

# POISSON-DIRICHLET DISTRIBUTION FOR RANDOM BELYI SURFACES

ALEX GAMBURD

*Dedicated to the memory of Bob Brooks*

ABSTRACT. Brooks and Makover introduced an approach to studying the global geometric quantities (in particular, the first eigenvalue of the Laplacian, injectivity radius and diameter) of a “typical” compact Riemann surface of large genus based on compactifying finite-area Riemann surfaces associated with random cubic graphs; by a theorem of Belyi these are “dense” in the space of compact Riemann surfaces. The question as to how these surfaces are distributed in the Teichmüller spaces depends on the study of oriented cycles in random cubic graphs with random orientation; Brooks and Makover conjectured that asymptotically normalized cycles lengths follow Poisson-Dirichlet distribution. We present a proof of this conjecture using representation theory of the symmetric group. Consequently we also make progress towards a conjecture of Pippenger and Schleich which arose in the study of topological characteristics of random surfaces generated by cubic interactions.

## 1. INTRODUCTION

Study of the first eigenvalue of the Laplace operator on compact Riemann surfaces of increasing genus has received considerable attention over the last thirty years; see [18] and references therein. On the one hand, we have a celebrated theorem of Selberg [46] (see [34] and [30] for refined estimates towards Selberg’s conjecture) and its generalization by Sarnak and Xue [45] (see [27] and [25] for related results), asserting that the first eigenvalue of the congruence surfaces of arbitrary genus is bounded away from zero; on the other hand there are examples due to Selberg [46], Randol [42], and Buser [17], showing that, in general, the first eigenvalue can be made arbitrarily small. The surfaces in the latter examples are “long and thin”, so in some sense live on the boundary of Teichmüller spaces, and it is a fascinating question to determine what happens for a “typical” Riemann surface of large genus.

---

The author was supported in part by the NSF postdoctoral fellowship.

The idea of using cubic graphs to study the first eigenvalue of Riemann surfaces originated in the work of Buser [15, 16]. As we discuss in section 3, one understands rather well the behavior of the first eigenvalue of the discrete Laplacian on a random cubic graph. In [14], Brooks and Makover introduced an approach to studying the first eigenvalue of the Laplacian of a “typical” compact Riemann surface of large genus based on compactifying finite-area Riemann surfaces associated with random cubic graphs with random orientation; we review Brooks-Makover construction in section 2. For each cubic graph  $\Gamma$  with an orientation  $\mathcal{O}$ , they associate two Riemann surfaces:  $S^{\mathcal{O}}(\Gamma, \mathcal{O})$ , a finite area noncompact surface, and  $S^C(\Gamma, \mathcal{O})$ , a compact surface. The surface  $S^{\mathcal{O}}(\Gamma, \mathcal{O})$  is an orbifold cover of  $\mathbb{H}/PSL(2, \mathbb{Z})$  described by  $(\Gamma, \mathcal{O})$  and therefore shares some of the global geometric properties with the graph  $\Gamma$ . The compact surface  $S^C(\Gamma, \mathcal{O})$  is a conformal compactification of  $S^{\mathcal{O}}(\Gamma, \mathcal{O})$ ; Brooks and Makover proved that almost always the global geometry of  $S^C(\Gamma, \mathcal{O})$  is controlled by the geometry of  $S^{\mathcal{O}}(\Gamma, \mathcal{O})$ . Moreover, according to Belyi’s theorem [4] the surfaces  $S^C(\Gamma, \mathcal{O})$  are precisely the Riemann surfaces which can be defined over some number field and so form a “dense” set in the space of all Riemann surfaces.

The question as to how these surfaces are distributed in the Teichmüller spaces depends on the study of oriented cycles in random cubic graphs with random orientation; Brooks and Makover conjectured that asymptotically normalized cycles lengths follow Poisson-Dirichlet distribution. We recall the definition of Poisson-Dirichlet distribution [3]. Let  $B_1, B_2, \dots$  be independent random variables uniformly distributed on  $[0, 1]$ . Define  $G = (G_1, G_2, \dots)$  as follows:

$$G_1 = B_1; \quad G_2 = (1 - B_1)B_2; \quad \dots \quad G_i = (1 - B_1)(1 - B_2) \dots (1 - B_{i-1})B_i.$$

The random sequence  $G$  can be viewed as a description of a random breaking of a stick of unit length into infinite sequence of subintervals. A stick of length  $B_1$  is broken off at the left, which leaves a piece of length  $1 - B_1$ . From this a piece of length  $(1 - B_1)(1 - B_2)$  is broken off, and so on.  $G$  is a distribution on the set

$$\Omega = \left[ x \in R^{\infty} : x_1, x_2, \dots, \geq 0, \sum_{i=1}^{\infty} x_i = 1 \right].$$

The ranked version of  $G$ ,  $(G_{(1)}, G_{(2)}, \dots)$  where  $G_{(1)} \geq G_{(2)} \geq \dots$  has Poisson-Dirichlet distribution. If, instead of uniform distribution,  $B_i$  has beta- $(1, \theta)$  density  $\theta(1 - x)^{\theta-1}$  on  $[0, 1]$  with  $\theta > 0$ , the resulting distribution is called Poisson-Dirichlet distribution with parameter  $\theta$ . Poisson-Dirichlet distribution arises in a great variety of problems, see [3] and references therein; in a recent breakthrough work [20] it was

proved that it is the unique invariant distribution for uniform split-merge transformations.

As we discuss in section 3, the distribution of cycle length for random regular graphs in any of the standard models is a rather difficult problem; adding random orientation, at first, seems to only complicate matters (see example in section 2). However, as we explain at the end of section 3, this extra randomness turns out to in fact help us: by giving a permutational model for a random regular graph with random orientation, we can convert the problem into one involving symmetric group. More precisely, the distribution of oriented cycles in random  $k$ -regular graph on  $n$  vertices with random orientation is the same as the distribution of cycles in the permutation  $\beta\alpha$  where  $\beta$  is chosen uniformly on the conjugacy class consisting of the product of  $k$ -cycles, and  $\alpha$  is chosen with uniform probability on the conjugacy class consisting of the product of 2-cycles in the symmetric group  $S_N$  with  $N = nk$ . In section 4 using Diaconis-Shahshahani upper-bound lemma [21], the estimate on the number of rim hook tableaux by Fomin and Lulov [22], and representation theory of the symmetric group (in particular, hook-length formula and Murnaghan-Nakayama rule), we show that as  $n \rightarrow \infty$  the distribution of  $\beta\alpha$  converges to uniform distribution. We then invoke what is perhaps the oldest occurrence of Poisson-Dirichlet distribution – the distribution of normalized cycle lengths for a random permutation in  $S_n$  as  $n$  tends to infinity – to prove Brooks-Makover conjecture.

It turns out that the number of oriented cycles in random cubic graphs with random orientation was also studied by Pippenger and Schleich [41] in connection with topological characteristics of random surfaces generated by cubic interactions. (I am grateful to Bálint Virág for bringing the paper [41] to my attention). The surfaces considered by Pippenger and Schleich are obtained by taking  $3n$  arcs of an even number of oriented triangles and randomly identifying them in pairs respecting the orientation; these surfaces arise in various contexts in two-dimensional quantum gravity and as world-sheets of string theory. Random cubic graphs with random orientation provide an alternative way of constructing the surfaces in [41]. Denoting by  $l(n)$  the number of oriented cycles in a random cubic graph on  $n$  vertices with a random orientation, Pippenger and Schleich proved that  $\mathbb{E}(l(n)) = \log n + O(1)$  and  $\text{Var}(l(n)) = O(\log n)$ . Further, based on empirical study of 10,000 random surfaces, each constructed from 80,000 triangles they conjectured that

