An Algebraic Approach To Rectangle Packing Problems

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Abstract

A method for converting the geometrical problem of rectangle packing to an algebraic problem of solving a system of polynomial equations is described.

1 Introduction

There are many interesting infinite rectangle packing problems which have been studied. Some of them are: Packing of rectangles with side lengths $(1/n, 1/(n+1))$ where $n = 1, 2, ...$ into the unit square, or rectangles of sides $1/n$ where $n = 1, 2, ...$ into a rectangle of area $\pi^2/6$ and so on [2, 3]. There are many improvements [1, 4] for these problems but some of them are still open. In this paper, we describe a method to transform a general type of rectangle packing problem into a system of polynomial equations.

2 Equivalence of Two Problems

Let us say we have a set of rectangles (finite or infinite) that we want to pack into a box (a bigger rectangle) of sides (A, B) which we put onto a coordinate frame where its left-bottom corner is on the origin. Let us denote the coordinates of the corners of n^{th} rectangle with $\{(x_n^-, y_n^-), (x_n^+, y_n^-), (x_n^-, y_n^+), (x_n^+, y_n^+)\}$ as in Figure 1.

If we have a perfect packing then we must have the following equality:

$$\sum_n \int_{y_n^-}^{y_n^+} \int_{x_n^-}^{x_n^+} f(x,y)dxdy = \int_0^B \int_0^A f(x,y)dxdy \tag{1}$$

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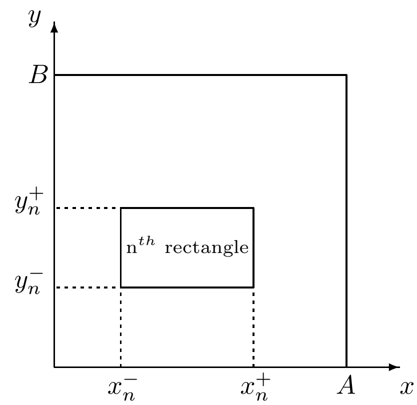

Figure 1: Coordinates of the corners of the n^{th} rectangle.

for any function $f(x, y)$, because a perfect packing is simply a covering of the box. This is a necessity condition for these rectangles to perfectly fit, but it is not obvious whether it is sufficient or not.

This necessity equation gives us some interesting equalities for some well known infinite packing problems. For example the problem of packing the rectangles with sides $(1/n, 1/(n+1))$ with $n = 1, 2, ...$ into the unit square. For this problem, if we let $f(x, y)$ to be linear and quadratic functions of x and y and define $x_n = (x_n^+ + x_n^-)/2$, $y_n = (y_n^+ + y_n^-)/2$ (center of mass coordinates) then we get these following relations:

$$\begin{aligned}
\sum_{n\geq 1} \frac{x_n}{n(n+1)} &= \frac{1}{2}, \quad (2)\\
\sum_{n\geq 1} \frac{y_n}{n(n+1)} &= \frac{1}{2}, \quad (3)\\
\sum_{n\geq 1} \frac{x_n y_n}{n(n+1)} &= \frac{1}{4}, \quad (4)\\
\sum_{n\geq 1} \frac{x_n^2 + y_n^2}{n(n+1)} &= \frac{1}{3} + \frac{\pi^2}{36}, \quad (5)\\
\sum_{n\geq 1} \frac{(x_n + y_n)^2}{n(n+1)} &= \frac{5}{6} + \frac{\pi^2}{36}, \quad (6)\\
\sum_{n\geq 1} \frac{(x_n - y_n)^2}{n(n+1)} &= \frac{\pi^2}{36} - \frac{1}{6}. \quad (7)
\end{aligned}$$

So a perfect packing must satisfy these equalities. Now by setting $f(x, y)$

more cleverly we can get a sufficient condition for these rectangles to perfectly pack.

Theorem 1 *Rectangles of sides $(w(n), l(n))$, where $n = 1, 2, ..,$ can perfectly pack a rectangle of sides (A, B) if and only if the following system of polynomial equations:*

$$\sum_n \Big\{(x_n^+)^{S_1} - (x_n^-)^{S_1}\Big\}\Big\{(y_n^+)^{S_2} - (y_n^-)^{S_2}\Big\} = A^{S_1} B^{S_2} \qquad S_1, S_2 = 1, 2, ...$$

has a solution with the following constraints:

$$\begin{aligned} \Delta x_n + \Delta y_n &= w(n) + l(n), \\ \Delta x_n \Delta y_n &= w(n) l(n) \end{aligned}$$

where $\Delta x_n = x_n^+ - x_n^-$ and $\Delta y_n = y_n^+ - y_n^-$.

