

EQUIVARIANT HARMONIC CYLINDERS

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ABSTRACT. We prove that a primitive harmonic map is equivariant if and only if it admits a holomorphic potential of degree one. We investigate when the equivariant harmonic map is periodic, and as an application discuss constant mean curvature cylinders with screw motion symmetries.

INTRODUCTION

The theory of harmonic maps, especially those from a Riemann surface to a Riemannian symmetric space, has been greatly enriched in recent years by the realisation that they constitute an integrable system [4, 9, 10, 13, 22, 23]. Thus these maps admit a spectral deformation (the *associated family*); algebro-geometric (finite type) solutions and an interpretation in terms of certain holomorphic maps to a loop group. This last culminates in the (somewhat indirect) description by Dorfmeister-Pedit-Wu of all primitive harmonic maps of a simply connected Riemann surface into a k -symmetric space in terms of *holomorphic potentials*: certain holomorphic 1-forms with values in a loop algebra.

In this paper, we study the simplest case of this theory: that of equivariant harmonic maps where the underlying PDE reduces to an ODE. We show that such maps are characterised as those with a holomorphic potential of the simplest kind: the 1-form has simple poles at zero and infinity and satisfies a natural reality condition. Along the way, we show that equivariance is a property which is preserved under spectral deformation.

A pleasant application of the foregoing theory lies in the fact that several types of surface of classical geometric interest are characterised by harmonicity of an appropriate Gauss map [3, 7, 17]. In particular, the classical theory of constant mean curvature surfaces in \mathbb{R}^3 amounts to the study of harmonic maps to a 2-sphere with the link between the loop group approach and the classical surfaces being provided by the Sym–Bobenko formula.

We apply our general theory to this case which means that we study constant mean curvature surfaces with screw motion symmetry. We find a very simple proof of a result of Do Carmo–Dajczer [8] which asserts that these surfaces are precisely those in the associated family of a Delaunay surface (that is, a constant mean curvature surface of revolution). For this, we provide an interpretation of the Sym–Bobenko formula in terms of a homomorphism from a loop group to the Euclidean group which may be of independent interest.

We then turn to a detailed study of the period problem for constant mean curvature surfaces with screw motion symmetry. Armed with our knowledge of the holomorphic potential, we are able to explicitly compute, in terms of elliptic functions, the corresponding map into the loop group (usually, this involves solving a Riemann–Hilbert problem). With this in hand, we can prove the existence of infinitely many non-congruent cylinders in the associated family of each Delaunay surface. Otherwise said, in each associated family of equivariant

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harmonic maps $\mathbb{C} \rightarrow S^2$ there are infinitely many equivariant harmonic tori. By our results in the first part, all equivariant harmonic tori in the two-sphere arise this way.

1. PRIMITIVE HARMONIC MAPS AND LOOP GROUPS

1.1. We study primitive harmonic maps of a Riemann surface into a k -symmetric space and so begin by recalling the ingredients of that story [5, 6].

Let G be a compact semisimple Lie group. A (regular) k -symmetric G -space [12] is a coset space $N = G/K$ where $(G^\tau)_0 \subset K \subset G^\tau$ for some automorphism $\tau : G \rightarrow G$ of finite order $k \geq 2$.

In particular, a Riemannian symmetric space of compact type is the same as a 2-symmetric space.

Let \mathfrak{g} be the Lie algebra of G . Then τ induces an automorphism, also called τ , of \mathfrak{g} whose eigenspace decomposition gives a K -stable \mathbb{Z}_k grading on $\mathfrak{g}^{\mathbb{C}}$: $\mathfrak{g}^{\mathbb{C}} = \sum_{\ell \in \mathbb{Z}_k} \mathfrak{g}_\ell$. Here \mathfrak{g}_ℓ is the ω^ℓ -eigenspace of τ for $\omega = e^{2\pi i/k}$.

Define $\mathfrak{m} \subset \mathfrak{g}$ by $\mathfrak{m}^{\mathbb{C}} = \sum_{\ell \neq 0} \mathfrak{g}_\ell$ to get a reductive decomposition

$$(1.1) \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}.$$

1.2. Let M be a Riemann surface. We study maps $\varphi : M \rightarrow G/K$ via their *frames*: that is, maps $F : M \rightarrow G$ for which $\varphi = FK$. Given a frame F , set $\alpha = F^{-1}dF$, a \mathfrak{g} -valued 1-form on M and write $\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}}$ according to the decomposition (1.1). Further write $\alpha_{\mathfrak{m}} = \alpha'_{\mathfrak{m}} + \alpha''_{\mathfrak{m}}$ according to the type decomposition $TM^{\mathbb{C}} = T^{1,0} \oplus T^{0,1}$. Thus $\alpha'_{\mathfrak{m}}$ is a $(1,0)$ -form with values in $\mathfrak{m}^{\mathbb{C}}$ and $\alpha''_{\mathfrak{m}} = \overline{\alpha'_{\mathfrak{m}}}$.

A map $\varphi : M \rightarrow G/K$ is *harmonic* if, for any (hence every) frame,

$$d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}] = 0$$

and *primitive* if $\alpha'_{\mathfrak{m}}$ takes values in \mathfrak{g}_{-1} . It is easy to see that when $k > 2$, a primitive map is automatically harmonic while, when $k = 2$, the primitivity condition is vacuous. We combine the cases of interest to us by saying that φ is *primitive harmonic* if $k = 2$ and φ is harmonic or $k > 2$ and φ is primitive.

1.3. The basic observation [14, 23, 24] is that if F frames a primitive harmonic map then defining

$$(1.2) \quad \alpha_\lambda = \alpha_{\mathfrak{k}} + \lambda^{-1}\alpha'_{\mathfrak{m}} + \lambda\alpha''_{\mathfrak{m}}$$

yields a solution of the Maurer–Cartan equations for each non-zero complex number $\lambda \in \mathbb{C}^\times$. Thus, we can (locally or after passage to the universal cover of M) integrate to find $F_\lambda : M \rightarrow G^{\mathbb{C}}$ such that $F_\lambda^{-1}dF_\lambda = \alpha_\lambda$. It is easy to see that α_λ has the following symmetries:

$$\overline{\alpha_\lambda} = \alpha_\lambda \quad \tau\alpha_\lambda = \alpha_{\omega\lambda}.$$

Thus we may choose the constants of integration to ensure:

- (i) $\overline{F_{1/\lambda}} = F_\lambda$, for all $\lambda \in \mathbb{C}^\times$. Here the conjugation on $G^{\mathbb{C}}$ has fixed set G so that, in particular, F_λ takes values in G when $\lambda \in S^1$;
- (ii) $\tau F_\lambda = F_{\omega\lambda}$, for all $\lambda \in \mathbb{C}^\times$;
- (iii) $F_1 = F$;

We encapsulate all this by viewing F_λ as a map from (the universal cover of) M into the loop group ΛG_τ given by

$$\Lambda G_\tau = \{g : S^1 \rightarrow G \text{ smooth} : g(\omega\lambda) = \tau g(\lambda)\}$$

and say that $F_\lambda : M \rightarrow \Lambda G_\tau$ is an *extended frame* if $\alpha_\lambda = F_\lambda^{-1} dF_\lambda$ is of the form (1.2).

We have just seen that any primitive harmonic map has an extended frame with $F_1 = F$. Conversely, given an extended frame F_λ , it is easy to see that, for each $\lambda \in S^1$, F_λ frames a primitive harmonic map: these constitute the *associated family* of such maps.

1.4. The loop group ΛG_τ participates in an Iwasawa decomposition of its complexification: set

$$\Lambda G_\tau^{\mathbb{C}} = \left\{ g : S^1 \rightarrow G^{\mathbb{C}} \text{ smooth} : g(\omega\lambda) = \tau g(\lambda) \right\}$$

$$\Lambda_+ G_\tau = \{g \in \Lambda G_\tau^{\mathbb{C}} : g \text{ extends holomorphically to } |\lambda| < 1 \text{ and } g(0) \in B\}$$

where $K^{\mathbb{C}} = KB$ is some fixed Iwasawa decomposition of $K^{\mathbb{C}}$.

