$

$i = 1,2,3$, with $n_1 = 0$ has no solution for any $E$, provided the integers $n_2, n_3$ are large and $\omega$ is outside of some exceptional set $\Omega(n_2, n_3)$ of small measure which does not depend on $E$; here $\{y\}$ stands for the fractional part of $y$. Moreover, we want the total measure of the union of the sets $\Omega(n_2, n_3)$ over a certain set of integers $n_2, n_3$ to be small. The variable $E$ can be eliminated simply by subtraction. Therefore, we consider the following functions

$$f(x,\omega) = E_2(x + \{n_2\omega\}, \omega) - E_1(x,\omega)$$
$$g(x,\omega) = E_3(x + \{n_3\omega\}, \omega) - E_1(x,\omega)$$

defined wherever the expressions on the right-hand side are defined. To make use of Theorem 8.7 one needs these functions to be smooth. Since the function $\{y\}$ is non-smooth only at the integer values of the variable $y$, one should define an appropriate cover of the domain of the functions $f(x,\omega), g(x,\omega)$ by rectangles with edges in the plane of the variables $x, \omega$ adjusted in such a way that the functions are smooth inside these rectangles. Note also that the conditions imposed on the functions $f$ and $g$ in Theorem 8.7 are not exactly symmetric with respect to $f$ and $g$ (the quantity $\overline{\mu}$ is much larger than $\mu$). For that reason we assume that the integer $n_3$ appearing in the expression for the function $g$ is much larger than $n_2$. Taking all that into account we define the aforementioned rectangles as follows: Let $x_i, \omega_i$ be as above, $0 < \tau \ll 1$ . Set $t = 8[\tau^{-1}]$. Let $m_i, k_i, n_i$, $i = 1, 2$ be integers such that $0 < m_i < n_i$, $0 < k_i < t$, $i = 1, 2$, and

$$\frac{m_i}{n_i} + \frac{k_i}{tn_i} < \omega_i < \frac{m_i}{n_i} + \frac{k_i+1}{tn_i} < \frac{m_i+1}{n_i}$$
$$\frac{m_i}{n_2} + \frac{k_2}{tn_2} < \frac{m_3}{m_3} < \frac{m_3}{n_3} + \frac{k_3}{tn_3} < \omega_3 < \frac{m_3}{n_3} + \frac{k_3+1}{tn_3} < \frac{m_2}{n_2} + \frac{k_2+1}{tn_2}$$
$$\frac{k_2}{t} < x_2 - x_2 < \frac{k_2+1}{t}, \quad \frac{k_3}{t} < x_3 - x_1 < \frac{k_3+1}{t}$$

Then

$$f(x,\omega) = E_2(x + \{n_2\omega\}, \omega) - E_1(x,\omega), \quad x \in (x_1 - \underline{\tau}, x_1 + \underline{\tau}),\ \omega \in (\omega_2 - \underline{\tau}_2, \omega_2 + \underline{\tau}_2)$$
$$g(x,\omega) = E_3(x + \{n_3\omega\}, \omega) - E_1(x,\omega), \quad x \in (x_1 - \underline{\tau}, x_1 + \underline{\tau}),\ \omega \in (\omega_3 - \underline{\tau}_3, \omega_3 + \underline{\tau}_3)$$

where $\underline{\tau} = t^{-1}$, $\underline{\tau}_i = t^{-1}n_i^{-1}$, $i = 2, 3$ are well-defined and $C^2$ smooth. Applying Theorem 8.7 with $\delta = (\log \varepsilon^{-1})^{-A}$ yields the following:

**Lemma 8.8.** *Assume that*

$$\underline{\mu} = \min_{i=2,3} \inf_{x,y} \partial_x E_i > 0$$

*and* $n_2 > 4\mu_1^{-1}T_1$, *where* $\mu_1 = \min(1, \mu)$,

$$T_1 = \sup\{|\partial^\alpha E_1| \,:\, (x,y) \in \mathcal{R}_i,\ 1 \le |\alpha| \le 2,\ i = 1,2,3\} + 2$$

$n_3 > 16\mu_1^{-1}n_1^2(\log\varepsilon^{-1})^{2A}$ *where* $\log\varepsilon^{-1} > \mu_1^{-1}T_1^2$. *Then there exist intervals*

$$(\underline{\zeta}_j, \overline{\zeta}_j) \subset (m_2/n_2, (m_2+1)/n_2), \quad (\underline{\eta}_k, \overline{\eta}_k) \subset (m_3/n_3, (m_3+1)/n_3),$$