$$\mathbb{E}(l(n)) = \log(3n) + \gamma + O(n^{-1})$$

and

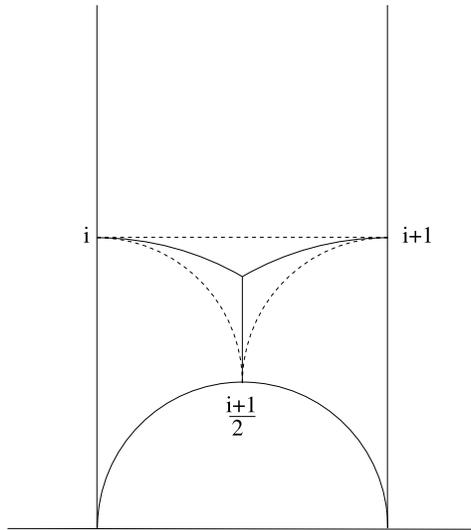
$$\text{Var}(l(n)) = \log(3n) + \gamma - \pi^2/6 + O(n^{-1}),$$

where  $\gamma = 0.5772\dots$  is an Euler constant. An immediate corollary of the main theorem, presented in section 5, establishes Pippenger-Schleich conjecture with an error term of  $O(n^{-1/6})$  instead of  $O(n^{-1})$ ; we also obtain asymptotic normal distribution for  $l(n)$ . Another immediate corollary is that the expected area of the largest embedded ball in a random Belyi surface converges to  $\frac{0.62}{2\pi}$  of the total surface area. We conclude section 5 by briefly discussing the conjectured Tracy-Widom distribution for the second largest eigenvalue of random regular graphs; we hope that our results will be useful in approaching this fascinating open problem.

**Acknowledgements.** I started working on this problem while visiting Bob Brooks at the Technion in August 2001; I would like to thank the Department of Mathematics at Technion for its hospitality. The interplay between spectral geometry of surfaces and graphs, particularly the issue of first eigenvalue, has been a recurrent theme in Bob's work; his papers, written with deep sympathy for the reader, and his continuing support and encouragement have been very important to me. The results presented in this paper were obtained in the spring of 2002 and I communicated them to Bob a few months before his untimely death in September 2002. I lectured on them in June 2002 at the spectral geometry conference in Lexington, Kentucky and at the Brooks Memorial Conference in Technion in December, 2003; I would like to thank the organizers of these conferences. After my talk at the Brooks Memorial Conference Alex Lubotzky told me that Martin Liebeck and Aner Shalev independently obtained a result similar to Proposition 2; I would like to thank Aner Shalev for sending me their preprint [31]. I would like to thank Persi Diaconis, Eran Makover, Anatoly Vershik, Ofer Zeitouni, and Martin Zerner for interest in this work and stimulating discussions.

## 2. BELYI SURFACES

In [4] Belyi proved a remarkable result asserting that a Riemann surface  $S$  can be defined over the field of algebraic numbers  $\overline{\mathbb{Q}}$  if and only if there exists a covering  $f : S \rightarrow \overline{\mathbb{C}}$  unramified outside  $\{0, 1, \infty\}$ . We call such surfaces Belyi surfaces. In this section we review Brooks-Makover construction of Belyi surfaces from cubic graphs. We remark that Mulase and Penakava [37] have given an alternative very interesting construction of Belyi surfaces; in their construction the edges of the graphs are allowed to have variable lengths.


 FIGURE 1. The marked ideal triangle  $T$ 

Let  $\Gamma$  be a cubic graph. An orientation  $\mathcal{O}$  on the graph  $\Gamma$  is an assignment for each vertex  $v$  of  $\Gamma$  of a cyclic ordering of the half-edges incident to that vertex. Given a cubic graph on  $n$  vertices, it will have  $2^n$  different orientations. A *left-hand-turn path* (LHT path) on  $\Gamma$  is a closed path on  $\Gamma$  such that, at each vertex, the path turns left in the orientation  $\mathcal{O}$ .

Given a pair  $(\Gamma, \mathcal{O})$  we construct a finite area Riemann surface  $S^{\mathcal{O}}(\Gamma, \mathcal{O})$  as follows. We take the ideal hyperbolic triangle  $T$  with vertices  $0, 1,$  and  $\infty$  shown in Figure 1. The solid lines in Figure 1 are geodesics joining the points  $i, i + 1,$  and  $\frac{i+1}{2}$  with the point  $\frac{1+i\sqrt{3}}{2}$ , while the dotted lines are horocycles joining pairs of points from the set  $\{i, i + 1, \frac{i+1}{2}\}$ . We may think of these points as “midpoints” of the corresponding sides of the ideal triangles, even though the sides are of infinite length. We may also think of the three solid lines as edges of a graph emanating from a vertex. We may then give them the cyclic ordering  $(i, i + 1, \frac{i+1}{2})$ . Given a cubic graph  $\Gamma$  with orientation  $\mathcal{O}$  we construct  $S^{\mathcal{O}}(\Gamma, \mathcal{O})$ , by associating to each vertex an ideal triangle, and gluing neighboring triangles. We glue two copies of  $T$  along the corresponding sides, subject to the following two conditions:

- a:** the “midpoints” of the two sides are glued together,  
and
- b:** the gluing preserves the orientation of the two copies of  $T$ .

The conditions (a) and (b) determine the gluing uniquely. It is easily seen that the surface  $S^O(\Gamma, \mathcal{O})$  is a complete Riemann surface with a finite area equal to  $\pi n$ , where  $n$  is the number of vertices of  $\Gamma$ .

It is easy to see that the horocycle pieces on each  $T$  glue together to give closed horocycles about a cusp of  $S^O(\Gamma, \mathcal{O})$ . The length of each closed horocycle is precisely the length of corresponding LHT path. The surface  $S^C(\Gamma, \mathcal{O})$  is the conformal compactification of  $S^O(\Gamma, \mathcal{O})$ . The oriented graph  $(\Gamma, \mathcal{O})$  describes  $S^O(\Gamma, \mathcal{O})$  as a covering space of  $\mathbb{H}^2/PSL_2(\mathbb{Z})$ , with each vertex being an orbifold point of order 3.

We are now ready to state the following result, whose proof was sketched by Brooks and Makover in [14].

**Lemma 1.**  *$S$  is a Belyi surface if and only if  $S = S^C(\Gamma, \mathcal{O})$  for some cubic graph  $\Gamma$ .*

**Proof:** We first show that if  $G$  be a torsion-free finite index subgroup of  $PSL_2(\mathbb{Z})$ , then  $\mathbb{H}^2/G = S^C(\Gamma, \mathcal{O})$  for some cubic graph  $\Gamma$ . Indeed, we can take as a fundamental domain for  $PSL_2(\mathbb{Z})$  a set  $F = \{0 < \Re(z) < 1, |z| > 1, |z - 1| > 1\}$ . Three copies of  $F$  fit together to give a marked ideal triangle as presented in figure 1, they are transformed by means of elliptic element  $w$  of order 3 in  $PSL_2(\mathbb{Z})$ .

Now since  $G$  is torsion free,  $F$ ,  $w(F)$  and  $w^2(F)$  are not equivalent under  $G$  and so they can all be included in the fundamental domain of  $G$ . In particular, there is a fundamental domain of  $G$  such that it consists of the copies of ideal triangle  $T$ . The graph dual to this triangulation, together with the boundary pairings and orientation of  $\mathbb{H}^2/G$  is exactly the pair  $(\Gamma, \mathcal{O})$ .

Now since  $S$  is a Belyi surface if and only if one can find finitely many points  $\{p_1, \dots, p_l\}$  on  $S$  such that  $S - \{p_1, \dots, p_l\}$  is isomorphic to  $\mathbb{H}^2/G$  where  $G$  is a torsion-free finite index subgroup of  $PSL_2(\mathbb{Z})$  [29], the lemma is proved.