Theorem 1.1 ([9, 15]). *Pointwise multiplication $\Lambda G_\tau \times \Lambda_+ G_\tau \rightarrow \Lambda G_\tau^{\mathbb{C}}$ is a diffeomorphism.*

Consequently, every $g \in \Lambda G_\tau^{\mathbb{C}}$ can be uniquely factored $g = Fb$ with $F \in \Lambda G_\tau$ and $b \in \Lambda_+ G_\tau$.

1.5. Theorem 1.1 underlies a construction of extended frames which is an infinite-dimensional version of a formula of Symes [21] from integrable systems theory. For this, let $\Lambda \mathfrak{g}_\tau, \Lambda \mathfrak{g}_\tau^{\mathbb{C}}$ be the Lie algebras of $\Lambda G_\tau, \Lambda G_\tau^{\mathbb{C}}$ respectively. Thus

$$\begin{aligned} \Lambda \mathfrak{g}_\tau^{\mathbb{C}} &= \left\{ \xi : S^1 \rightarrow \mathfrak{g}^{\mathbb{C}} \text{ smooth} : \xi(\omega\lambda) = \tau \xi(\lambda) \right\}; \\ \Lambda \mathfrak{g}_\tau &= \left\{ \xi \in \Lambda \mathfrak{g}_\tau^{\mathbb{C}} : \xi : S^1 \rightarrow \mathfrak{g} \right\}. \end{aligned}$$

Further let $\Lambda_{-1, \infty} \mathfrak{g}^{\mathbb{C}}$ be the vector subspace of $\Lambda \mathfrak{g}_\tau^{\mathbb{C}}$ given by

$$\Lambda_{-1, \infty} \mathfrak{g}^{\mathbb{C}} = \left\{ \xi \in \Lambda \mathfrak{g}_\tau^{\mathbb{C}} : \xi \text{ extends holomorphically to } 0 < |\lambda| < 1 \text{ with a simple pole at } 0 \right\}.$$

We have:

Theorem 1.2 ([5]). *For $z \in \mathbb{C}$ and $\xi \in \Lambda_{-1, \infty} \mathfrak{g}^{\mathbb{C}}$, let $\exp(z\xi) = F_\lambda(z)b(z)$ be the Iwasawa decomposition of $\exp(z\xi)$. Then $F_\lambda : \mathbb{C} = \mathbb{R}^2 \rightarrow \Lambda G_\tau$ is an extended frame.*

This construction accounts for all primitive harmonic maps of semisimple finite type [5] and is a special case of the method of Dorfmeister–Pedit–Wu [9] which describes all primitive harmonic maps of a simply connected Riemann surface into a k -symmetric space. Following [9], we say that the *holomorphic potential* ξdz generates the associated family of primitive harmonic maps.

2. EQUIVARIANT PRIMITIVE HARMONIC MAPS

2.1. Consider a map $\varphi : \mathbb{R}^2 \rightarrow G/K$ that is \mathbb{R} -equivariant, so that

$$(2.1) \quad \varphi(x, y) = \exp(x A_0) \psi(y)$$

for some $A_0 \in \mathfrak{g}$ and map $\psi : \mathbb{R} \rightarrow G/K$. If g frames ψ , then $F : \mathbb{R}^2 \rightarrow G$ defined by $F(x, y) = \exp(x A_0) g(y)$ frames φ and

$$(2.2) \quad \begin{aligned} F^{-1} dF &= \text{Ad} g^{-1} A_0 dx + g^{-1} g' dy \\ &= A(y) dx + B(y) dy. \end{aligned}$$

In particular, both components of $F^{-1}dF$ depend only on y .

Conversely, given $F : \mathbb{R}^2 \rightarrow G$ with $F^{-1}dF$ of the form (2.2), set $g(y) := F(0, y)$ and $h(x, y) := F(x, y)g^{-1}(y)$, and note the following properties:

- (i) $g^{-1}dg = B dy$
- (ii) $h(0, y) = \mathbb{1}$ for all $y \in \mathbb{R}$
- (iii) $h^{-1}dh = \text{Adg}(F^{-1}dF - g^{-1}dg) = \text{Adg} A dx$

The Maurer–Cartan equation for $h^{-1}dh$ reads $\partial/\partial y(\text{Adg} A) = 0$. Thus $\text{Adg} A =: A_0$ is constant, which with (ii) above yields $h(x, y) = \exp(x A_0)$. Hence $F(x, y) = \exp(x A_0)g(y)$, so that F frames an equivariant map. This proves

Proposition 2.1. *$F(x, y) : \mathbb{R}^2 \rightarrow G$ frames an equivariant map if and only if $F^{-1}dF$ depends only on y .*

2.2. We now come to the first of our main results: we consider equivariant primitive harmonic maps and show that these are all generated by holomorphic potentials of the simplest kind. Moreover, we find that equivariance is shared by all members of an associated family.

For this, contemplate the *degree one* elements of $\Lambda_{\mathfrak{g}_\tau}$:

$$\Lambda_1 \mathfrak{g}_\tau = \{\xi \in \Lambda \mathfrak{g}_\tau : \xi = \lambda \xi_1 + \xi_0 + \lambda^{-1} \xi_{-1}\} = \Lambda_{-1, \infty} \mathfrak{g}^{\mathbb{C}} \cap \Lambda \mathfrak{g}_\tau.$$

Note that $\xi_\ell \in \mathfrak{g}_\ell$ and $\overline{\xi_\ell} = \xi_{-\ell}$. We call a holomorphic potential ξdz with $\xi \in \Lambda_1 \mathfrak{g}_\tau$ a *degree one potential*.

Given a degree one potential ξdz with $\xi \in \Lambda_1 \mathfrak{g}_\tau$, we examine the extended frame $F_\lambda : \mathbb{R}^2 \rightarrow \Lambda G_\tau$ that results from Theorem 1.2. With $z = x + iy$, we have $\exp(z\xi) = \exp(x\xi) \exp(iy\xi)$ and since $\exp(x\xi) \in \Lambda G_\tau$, for all $x \in \mathbb{R}$, we conclude that

$$(2.3) \quad F_\lambda(z) = \exp(x\xi) g_\lambda(y)$$

for some $g_\lambda : \mathbb{R} \rightarrow \Lambda G_\tau$ resulting from the Iwasawa decomposition of $y \mapsto \exp(iy\xi)$. Hence F_λ is the frame of an equivariant primitive harmonic map for each $\lambda \in S^1$. We have therefore proved

Proposition 2.2. *A degree one potential generates an associated family of equivariant primitive harmonic maps.*

2.3. We next prove the converse of Proposition 2.2, that any equivariant primitive harmonic map is generated by a degree one potential. So let $\varphi : \mathbb{R}^2 \rightarrow G/K$ be an equivariant primitive harmonic map. After translation by an element of G , we may assume that $\varphi(0) = eK$ and then we can find an equivariant frame F as in paragraph 2.1 with $F(0) = \mathbb{1}$.

From Proposition 2.1 we know that $\alpha = F^{-1}dF$ is of the form $\alpha = A(y) dx + B(y) dy$, so that $\alpha_\lambda = A_\lambda(y) dx + B_\lambda(y) dy$ with

- (a) $A_\lambda(y), B_\lambda(y) \in \Lambda_1 \mathfrak{g}_\tau$,
- (b) $\alpha_\lambda(\partial/\partial \bar{z})$ is holomorphic (indeed affine) in λ on \mathbb{C} .