$1 \le j \le j_0$, $1 \le k \le k_0$ *such that*

- $\sum_j(\overline{\zeta}_j - \underline{\zeta}_j) \le (\log\varepsilon^{-1})^{-A/8}$, $\sum_k(\overline{\eta}_k - \underline{\eta}_k) \le \varepsilon^{\frac{1}{4}}$
- *the intervals* $(\underline{\zeta}_j, \overline{\zeta}_j)$ *do not depend on* $n_3$
- $j_0, k_0 \le \varepsilon^{-\frac{3}{8}}$
- *for any*

$$\omega \in \Big((\omega_2 - \underline{\tau}_2, \omega_2 + \tau_2) \setminus \bigcup_j (\underline{\zeta}_j, \overline{\zeta}_j)\Big) \cap \Big((\omega_3 - \underline{\tau}_3, \omega_3 + \underline{\tau}_3) \setminus \bigcup_k (\underline{\eta}_k, \overline{\eta}_k)\Big)$$

  *the system*

$$|E_2(x + n_2\omega, \omega) - E_1(x,\omega)| < \varepsilon, \quad |E_3(x + n_3\omega, \omega) - E_1(x,\omega)| < \varepsilon$$

  *has no solution with* $x \in (x_1 - \underline{\tau}, x_1 + \underline{\tau})$.

Note that since the intervals $(\underline{\zeta}_j, \overline{\zeta}_j)$ do not depend on $n_3$ in the previous lemma, one can sum there over $n_3$ in the interval $(\log\varepsilon^{-1})^{4A} < n_3 < \varepsilon^{-1/16}$, and then over $n_2$ in the interval $(\log\varepsilon^{-1})^{A/16} < n_2 < (\log\varepsilon^{-1})^{2A}$. That leads to the following main result of this section.

**Proposition 8.9.** *Let $E_i(x,\omega)$ be $C^2$ functions with $(x,\omega) \in \mathcal{R}_i = (a_i, b_i)\times(c_i, d_i) \subset [0,1]\times[0,1]$, $i = 1,2,3$. Assume*

$$\underline{\mu} = \min_{i=2,3} \inf_{x,y} \partial_x E_i > 0$$

*Given $\varepsilon > 0$ such that $\log\varepsilon^{-1} > \mu_1^{-1} T_1^2 \tau^{-1}$, $\mu_1 = \min(1, \mu)$, $\tau = \min_i \min(b_i - a_i, d_i - c_i)$,*

$$T_1 = \sup\{|\partial^\alpha E_i(x,y)| \;:\; (x,y) \in \mathcal{R}_i, ; 1 \le |\alpha| \le 2,\ 1 \le i \le 3\}$$

*there exist intervals $(\underline{\theta}_j, \overline{\theta}_j)$ with $1 \le j \le \overline{j}$ so that*

- $\sum_j (\overline{\theta}_j - \underline{\theta}_j) \le (\log\varepsilon^{-1})^{-A/16}$
- $\overline{j} \le \varepsilon^{-\frac{1}{8}}$
- *for any $(\log\varepsilon^{-1})^{\frac{A}{16}} < n_2 < (\log\varepsilon^{-1})^{2A}$, $(\log\varepsilon^{-1})^{4A} < n_3 < \varepsilon^{-1/16}$, $\omega \notin \bigcup_j (\underline{\theta}_j, \overline{\theta}_j)$ the following system*

  $$|E_2(x + n_2\omega, \omega) - E_1(x,\omega)| < \varepsilon, \quad |E_3(x + n_3\omega, \omega) - E_1(x,\omega)| < \varepsilon$$

  *has no solution.*

## 9. Existence of resonances and a proof of Theorem 1.1

To locate double resonances we will use positive-slope and negative slope segments $\left\{E_j^{(\underline{N})}(x), \underline{x}, \bar{x}\right\}$ as it was explained in the introduction.

**Lemma 9.1.** *Let $\left\{E_{j_1}^{(\underline{N})}(x), \underline{x}_1, \bar{x}_1\right\}$ and $\left\{E_{j_2}^{(\underline{N})}(x), \underline{x}_2, \bar{x}_2\right\}$ be a positive-slope and a negative-slope $I$–segment, respectively, where $I = \big[\underline{E}, \overline{E}\big]$, with $\overline{E} - \underline{E} > \exp\big(-\underline{N}^\delta\big)$. Then there exists an integer $m \in \Big[1, \exp\big(2\underline{N}^\delta\big)\Big]$, and $x_0 \in (\underline{x}_1, \bar{x}_1)$ such that*

$$E_{j_1}^{(\underline{N})}(x_0) = E_{j_2}^{(\underline{N})}(x_0 + m\omega)\ . \tag{9.1}$$