We define probability on the space of oriented graphs with  $n$ -vertices  $(\Gamma_n, \mathcal{O})$  as follow: We pick a random cubic graph with  $n$  vertices using the Bollobas model, described in the next section, then we pick an orientation  $\mathcal{O}$  with equal probability from all the  $2^n$  possible orientations on the given graph. Let  $Q$  be a property of graphs, we denote by  $\text{Prob}_n[Q]$  the probability that an oriented graph  $(\Gamma_n, \mathcal{O})$  picked from our probability space has property  $Q$ . Brooks and Makover proved [14] the following result on Belyi surfaces constructed from random cubic graphs with random orientation:

**Theorem 1.** *There exist positive constants  $C_1, C_2, C_3$ , and  $C_4$  such that, as  $n \rightarrow \infty$ :*

**a:** The first eigenvalue  $\lambda_1(S^C(\Gamma, \mathcal{O}))$  satisfies

$$\text{Prob}_n[\lambda_1(S^C(\Gamma, \mathcal{O})) \geq C_1] \rightarrow 1.$$

**b:** The Cheeger constant  $h(S^C(\Gamma, \mathcal{O}))$  satisfies

$$\text{Prob}_n[h(S^C(\Gamma, \mathcal{O})) \geq C_2] \rightarrow 1.$$

**c:** The shortest geodesic  $\text{syst}(S^C(\Gamma, \mathcal{O}))$  satisfies

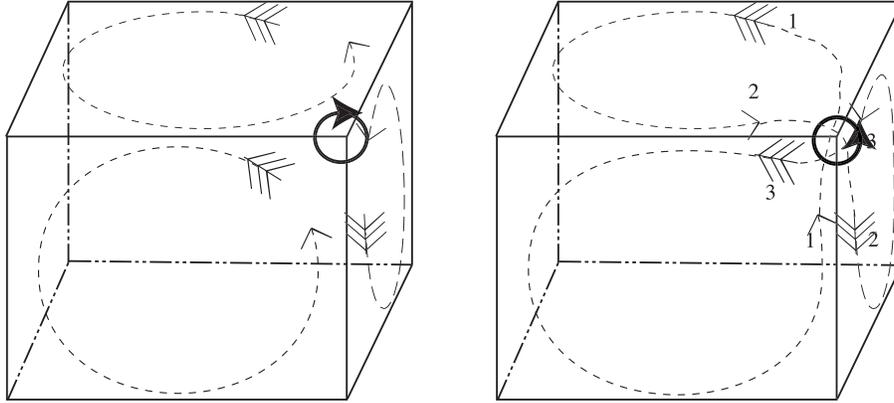
$$\text{Prob}_n[\text{syst}(S^C(\Gamma, \mathcal{O})) \geq C_3] \rightarrow 1.$$

**d:** The diameter  $\text{diam}(S^C(\Gamma, \mathcal{O}))$  satisfies

$$\text{Prob}_n[\text{diam}(S^C(\Gamma, \mathcal{O})) \leq C_4 \log(\text{genus}(S^C(\Gamma, \mathcal{O})))] \rightarrow 1.$$

The theorem is proved, roughly, as follows. First one establishes that with probability tending to 1 as  $n \rightarrow \infty$  a property in question holds for random cubic graphs (see section 3 for results pertaining to parts [a] and [b]). Then, using the fact that  $(\Gamma, \mathcal{O})$  describes  $S^O(\Gamma, \mathcal{O})$  as an orbifold covering, one transfers this information to open surfaces  $S^O(\Gamma, \mathcal{O})$ , using the results of Brooks in [8, 9]. One then transfers the desired property to the surfaces  $S^C(\Gamma, \mathcal{O})$  by using Ahlfors-Schwarz - Wolpert Lemma as developed by Brooks in [10, 11] and extended by Brooks and Makover in [12, 13, 14].

The topology of the surface can be read of from  $(\Gamma, \mathcal{O})$ , using LHT paths. In particular, the genus is given by Euler's formula; as is customary when using Euler's formula we refer LHT path as a *face* of the oriented graph.



A

B

FIGURE 2. Changing orientation on the cube

Let  $n$  be the number of vertices of  $\Gamma$  and let  $l(\Gamma, \mathcal{O})$  be the number of disjoint left-hand-turn paths. For a cubic graph the number of edges is  $\mathcal{E}(\Gamma) = \frac{3n}{2}$  and the number of faces is  $\mathcal{F} = l(\Gamma, \mathcal{O})$ . Therefore by Euler's formula we have:

$$(1) \quad \text{genus}(S^C(\Gamma_n, \mathcal{O})) = \text{genus}(S^O(\Gamma_n, \mathcal{O})) = 1 + \frac{n - 2l}{4}.$$

According to Euler's formula in order to find the genus of a surface constructed from a cubic graph with  $n$ -vertices, we need to estimate the number of faces, i.e., left-hand-turn paths. As explained in [14], the length of the largest LHT path  $L$  also determines the area of the largest embedded ball.

As we discuss in section 3, computing the distribution of the closed paths in random regular graphs is a very difficult problem. An additional complication is that the left-hand-turn paths are not necessarily simple closed paths on the graph, but can self-intersect in a very complex pattern. We would like to show these complications in a simple example of the 1-skeleton of the cube, with the usual orientation (Figure 2 **A**). We have  $\mathcal{F} = 6$ , and all the faces are simple paths of the graph, therefore the genus is 0. In Figure 2 **B** we changed the orientation of the right upper vertex. With the new orientation, the three simple faces that were adjacent to the upper right vertex before we changed the orientation are now joined to one composite face while the other three faces are unchanged hence,  $\mathcal{F} = 4$ , and the surface will have genus 1 .

### 3. RANDOM REGULAR GRAPHS

In this section we briefly review the pertinent facts on random  $k$ -regular graphs; see Wormald's paper [53] for an excellent survey. Given a  $k$ -regular graph  $\mathcal{G}$  and a subset  $X$  of  $V$ , the *expansion* of  $X$ ,  $c(X)$ , is defined to be the ratio  $|\partial(X)|/|X|$ , where  $\partial(X) = \{y \in \mathcal{G} : \text{distance}(y, X) = 1\}$ . The *expansion coefficient* of a graph  $\mathcal{G}$  is an analogue of the Cheeger's constant for Riemann surfaces and is defined as follows

$$(2) \quad c(\mathcal{G}) = \inf \left\{ c(X) \mid |X| < \frac{1}{2}|\mathcal{G}| \right\}.$$

A family of  $k$ -regular graphs  $X_{n,k}$  forms a family of  $C$ -expanders [32, 44] if there is a fixed positive constant  $C$ , such that

$$(3) \quad \liminf_{n \rightarrow \infty} c(X_{n,k}) \geq C.$$

The *adjacency matrix* of  $\mathcal{G}$ ,  $A(\mathcal{G})$  is the  $|\mathcal{G}|$  by  $|\mathcal{G}|$  matrix, with rows and columns indexed by vertices of  $\mathcal{G}$ , such that the  $x, y$  entry is 1 if and only if  $x$  and  $y$  are adjacent and 0 otherwise. For a  $k$ -regular graph the adjacency matrix is related to the combinatorial laplacian  $\Delta$  by  $A = kI - \Delta$ . Using the discrete Cheeger-Buser inequality, the condition (3) can be rewritten in terms of of the second largest eigenvalue of the adjacency matrix  $A(\mathcal{G})$  as follows:

$$(4) \quad \limsup_{n \rightarrow \infty} \lambda_1(A_{n,k}) < k.$$

In 1973 Pinsker [40] observed that a random regular graph is a good expander. This corresponds to the following fact about random matrices: random symmetric matrix of size  $N$  with  $k$  ones in each row and column and all other entries zero has the biggest eigenvalue equal to  $k$ , but the next eigenvalue will be bounded away from  $k$  by a fixed amount independent of  $N$ . The result of Pinsker on expansion coefficient of random regular graphs was considerably strengthened by Bollobás [7] who also introduced a widely used configuration model model for random regular graphs [6]. In this model random  $k$ -regular graphs on  $N$  vertices are represented as the images of so-called configurations. Let  $W = \bigcup_{j=1}^n W_j$  be a fixed set of  $2m = nd$  vertices, where  $|W_j| = d$ . A *configuration*  $F$  is a partition of  $W$  into  $m$  pairs of vertices called *edges* of  $F$ . Clearly there are

$$(5) \quad N = N(m) = \binom{2m}{2} \binom{2m-2}{2} \dots \binom{2}{2} / m! = \frac{(2m)_m}{2^m}$$

configurations. (We write  $(a)_b = a(a-1)\dots(a-b+1)$ ).