Let F_λ be the corresponding extended frame. An immediate consequence of (a) and Proposition 2.1 is that, for each $\lambda \in S^1$, F_λ frames an equivariant map so that we may conclude:

Proposition 2.3. *If a primitive harmonic map is equivariant, then so are all members of its associated family.*

2.4. To show that an extended frame F_λ of an associated family of equivariant primitive harmonic maps is generated by a degree one potential, we argue as in the proof of Proposition 2.1: Set $g_\lambda(y) = F_\lambda(0, y)$ and conclude that $\text{Ad}g_\lambda A_\lambda$ is constant. Since $g_\lambda(0) = \mathbb{1}$, that constant is $A_\lambda(0) \in \Lambda_1 \mathfrak{g}_\tau$. Set

$$\xi = A_\lambda(0) = \text{Ad}g_\lambda A_\lambda(y),$$

to obtain $F_\lambda(x, y) = \exp(x\xi)g_\lambda(y)$. It remains to show that F_λ arises from the holomorphic potential ξdz , that is $\exp(z\xi) = F_\lambda b_\lambda$ for some smooth map $b_\lambda : \mathbb{R} \rightarrow \Lambda_+ G_\tau^\mathbb{C}$. For this it suffices to check that $b_\lambda(y) = g_\lambda^{-1}(y) \exp(iy\xi)$ is holomorphic in λ near 0. This is certainly true at $z = 0$, so we need only check holomorphicity in λ of its right(!) Maurer–Cartan form

$$db_\lambda b_\lambda^{-1} = (\text{Ad}g^{-1} i\xi - g^{-1}g') dy = i(A_\lambda + iB_\lambda) dy = \frac{i}{2}\alpha_\lambda(\partial/\partial\bar{z}),$$

which is clearly holomorphic in λ by (b) above. Thus $F_\lambda(z)$ is the ΛG_τ -factor of $\exp(z\xi)$, and we have proven that equivariant harmonic maps are generated by degree one potentials. Together with Proposition 2.2, we have proven

Theorem 2.4. *A primitive harmonic map is equivariant if and only if it is generated by a degree one potential.*

3. THE EUCLIDEAN GROUP AND THE SYM–BOBENKO FORMULA

3.1. Consider the semi-direct product $G \ltimes \mathfrak{g}$ where G acts on \mathfrak{g} via the adjoint action. Thus

$$(g, \zeta)(h, \eta) = (gh, \text{Ad}g \eta + \zeta).$$

$G \ltimes \mathfrak{g}$ has an affine action on \mathfrak{g} via

$$(3.1) \quad (g, \zeta) \cdot \eta = \text{Ad}g \eta + \zeta.$$

Proposition 3.1. *For $\mu \in S^1$ we have a homomorphism*

$$(3.2) \quad \Phi_\mu : \Lambda G_\tau \rightarrow G \ltimes \mathfrak{g}, \quad F \mapsto (F, F'F^{-1})|_\mu$$

where, for $F \in \Lambda G_\tau$ and $\lambda = e^{it}$, F' denotes the derivative of F with respect to $t \in \mathbb{R}$.

Proof. Let $F_1, F_2 \in \Lambda G_\tau$. Then

$$\Phi_\mu(F_1 F_2) = (F_1 F_2, \text{Ad}F_1(F_2'F_2^{-1}) + F_1'F_1^{-1})|_\mu = \Phi_\mu(F_1) \Phi_\mu(F_2).$$

□

Differentiating Φ_μ at $\mathbb{1} \in \Lambda G_\tau$ provides a Lie algebra homomorphism, also denoted by $\Phi_\mu : \Lambda \mathfrak{g}_\tau \rightarrow \mathfrak{g} \ltimes \mathfrak{g}$ and given by

$$\Phi_\mu(\xi) = (\xi, \xi')|_\mu,$$

where we again write $\xi' = (\partial/\partial t)\xi$, for $\xi \in \Lambda \mathfrak{g}_\tau$.

3.2. All this comes alive when $G = \text{SU}(2)$. Then $\mathfrak{su}(2) \cong \mathbb{R}^3$ and the affine action (3.1) gives a double cover $\text{SU}(2) \ltimes \mathfrak{su}(2) \rightarrow \text{Euc}(3)$ of the Euclidean group. An extended frame $F(z, \lambda)$ gives rise to an associated family of parallel pairs f_λ^\pm of conformal constant mean curvature immersions via the Sym–Bobenko formula [2, 20] which, in our formalism, reads

$$(3.3) \quad f_\lambda^\pm = -\frac{1}{2H} \Phi_\lambda(F) \cdot (\mp e_1).$$

Here $e_1 \in \mathfrak{su}(2)$ is the normal to $f = f_1^+$ at $z = 0$, and $H \in \mathbb{R}^*$ is the mean curvature. Since the normal to the surface f^- in (3.3) is $N_\lambda = \text{Ad}F_\lambda e_1$, the parallel surfaces satisfy

$$(3.4) \quad f_\lambda^- + \frac{1}{H} N_\lambda = f_\lambda^+.$$

Suppose that $F_\lambda(z)$ is generated by ξdz with $\xi \in \Lambda_1 \mathfrak{su}(2)_\tau$. Then, as in (2.3) we have that $F_\lambda(x, y) = \exp(x\xi) g_\lambda(y)$, and the associated families are

$$\begin{aligned} f_\lambda^\pm(x, y) &= -\frac{1}{H} \Phi_\lambda(\exp(x\xi)) \Phi_\lambda(g_\lambda(y)) \cdot (\mp \frac{1}{2} e_1) \\ &= -\frac{1}{H} \exp(x\Phi_\lambda(\xi)) \cdot (-H f_\lambda^\pm(0, y)). \end{aligned}$$

Thus at $\lambda \in S^1$ the surfaces f_λ^\pm have screw-motion symmetry generated by $\Phi_\lambda(\xi) \in \mathfrak{euc}(3) = \mathfrak{su}(2) \times \mathfrak{su}(2)$, and we have proven

Proposition 3.2. *A degree one potential ξdz generates associated families f_λ^\pm of constant mean curvature surfaces, of which each member has screw-motion symmetry generated by $\Phi_\lambda(\xi)$.*

We will show that at specific $\mu \in S^1$ we obtain surfaces of revolution and thus Delaunay surfaces. This will be the case exactly when the orbits of the 1-parameter subgroup generated by $\Phi_\mu(\xi)$ are co-axial circles.

Lemma 3.3. *An element $(\xi, \eta) \in \mathfrak{su}(2) \times \mathfrak{su}(2)$ generates a rotation about an axis if and only if $\xi \perp \eta$ with respect to the Killing form.*

Proof. Let $(\xi, \eta) \in \mathfrak{su}(2) \times \mathfrak{su}(2)$ and write $\eta = \eta^\top + \eta^\perp$ with $\eta^\top \in \text{Im}(\text{ad } \xi)$ and $\eta^\top \perp \eta^\perp$ (and thus $\eta^\perp \parallel \xi$). Hence there exists a unique $\zeta \in \text{Im}(\text{ad } \xi)$ with $[\xi, \zeta] = -\eta^\top$. The adjoint action of $\text{SU}(2) \times \mathfrak{su}(2)$ reads

$$\text{Ad}(g, \zeta)(\xi, \eta) = (\text{Ad}_g \xi, \text{Ad}_g \eta - [\text{Ad}_g \xi, \eta]),$$

and we compute

$$\begin{aligned} (\mathbb{1}, \zeta)^{-1} \exp(t(\xi, \eta)) (\mathbb{1}, \zeta) &= \exp(t \text{Ad}(\mathbb{1}, -\zeta)(\xi, \eta)) = \exp(t(\xi, \eta + [\xi, \zeta])) \\ &= \exp\left(t(\xi, \eta^\perp)\right) = (\exp(t\xi), t\eta^\perp), \end{aligned}$$

since $[\xi, \eta^\perp] = 0$. Thus $(\mathbb{1}, \zeta)^{-1} \exp(t(\xi, \eta)) (\mathbb{1}, \zeta)$ is a screw-motion symmetry with axis $\langle \xi \rangle$ and translational part given by η^\perp . It follows that $\exp(t(\xi, \eta))$ is a screw-motion symmetry with axis $\langle \xi \rangle + \zeta$ and the same translational part. In particular, if $\xi \perp \eta$ and thus $\eta^\perp = 0$, then $\exp(t(\xi, \eta))$ consists of rotations in ξ^\perp with axis $\langle \xi \rangle + \zeta$. \square

Theorem 3.4. *A degree one potential ξdz with $\xi \in \Lambda_1 \mathfrak{su}(2)_\tau$ generates an associated family of a Delaunay surface.*