Proof 1 *In the first equality, let $f(x, y) = e^{px+qy}$, so we get:*

$$\sum_n \int_{y_n^-}^{y_n^+} \int_{x_n^-}^{x_n^+} e^{px+qy} dx dy = \int_0^B \int_0^A e^{px+qy} dx dy \qquad \forall p, q \in \mathbb{R}. \tag{8}$$

If there exist a perfect packing then we must have the equality above. Now multiply both sides with $e^{-(pa+qb)}$ where $a, b \in \mathbb{R}$, replace p & q with ip & iq and integrate with respect to p and q from $-\infty$ to ∞ to get:

$$\sum_n \int_{y_n^-}^{y_n^+} \int_{x_n^-}^{x_n^+} \delta(x-a)\delta(y-b) dx dy = \int_0^B \int_0^A \delta(x-a)\delta(y-b) dx dy. \tag{9}$$

One can easily see that the right hand side is equal to 1 if the point (a, b) is inside the box and 0 if it is outside. But each of the integrals inside the summation on the left hand side can only give 0 or 1, so if the right hand side is 0 then none of the rectangles can contain the point (a, b) which means that all the rectangles are inside the box and if it is 1 then only one rectangle may contain the point (a, b) which means that the rectangles do not overlap. This is true for all points (a, b), hence we must have a perfect packing.

Now if we take the integrals in (8) and cancel some common terms we get:

$$\sum_n \Big\{ e^{px_n^+ + qy_n^+} - e^{px_n^+ + qy_n^-} - e^{px_n^- + qy_n^+} + e^{px_n^- + qy_n^-} \Big\} = e^{pA+qB} - e^{pA} - e^{qB} + 1. \tag{10}$$

Let's expand this into a series with respect to p and q to get:

$$\sum_n \sum_{k \geq 0} \frac{1}{k!} \Big\{ (px_n^+ + qy_n^+)^k - (px_n^+ + qy_n^-)^k - (px_n^- + qy_n^+)^k + (px_n^- + qy_n^-)^k \Big\} = \sum_{k \geq 0} \frac{1}{k!} \Big\{ (pA + qB)^k - (pA)^k - (qB)^k \Big\} + 1, \tag{11}$$

and this is equal to:

$$\sum_n \sum_{k\geq 0} \sum_{r=0}^{k} \frac{1}{k!}\binom{k}{r} p^r q^{k-r} \Big\{ (x_n^+)^r (y_n^+)^{k-r} - (x_n^+)^r (y_n^-)^{k-r} - (x_n^-)^r (y_n^+)^{k-r} + (x_n^-)^r (y_n^-)^{k-r} \Big\}$$

$$= \sum_{k\geq 0} \frac{1}{k!} \Big\{ \sum_{r=0}^{k} \binom{k}{r} p^r q^{k-r} A^r B^{k-r} - (pA)^k - (qB)^k \Big\} + 1. \quad (12)$$

Now, $k = 0$ term cancel the $+1$ 0n the right hand side, and $r = 0$ and $r = k$ terms cancel out from both sides. If we collect similar terms we get the following equation:

$$\sum_n \sum_{k\geq 2} \sum_{r=1}^{k-1} \frac{1}{k!}\binom{k}{r} \; p^r q^{k-r} \Big\{ (x_n^+)^r - (x_n^-)^r \Big\} \Big\{ (y_n^+)^{k-r} - (y_n^-)^{k-r} \Big\}$$

$$= \sum_{k\geq 2} \sum_{r=1}^{k-1} \frac{1}{k!}\binom{k}{r} p^r q^{k-r} A^r B^{k-r}. \quad (13)$$

Here p and q are continuous variables, so the above equality holds if and only if for both sides of the equality the coefficients of the product $p^{S_1} q^{S_2}$ are equal for all integers $S_1, S_2 \geq 1$. Hence at the end we see that we have these equations:

$$\sum_n \Big\{ (x_n^+)^{S_1} - (x_n^-)^{S_1} \Big\} \Big\{ (y_n^+)^{S_2} - (y_n^-)^{S_2} \Big\} = A^{S_1} B^{S_2} \qquad S_1, S_2 = 1, 2, ... \quad (14)$$

Without the constraints the above equation give all possible covering of the box, one must use the constraints to obtain the desired covering. Hence the above equations with the constraints are equivalent to the packing problem.

□

This equation has also a very simple geometrical meaning. Observe that when we take the product inside the summation we get 4 terms corresponding to 4 corners of the rectangles. The left-bottom and right-top corners come with a plus sign and left-top and right-bottom corners come with a minus sign. For a perfect packing, rectangles are in contact and for each corner except the corners of the box there exist another corner which is on the same point. The terms corresponding to these corners come with opposite signs and cancel each other. Thus the only remaining terms may come from the corners of the box, and 3 of them are 0 because of the corresponding x and y coordinates, and it remains only the term coming from the right-top corner and this one is exactly equal to right hand side of the above equation.

3 Discussion

By using numerical methods, one can solve for a given set of rectangles their positions corresponding to a perfect packing (if it exists of course, otherwise we get an empty set) by using a finite set of equations. This is not an easy task in general but it can be done as one can try. This numerical solution is not efficient and impossible for an infinite set of rectangles but the point is that one can show that a polynomial equation has a root without explicitly finding it. So we may show that there exist a perfect packing by showing that these polynomials have a zero (or vice-versa). A multi-dimensional version of the intermediate value theorem can be used for this purpose.

References

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