Let  $\Phi$  be a set of configurations; we turn it into a probability space by giving all configurations the same probability. We now define a map  $\Phi \rightarrow \mathcal{G}_{n,k}$  as follows. Given a configuration  $F$  let  $\phi(F)$  be the graph with vertex set  $V = 1, \dots, n$  in which  $ij$  is an edge iff  $F$  has a pair with one end in  $W_i$  and the other in  $W_j$ . Every  $G \in \mathcal{G}_{n,d}$  is the image of  $G = \phi(F)$  for  $(d!)^n$  configurations. The number of configurations containing a given fixed set of  $l$  edges is

$$(6) \quad N_l(m) = \frac{\binom{2m-2l}{2} \binom{2m-2l-2}{2} \dots \binom{2}{2} / (m-l)!}{N(m)} = \frac{1}{(2m-1)(2m-3)\dots(2m-2l+1)}$$

Using configuration model and in particular (6) Bollobás proved the following result.

**Theorem 2** (Bollobás [5]). *Let  $X_i$  denote the number of closed walks in  $\mathcal{G} \in \mathcal{G}_{n,k}$  of length  $i$ . Then for  $i = o(\log_{k-1} n)$  as  $n \rightarrow \infty$  random variables  $X_i$  converge to independent Poisson random variables with mean  $\frac{(k-1)^i}{2^i}$ .*

Counting cycles of length greater than  $\log n$  is substantially more difficult. In a recent breakthrough work [24] Friedman estimates the number of cycles of length  $O(\log^2 n)$  and uses this estimates (among other things) to prove that  $k$ -regular graphs on  $n$  vertices  $\mathcal{G}_{n,k}$  are asymptotically Ramanujan: for  $k$  fixed and  $\varepsilon > 0$  the probability that  $\lambda_1(X_{n,k}) \leq 2\sqrt{k-1} + \varepsilon$  tends to 1 as  $n \rightarrow \infty$ .

The bound of  $2\sqrt{k-1}$  is optimal in view of the result of Alon-Boppana [1, 33]. We also mention an early result of McKay [35] who showed that spectral density of random  $k$ -regular graphs converges to Kesten's measure, that is a measure supported on  $[-2\sqrt{k-1}, 2\sqrt{k-1}]$  and given by

$$(7) \quad \nu_k = \frac{k}{2\pi} \frac{\sqrt{4(k-1) - t^2}}{k^2 - t^2}.$$

We now introduce the permutational model for random regular graphs with random orientation. Consider a neighborhood of a vertex of a cubic graph. It contains three half-edges incident to it. Denote the set of half-edges by  $H$ ;  $|H| = 2|E| = 3|V|$ . Now the cyclic ordering of half-edges at each vertex is specified by a 3-cycle and a collection of all cyclic orders on the half-edges yields a permutation of  $H$  consisting of the product of 3-cycles. The structure of the underlying graph is given by the way half-edges couple to each other to form an edge; this is described by a permutation all of whose cycles are of order 2.

In the figure 3 we have

$$\begin{aligned} \beta &= (1, 3, 5)(2, 12, 8)(4, 7, 9)(6, 10, 11) \\ \alpha &= (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12) \end{aligned}$$

and

$$\varphi = \sigma\alpha = (1, 12, 6)(2, 3, 7)(4, 5, 10)(8, 9, 11).$$

Let  $\mathcal{G} \in \mathcal{G}_{n,k}$  be a  $k$  regular graph on  $n$  vertices; let  $\mathcal{O}$  be its orientation. Associated with a  $(\mathcal{G}, \mathcal{O})$  we have a pair of permutations  $(\beta, \alpha)$  in  $S_N$ , where  $N = kn$  with cycles of  $\beta$  encoding the information about the vertices with their cyclic orientation and  $\alpha$  encoding the information about the edges. The cycles of  $\varphi = \beta\alpha$  encode the information about the faces.

Now the problem of the distribution of the lengths of LHTP can be given the following equivalent formulation. Denote by  $C_r$  a conjugacy

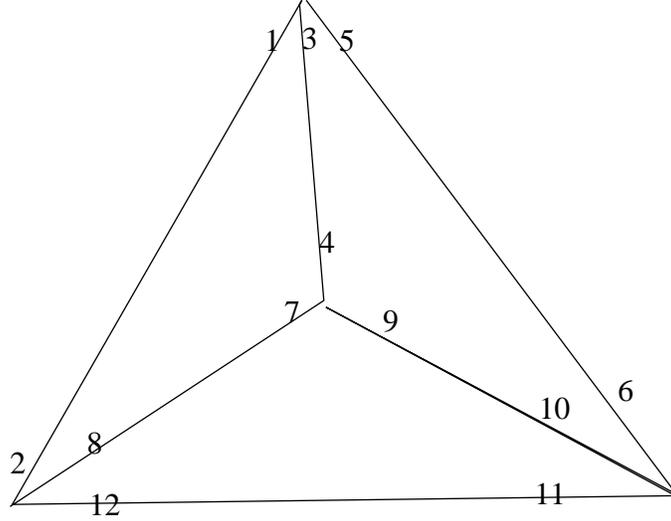


FIGURE 3. Permutational representation

class of  $A_N$  consisting of the product of  $r$ -cycles. Choose  $\beta$  with uniform probability on  $C_k$  and choose  $\alpha$  with uniform probability on  $C_2$  then the distribution of LHT paths in random  $k$ -regular graph on  $n$  vertices with random orientation is the same as the distribution of cycles in the permutation  $\beta\alpha$ .

#### 4. PROOF OF BROOKS-MAKOVER CONJECTURE

This section is devoted to the proof of the following result.

**Theorem 3.** *Let  $P_r$  denote the probability measure on  $A_N$  supported on  $C_r$ ; let  $U$  denote the uniform distribution on  $A_N$ . Then for  $k \geq 3$*

$$(8) \quad \|P_k * P_2 - U\| \xrightarrow{n \rightarrow \infty} 0.$$

Here

$$(9) \quad \|f - h\| = \max_{A \in G} |f(A) - g(A)|$$

is a total variation distance.

**Corollary 1.** *The distribution of LHTP for random regular graphs with random orientation converges to Poisson-Dirichlet distribution; in particular Brooks-Makover conjecture is true.*

**Proof:** Let  $\pi$  denote the cycle type of a permutation; denote by  $C^S(\pi)$  the number of permutations with a given cycle type  $\pi$  in the

symmetric group  $S_N$  and by  $C^A(\pi)$  the number of permutations with a cycle type  $\pi$  in the alternating group  $A_N$ . As is well-known

$$(10) \quad C^S(\pi) = C^S(a_1, \dots, a_N) = \frac{N!}{\prod_i i^{a_i(\pi)} a_i(\pi)!}$$

for  $\pi = 1^{a_1} \dots N^{a_N}$ . The cycle index polynomials of  $S_N$  is given by

$$p_N(x_1, \dots, x_N) = \sum_{1a_1 + \dots + Na_N = N} C^S(a_1, \dots, a_N) x^{a_1} \dots x^{a_N}.$$

It is easy to see that a cycle index polynomial of  $A_N$ ,  $q_N(x_1, \dots, x_N)$  is given by

$$q_N(x_1, \dots, x_N) = p_N(x_1, \dots, x_N) + p_N(x_1, -x_2, \dots, (-1)^{N-1} x_N).$$

It follows that

$$(11) \quad C^A(\pi) = \begin{cases} 2C^S(\pi) & \text{if } a_2 + \dots + a_N \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

Now the condition that  $a_2 + \dots + a_N$  is even, is exactly the condition for a permutation to be in  $A_N$ , and, subject to this condition, we see that the distribution of cycles is given by exactly the same formula as in the case of the symmetric group. Consequently the proofs of the asymptotic Poisson-Dirichlet distribution for normalized cycle lengths in the symmetric group apply also in the case of alternating group. The proof of the corollary is completed by applying theorem 3 and triangle inequality.

We now turn to the proof of theorem 3. The basic tool is the following result, known as Diaconis-Shahshahani upper-bound lemma; see [19] for a recent survey of its applications and ramifications.