Proof. Write $\xi = \lambda^{-1} \xi_- + \xi_0 + \lambda \xi_+$ so that $\Phi_\mu(\xi) = (\mu^{-1} \xi_- + \xi_0 + \mu \xi_+, i\mu \xi_+ - i\mu^{-1} \xi_-)$. Now $\xi_\pm \perp \xi_0$ since $\mathfrak{g}_\pm \perp \mathfrak{g}_0$, so with $\langle \cdot, \cdot \rangle$ denoting the Killing form, we have $\langle \mu^{-1} \xi_- + \xi_0 + \mu \xi_+, i\mu \xi_+ - i\mu^{-1} \xi_- \rangle = i\mu^2 \langle \xi_+, \xi_+ \rangle - i\mu^{-2} \langle \xi_-, \xi_- \rangle$, which vanishes for $\mu^4 = \langle \xi_-, \xi_- \rangle / \langle \xi_+, \xi_+ \rangle$. Clearly the fourth roots are on S^1 since $\overline{\xi_-} = \xi_+$. Hence at these points, by Lemma 3.3, the surface is a surface of revolution and thus a Delaunay surface. Hence a degree one potential generates an associated family of a Delaunay surface. \square

With the above results we are now in a position to present a new and elementary proof of a theorem by DoCarmo and Dajczer [8].

Theorem 3.5. [8] *A constant mean curvature surface has screw-motion symmetry if and only if it lies in an associated family of a Delaunay surface.*

Proof. Let $y \mapsto (u(y), v(y), w(y))^t$ be the generating curve in \mathbb{R}^3 of a helicoidal surface such that we have a conformal parametrisation

$$(3.5) \quad f(x, y) = \begin{pmatrix} \cos(x) & \sin(x) & 0 \\ -\sin(x) & \cos(x) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u(y) \\ v(y) \\ w(y) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ ax \end{pmatrix}, \quad a \in \mathbb{R}.$$

The Gauss map $N = |f_x \times f_y|^{-1} f_x \times f_y$ is equivariant, since

$$f_x \times f_y = \begin{pmatrix} \cos(x) & \sin(x) & 0 \\ -\sin(x) & \cos(x) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -uw' - av' \\ -vw' + au' \\ uu' + vv' \end{pmatrix},$$

and $|f_x \times f_y|$ is independent of the parameter x . Since the mean curvature is constant, the Gauss map is harmonic [17], and hence by Theorem 2.4 admits a holomorphic potential ξdz with $\xi \in \Lambda_1 \mathfrak{su}(2)_\tau$ of degree one. By Theorem 3.4 this generates the harmonic map of an associated family of Delaunay surfaces.

Conversely, let $f_{\lambda_0} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be some member of an associated family of a Delaunay surface. Then at some $\lambda_1 \in S^1$ we have that f_{λ_1} is a surface of revolution, and thus has an equivariant Gauss map generated by some degree one potential. By Proposition 3.2, the surface has screw-motion symmetry. \square

4. A NORMAL FORM

Our main objective in this second part is to solve period problems for equivariant harmonic maps $\mathbb{C} \rightarrow S^2$ and the resulting helicoidal constant mean curvature surfaces. We proceed by deriving a normal form for the corresponding potentials of degree one, and then compute the surface invariants generated by such a potential. These are necessary ingredients for the subsequent explicit extended frame and its monodromy.

We saw above that all equivariant harmonic maps $\mathbb{C} \rightarrow S^2$ come from holomorphic degree one potentials on \mathbb{C} with values in $\Lambda_1 \mathfrak{su}(2)_\tau$. Such a potential is of the form

$$(4.1) \quad \begin{pmatrix} c & a\lambda^{-1} + \bar{b}\lambda \\ b\lambda^{-1} + \bar{a}\lambda & -c \end{pmatrix} i dz$$

with $a, b \in \mathbb{C}$ and $c \in \mathbb{R}$. It turns out that up to rigid motions and a translation of y , all equivariant harmonic maps $\mathbb{C} \rightarrow S^2$ can be generated by a two-parameter subfamily of such potentials. This prompts the following

Definition 4.1. *A $\Lambda_1 \mathfrak{su}(2)_\tau$ -valued holomorphic 1-form on \mathbb{C} of the form*

$$(4.2) \quad \begin{pmatrix} 0 & a\lambda^{-1} + b\lambda \\ b\lambda^{-1} + a\lambda & 0 \end{pmatrix} i dz$$

with $a, b \in \mathbb{R}$, is said to be in normal form.

Lemma 4.2. *Up to rigid motions, any Delaunay surface can be generated by a potential in normal form.*

Proof. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a conformal immersion of a Delaunay surface. Then its Gauss map N is equivariant and thus generated by a degree one potential ηdz with $\eta \in \Lambda_1 \mathfrak{su}(2)_\tau$ such that for the associated family of Gauss maps N_λ we have $N_1 = N$. Then up to rigid motion we have for the associated family of surfaces f_λ that $f_1 = f$. Apriori, ηdz is of the form (4.1) with $a, b \in \mathbb{C}$ and $c \in \mathbb{R}$. Since f_1 is a surface of revolution, Lemma 3.3 implies that $\eta \perp \eta'$ at $\lambda = 1$. A computation reveals that $\langle \eta, \eta' \rangle|_{\lambda=1} = 0$ if and only if the product $ab \in \mathbb{R}$.

After a diagonal unitary gauge (rotation in the tangent plane), we may assume that $a \in \mathbb{R}$, and consequently also $b \in \mathbb{R}$. Finally, after a possible translation in the domain, we may assume that the base point lies on a circle of maximal or minimal radius. This condition is equivalent to $N_1(0) = \mathbf{e}_1$ being perpendicular to the axis of revolution $\eta_{\lambda=1}$. A quick computation shows that this holds if and only if $c = 0$. \square

Corollary 4.3. *Up to isometry, any equivariant harmonic map $\mathbb{C} \rightarrow S^2$ is generated by a $\Lambda_1 \mathfrak{su}(2)_\tau$ -valued holomorphic 1-form in normal form.*

We will compute the metric and Hopf differential of the associated Delaunay surface generated by a potential in normal form. Since the parameter in all further elliptic functions and integrals is in terms of a^2/b^2 , we set

$$(4.3) \quad \kappa = a^2/b^2 \text{ and } \kappa' = 1 - \kappa.$$

Theorem 4.4. *Let $H \in \mathbb{R}^*$ and consider an associated family f_λ generated by a potential in normal form with constants $a, b \in \mathbb{R}$ via the Sym–Bobenko formula (3.3). The Hopf differential of f_λ is $\lambda^{-2} Q dz^2$ with*

$$(4.4) \quad Q = 2 a b H^{-1}.$$

The metric is given by $v_0^2 (dz \otimes d\bar{z})$ with $v_0 = v_0(y)$ the Jacobian elliptic function

$$(4.5) \quad v_0(y) = 2bH^{-1} \operatorname{dn}(2by | \kappa').$$

Proof. Let $F : \mathbb{R}^2 \rightarrow \Lambda \operatorname{SU}(2)_\tau$ be the extended framing obtained from a potential ξdz in normal form and let f be one of the immersions given by (3.3). This means that F is the $\Lambda \operatorname{SU}(2)_\tau$ -factor of $\exp(z\xi) = FB$. Then $\alpha = F^{-1}dF$ is of the form

$$(4.6) \quad \alpha = \frac{1}{2v} \begin{pmatrix} -v_z dz + v_{\bar{z}} d\bar{z} & 2i\lambda^{-1} Q dz + iv^2 \lambda H d\bar{z} \\ iv^2 \lambda^{-1} H dz + 2i\lambda \bar{Q} d\bar{z} & v_z dz - v_{\bar{z}} d\bar{z} \end{pmatrix}$$

for functions v, Q, H . The smooth function $v^2 : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ is the conformal factor, the quadratic differential $\lambda^{-2} Q dz^2$ the Hopf differential of f_λ .