*Here $0 < \delta \ll 1$ is arbitrary but fixed and $\underline{N} > \underline{N}_0(\delta)$.*

*Proof.* Assume for instance, $\underline{x}_1 < \underline{x}_2$. Then necessarily also $\bar{x}_1 < \bar{x}_2$. Let $y_1 = \bar{x}_2 - \bar{x}_1$ and $y_2 = \underline{x}_2 - \underline{x}_1$. The function

$$h(E) = \left(E_{j_2}^{(\underline{N})}\right)^{-1}(E) - \left(E_{j_1}^{(\underline{N})}\right)^{-1}(E)$$

satisfies

$$h(\overline{E}) = y_1, \quad h(\underline{E}) = y_2$$

It follows that

$$y_1 < y_2 - C^{-1}(\overline{E} - \underline{E}) < y_2 - \exp(-2\underline{N}^\delta)$$

Hence, by the Diophantine nature of $\omega$, there exists $m \le \exp\big(2\underline{N}^\delta\big)$ so that $\{m\omega\} \in (y_1, y_2)$. Consequently, there is a unique $E' \in (\underline{E}, \overline{E})$ so that $h(E') = \{m\omega\}$. Set $x_0 := \left(E_{j_1}^{(\underline{N})}\right)^{-1}(E')$. By construction, $\underline{x}_1 < x_0 < \bar{x}_1$ and

$$E_{j_2}^{(\underline{N})}(x_0 + m\omega) = E' = E_{j_1}^{(\underline{N})}(x_0)$$

as desired. □

Given $N$ let $\underline{N} = \Big[\big(\log N\big)^A\Big]$, $A \ll \delta^{-1}$ and let $\Omega_{\underline{N}}$, $\mathcal{E}_{\underline{N},\omega}$ stand for the sets defined in Corollary 5.5. Fix $\omega \in \mathbb{T}_{c,a} \setminus \Omega_{\underline{N}}$. Let $E_j^{(\underline{N})}(x,\omega)$ stand for the Rellich functions of $H_N(x,\omega)$. Denote

$$\tilde{I}_k^{(\underline{N})} = [k/\underline{N}, (k+1)/\underline{N}], \quad k \in [-K(\underline{N}), K(\underline{N})],\ K(\underline{N}) = C\underline{N},\ C \gg 1$$

For each such $\tilde{I}_k^{(\underline{N})}$ let

$$(E^{(\underline{N})}_{j(s,k)}(x,\omega), \underline{x}_{j(s,k)}, \bar{x}_{j(s,k)}), \quad s = 1, 2, \ldots$$

denote the $I_k^{(\underline{N})}$-segments with $I_k^{(\underline{N})} \subset \tilde{I}_k^{(\underline{N})} \setminus \mathcal{E}_{\underline{N},\omega}$ arbitrary but fixed (provided such a segment exists). For each triple of segments

$$(E^{(\underline{N})}_{j(s_i,k)}(x,\omega), \underline{x}_{j(s_i,k)}, \bar{x}_{j(s_i,k)}), \quad i = 1, 2, 3$$

one can apply Proposition 8.9 with $\varepsilon_{\underline{N}} = \exp(-\underline{N}^\delta)$, $\underline{\mu} = \mu_{\underline{N}} = \exp(-(\log \underline{N})^B)$. Let $\Omega(s_1, s_2, s_3, k, \underline{N})$ be the set of exceptional $\omega$ defined by that proposition for this triple. Set

$$\mathcal{T}_N = \bigcup_k \bigcup_{s_1,s_2,s_3} \Omega(s_1, s_2, s_3, k, \underline{N})$$

Then

$$\operatorname{mes}(\mathcal{T}_{\underline{N}}) \le \underline{N}^{-A}$$

where $A \gg 1$ is some constant, and for any $\omega \notin \left(\mathcal{T}_N \cup \Omega_{\underline{N}}\right)$ the system of inequalities

$$|E_{j(s_2,k)}(x + n_2\omega, \omega) - E_{j(s_1,k)}(x,\omega)| < \varepsilon_{\underline{N}}, \quad |E_{j(s_3,k)}(x + n_3\omega, \omega) - E_{j(s_1,k)}(x,\omega)| < \varepsilon_{\underline{N}}$$

has no solution with

$$x \in (\min(\underline{x}_{j(s_1,k)}, \bar{x}_{j(s_1,k)}), \max(\underline{x}_{j(s_1,k)}, \bar{x}_{j(s_1,k)}))$$

for any $k, s_1, s_2, s_3$.