**Proposition 1** (Diaconis-Shahshahani [21]). *Let  $G$  be a finite group and denote by  $\hat{G}$  the set of irreducible unitary representations of  $G$ . Let  $P$  be a probability measure on  $G$  and denote by  $\hat{P}(\rho)$  its Fourier transform at the representation  $\rho \in \hat{G}$ . Let  $U$  be uniform probability measure on  $G$ . Then*

$$(12) \quad \|P - U\|^2 \leq \frac{1}{4} \sum_{\substack{\rho \in \hat{G} \\ \rho \neq \text{id}}} \dim(\rho) \text{tr} \left( \hat{P}(\rho) \overline{\hat{P}(\rho)} \right)$$

Here  $\|P - U\|$  is total variation distance defined in (9) and  $\text{id}$  denotes the trivial representation.

We apply proposition 1 to the situation where  $G = A_N$  and  $P = P_k * P_2$ . Since  $P_k$  is a class function,  $\hat{P}_k(\rho)$  is a scalar matrix,  $\hat{P}_k(\rho) =$

$\chi^\rho(C_k)I_{\dim(\rho)}$ , where  $\chi^\rho(C_k)$  is a character of the alternating group associated with representation  $\rho$  evaluated on the conjugacy class  $C_k$  and  $I_{\dim(\rho)}$  is an identity matrix of dimension  $\dim(\rho)$ . Now Fourier transform maps convolution of functions  $P_k * P_2$  into their product  $\widehat{P_k * P_2} = \widehat{P_k} \widehat{P_2}$  and consequently (12) implies

$$(13) \quad \|P_k * P_2 - U\|^2 \leq \frac{1}{4} \sum_{\substack{\rho \in \widehat{A_N} \\ \rho \neq \text{id}}} \left( \frac{\chi^\rho(C_k)\chi^\rho(C_2)}{\dim(\rho)} \right)^2.$$

The representation theory of the alternating group  $A_N$  is closely allied with the representation theory of the symmetric group  $S_N$  [28]. Representations of the symmetric group  $S_N$  are labelled by partitions  $\lambda \vdash N$ . A partition  $\lambda$  of a nonnegative integer  $N$  is a sequence  $(\lambda_1, \dots, \lambda_r) \in \mathbb{N}^r$  satisfying  $\lambda_1 \geq \dots \geq \lambda_r$  and  $\sum \lambda_i = N$ . We call  $|\lambda| = \sum \lambda_i$  the size of  $\lambda$ . The number of parts of  $\lambda$  is the length of  $\lambda$ , denoted  $l(\lambda)$ . Write  $m_i = m_i(\lambda)$  for the number of parts of  $\lambda$  that are equal to  $i$ , so we have  $\lambda = \langle 1^{m_1} 2^{m_2} \dots \rangle$ .

The Young diagram of a partition  $\lambda$  is defined as the set of points  $(i, j) \in \mathbb{Z}^2$  such that  $1 \leq i \leq \lambda_j$ ; it is often convenient to replace the set of points above by squares. The conjugate partition  $\lambda'$  of  $\lambda$  is defined by the condition that the Young diagram of  $\lambda'$  is the transpose of the Young diagram of  $\lambda$ ; equivalently  $m_i(\lambda') = \lambda_i - \lambda_{i+1}$ .

A standard Young tableau (SYT) of shape  $\lambda$  is a filling of the boxes of  $\lambda$  with positive integers  $1, \dots, N$  such that the rows and the columns are strictly increasing.

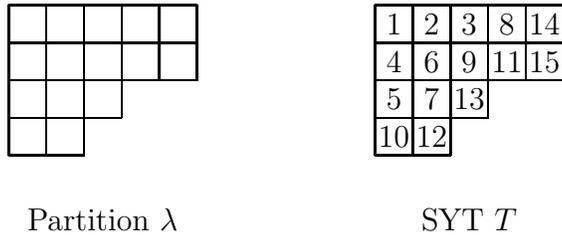


FIGURE 4

In the figure 4 we exhibited a partition  $\lambda = (5, 5, 3, 2) = \langle 1^0 2^1 3^1 5^2 \rangle$ , and a SYT  $T$  of shape  $\lambda$ .

We denote by  $f^\lambda$  the dimension of representation of  $S_N$  indexed by  $\lambda$ ; it is equal to the number of standard Young tableaux of shape  $\lambda$  [28]. We denote by  $\chi^\lambda(\mu)$  the value of the character indexed by  $\lambda$  on the conjugacy class  $C_\mu$ . For example, for the conjugacy class  $C_k$  the corresponding partition is  $\mu = k^n$ .

The partition  $\lambda$  is called self-associated if its Young diagram is symmetric with respect to the main diagonal, equivalently if  $\lambda = \lambda'$ ; partition displayed in figure 4 is an example. Now for an irreducible representation of  $S_N$  indexed by  $\lambda$  which is not self-associated, the restriction of  $\lambda$  to  $A_N$  is irreducible [28, p.67]. If  $\lambda$  is self-associated then restriction to  $A_N$  splits into two irreducible representations  $\lambda^+$  and  $\lambda^-$ . The character value of the irreducible representations  $\lambda^+$  and  $\lambda^-$  can be expressed in terms of character values of  $\chi^\lambda(\mu)$ ; for the conjugacy class  $\mu$  not equal to the set of main diagonal hooks of  $\lambda$  (see the definition of hook below) it is given by  $\frac{\chi^\lambda(\mu)}{2}$ . Consequently we obtain the following lemma:

**Lemma 2.** *Notation being as above we have*

$$(14) \quad \sum_{\substack{\rho \in \widehat{A_N} \\ \rho \neq \text{id}}} \left( \frac{\chi^\rho(C_k) \chi^\rho(C_2)}{\dim(\rho)} \right)^2 \leq 2 \sum_{\substack{\lambda \in \widehat{S_N} \\ \lambda \neq \langle N \rangle, \langle 1^N \rangle}} \left( \frac{\chi^\lambda(C_k) \chi^\lambda(C_2)}{f^\lambda} \right)^2.$$

**Proof:** Indeed, if  $\lambda$  is not a self-associated partition, we have  $\dim_{A_N}(\lambda) = f^\lambda$  and  $\chi_{A_N}^\lambda(\mu) = \chi^\lambda(\mu)$ . If  $\lambda$  is a self-associated partition, then since conjugacy class  $C_k$  is not an array of main diagonal hooks of  $\lambda$ , we have  $\dim_{A_N}(\lambda^\pm) = \frac{f^\lambda}{2}$  and  $\chi_{A_N}^{\lambda^\pm}(C_k) = \frac{\chi^\lambda(C_k)}{2}$ . This completes the proof of Lemma 2.

We thus have to analyze the sum

$$(15) \quad \sum_{\substack{\lambda \vdash N \\ \lambda \neq \langle N \rangle, \langle 1^N \rangle}} \left( \frac{\chi^\lambda(C_k) \chi^\lambda(C_2)}{f^\lambda} \right)^2.$$

We briefly recall the basic facts pertaining to  $f^\lambda$  and  $\chi^\lambda(\mu)$  referring to [28] for more details. Given a diagram  $\lambda$  and a square  $u = (i, j) \in \lambda$ , a hook with a vertex  $u$  is a set of squares in  $\lambda$  directly to the right or directly below  $u$ . We define hook-length (also referred to as hook number)  $h(u)$  of  $\lambda$  at  $u$  by

$$h(u) = \lambda_i + \lambda'_j - i - j - 1.$$

Equivalently,  $h(u)$  is the number of squares directly to the right or directly below  $u$ , counting  $u$  itself once. For instance in figure 5 we display hook lengths for partition  $\lambda = (5, 5, 3, 2)$ .

**Theorem 4** (Hook-length formula). *Notation being as above, we have*

$$(16) \quad f^\lambda = \frac{N!}{\prod_{u \in \lambda} h(u)}.$$

8	7	5	3	2
7	6	4	2	1
4	3	1		
2	1			

FIGURE 5

Projecting a hook with a vertex  $u$  onto the boundary (rim) of  $\lambda$  yields a rim hook. By definition a rim hook is a connected skew shape with no  $2 \times 2$  square. The height of rim hook,  $\text{ht}(R)$  is defined to be one less than its number of rows.