Since $\xi dz = B^{-1} \alpha B + B^{-1} d B$, comparison of λ^{-1} coefficients yields $\alpha_{-1} = B_0 \xi_{-1} B_0^{-1}$. Writing

$$B_0 = \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}$$

for smooth $r : \mathbb{R} \rightarrow \mathbb{R}_+$ this gives $iv^{-1}Q = ir^2a$ and $ivH = 2r^{-2}b$. Eliminating v from these two equations yields the formula for the Hopf differential (4.4). Furthermore, the second of these equations evaluated at the basepoint $z = 0$ reads $v(0)H = 2r^{-2}(0)b$. From $F(0) = B(0) = \mathbb{1}$ it follows that $r(0) = 1$ and consequently the function $v : \mathbb{R} \rightarrow \mathbb{R}^*$ generated by the potential ξdz satisfies

$$(4.7) \quad v(0) = 2b/H.$$

Since the extended frame splits as in (2.3), the conformal factor $v^2 = v^2(y)$ depends only on the variable $y \in \mathbb{R}$, and the Gauß equation is

$$(4.8) \quad v^{-1}\ddot{v} - v^{-2}\dot{v}^2 + v^2H^2 - 4v^{-2}|Q|^2 = 0.$$

Since H and Q are constant, we can find a first integral of equation (4.8) of the form $\dot{v}^2 = A(v^2 - B)(v^2 - C)$. Differentiating this, and using $A(B+C) = Av^2 - v^{-2}\dot{v}^2 + v^{-2}ABC$, gives $v^{-1}\ddot{v} - v^{-2}\dot{v}^2 - Av^2 + v^{-2}ABC = 0$. Comparing this with (4.8) implies that the constants A, B and C must satisfy $A = -H^2$ and $BC = 4H^{-2}|Q|^2$. Set $B = 4H^{-2}a^2$ and $C = 4H^{-2}b^2$. Hence

$$(4.9) \quad \dot{v}^2 = -H^2(v^2 - 4H^{-2}a^2)(v^2 - 4H^{-2}b^2)$$

Remark 4.5. *Two properties follow from the differential equation (4.9):*

(i) $\dot{v}(y) = 0$ if and only if $v(y) \in \{\pm 2a/H, \pm 2b/H\}$.

(ii) *Since the left hand side in (4.9) is positive, the signs of the two factors on the right hand side in (4.9) must be different, and thus any solution v oscillates either between $-2|a/H|$ and $-2|b/H|$, or between $2|a/H|$ and $2|b/H|$.*

The general solution of (4.8) is given in terms of a Jacobi elliptic function as follows: Taking the square root, rewrite (4.9) as

$$(4.10) \quad dy = \pm (-H^2(v^2 - 4H^{-2}a^2)(v^2 - 4H^{-2}b^2))^{-1/2} dv.$$

Integrating, we pick up a constant $C \in \mathbb{R}$, and substituting $2|a|t = v|H|$ gives

$$y = C \pm \frac{1}{2i|b|} \int_0^{\frac{|H|v}{2|a|}} \frac{dt}{\sqrt{(1-t^2)(1-\kappa t^2)}} = C \pm \frac{1}{2i|b|} F\left(\arcsin\left(\frac{|H|v}{2|a|}\right) \middle| \kappa\right),$$

where F denotes the elliptic integral of the first kind

$$(4.11) \quad F(\varphi | m) = \int_0^{\sin \varphi} \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}}.$$

Thus the general solution of (4.8) is

$$(4.12) \quad v(y) = 2|a/H| \operatorname{sn}(\pm 2i|b|(y-C) | \kappa).$$

It remains to determine the solution v_0 with initial condition $v(0) = 2b/H$. The solution with $v(0) = 2|b/H|$ is given by $v(y) = \pm 2|a/H| \operatorname{sn}(2i|b|y + F(\arcsin(\kappa^{-1/2}) | \kappa) | \kappa)$. By the *complex arguments* formula for sn , see 16.21.1 in [1], this simplifies to $v(y) = \pm 2|b/H| \operatorname{dn}(2|b|y | \kappa')$. Choosing the sign in (4.10) so that $\pm|b/H| = b/H$, and using the fact that dn is an even function 16.8.3 [1], proves (4.5) and concludes the proof of Theorem 4.4. \square

Remark 4.6. *For parameter $\kappa \in (0, 1)$, the Jacobian elliptic function $\operatorname{dn}(y | \kappa')$ has the real half-period*

$$(4.13) \quad K' = F\left(\frac{\pi}{2} \middle| \kappa'\right).$$

Lemma 4.7. *Given a potential ξdz in normal form with $a, b \in \mathbb{R}$, then special choices for a, b have the following effect:*

- (i) *If $ab \neq 0$, then the surfaces generated by ξ and ξ^t differ by a rigid motion. Hence, swapping the roles of a and b gives the same surface up to isometry.*
- (ii) *The case $ab = 0$ gives a twice punctured round sphere or a degenerate 'surface' consisting of a point.*
- (iii) *The case $a = \pm b \neq 0$ results in an associated family of a right circular cylinder.*

Proof. (i) Swapping the roles of the constants a, b has no effect on the Hopf differential. We will show that the resulting surfaces have the same metric after a conformal change of coordinate (in fact a translation by a half-period). Let $a, b \in \mathbb{R}^*$, and let v_1 and v_2 be the conformal factors generated by ξdz respectively $\xi^t dz$, and κ as in (4.3). Then $v_1(y) = 2b/H \operatorname{dn}(2by | 1 - \kappa)$, $v_2(y) = \sigma 2a/H \operatorname{dn}(2ay | 1 - \kappa^{-1})$ by Theorem 4.4, with the

sign $\sigma = \pm 1$ in v_2 such that σa and b have the same sign. Then

$$\begin{aligned} v_1\left(y + \frac{K'}{2b}\right) &= 2bH^{-1} \operatorname{dn}(2by + K' | \kappa') \\ &= 2bH^{-1} \sqrt{1 - \kappa'} \operatorname{nd}(2by | \kappa') \text{ using 16.8.3 [1]} \\ &= \sigma 2aH^{-1} \operatorname{dn}(2ay | -\frac{\kappa'}{1-\kappa'}) \text{ using 16.10.4 [1]} \\ &= \sigma 2aH^{-1} \operatorname{dn}(2ay | 1 - \kappa^{-1}) = v_2(y). \end{aligned}$$

Thus after the change of coordinate $z \mapsto z + \frac{iK'}{2|b|}$ the surfaces have same mean curvature, Hopf differential and metric, and thus differ by a rigid motion. Hence swapping constants a, b in ξdz gives the same surface up to a rigid motion.

(ii) In case $a = 0$ and $b \neq 0$ the resulting immersion is either totally umbilic, or in the case of the parallel ‘surface’ a point. In case $a = b = 0$ the parallel surfaces both degenerate to a point, since then the corresponding frame is $F \equiv \mathbb{1}$.

(iii) Since $\operatorname{dn}(u | 0) \equiv 1$, the resulting conformal factor is $v(y) \equiv 2|b/H|$. Hence the associated family surface is that of a right circular cylinder. \square

If we omitt the case $a = \pm b$, then as a consequence of Lemma 4.7 (i) we may assume without loss of generality that $a^2 < b^2$ and thus that the parameter satisfies

$$(4.14) \quad \kappa = \frac{a^2}{b^2} \in (0, 1).$$

We will need some properties of the square root v_0 of the Delaunay conformal factor.

Proposition 4.8. *For $\kappa \in (0, 1)$, the function $v_0(y) = 2bH^{-1} \operatorname{dn}(2by | \kappa')$ satisfies*

$$(4.15) \quad v_0(y) = v_0(y + b^{-1}K') \text{ for all } y \in \mathbb{R},$$

$$(4.16) \quad v_0\left(y + \frac{K'}{2b}\right) = v_0\left(\frac{K'}{2b} - y\right) \text{ for all } y \in \mathbb{R},$$

$$(4.17) \quad \dot{v}_0(y) = 0 \iff y \in \frac{K'}{2b} \mathbb{Z},$$

$$(4.18) \quad v_0(y) = 2bH^{-1} \iff y \in b^{-1}K' \mathbb{Z}.$$

Proof. Using Remark 4.6 shows that v_0 has period $b^{-1}K'$, proving (4.15). Assertion (4.16) follows from the change of argument formula 16.8.3 in [1]. Recall from Remark 4.5 (i) that the derivative of v_0 vanishes at values $v_0 = \pm 2a/H, \pm 2b/H$, and with the above this proves (4.17) and (4.18). \square

5. THE EXTENDED DELAUNAY FRAME

To solve the period problem for a helicoidal CMC surface, we need to compute the extended framing of an associated family of a Delaunay surface. This was obtained in [19] in an untwisted setting, but since we need most of the ingredients, we provide the reader with all the details.