**Lemma 9.2.** *Let $I = I_k^{(\underline{N})}$ be as above. Either $I \cap \bigcup_x \operatorname{sp} H_{\underline{N}}(x,\omega) = \emptyset$, or $I$ contains a pre-gap $\left(\underline{E}^{(1)}, \overline{E}^{(1)}\right)$ interval at scale $N$.*

*Proof.* Assume $I \cap \bigcup_x \operatorname{sp} H_N(x,\omega) \neq \emptyset$. Then by Lemma 6.1 there exists an $I$–segment $\left\{E^{(\underline{N})}_{j_1}(x), \underline{x}_1, \bar{x}_1\right\}$. Assume for instance that it is a positive-slope $I$–segment. By Lemma 6.5 there exists also a negative-slope $I$–segment $\left\{E^{(\underline{N})}_{j_2}(x), \underline{x}_2, \bar{x}_2\right\}$. By Lemma 9.1 there exists $\underline{N} < m < N_1 - \underline{N}$, $N_1 = \left[\exp\left(\underline{N}^\delta\right)\right]$ and $x_0'$ such that $E^{(\underline{N})}_{j_1}(x_0') = E^{(\underline{N})}_{j_1}(x_0' + m\omega)$. Note that $N_1 < N^\varepsilon$ for any $0 < \varepsilon < 1$ provided $N$ is large because of $\delta A \ll 1$. Set $N_1' = \left[N^{1/2}\right]$, $\Lambda_1 = \left[N_1' + 1, N_1' + \underline{N}\right]$, $\Lambda_2 = m + \Lambda_1$, $x_0 = x_0' + N_1'\omega$. Then conditions (a), (b) of the Definition 7.1 of double resonances are valid. Condition (c) is valid due to Proposition 8.9, since $\omega \notin \mathcal{F}_N$. So, $(x_0, E_0)$ is a point of a double resonance for $H_N(\cdot)$. Moreover, due to the properties of the $I$–segments, $\left|\partial E^{(\underline{N})}_{j_k}\right| > \exp\left(-\underline{N}^\sigma\right)$ for $x \in (\min(\underline{x}_k, \bar{x}_k), \max(\underline{x}_k, \bar{x}_k))$, $k = 1, 2$. So, condition (d) is also valid, see (7.3). Finally condition (e) of the Definition 7.1 is clearly valid. Thus $(x_0, E_0)$ is a regular double resonance point for $H_N(\cdot)$. Therefore the assertion follows from Proposition 7.11. □

Let $N^{(t)}$ be arbitrary integers such that $N^{(t-1)} \asymp \left(\log N^{(t)}\right)^K$, $K \gg 1$. Let $I^{N^{(s)}}_{k^{(s)}}$ be the intervals defined before Lemma 9.2, with $s = 1, 2, \ldots$. Assume that

- $I^{N^{(s)}}_{k^{(s)}} \cap \bigcup_x \operatorname{sp} H_{N^{(s-1)}}(x,\omega) \neq \emptyset$ for all $s_1 \le s \le s_1 + r$ where $r$ is a positive integer.
- $I^{N^{(s+1)}}_{k^{(s+1)}}$ is a subset of a pre-gap defined for $I^{N^{(s)}}_{k^{(s)}}$ by means of Lemma 9.2.

**Lemma 9.3.** *Using the above notations, one has $r \le C(\lambda, V)$.*

*Proof.* It follows from Proposition 7.11 that

$$\begin{aligned} &\frac{1}{N^{(s+k)}} \#\left\{z \in \mathcal{A}_{\rho^{(s+k)}} : f_{N^{(s+k)}}(z,\omega,E) = 0\right\} \\ &\le \frac{1}{N^{(s+k)}} \#\left\{z \in \mathcal{A}_{\rho^{(s+k-1)}/2} : f_{N^{(s+k)}}(z,\omega,E) = 0\right\} - 2 + 4\left(N^{(s+k)}\right)^{-1/2} \end{aligned}$$

where $\rho^{(t)} \asymp \exp\left(-\left(N^{(t)}\right)^{\delta}\right)$, $k = 1, 2, \ldots, r$. On the other hand, by Lemma 2.24,

$$\frac{1}{N^{(s+k)}} \# \left\{ z \in \mathcal{A}_{\rho^{(s+k-1)}/2} : f_{N^{(s+k)}}(z, \omega, E) = 0 \right\}$$
$$\leq \frac{1}{N^{(s+k-1)}} \# \left\{ z \in \mathcal{A}_{\rho^{(s+k-1)}} : f_{N^{(s+k-1)}}(z, \omega, E) = 0 \right\} + \left(N^{(s+k-1)}\right)^{-1/4}$$

provided the $\rho^{(s+k)}$ are chosen appropriately. Thus

$$\frac{1}{N^{(s+r)}} \# \left\{ z \in \mathcal{A}_{\rho^{(s+r)}} : f_{N^{(s+r)}}(z, \omega, E) = 0 \right\} \leq \frac{1}{N^{(s)}} \# \left\{ z \in \mathcal{A}_{\rho^{(s)}} : f_{N^{(s)}}(z, \omega, E) = 0 \right\} - r$$

and the assertion follows. □

The proof of Theorem 1.1 follows from Lemmas 9.2, 9.3.