A rim hook tableaux  $T$  of shape  $\lambda$  and type  $\mu$  where  $\mu = (\mu_1, \dots, \mu_n)$  is an assignment of positive integers to the squares of  $\lambda$  such that every row and column is weakly increasing; the integer  $i$  appears  $\mu_i$  times and the set of squares occupied by  $i$  forms a rim hook. The height of rim hook tableaux is defined to be the sum of the height of rim-hooks appearing in  $T$ .

In the figure 6 we exhibited a rim hook tableaux of shape  $\lambda = (5, 5, 3, 2)$  and type  $\mu = (4, 4, 4, 3)$ .

1	1	3	3	4
1	2	3	4	4
1	2	3		
2	2			

FIGURE 6

In a particular case when all the parts of  $\mu$  have size  $k$  the tableaux described above is referred to as a  $k$ -rim hook tableaux; we denote the number of  $k$ -rim hook tableaux by  $f_k^\lambda$ . In particular  $f_1^\lambda = f^\lambda$  is the number of standard Young tableaux.

**Theorem 5** (Murnaghan-Nakayma Rule). *Notation being as above, we have*

$$(17) \quad \chi^\lambda(\mu) = \sum_T (-1)^{\text{ht}(T)},$$

where the sum is over all rim hook tableaux of shape  $\lambda$  and type  $\mu$ .

For  $\mu = k^n$  all signs in the formula (17) are the same [28, p.82]; consequently

$$(18) \quad |\chi^\lambda(C_k)| = f_k^\lambda.$$

In the proof of Theorem 3 we will make crucial use of the following estimate for  $f_k^\lambda$ :

**Theorem 6** (Fomin-Lulov [22]). *Suppose  $N = kn$ . Then*

$$(19) \quad f_k^\lambda \leq \frac{n!k^n}{n!^{1/k}} (f^\lambda)^{\frac{1}{k}} = O\left(N^{\frac{1}{2} - \frac{1}{2k}}\right) (f^\lambda)^{\frac{1}{k}},$$

where the implied constant depends only on  $k$  and not on  $\lambda$ .

Another crucial ingredient in the proof of Theorem 3 is furnished by the following proposition.

**Proposition 2.** *For any  $t > 0$  we have*

$$(20) \quad \sum_{\substack{\lambda \vdash N \\ \lambda_1, \lambda'_1 \leq N-m}} (f^\lambda)^{-t} = O(N^{-mt}),$$

where implied constant depends only on  $m$ .

We remark that the sums of the form  $\sum_{\lambda \vdash N} (f^\lambda)^\beta$  for  $\beta > 0$  have been studied by Vershik and Kerov [52] and by Regev [43]. In particular, Regev relates the asymptotic computations of such sums to the matrix integral of the form

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \prod_{i < j} |x_i - x_j|^\beta e^{-(\beta/2)(x_1^2 + \dots + x_N^2)} dx_1 \dots dx_N,$$

this being perhaps the first hint of the deep connection between random matrices and random permutations; see [39] and references therein for a recent survey.

**Proof of Proposition 2:** First of all we observe that since  $f^\lambda = f^{\lambda'}$ , it suffices to prove proposition 2 for the sum

$$\sum_{\substack{\lambda \vdash N \\ \lambda'_1 < \lambda_1 \leq N-m}} (f^\lambda)^{-t}.$$

Now we split this sum into three parts:

$$(21) \quad \Sigma_1 = \sum_{\substack{\lambda \vdash N \\ \lambda'_1 < \lambda_1 \\ \frac{3N}{4} < \lambda_1 \leq N-m}} (f^\lambda)^{-t};$$

$$(22) \quad \Sigma_2 = \sum_{\substack{\lambda \vdash N \\ \lambda'_1 < \lambda_1 \\ \frac{N}{8} < \lambda_1 \leq \frac{3N}{4}}} (f^\lambda)^{-t};$$

and

$$(23) \quad \Sigma_3 = \sum_{\substack{\lambda_1 - N \\ \lambda_1 < \lambda_1 \\ \lambda_1 \leq \frac{N}{8}}} (f^\lambda)^{-t}.$$

To analyze  $\Sigma_1$  we first note that

$$(24) \quad f^\lambda \geq \binom{\lambda_1}{N - \lambda_1} \quad \text{if} \quad \lambda_1 > \frac{N}{2}.$$

Indeed, since the product of hook numbers for a partition of size  $l$  is at most  $l!$ , the product of hook numbers in the rows below the top one is at most  $(N - \lambda_1)!$ . The set of hook numbers in the top row consist of distinct numbers not exceeding  $N$ . Since the length of the second row is at most  $r = N - \lambda_1$ , the hook numbers in the leftmost  $\lambda_1 - r$  cells of the first row are  $1, \dots, 2\lambda_1 - N$ . The product of the remaining  $r$  hook numbers in the first row is no greater than  $N(N - 1) \dots (N - r) = \frac{N!}{\lambda_1!}$ . Now application of hook-length formula (16) completes the proof of the estimate (24).

Continuing to denote  $r = N - \lambda_1$ , we can now estimate  $\Sigma_1$  using (24):

$$(25) \quad \Sigma_1 \leq \sum_{m \leq r \leq \frac{N}{4}} \frac{p(r)}{\binom{N-r}{r}^t},$$

where  $p(r)$  is the number of partitions of  $r$ . Since the number of partitions  $p(r)$  satisfies the following inequality [2] valid for all  $r \geq 1$

$$(26) \quad p(r) \leq \exp(\pi\sqrt{2r/3}),$$

we have

$$(27) \quad \Sigma_1 \leq \sum_{m \leq r \leq \frac{N}{4}} \frac{c_1^{\sqrt{r}}}{\binom{N-r}{r}^t}$$

for absolute constant  $c_1 = e^{\pi\sqrt{2/3}}$ . Let  $a_r = \frac{1}{\binom{N-r}{r}}$ . We have

$$a_{r+1} = \frac{(N-r)(r+1)}{(N-2r)(N-2r-1)} a_r.$$

For large  $N$ , as  $r$  increases from  $m$  to  $\frac{N}{4}$  the function  $\frac{(N-r)(r+1)}{(N-2r)(N-2r-1)}$  monotonically decreases from roughly  $\frac{1}{N}$  to  $\frac{3}{4}$ . Consequently

$$\Sigma_1 \leq \left( \frac{1}{(N-m) \dots (N-2m)} \right)^t \sum_{m \leq r \leq \frac{N}{4}} \frac{c_1^{\sqrt{r}}}{c_2^r},$$

where  $c_2 = \left(\frac{4}{3}\right)^t > 1$ . Now

$$\sum_{m \leq r \leq \frac{N}{4}} \frac{c_1^{\sqrt{r}}}{c_2^r} < \sum_{m \leq r \leq \infty} \frac{c_1^{\sqrt{r}}}{c_2^r},$$

which converges by Cauchy criterion and consequently we obtain the desired estimate  $\Sigma_1 = O(N^{-mt})$ .

To analyze  $\Sigma_2$  we first note that

$$(28) \quad f^\lambda \geq \frac{(17N/16 - \lambda_1)!}{(N - \lambda_1)!(N/16 + 16)!} \quad \text{if} \quad \lambda_1 \geq \frac{N}{8}.$$

The proof of this estimate is similar to the proof of the estimate (24). The product of hook numbers outside of the first row is at most  $(N - \lambda_1)!$ . Suppose there is a total of  $v$  rows. Then

$$\lambda_2 + \dots + \lambda_v < \frac{7N}{8}.$$

Consequently

$$\left(\lambda_2 - \frac{N}{16}\right) + \dots + \left(\lambda_{15} - \frac{N}{16}\right) < 0,$$

therefore the rows below the 16th one are of size at most  $\frac{N}{16}$ . That means that the product of hook numbers in the  $\frac{N}{16}$  leftmost boxes of the first row is no greater than  $(\frac{N}{16} + 16)!$ . The remaining hook lengths in the first row constitute  $\lambda_1 - \frac{N}{16}$  distinct numbers less than or equal to  $N$ ; their product is therefore does not exceed  $\frac{N!}{(N + N/16 - \lambda_1)!}$ . Now application of hook-length formula (16) completes the proof of the estimate (28).