Let ξdz be a potential in normal form with $a, b \in \mathbb{R}^*$ and $a \neq \pm b$, and Q and v_0 as in Theorem 4.4 with $\kappa < 1$. To find the unitary factor of the map $z \mapsto \exp(z\xi)$ it suffices to factorise $\exp(iy\xi)$, since $\exp(z\xi)$ is $\Lambda\mathrm{SU}(2)_\tau$ valued along $y = 0$. Off $y = 0$, we make the following Ansatz:

$$F = \exp(\mathbf{f}\xi)T$$

where $\mathbf{f} = \mathbf{f}(x, y, \lambda) = x + iy + f(y, \lambda)$ for a function f with $f(0, \lambda) = 0$, and $T = T(y, \lambda)$ an upper-triangular matrix

$$T = \begin{pmatrix} A & B \\ 0 & A^{-1} \end{pmatrix},$$

with $A(0, \lambda) = 1$ and $B(0, \lambda) = 0$. The Maurer–Cartan form $\alpha = \alpha_1 dx + \alpha_2 dy$ of the unitary frame is given by

$$(5.1) \quad \alpha_1 = \frac{i}{2} \begin{pmatrix} v_0 v_0^{-1} & 2v_0^{-1} \lambda^{-1} Q + v_0 \lambda H \\ v_0 \lambda^{-1} H + 2v_0^{-1} \lambda \bar{Q} + & -v_0 v_0^{-1} \end{pmatrix},$$

$$(5.2) \quad \alpha_2 = \frac{1}{2} \begin{pmatrix} 0 & -2v_0^{-1} \lambda^{-1} Q + v_0 \lambda H \\ -v_0 \lambda^{-1} H + 2v_0^{-1} \lambda \bar{Q} & 0 \end{pmatrix}.$$

For $F = \exp(\mathbf{f} \xi) T$ we then have $F^{-1} dF = T^{-1} \xi T dx + (\dot{\mathbf{f}} T^{-1} \xi T + T^{-1} \dot{T}) dy$, so \mathbf{f} , T must simultaneously solve $\alpha_1 = T^{-1} \xi T$ and $\alpha_2 = \dot{\mathbf{f}} T^{-1} \xi T + T^{-1} \dot{T}$.

We denote the lower left entry of α_1 by Ω_1 , and since $Q = 2abH^{-1}$, we have

$$(5.3) \quad \begin{aligned} \Omega_1(y, \lambda) &= \frac{i}{2} v_0(y) H \lambda^{-1} + 2i v_0(y)^{-1} ab H^{-1} \lambda \\ &\neq 0 \text{ for all } \lambda \in S^1, y \in \mathbb{R}. \end{aligned}$$

The claim $\Omega_1 \neq 0$ follows from remark 4.5 (ii), and $a \neq \pm b$.

Let us denote the lower left entry of ξ by ω , and note that $a \neq \pm b$ ensures

$$(5.4) \quad \omega = ib\lambda^{-1} + ia\lambda \neq 0 \text{ for all } \lambda \in S^1.$$

Then $T^{-1} \xi T = \alpha_1$ is equivalent to

$$\begin{pmatrix} -AB\omega & -A^{-2}\omega^* - B^2\omega \\ A^2\omega & AB\omega \end{pmatrix} = \begin{pmatrix} \frac{i}{2}v_0^{-1}v_0 & -\Omega_1^* \\ \Omega_1 & -\frac{i}{2}v_0^{-1}v_0 \end{pmatrix},$$

with unique (up to sign) solution

$$(5.5) \quad T = \frac{1}{\sqrt{\omega \Omega_1}} \begin{pmatrix} \Omega_1 & -\frac{i}{2}v_0^{-1}v_0 \\ 0 & \omega \end{pmatrix}.$$

Hence $T(y, \lambda)$ is defined for all $y \in \mathbb{R}$ and $\lambda \in S^1$. For v_0 as in (4.5), we have

$$(5.6) \quad \Omega_1(0, \lambda) = \omega(\lambda),$$

and consequently $T(0, \lambda) = \mathbb{1}$. It remains to determine the function \mathbf{f} such that

$$(5.7) \quad \alpha_2 = \dot{\mathbf{f}} \alpha_1 + T^{-1} \dot{T}.$$

To this end we compute

$$T^{-1} \dot{T} = \frac{1}{2\Omega_1} \begin{pmatrix} \dot{\Omega}_1 & -i(v_0 v_0^{-1}) \cdot \\ 0 & -\dot{\Omega}_1 \end{pmatrix} = \frac{1}{2\Omega_1} \begin{pmatrix} \dot{\Omega}_1 & iH^2 v_0^2 - 16iv_0^{-2} a^2 b^2 H^{-2} \\ 0 & -\dot{\Omega}_1 \end{pmatrix}.$$

We denote the lower left entry of α_2 by

$$(5.8) \quad \Omega_2 = -\frac{1}{2}v_0 H \lambda^{-1} + 2v_0^{-1} ab H^{-1} \lambda.$$

The bottom left of (5.7) immediately yields $\dot{\mathbf{f}} = i + \dot{f} = \Omega_1^{-1} \Omega_2$. A quick computation shows that this solution is miraculously compatible with the other three equations in (5.7). Thus $\dot{f} = \Omega_1^{-1}(\Omega_2 - i\Omega_1)$, and we define

$$(5.9) \quad f(y, \lambda) = \int_0^y \frac{\Omega_2(\zeta, \lambda) - i\Omega_1(\zeta, \lambda)}{\Omega_1(\zeta, \lambda)} d\zeta = \int_0^y \frac{8ab\lambda d\zeta}{4iab\lambda + iH^2\lambda^{-1}v_0^2(\zeta)},$$

and now set

$$(5.10) \quad \mathbf{f}(x, y, \lambda) = x + iy + f(y, \lambda).$$

Clearly $f(0, \lambda) = 0$ and we thus have a solution of $dF = F \alpha_1 dx + F \alpha_2 dy$ with α_1, α_2 as in (5.1) and (5.2) of the form $F = \exp(\mathbf{f} \xi) T$.

For v_0 as in (4.5), it is easy to verify that $T(0, \lambda) = \mathbb{1}$, and consequently $F(0, 0, \lambda) = \mathbb{1}$. To show that F is the extended framing of ξ , it remains to show that $B(y, \lambda) = T^{-1} \exp(-\mathbf{f} \xi) \in \Lambda_+ \mathrm{SL}(2, \mathbb{C})_\tau$ for all $y \in \mathbb{R}$. From $\Phi = FB$, it follows that $B = \dot{B}(y, \lambda)$ solves $dB + \alpha B = B \xi dz$ with $B(0, \lambda) = \mathrm{Id}$. This decouples into $\dot{B} + \alpha_2 B = i B \xi$ and $\alpha_1 B = B \xi$. It thus remains to verify that

$$(5.11) \quad \dot{B} B^{-1} = i \alpha_1 - \alpha_2.$$

On the one hand, $\dot{B} B^{-1} = -T^{-1} \dot{T} - \dot{\mathbf{f}} \alpha_1$, while (5.1) and (5.2) yield

$$i \alpha_1 - \alpha_2 = \frac{1}{2} \begin{pmatrix} -v_0 v_0^{-1} & -2v_0 \lambda H \\ -4v_0^{-1} \lambda \bar{Q} & v_0 v_0^{-1} \end{pmatrix}.$$

A computation verifies that indeed B solves (5.11) and hence is positive and smooth. Further, (5.11) can be integrated at $\lambda = 0$ and gives $B_0 = B(y, 0) = \mathrm{diag}[v_0^{-1/2}, v_0^{1/2}]$. Hence $B : \mathbb{R} \rightarrow \Lambda_+ \mathrm{SL}(2, \mathbb{C})_\tau$. Altogether, this proves

Theorem 5.1. [19] *The extended framing for the Delaunay surface generated by a potential ξdz in normal form with $a \neq \pm b$ is given by*

$$(5.12) \quad F = \exp(\mathbf{f} \xi) T$$

with $\mathbf{f} = \mathbf{f}(x, y, \lambda)$ as in (5.10) and $T = T(y, \lambda)$ as in (5.5) and v_0 as in (4.5).