## References


[AviJit] Avila, A., Jitomirskaya, S. *Solving the ten martini problem*, preprint 2005.

[BelSim] Béllissard, J., Simon, B. *Cantor spectrum for the almost Mathieu equation.* J. Funct. Anal. 48 (1982), no. 3, 408–419.

[Bha] Bhatia, R. *Perturbation bounds for matrix eigenvalues.* Pitman research notes in mathematics series 162, Longman, 1987.

[Bou] Bourgain, J. *Green's function estimates for lattice Schrödinger operators and applications.* Annals of Mathematics Studies, 158. Princeton University Press, Princeton, NJ, 2005.

[BouGol] Bourgain, J., Goldstein, M. *On nonperturbative localization with quasi-periodic potential.* Ann. of Math. (2) 152 (2000), no. 3, 835–879.

[Cha] Chan, J. *Method of variations of potential of quasi-periodic Schrödinger equation*, preprint 2005.

[ChoEllYui] Choi, M.D., Elliott, G.A., Yui, N. *Gauss polynomials and the rotation algebra.* Invent. Math. 99(2), 225–246 (1990)

[Fed] Federer, H. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer-Verlag New York Inc., New York 1969.

[FroSpe1] Fröhlich, J., Spencer, T. *Absence of diffusion in the Anderson tight binding model for large disorder or low energy.* Comm. Math. Phys. **88** (1983), 151–189.

[FroSpe2] Fröhlich, J., Spencer, T. *A rigorous approach to Anderson localization.* Phys. Rep. **103** (1984), no. 1–4, 9–25.

[FroSpeWit] Fröhlich, J., Spencer, T., Wittwer, P. *Localization for a class of one dimensional quasi-periodic Schrödinger operators.* Commun. Math. Phys. **132** (1990), 5–25.

[FurKes] Fürstenberg, H., Kesten, H. *Products of random matrices.* Ann. Math. Statist 31 (1960), 457–469.

[GolSch1] Goldstein, M., Schlag, W. *Hölder continuity of the integrated density of states for quasiperiodic Schrödinger equations and averages of shifts of subharmonic functions.* Ann. of Math. (2) 154 (2001), no. 1, 155–203.

[GolSch2] Goldstein, M., Schlag, W. *Fine properties of the integrated density of states and a quantitative separation property of the Dirichlet eigenvalues*, Preprint 2005.

[Her] Herman, M. *Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un theoreme d'Arnold et de Moser sur le tore de dimension* 2. Comment. Math. Helv. 58 (1983), no. 3, 453–502.

[Joh] Johnson, R. *Cantor spectrum for the quasi-periodic Schrödinger equation.* J. Diff. Eq. 91, 88–110 (1991)

[Lan] Lang, S. *Algebra*, Third Edition, Addison Wesley, 1993.

[Las] Last, Y. *Zero measure spectrum for the almost Mathieu operator.* Comm. Math. Phys. 164 (1994), 421–432.

[Lev] Levin, B. Ya. *Lectures on entire functions.* Transl. of Math. Monographs, vol. 150. AMS, Providence, RI, 1996.

[Pui] Puig, J. *Cantor spectrum for the almost Mathieu operator.* Comm. Math. Phys. 244 (2004), no. 2, 297–309.

[ReeSim4] Reed, M., Simon, B. *Methods of modern mathematical physics. IV.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1979.

[Sin] Sinai, Y. G. *Anderson localization for one-dimensional difference Schrödinger operator with quasi-periodic potential.* J. Stat. Phys. **46** (1987), 861–909.

[Tod] Toda, M. *Theory of nonlinear lattices.* Second edition. Springer Series in Solid-State Sciences, 20. Springer-Verlag, Berlin, 1989.


Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 1A1
*E-mail address*: gold@math.toronto.edu

The University of Chicago, Department of Mathematics, 5734 South University Avenue, Chicago, IL 60637, U.S.A.

*E-mail address*: schlag@math.uchicago.edu