Now as  $\lambda_1$  increases from  $\frac{N}{4}$  to  $\frac{3N}{4}$  the expression on the right-hand side of (28) decreases; therefore in the whole range of the sum  $\Sigma_2$  we have

$$f^\lambda \geq \frac{(17N/16 - 3N/4)!}{(N - 3N/4)!(N/16 + 16)!} > \frac{(5N/16)!}{(N/4)!(N/16)!},$$

and the latter expression is greater than  $5^{N/16}$  for sufficiently large  $N$ . Consequently

$$\Sigma_2 \leq p(N)5^{-tN/16} \leq c_1^{\sqrt{N}} c_3^{-N} = O(c_4^{-N})$$

for  $c_3 = 5^{t/16}$  and some  $c_4 > 1$ .

Turning now to the sum  $\Sigma_3$  we note that

$$(29) \quad f^\lambda \geq \left(\frac{4}{e}\right)^N \quad \text{if} \quad \lambda'_1 \leq \lambda_1 < \frac{N}{8}.$$

Indeed for any box  $u$  in  $\lambda$  we have  $h(u) \leq \lambda_1 + \lambda'_1 < \frac{N}{4}$ . Consequently using hook-length formula (16) we obtain

$$f^\lambda > \frac{N!}{(N/4)^N} > \frac{(N/e)^N}{(N/4)^N} > \left(\frac{4}{e}\right)^N,$$

proving the estimate (29). Consequently

$$\Sigma_3 \leq c_1^{\sqrt{N}} c_5^{-N} = O(c_6^{-N})$$

for  $c_5 = \left(\frac{4}{e}\right)^t$  and some  $c_6 > 1$ .

This completes the proof of Proposition 2.

Now the Theorem 3 is proved by combining inequality (13) with Lemma 2, Theorem 6, and Proposition 2 with  $m = 4$ , together with the values of the remaining characters and dimensions, which are computed using Theorem 4 and Theorem 5 and summarized in Table 1. We denote by  $\Lambda$  the set of partitions in the first column of Table 1.

$$\begin{aligned} \|P_k * P_2 - U\|^2 &\leq \frac{1}{4} \sum_{\substack{\rho \in \widehat{A}_N \\ \rho \neq \text{id}}} \left( \frac{\chi^\rho(C_k) \chi^\rho(C_2)}{\dim(\rho)} \right)^2 \quad \text{by inequality (13)} \\ &\leq \frac{1}{2} \sum_{\substack{\lambda \in \widehat{S}_N \\ \lambda \neq \langle N \rangle, \langle 1^N \rangle}} \left( \frac{\chi^\lambda(C_k) \chi^\lambda(C_2)}{f^\lambda} \right)^2 \quad \text{by Lemma 2} \\ &= \frac{1}{2} \sum_{\substack{\lambda \vdash N \\ \lambda_1, \lambda'_1 \leq N-4}} \left( \frac{\chi^\lambda(C_k) \chi^\lambda(C_2)}{f^\lambda} \right)^2 + \sum_{\substack{\lambda \vdash N \\ \lambda \in \Lambda}} \left( \frac{\chi^\lambda(C_k) \chi^\lambda(C_2)}{f^\lambda} \right)^2 \\ &= O(N^{\frac{3}{2} - \frac{1}{k}}) \sum_{\substack{\lambda \vdash N \\ \lambda_1, \lambda'_1 \leq N-4}} (f^\lambda)^{-(1 - \frac{2}{k})} + O(N^{-2}) \quad \text{by Theorem 6 and Table 1} \\ &= O(N^{\frac{5}{2} - \frac{7}{k}}) + O(N^{-2}) \quad \text{by Proposition 2;} \end{aligned}$$

the first term in the last expression is at most  $O(N^{-\frac{1}{6}})$  since  $k \geq 3$ . This completes the proof of Theorem 3.

$\lambda$	$f^\lambda$	$ \chi^\lambda(C_2) $	$ \chi^\lambda(C_3) $	$ \chi^\lambda(C_4) $	$ \chi^\lambda(C_k) $ $k \geq 5$
$(N-1, 1)$	$N-1$	1	1	1	1
$(N-2, 2)$	$\frac{N(N-3)}{2}$	$\frac{N}{2}$	0	1	1
$(N-2, 1, 1)$	$\frac{(N-1)(N-2)}{2}$	$\frac{N}{2} + 1$	1	1	1
$(N-3, 2, 1)$	$\frac{N(N-2)(N-4)}{3}$	0	$\frac{N}{3} + 1$	0	1
$(N-3, 1, 1, 1)$	$\frac{(N-1)(N-2)(N-3)}{3}$	$\frac{N}{2} + 1$	$\frac{N}{3} - 1$	1	1
$(N-3, 3)$	$\frac{(N)(N-1)(N-5)}{6}$	$\frac{N}{2} + 2$	$\frac{N}{3} + 1$	0	$\begin{cases} 0 & \text{if } k = 5; \\ 1 & \text{if } k > 5 \end{cases}$

TABLE 1. Character and dimension values

## 5. CONCLUDING REMARKS

Extensive information is available on the distribution of cycles in random permutations and on Poisson-Dirichlet distribution [3] and we will pursue the exhaustive exploitation of the consequences of Theorem 3 in a subsequent publication. Here we note just two immediate corollaries which substantially refine results in [14], [26], and [41].

**Corollary 2.** *Let  $l(n)$  denote the number of LHT paths in a random cubic graph on  $n$  vertices with random orientation. Then as  $n \rightarrow \infty$*

$$\mathbb{E}(l(n)) = \log(3n) + \gamma + O(n^{-1/6}),$$

$$\text{Var}(l(n)) = \log(3n) + \gamma - \pi^2/6 + O(n^{-1/6}),$$

where  $\gamma = 0.5772\dots$  is an Euler constant. Further,  $\frac{l(n) - \log n}{\sqrt{\log n}}$  converges to standard normal distribution  $\mathcal{N}(0, 1)$ . By formula (1) we obtain that the genus is distributed as  $1 + \frac{n}{4} - \mathcal{N}(\log n, \sqrt{\log n})$ .

**Corollary 3.** *Let  $L(n)$  be the length of the largest LHT path. Then*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left( \frac{L(n)}{n} \right) = \int_0^\infty \exp \left( -x - \int_x^\infty (e^{-y}/y) dy \right) dx \sim 0.6243.$$

Consequently the expected area of the largest embedded ball in  $S^C(\Gamma, \mathcal{O})$  converges to  $\frac{0.62}{2\pi}$  of the total surface area.

In fact, the limiting distribution of  $L$  is also known, but we do not pursue it here.

Recent numerical experiments of T. Novikoff [38] present convincing evidence in favor of the following conjecture.

**Conjecture 1.** The distribution of the second largest eigenvalue of the adjacency matrix of random regular graph, suitably rescaled, follows Tracy-Widom GOE distribution.

Tracy and Widom [49, 50] computed the limiting distribution function for the largest eigenvalue in the classical Gaussian ensembles; these distributions functions are expressed in terms of certain Painlevé II function and are now believed to describe new universal limit laws for a wide variety of process arising in mathematical physics and interacting particle systems [51].

A dramatic consequence of Conjecture 1 would be that the probability of random regular graph being Ramanujan approaches 0.52 as the size of the graph tends to infinity, corresponding to the skewness in the Tracy-Widom GOE distribution.

To approach Conjecture 1 following the method Sinai and Soshnikov [47] and Soshnikov [48] in his breakthrough proof of the universality at the edge of the spectrum in Wigner matrices, one needs precise information for the number of closed walks of size up to  $n^{2/3}$ , where  $n$  is the size of the graph. We hope that the results of this paper would be useful in pursuing such an approach.