6. HELICOIDAL CMC CYLINDERS IN \mathbb{R}^3

We now arrive at our main objective which is to solve period problems for equivariant harmonic maps $\mathbb{C} \rightarrow S^2$ as well as for the corresponding constant mean curvature surfaces. We will show that every equivariant harmonic map $\mathbb{C} \rightarrow S^2$ is periodic. Further, we will discuss the closing conditions for helicoidal constant mean curvature cylinders.

Other methods have been used to study helicoidal CMC surfaces, as in the investigations of Roussos et al., see [11] and [16] and the references therein.

Consider translations

$$\delta : z \mapsto z + p + iq \text{ with } p, q \in \mathbb{R}.$$

If an extended framing $F(z, \lambda)$ with $F(0, \lambda) = \mathbb{1}$ has a well defined monodromy

$$(6.1) \quad M(\delta, \lambda) = \delta^* F F^{-1} = F(\delta(0), \lambda)$$

with respect to δ , then $\delta^* f_{\mu_0} = f_{\mu_0}$ for $\mu_0 \in S^1$ if and only if $\Phi_{\mu_0}(M) = \mathbb{1}$, which in turn is equivalent to the two *closing conditions*:

$$(6.2) \quad M(\delta, \mu_0) = \pm \mathbb{1} \text{ and } M'(\delta, \mu_0) = 0.$$

We first consider the case of a right circular cylinder, generated by a potential in normal form with $a = \pm b \neq 0$.

Lemma 6.1. *Every member in the associated family of a right circular cylinder has a period.*

Proof. Up to rigid motion, an associated family f_λ of a right circular cylinder is generated by a potential ξdz in normal form with $a = \pm b$. Consider the case $a = b$. The corresponding extended frame is $\exp(ia(z\lambda^{-1} + \bar{z}\lambda)) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with monodromy

$$M(\delta, \lambda) = \cosh(ia(\delta \lambda^{-1} + \bar{\delta} \lambda)) \mathbb{1} + \sinh(ia(\delta \lambda^{-1} + \bar{\delta} \lambda)) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence $M(\delta, \lambda_0)$ satisfies the two closing conditions if and only if $\delta\lambda_0^{-1} + \bar{\delta}\lambda_0 \in \frac{\pi}{a}\mathbb{Z}$ and $\bar{\delta}\lambda_0 - \delta\lambda_0^{-1} = 0$, and thus if and only if $\delta \in \frac{\pi}{2a}\lambda_0\mathbb{Z}$. Then for all $k \in \mathbb{Z}$

$$f_{\lambda_0}(z + \frac{\pi}{2a}\lambda_0 k) = f_{\lambda_0}(z).$$

The case $a = -b$ is proven similarly. \square

Let ξ as in (4.2) with $a, b \in \mathbb{R}^*$. It is easily verified that $0 \leq \det \xi(\lambda)$ for $\lambda \in S^1$, since $\min\{(a-b)^2, (a+b)^2\}$ is a lower bound. Hence

$$(6.3) \quad \mu(\lambda) = \sqrt{-\det \xi(\lambda)}.$$

is purely imaginary for all $\lambda \in S^1$. The roots of μ are $\pm i\sqrt{a/b}, \pm i\sqrt{b/a}$. Since we have discussed the case $a = \pm b$ of a right circular cylinder, we will from now assume that $a \neq \pm b$ and thus ensure that these roots are off S^1 .

To study the monodromy behaviour of extended frames generated by potentials in normal form, define

$$(6.4) \quad \mathbf{g} = \mathbf{g}(\delta, \lambda) = \mathbf{f}(p, q, \lambda) \mu(\lambda),$$

with \mathbf{f} as in (5.10) and $\mu(\lambda)$ as in (6.3). The monodromy of F in (5.12) with respect to δ is then $M(\delta, \lambda) = F(p, q, \lambda) = \exp(\mathbf{f}(p, q, \lambda) \xi(\lambda)) T(q, \lambda)$, which we may write as

$$(6.5) \quad M(\delta, \lambda) = \cosh(\mathbf{g}) T + \mu^{-1} \sinh(\mathbf{g}) \xi T.$$

We have seen in Lemma 4.2 that at $\lambda^4 = 1$ we obtain surfaces of revolution. We omit these fourth roots of unity in our further study of helicoidal cylinders.

Lemma 6.2. *Let $M(\delta, \lambda)$ as in (6.5) be the monodromy of an extended frame generated by a potential in normal form with $a \neq \pm b$, and $\lambda_0 \in S^1 \setminus \{\pm 1, \pm i\}$.*

(i) $M(\delta, \lambda_0) = \pm \mathbb{1}$ if and only if $T(q, \lambda_0) = \mathbb{1}$ and

$$(6.6) \quad \mathbf{g}(\delta, \lambda_0) \in \pi i\mathbb{Z},$$

(ii) $T(q, \lambda_0) = \mathbb{1}$ holds if and only if $v_0(q) = v_0(0)$ and $\dot{v}_0(q) = 0$.

(iii) If we assume $M(\delta, \lambda_0) = \pm \mathbb{1}$, then $M'(\delta, \lambda_0) = 0$ if and only if

$$(6.7) \quad \mathbf{g}'(\delta, \lambda_0) = 0.$$

Proof. (i) If $M(\delta, \lambda_0) = \pm \mathbb{1}$, then the lower left entry in (6.5) is

$$\mu(\lambda_0)^{-1} \sinh(\mathbf{g}(\lambda_0)) \sqrt{\omega(\lambda_0) \Omega_1(q, \lambda_0)} = 0.$$

Hence either $\mathbf{g}(\delta, \lambda_0) \in \pi i\mathbb{Z}$ or $\omega(\lambda_0) \Omega_1(q, \lambda_0) = 0$. But the latter is not possible by (5.3) and (5.4). Hence $\mathbf{g}(\delta, \lambda_0) \in \pi i\mathbb{Z}$, and consequently $T(q, \lambda_0) = \mathbb{1}$. The converse is obvious.

(ii) Recall the definition of T from equation (5.5). Clearly $T(q, \lambda_0) = \mathbb{1}$ holds if and only if $\omega(\lambda_0) = \Omega_1(q, \lambda_0)$ and $\dot{v}_0(q) = 0$. Now $\omega(\lambda_0) = \Omega_1(q, \lambda_0)$ is a quadratic equation in $v_0(q)$ with solutions $v_0(q) = 2b/H, \lambda_0^2/(5H)$. Since $\dot{v}_0(q) = 0$, we must have $v_0(q) = 2b/H = v_0(0)$ by (4.9).

(iii) Differentiating the monodromy in (6.5) with respect to t , and making use of $T(q, \lambda_0) = \mathbb{1}$ and $\mathbf{g}(\delta, \lambda_0) \in \pi i\mathbb{Z}$, we obtain

$$(6.8) \quad M'(\delta, \lambda_0) = \pm T'(q, \lambda_0) \pm \mu(\lambda_0)^{-1} \mathbf{g}'(\delta, \lambda_0) \xi(\lambda_0).$$

In part (ii) we saw that $T(q, \lambda_0) = \mathbb{1}$ implies that $\dot{v}_0(q) = 0$ and $\omega(\lambda_0) = \Omega_1(q, \lambda_0)$. Thus the first term on the right hand side in (6.8) is diagonal and

$$T'(q, \lambda_0) = \frac{1}{2\omega}(\Omega_1'(q, \lambda_0) - \omega'(\lambda_0)) \mathbb{1}.$$

On the other hand, the second term on the right hand side in (6.8) is off-diagonal, and since the entries of ξ have no roots on S^1 , the vanishing of $M'(\delta, \lambda_0)$ is equivalent to the simultaneous vanishing of $\mathbf{g}'(\delta, \lambda_0)$ and $(\Omega'_1(q, \lambda_0) - \omega'(\lambda_0))$. A quick computation reveals that $v_0(q) = 2b/H$ ensures that $\Omega'_1(q, \lambda_0) = \omega'(\lambda_0)$ holds. Hence $M'(\delta, \lambda_0) = \pm \mathbf{g}'(\delta, \lambda_0) \xi(\lambda_0)/\mu(\lambda_0)$ and the claim follows. \square

6.1. The rotational period. Solving a period problem for a constant mean curvature surface is tantamount to ensuring the two closing conditions (6.2). The closing conditions are equivalent to killing rotational and translational periods respectively. We first consider rotational periods, and will learn that one of the generators of the period lattice has to be a period of the metric. We also show that every equivariant harmonic map $\mathbb{C} \rightarrow S^2$ is periodic.

Consider the function v_0 from (4.5). We have seen in (4.17) and (4.18) that the conditions $v'_0(q) = 0$ and $v_0(q) = v_0(0)$ hold if and only if $q \in b^{-1}\mathbf{K}'\mathbb{Z}$. So let

$$(6.9) \quad q = |b|^{-1}\mathbf{K}'k \text{ for some } k \in \mathbb{Z}.$$

For any such q we have $v_0(y+q) = v_0(y)$ and $T(y+q, \lambda) = T(y, \lambda)$. Consequently,

$$\begin{aligned} f(y+q) &= \int_0^{y+q} \frac{8ab\lambda d\zeta}{4iab\lambda + iH^2\lambda^{-1}v_0^2(\zeta)} \\ &= \int_0^y \frac{8ab\lambda d\zeta}{4iab\lambda + iH^2\lambda^{-1}v_0^2(\zeta)} + \int_0^q \frac{8ab\lambda d\zeta}{4iab\lambda + iH^2\lambda^{-1}v_0^2(\zeta)} \\ &= f(y) + f(q), \end{aligned}$$

and therefore

$$(6.10) \quad F(x+p, y+q, \lambda) = \exp((p+iq+f(q))\xi) F(x, y, \lambda).$$

Since F takes values in $\Lambda\text{SU}(2)_\tau$, it follows that

$$(6.11) \quad \text{Im}(f(q, \lambda)) = -q \text{ for all } \lambda \in S^1.$$

We now proceed to find values for $p \in \mathbb{R}$ such that for some fixed $\lambda_0 \in S^1 \setminus \{\pm 1, \pm i\}$ both (6.6) and (6.7) hold.

Recall the definitions of $\mathbf{g}(\delta, \lambda_0)$ from (6.4) and $f(q, \lambda_0)$ from (5.9). Now $\mathbf{g}(\delta, \lambda_0) \in \pi i\mathbb{Z}$ in (6.6) means that $(p+iq+f(q, \lambda_0))\mu(\lambda_0) = \pi il$ for some $l \in \mathbb{Z}$. Solving this for p , and using (6.11), we have that $\mathbf{g}(\delta, \lambda_0) \in \pi i\mathbb{Z}$ if and only if p is of the form $p = \pi il \mu^{-1}(\lambda_0) - \text{Re}(f(q, \lambda_0))$ for some $l \in \mathbb{Z}$. We have proven

Lemma 6.3. *The monodromy $M(\delta, \lambda)$ of an extended frame generated by a potential in normal form satisfies $M(\delta, \lambda_0) = \pm \mathbb{1}$ if and only if $\delta : z \mapsto z + p + iq$ with*

$$(6.12) \quad p = \pi il \mu^{-1}(\lambda_0) - \text{Re}(f(q, \lambda_0)) \text{ for some } l \in \mathbb{Z},$$

$$(6.13) \quad q = |b|^{-1}\mathbf{K}k \text{ for some } k \in \mathbb{Z}.$$

Theorem 6.4. *Every equivariant harmonic map $N : \mathbb{C} \rightarrow S^2$ is periodic.*

Proof. By Corollary 4.3, up to isometry, every associated family of equivariant harmonic maps $N_\lambda : \mathbb{C} \rightarrow S^2$ can be generated by a potential ξdz in normal form. Assume, possibly after a rigid motion, that $N = N_{\lambda_0}$ for some $\lambda_0 \in S^1$. The extended framing then has monodromy $M(\delta, \lambda)$ as in (6.5). The periodicity of the harmonic map $N(z+\delta) = N(z)$ is equivalent to the first closing condition $M(\delta, \lambda_0) = \mathbb{1}$, which holds for $\delta = p+iq$ with p as in (6.12) and q as in (6.13). \square

6.2. The translational period. Amongst equivariant harmonic maps, we now single out those which generate periodic immersions.

It remains to realize the constraint of (6.7), as a condition on the spectral parameter. For $\lambda_0 = e^{it_0} \in S^1$ and p, q as in (6.12) and (6.13) we have that $(\mathbf{f} \mu)(p, q, \lambda_0) = \pi i l$ for some $l \in \mathbb{Z}$, and compute

$$\begin{aligned} \mathbf{g}'(p, q, \lambda_0) &= f'(q, \lambda_0) \mu(\lambda_0) + \mathbf{f}(p, q, \lambda_0) \mu'(\lambda_0) \\ &= f'(q, \lambda_0) \mu(\lambda_0) + \pi i l \mu^{-1}(\lambda_0) \mu'(\lambda_0). \end{aligned}$$

Hence $\mathbf{g}'(p, q, \lambda_0) = 0$ is equivalent to

$$(6.14) \quad l = -\frac{\mu^2(\lambda_0)}{\pi i \mu'(\lambda_0)} f'(q, \lambda_0).$$

Since it is not apparent for which $\lambda \in S^1$, if indeed any, the right hand side in (6.14) is integer-valued, we view it as a function of λ and define

$$(6.15) \quad L(\lambda) = -\frac{\mu^2(\lambda)}{\pi i \mu'(\lambda)} f'(q, \lambda).$$

Proposition 6.5. *The function $L(\lambda)$ is real and meromorphic on S^1 with only simple poles at the fourth roots of unity.*

Proof. Consider first the function $f' : S^1 \rightarrow \mathbb{R}^*$ given by

$$(6.16) \quad f'(q, \lambda) = \int_0^q \frac{16ab H^2 v_0^2(\zeta) d\zeta}{(4ab\lambda + H^2 \lambda^{-1} v_0^2(\zeta))^2}.$$

By (5.3), the denominator in the integrand in (6.16) is non-zero for all $\lambda \in S^1$. By (6.11) we have that $f'(q, \lambda) = \operatorname{Re} f'(q, \lambda)$ for each $\lambda \in S^1$, and conclude that f' is real analytic and non-zero. The first factor of $L(\lambda)$ computes to

$$(6.17) \quad -\frac{\mu^2(\lambda)}{\pi i \mu'(\lambda)} = \frac{\mu^3(\lambda)}{\pi ab (\lambda^{-2} - \lambda^2)},$$

which is clearly real for $\lambda \in S^1$, since $\mu(\lambda) \in i\mathbb{R}^*$ and $\lambda^{-2} - \lambda^2 \in i\mathbb{R}$. Hence $L(\lambda)$ is real for $\lambda \in S^1$. The only poles of L are those of the first factor (6.17), namely simple poles at the fourth roots of unity. This concludes the proof. \square

As a consequence of the last proposition, we are assured of infinitely many values $\lambda \in S^1 \setminus \{\lambda^4 = 1\}$ at which $L(\lambda) \in \mathbb{Z}$. This proves the existence of infinitely many helicoidal constant mean curvature cylinders, and we conclude

Theorem 6.6. *In each associated family of a Delaunay surface there are infinitely many non-congruent cylinders with screw-motion symmetry.*

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