#### REFERENCES

- [1] N. Alon, *Eigenvalues and expanders*, *Combinatorica*, **6**, 1986, 83-96.
- [2] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, Berlin, 1976.
- [3] R. Arratia, A. D. Barbour, S. Tavaré, *Logarithmic combinatorial structures: a probabilistic approach*, EMS Monographs in Mathematics **1**, 2003.
- [4] G. V. Belyi, *Galois extensions of a maximal cyclotomic field*, *Izv. Akad. Nauk SSSR Ser. Mat.* **43**, 1979, 267-276.
- [5] B. Bollobás, *A probabilistic proof of the asymptotic formula for the number of labelled regular graphs*, *Europ. Journal of Combinatorics*, **1**, 1980, 311-316.
- [6] B. Bollobás, *Random graphs*, Academic Press Inc., London, 1985.
- [7] B. Bollobás, *The isoperimetric number of random regular graphs*, *European J. Combin.* **9** (1988), no. 3, 241-244.
- [8] R. Brooks, *The spectral geometry of a tower of coverings*, *J. Diff. Geom.*, **23**, 1986, 97-107.
- [9] R. Brooks, *Some remarks on volume and diameter of Riemannian manifolds*, *J. Diff. Geom.*, **27**, 1988, 81-86.
- [10] R. Brooks, *Platonic Surfaces*, *Comment. Math. Helv.*, **74**, 1999, 156-170.
- [11] R. Brooks, *Some geometric aspects of the work of Lars Ahlfors*, *IMCP*, **14**, 2000, 31-39.
- [12] R. Brooks and E. Makover, *Riemann surfaces with large first eigenvalue*, *J. Anal. Math.*, **83**, 2001, 243-258.
- [13] R. Brooks and E. Makover, *Belyi surfaces*, *IMCP*, **15**, 2001, 37-46.
- [14] R. Brooks and E. Makover, *Random Construction of Riemann Surfaces*, [math.DG/0106251](https://arxiv.org/abs/math/0106251).
- [15] P. Buser, *Cubic graphs and the first eigenvalue of a Riemann surface*, *Math. Z.* **162**, 1978, 87-99.
- [16] P. Buser, *On the bipartition of graphs*, *Discrete Appl. Math.* **9**, 1984, 105-109.

- [17] P. Buser, *On Cheeger's inequality  $\lambda_1 \geq h^2/4$* , Proc. Symp. Pure. Math. **36**, 1980, 29-77.
- [18] P. Buser, *Geometry and Spectra of Compact Riemann Surfaces*, Progress in Mathematics, **106**, Birkhäuser, Boston, 1992.
- [19] P. Diaconis, *Random walks on groups: characters and geometry*, preprint.
- [20] P. Diaconis, E. Mayer-Wolf, O. Zeitouni, M. Zerner, *The Poisson Dirichlet law is the unique invariant distribution for uniform split-merge transformations*, Ann. Probab. **32**, 2004, 915-938.
- [21] P. Diaconis and M. Shahshahani, *Generating a random permutation with random transpositions*, Z. Wahrscheinlichkeitstheorie verw. Gebiete, **57**, 1981, 159-179.
- [22] S. V. Fomin and N. Lulov, *On the number of rim hook tableaux*, Journal of Mathematical Sciences, **87**, 1997, 4118-4123.
- [23] J. Friedman, *On the second eigenvalue and random walk in random  $d$ -regular graphs*, Combinatorica, **11**, 1991, 331-362.
- [24] J. Friedman, *A proof of Alon's second eigenvalue conjecture*, to appear in the Memoirs of the A.M.S.
- [25] A. Gamburd, *Spectral gap for infinite index "congruence" subgroups of  $SL_2(\mathbb{Z})$* , Israel Journal of Mathematics, **127**, 2002, 157-200.
- [26] A. Gamburd and E. Makover, *On the genus of a random Riemann surface*, Contemporary Mathematics, **311**, 2002, 133-140.
- [27] M. Huxley, *Eigenvalues of congruence subgroups*, Contemp. Math., vol. **53**, (1986), 341-349.
- [28] G. James and A. Kerber, *The representation theory of the symmetric group*, Addison-Wesley Publishing Company, Reading, MA, 1981.
- [29] G. Jones and S. Singerman, *Belyi functions, hypermaps and Galois groups*, Bull. London Math. Soc., **28**, 1996, 561-590.
- [30] H. Kim and P. Sarnak, *Refined estimates towards the Ramanujan and Selberg conjectures*, JAMS, **16**, 1 (2003), 139-183.
- [31] M. E. Liebeck and A. Shalev, *Fuchsian groups, coverings of Riemann surfaces, subgroups growth, random quotients and random walks*, preprint.
- [32] A. Lubotzky, *Discrete Groups, Expanding Graphs and Invariant Measures*, Progress in Mathematics Vol. 195, Birkhäuser, 1994.
- [33] A. Lubotzky, R. Phillips, P. Sarnak, *Ramanujan graphs*, Combinatorica **8** 1988, 261-277.
- [34] W. Luo, Z. Rudnick and P. Sarnak, *On Selberg's eigenvalue conjecture*, GAFA **5** (1995), 387-401.
- [35] B. McKay, *The expected eigenvalue distribution of a large regular graph*, Linear Algebra and its Applications **40**, (1981) 203-216.
- [36] N. Lulov and I. Pak, *Rapidly mixing random walks and bounds on characters of the symmetric group*, Journal of Algebraic Combinatorics, **16**, 2002, 151-163.
- [37] M. Mulase and M. Penkava, *Ribbon graphs, quadratic differentials on Riemann surfaces, and algebraic curves defined over  $\bar{\mathbb{Q}}$* , Asian J. Math., **2**, 1998, 875-919.
- [38] T. Novikoff, *Asymptotic behavior of the random 3-regular bipartite graph*, preprint.
- [39] A. Okounkov, *Symmetric functions and random partitions*, in Symmetric Functions 2001: Surveys of Developments and Perspectives, edited by S. Fomin, Kluwer, 2002.

- [40] M. Pinsky, *On the complexity of concentrator*, 7th Annual Teletraffic Conference, 318/1-318/4, Stockholm, 1973.
- [41] N. Pippenger and K. Schleich, *Topological characteristics of random surfaces generated by cubic interactions*, [gr-qc/0306049](#).
- [42] B. Randol, *Small eigenvalues of the Laplace operator on compact Riemann surfaces*, Bull. Amer. Math. Soc. **80** (1974), 996-1008.
- [43] A. Regev, *Asymptotic values for degrees associated with strips of Young diagrams*, Adv. in Math., **41**, 1981, 115-136.
- [44] P. Sarnak, *What is an expander?*, Notices of the American Mathematical Society, **51**, 2004, 762-763.
- [45] P. Sarnak and X. Xue, *Bounds for multiplicities of automorphic representations*, Duke Math. J. **64**, (1991), 207-227.
- [46] A. Selberg, *On the estimation of Fourier coefficients of modular forms*, Proc. Symp. Pure Math. **VII**, Amer. Math. Soc.(1965), 1-15.
- [47] Ya. Sinai and A. Soshnikov, *A refinement of Wigner's semicircle law in a neighborhood of the spectrum edge for random symmetric matrices*, Functional Analysis and its Applications, **32**, 1998, 114-131.
- [48] A. Soshnikov, *Universality at the edge of the spectrum in Wigner random matrices*, Commun. Math. Phys. **207**, 1999, 697-733.
- [49] C. A. Tracy and H. Widom, *Level-spacing distribution and the Airy kernel*, Commun. Math. Phys. **159**, 1994, 151-174.
- [50] C. A. Tracy and H. Widom, *On orthogonal and symplectic matrix ensembles*, Commun. Math. Phys. **177**, 1996, 727-754.
- [51] C. A. Tracy and H. Widom, *Distribution functions for largest eigenvalues and their applications*, Proceedings of the International Congress of Mathematicians, Beijing 2002, Vol. I, ed. LI Tatsien, Higher Education Press, Beijing 2002, 587-596.
- [52] A. Vershik and S. Kerov, *Asymptotics of the largest and typical dimensions of irreducible representations of symmetric group*, Functional. Anal. Appl. **19** (1) (1985), 21-31.
- [53] N. C. Wormald, *Models of random regular graphs*, Survey in Combinatorics 1999, 239-298, Cambridge University Press, 1999.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA CRUZ  
AND DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CA  
94305

*E-mail address:* [agamburd@math.stanford.edu](mailto:agamburd@math.stanford.edu)